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GENERAL CONSTRUCTIONS OF NORMAL NUMBERS OF KOROBOV TYPE

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§1. Introduction

Let $b\geqq 2$ be an integer given arbitrarily, and let $x=[x].a_1a_2\cdots=[x]+a_1b^{-1}+a_2b^{-2}+\cdots$, $a_i\in\{0,1,\cdots,b-1\}$ be the b-adic exapnsion of a real number x, where $[x]$ is the integral part of x. For any block $d_1\cdots d_s$ of ($b-$adic) digits $d_1,\cdots,d_s\in\{0,1,\cdots,b-1\}$ of length S, $A(x,b,d_1\cdots d_s;N)$ denotes the number of $n(1\leqq n\leqq N)$ such that $a_{n+i-1}=d_i$ $(i=1,\cdots,s)$. Then x is said to be normal to base b if $\lim_{N\to\infty}A(x,b,d_1\cdots d_s;N)/N=b^{-s}$ for any s and any block $d_1\cdots d_s$. Normality can be defined also in terms of uniform distirbution. For a set $E\subset[0,1)$ with N elements we define the discrepancy $D(E)$ as

$$D(E)=\sup_{0\leqq u<v\leqq 1}|\#(E\cap[u,v))-N(v-u)|.$$

Further we write

$$D(N)=D(\{\{x\},\{xb\},\cdots,\{xb^{n-1}\}\})$$

and

$$D(N,H)=D(\{\{xb^N\},\{xb^{N+1}\},\cdots,\{xb^{N+H-1}\}\}),$$

where $\{x\}=x-[x]$. Then we note that

$$|A(x,b,d_1\cdots d_s:N)-Nb^{-s}|\leqq D(N),$$

and that x is normal if and only if $D(N)=o(N)$ (cf. [6]). $\tau_b(a)$ denotes the order of an integer b mod a for any integers a and b with $(a,b)=1$.

It is known that almost all real numbers are normal, however only few methods have been known to generate normal numbers. Among them, we mention arithmetic constructions of Stoneham and of Korobov. Historical surveys for another type of constructions of normal numbers can be found in [6], [8], and [9]. Stoneham ([10] Theorem 1) found the following normal numbers: Let a,b be relatively prime integers greater than 1, and let $\{Z_n\}_{n\geqq 1}$ and $\{\alpha_n\}_{n\geqq 1}$ be sequences of positive integers with $Z_n<a^n$, $(Z_n,a)=1$, and $\lim_{n\to\infty}\alpha_n=\infty$. Assume

that $\alpha_0=Z_0=0$, and put $S(n, a)=\sum_{i=1}^{n} \alpha_i \tau_b(a^i)$ with $S(0, a)=0$. Then the number

$$\sum_{n=1}^{\infty} \frac{Z_n-aZ_{n-1}}{a^n b^{s(n-1,a)}} \tag{1}$$

is normal to base b. In particular

$$(a-1)\sum_{n=1}^{\infty} \frac{1}{a^n b^{\{(n-1)a^n-na^{n-1}+1\}/(a-1)}}$$

is normal to base b. A.N. Korobov ([3] Theorem 1) gave, independently of Stoneham, similar constructions: Let a,b be relatively prime integers greater than 1, and let $\{\lambda_n\}_{n\geqq 1}$ and $\{\mu_n\}_{n\geqq 1}$ be strictly increasing sequences of positive integers with $\mu_n \geqq a^{\lambda_n}$. Then the number

$$\sum_{n=1}^{\infty} \frac{1}{a^{\lambda_n} b^{\mu_n}}$$

is normal to base b. Recently Wagner ([12] Theorem) constructed rings of normal numbers for the first time: Let a be an odd prime, b be an integer with $b\geqq 2$ and $a \nmid b$, and let $\{\lambda_n\}_{n\geqq 1}$ and $\{\mu_n\}_{n\geqq 1}$ be strictly increasing sequences of pgsitive integers with $\lim_{n\to\infty} \lambda_n/(n\mu_{n-1})=\lim_{n\to\infty} (\log \mu_n)/\lambda_n=\infty$. Then any nonzero element of the ring generated by

$$\prod_{n=1}^{\infty}\left(1+\frac{\varepsilon_n}{a^{\lambda_n} b^{\mu_n}}\right) (\varepsilon_n = \pm 1)$$

is normal to base b and nonnormal to base ab. The conditions in Wagner's theorem can be weakened, namely we need to assume only that a, b are relatively prime integers greater than 1 and $\lim_{n\to\infty} \lambda_n/\mu_{n-1}=\lim_{n\to\infty}(\log \mu_n)/\lambda_n=\infty$ (cf. [1] Theorem). More recently the author jointly with Shiokawa [2] gave rings of normal numbers of another type. To prove the normality of nonzero elements of the rings therein, we need a criterion ([2] Theorem 3) of normal numbers for numbers of the form given below, which can be rewritten as in the following: Let a, b be relatively prime integers greater than 1, and let $\{\lambda_n\}_{n\geqq 1}$, $\{\mu_n\}_{n\geqq 1}$, and $\{A_n\}_{n\geqq 1}$ be sequences of integers such that $\lambda_{n+1}>\lambda_n$, $\mu_{n+1}>\mu_n$, $a^{\lambda_n}\leqq \mu_n$ and

$$0<|A_n|<a^{\lambda_n-\lambda_{n-1}} \tag{2}$$

for sufficiently large n. Then the number

$$x=\sum_{n=1}^{\infty} \frac{A_n}{a^{\lambda_n} b^{\mu_n}} \tag{3}$$

is normal to base b and nonnormal to base ab. In this paper, we give a wide class of normal numbers which contains all the normal numbers of Stoneham, Korobov, Wagner, and ours mentioned above. Moreover, we shall discuss the disrepancy estimates, transcendency, irrationality measures, and non-Liouville property of these numbers. As for the proof of the normality, Stoneham's approach is considerably involved. Korobov's proof is quite different from that of Stoheham. Our method is an improvement of that of Korobov, by using Erdös-Turán inequality, as developed in [2].

§2. A class of nromal numbers

Theorem 1. *Let* a, b *be relatively prime integers greater than* 1, *and let* $\{\lambda_n\}_{n\geqq 1}$, $\{\mu_n\}_{n\geqq 1}$, *and* $\{A_n\}_{n\geqq 1}$ *be sequences of integers such that* $\lambda_{n+1}>\lambda_n$, $\mu_{n+1}>\mu_n$, $a^{\lambda_n}\ll\mu_n$, *and*

$$|A_n|\ll a^{\lambda_n},\ a^{\lambda_n-\lambda_{n-1}}\not| A_n \tag{4}$$

for sufficiently large n. *Then the number defined by* (3) *is normal to base* b *and nonnormal to base* ab.

REMARK 1. Putting $A_n=1$, we have Korobov's normal numbers. If $\lambda_n=n$, $\mu_n=\sum_{i=1}^{n-1}\alpha_i\tau_b(a^i)$, and $A_n=Z_n-aZ_{n-1}$, we get Stoneham's Theorem mentioned above, even without the condition $\lim_{n\to\infty}\alpha_n=\infty$ (using Lemma 1 below). We remark that if $Z_n=1$ (n: odd), $=a^n-1$ (n: even), $A_n=Z_n-aZ_{n-1}$ does not satisfy (2), but satisfies (4). Any nonzero element in the rings constructed by Wagner [12] and also by the author and Shiokawa [2] can be written in the form (3) with $\{A_n\}$ satisfying the conditions in Theorem 1 (cf. [1], [2]).

To prove Theorem 1 we hsall need the following lemmas.

Lemma 1. (cf. [4] Lemma 1 (Remark)). *For any relatively prime integers* a *and* b *greater than one, there exists a positive integer* n_0 *and a rational number* C *such that*

$$\tau_b(a^n)=Ca^n$$

for all integers n *with* $n\geqq n_0$.

Lemma 2. (cf. [4] Theorem 2). *Let* a, b, *and* n_0 *be as above. For any integers* $n\geqq n_0$ *and* c *with* $a^{n-n_0}\not| c$, *we have*

$$\sum_{j=1}^{\tau_b(a^n)} e^{2\pi i(cb^j/a^n)}=0\,.$$

Lemma 3. (cf. [5] Lemma 2). *With the same condtions as in Lemma* 2, *we have for any positive integer* $N\leqq\tau_b(a^n)$,

$$\left|\sum_{j=1}^{N} e^{2\pi i(cb^j/a^n)}\right| \ll na^{n/2}.$$

Lemma 4. (cf. [6] Theorem 2.5 (Erdös-Turán inequality)). *Let* $E=\{x_1, \cdots, x_N\}\in[0, 1)$. *Then for any positiev integer* M,

$$D(E)\ll\frac{N}{M}+\sum_{\nu=1}^{M}\frac{1}{\nu}\left|\sum_{j=1}^{N} e^{2\pi i\nu x_j}\right|.$$

Proof of Theorem 1. We shall prove the normality. We may assume that

$$\lambda_{n-1}>\max(\lambda_1, \cdots, \lambda_{n-2})>2n_0,$$
$$\mu_{n-1}>\max(\mu_1, \cdots, \mu_{n-2}), \text{ and } a^{\lambda_n-\lambda_{n-1}}\nmid A_n$$

for all $n\geqq n_1$ for some n_1. Let $n\geqq n_1$. We put

$$x_n=\sum_{i=1}^{n} A_i a^{-\lambda_i} b^{-\mu_i}=B_n a^{-\lambda_n} b^{-\mu_n},$$

where

$$B_n=\sum_{i=1}^{n-1} A_i a^{\lambda_n-\lambda_i} b^{\mu_n-\mu_i} A_n,$$

and $\tau_m=\tau_b(a^{\lambda_m})$. For any integer $N>\mu_{n_1+1}$ we define n, h_m, and r_m by

$$\mu_n<N\leqq\mu_{n+1},$$
$$\mu_{m+1}-\mu_m=h_m\tau_m+r_m(n_1\leqq m\leqq n-1,\ 1\leqq r_m\leqq\tau_m),$$

and

$$N-\mu_n=h_n\tau_n+r_n(1\leqq r_n\leqq\tau_n),$$

so that we have

$$h_m<(\mu_{m+1}-\mu_m)/\tau_m(m<n),\ h_n<(N-\mu_n)/\tau_n. \tag{5}$$

Then

$$D(N)\ll\sum_{m=n_1}^{n-1} D(\mu_m, \mu_{m+1}-\mu_m)+D(\mu_n, N-\mu_n)+1$$
$$\ll\sum_{m=n_1}^{n}\left\{\sum_{h=0}^{h_m-1} D(\mu_m+h\tau_m, \tau_m)+D(\mu_m+h_m\tau_m, r_m)\right\}+1.$$

Here we write $e(u)=e^{2\pi iu}$ and

$$E_{\nu mh}(u, v)=\sum_{j=1}^{u} e(\nu v b^{\mu_m+h\tau_m+j}).$$

By Lemma 4 with $M=M_m=2^{[\lambda_{m-1}/2]-n_0}$, we have

$$D(N)\ll\sum_{m=n_1}^{n}\Big[\sum_{h=0}^{h_m-1}\{\tau_m/M_m+\sum_{\nu=1}^{M_m}\nu^{-1}|E_{\nu mh}(\tau_m, x)|\}+r_m/M_m$$
$$+\sum_{\nu=1}^{M_m}\nu^{-1}|E_{\nu mh_m}(r_m, x)|\Big]+1$$

$$\leqq \sum_{m=n_1}^{n} (h_m\tau_m+r_m)/M_m+\sum_{m=n_1}^{n}\sum_{\nu=1}^{M_m}\nu^{-1}\sum_{h=0}^{h_m-1}|E_{\nu mh}(\tau_m, x)|$$

$$+\sum_{m=n_1}^{n}\sum_{\nu=1}^{M_m}\nu^{-1}|E_{\nu mh_m}(r_m, x)|+1$$

$$\leqq \Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5+1\,, \tag{6}$$

where

$$\Sigma_1=\sum_{m=u_1}^{n-1}(h_m\tau_m+r_m)/M_m\,,\quad \Sigma_2=(N-\mu_n)/M_n\,,$$

$$\Sigma_3=\sum_{m=n_1}^{n}\sum_{\nu=1}^{M_m}\nu^{-1}\sum_{h=0}^{h_m-1}|E_{\nu mh}(\tau_m, x_m)|\,,$$

$$\Sigma_4=\sum_{m=n_1}^{n}\sum_{\nu=1}^{M_m}\nu^{-1}|E_{\nu mh_m}(r_m, x_m)|\,,$$

$$\Sigma_5=\sum_{m=n_1}^{n}\sum_{\nu=1}^{M_m}\nu^{-1}\{\sum_{h=0}^{h_m-1}|E_{\nu mh_m}(\tau_m, x)-E_{\nu mh}(\tau_m, x_m)|$$

$$+|E_{\nu mh_m}(r_m, x)-E_{\nu mh_m}(r_m, x_m)|\}\,.$$

It is easily seen that

$$\Sigma_1\ll\sum_{m=n_1}^{n-1}(\mu_{m+1}-\mu_m)2^{-\lambda_{m-1}/2}\,, \tag{7}$$

$$\Sigma_2\ll(N-\mu_n)2^{-\lambda_{n-1}/2}\,. \tag{8}$$

We shall estimate Σ_3 using Lemma 2. Since $a^{\lambda_m-\lambda_{m-1}}\nmid A_m$ and $a^{\lambda_m-\lambda_{m-1}}|A_i a^{\lambda_m-\lambda_i}b^{\mu_m-\mu_i}(i<m)$, we have $a^{\lambda_m-\lambda_{m-1}}\nmid B_m$. For a prime p and an integer n, we denote by $v_p(m)$ the integer h for which $p^h|n$ and $p^{h+1}\nmid n$. Then there is a prime $p|a$ such that $\{p^{v_p(a)}\}^{\lambda_m-\lambda_{m-1}}\nmid B_m$, so that

$$v_p(B_m)<v_p(a)(\lambda_m-\lambda_{m-1})\,.$$

Furthermor we have

$$v_p(\nu)\leqq\log_2 M_m<\lambda_{m-1}-n_0\,.$$

Hence, sihce $(a, b)=1$, we have

$$v_p(\nu B_m b^{h\tau_m+j})<v_p(a)(\lambda_m-n_0)\,.$$

Namely $p^{v_p(a)(\lambda_m-n_0)}\nmid \nu B_m b^{h\tau_m+j}$, and so

$$a^{\lambda_m-n_0}\nmid \nu B_m b^{h\tau_m+j}\,. \tag{9}$$

Therefore we get

$$E_{\nu mh}(\tau_m, x_m)=0$$

by Lemma 2, which leads to

$$\Sigma_3=0\,. \tag{10}$$

Similarly we have by Lemma 3 with (9)

$$E_{\nu m h_m}(r_m, x_m) \ll \lambda_m a^{\lambda_m/2},$$

and hence

$$\Sigma_4 \ll \sum_{m=n_1}^{n} \lambda_m a^{\lambda_m/2} \log M_m \ll \lambda_n^2 a^{\lambda_n/2}. \tag{11}$$

Finally we shall estimate Σ_5. Since $|e(x)-e(x_m)| \ll |x-x_m| \ll b^{-\mu_{m+1}}$, we have

$$|E_{\nu m h}(\tau_m, x)-E_{\nu m h}(\tau_m, x_m)| \leqq \sum_{j=1}^{\tau_m} \nu |x-x_m| b^{\mu_m+h\tau_m+j}$$

$$\ll \nu b^{\mu_m+h_m\tau_m-\mu_{m+1}} \sum_{j=1}^{\tau_m} b^{-j} \ll \nu$$

and similarly

$$|E_{\nu m h_m}(r_m, x)-E_{\nu m h_m}(r_m, x_m)| \ll \nu .$$

Hence we obtain by Lemma 1 with (5) that

$$\Sigma_5 \ll \sum_{m=n_1}^{n} M_m h_m + \sum_{m=n_1}^{n} M_m$$

$$\ll \sum_{m=n_1}^{n-1} (\mu_{m+1}-\mu_m) 2^{-(\lambda_m-\lambda_{m-1}/2)} + (N-\mu_n) 2^{-(\lambda_n-\lambda_{n-1}/2)} + 2^{\lambda_{n-1}/2}. \tag{12}$$

Therefore it follows from (6), (7), (8), (10), (11), and (12) that

$$D(N) \ll \sum_{m=n_1}^{n-1} (\mu_{m+1}-\mu_m) 2^{-\lambda_{m-1}/2} + (N-\mu_n) 2^{-\lambda_{m-1}/2} + \lambda_n^2 a^{\lambda_n/2}. \tag{13}$$

Here the first term is $o(\sum_{m=n_1}^{n-1} (\mu_{m+1}-\mu_n)) = o(\mu_n) = o(N)$, and the second term is also $o(N)$. Furthermore, since $a^{\lambda_n} \ll \mu_n$, the third term is $o(N)$. Therefore we obtain $D(N)=o(N)$, and the normality to base b is proved. Nonnormality to base ab can be proved similarly as in the proof of Theorem 2 in [2].

REMARK 2. We can estimate as in [2] the discrepancy of normal numbers in Theorem 1. Let x be given in Theorem 1. Then for any positive integer N with $\mu_n < N \leqq \mu_{n+1}$ and any block $d_1 \cdots d_s \in \{0, 1, \cdots, b-1\}^s$ we have by (13)

$$\left| A(x, b, d_1 \cdots d_s; N) - \frac{N}{b^s} \right| (\ll D(N))$$

$$\ll \sum_{m=2}^{n-1} \frac{\mu_{m+1}-\mu_m}{(\sqrt{2})^{\lambda_{m-1}}} + \frac{N-\mu_n}{(\sqrt{2})^{\lambda_{n-1}}} + \lambda_n^2 a^{\lambda_n/2}.$$

On the other hand, if s is sufficiently large, we have for any $\varepsilon < 0$

$$(D(N) \gg) \left| A(x, b, d_1 \cdots d_s; N) - \frac{N}{b^s} \right| \gg \frac{\mu_{n+1}-(1+\varepsilon)\mu_n}{a^{\lambda_n}}$$

for infinitely many n. This can be proved similarly as in [2] Theorem 1.

EXAMPLE 1. As these estimates are implicit, we give here an example: Under the same conditions as in Theorem 1, assume further that $\lim_{n\to\infty}\lambda_{n+1}/\lambda_n=\alpha$ and $\mu_n=f(\lambda_n)\beta^{\lambda_n}$ where $f(x)$ is any polynomial with $f(m)\in \boldsymbol{N}$ $(m\in \boldsymbol{N})$. Then for any $\varepsilon>0$

$$D(N)\ll N^{1-(\log_\beta 2)/2\alpha^2+\varepsilon}$$

and , if s is sufficiently large,

$$|A(x, b, d_1\cdots d_s; N)-Nb^{-s}|\gg N^{1-(\log_\beta \alpha)/\alpha^2-\varepsilon}$$

for infinitely many n.

§3. Irrationality measure and transcendency

Stoneham ([10] Theorem 2) proved that the normal number (1) is non-Liouville and transcendental, if there exist positive constants δ and β such that $\delta<\alpha_n\tau_b(a^n)/S(n-1, a)<\beta$. And it is remarked in [2] that

$$\sum_{n=1}^{\infty}\frac{1}{a^n b^{c^n}}$$

$(a, b, c\in \boldsymbol{N}; a, b\geqq 2; (a, b)=1; c\geqq a)$ has the same properties. In this section we shall give a class of non-Liouville normal numbers containing these examples.

Theorem 2. *Make the same assumptions as in Theorem* 1. *Put* $c=\overline{\lim_{n\to\infty}}\,\mu_{n+1}/\mu_n$ *and* $d=\underline{\lim_{n\to\infty}}\,\mu_{n+1}/\mu_n$. *Assume that* $c<\infty$ *and* $d>1$, *then* x *is non-Liouville and transcendental. More precisely for any* $\varepsilon>0$, *we have*

$$\left|x-\frac{P}{Q}\right|\gg\frac{1}{Q^{\max(c,1+c/(d-1))+\varepsilon}} \tag{14}$$

for all integers P, Q $(\geqq 1)$, *and*

$$\left|x-\frac{P}{Q}\right|\ll\frac{1}{Q^{c-\varepsilon}} \tag{15}$$

for infinitely many integers $P, Q(\geqq 1)$ *with* $(P, Q)=1$. *On the other hand if* $c=\infty$, x is a Liouville number.

REMARK 3. Putting $\mu_n=S(n-1, a)=\sum_{i=1}^{n-1}a_i\tau_b(a^i)$ in Theorem 2, we have the Stoneham's result ([10] Theorem 2) mentioned above.

EXAMPLE 2. Let a, b, c be integers greater than 1 with $(a, b)=1$ and $c\geqq a$.

Then, for any polynomial $f(x)$ with $f(m)\in \boldsymbol{N}$ $(m\in \boldsymbol{N})$, the number

$$x=\sum_{n=1}^{\infty}\frac{1}{a^{n}b^{f(n)c^{n}}}$$

is normal to base b, nonnormal to base ab, non-Liouville, and transcendental. Furthermore we have for any $\varepsilon>0$,

$$|x-P/Q|\gg Q^{-\max(c,3)-\varepsilon}$$

for all integers P, $Q(\geqq 1)$, and

$$|x-P/Q|\ll Q^{-c+\varepsilon}$$

for infinitely many integers P, $Q(\geqq 1)$ with $(P, Q)=1$. In this sence Theorem 2, is the best possible.

To prove Theorem 2 we need the following lemmas.

Lemma 5 ([11] Lemma 5). *Let θ be real. Suppose that there exist sequences of integers $\{p_n\}$ and $\{q_n\}$ with $\lim_{n\to\infty} q_n=\infty$, $q_n\leqq \kappa_1 q_{n-1}^{\gamma}$ for some constant $\kappa_1>0$ and $\gamma>1$ such that for any $\delta>0$*

$$\left|\theta-\frac{p_n}{q_n}\right|<\frac{\kappa_2}{q_n^{\xi-\delta}}$$

for some constants $\kappa_2>0$ and $\xi>1$. Then for any $\varepsilon>0$

$$\left|\theta-\frac{x}{y}\right|>\frac{\kappa_3}{y^{1+\gamma/(\xi-1)+\varepsilon}}$$

holds for all integers $x, y(\geqq 1)$ with $x/y\notin\{p_n/q_n\}$ $(n\in N)$ and for some constant $\kappa_3>0$ independent of n.

For completeness we give here the proof of this lemma.

Proof. Let y be large, and let $n=n(y)$ the least positive integer such that

$$|q_m\theta-p_m|\leqq 1/(2y)$$

for all $m\leqq n$. Let x/y $(x, y\in \boldsymbol{N})$ be a rational number with $x/y\notin\{p_n/q_n\}$ $(n\in \boldsymbol{N})$. Then

$$\begin{aligned}|\theta-x/y|&\geqq|p_n/q_n-x/y|-|\theta-p_n/q_n|\\&\geqq 1/(yq_n)-1/(2yq_n)=1/(2yq_n)\,.\end{aligned}$$

By the minimality of n, we get

$$1/(2y)<|q_{n-1}\theta-p_{n-1}|<\kappa_2/q_{n-1}^{\xi-1-\delta}\,,$$

which implies $2\kappa_2 y > q_{n-1}^{\xi-1-\delta}$, so that

$$q_{n-1} < (2\kappa_2 y)^{1/(\xi-1-\delta)},$$

for any $\delta>0$. Hence

$$q_n \leqq \kappa_1 q_{n-1}^{\gamma} < \kappa_4 y^{\gamma/(\xi-1)+\varepsilon}$$

for any $\varepsilon>0$ and a constant $\kappa_4>0$. Therefore, we obtain

$$|\theta - x/y| \geqq 1/(2yq_n) > \kappa_3/y^{1+\gamma/(\xi-1)+\varepsilon}$$

for any $\varepsilon>0$ and a constant $\kappa_2>0$.

Lemma 6 (cf. [7] §5 p. 427). *Let θ be any real number. If there exist some constant $\kappa>1$ and infinite sequence $\{p_n/q_n\}$ $(n\in\boldsymbol{N}, p_n\in\boldsymbol{Z}, q_n\in\boldsymbol{N})$ such that $q_n = q_n' q_n''$ where each of q_n' and q_n'' is a power of an integer independent of n, $q_n<q_{n+1}$,*

$$0<|\theta - p_n/q_n| < q_n^{-\kappa},$$
$$\varlimsup_{n\to\infty} \log q_{n+1}/\log q_n < \infty, \quad \text{and} \quad \lim_{n\to\infty} \log q_n'/\log q_n = 0,$$

then θ is transcendental.

Proof of Theorem 2. We write

$$x_n = \sum_{i=1}^{n} A_n a^{-\lambda_i} b^{-\mu_i} = B_n a^{-\lambda_n} b^{-\mu_n} = p_n/q_n,$$

where $p_n = B_n$, $q_n = a^{\lambda_n} b^{\mu_n}$, and set $q_n' = a^{\lambda_n}$, $q_n'' = b^{\mu_n}$. Then for sufficiently large n we have

$$q_n < q_{n+1},\ q_{n+1} \leqq a^{\lambda_{n+1}} b^{(c+\varepsilon/2)\mu_n} \leqq q_n^{c+\varepsilon}, \tag{16}$$

and

$$|x - p_n/q_n| = \sum_{i=n+1}^{\infty} A_i a^{-\lambda_i} b^{-\mu_i} \ll b^{-\mu_{n+1}}. \tag{17}$$

Here for any $\varepsilon>0$ it follows that

$$\log b^{\mu_{n+1}}/\log q_n \geqq \mu_{n+1}/\mu_n - \varepsilon/2 \geqq d - \varepsilon.$$

By (17) we have

$$|x - p_n/q_n| \ll q_n^{-(d-\varepsilon)} \tag{18}$$

for all n and

$$|x - p_n/q_n| \ll q_n^{-(c-\varepsilon)} \tag{19}$$

for infinitely many n. (15) follows from the last inequality. Since $d>1$, we get for $P/Q \notin \{p_n/q_n\}$ $(P\in\boldsymbol{Z}, Q\in\boldsymbol{N})$,

$$|x - P/Q| \gg Q^{-\{1+c/(d-1)+\varepsilon\}}, \tag{20}$$

using Lemma 5. On the other hand, since

$$|x-p_n/q_n| \gg q_{n+1}^{-1}$$

we get by (16)

$$|x-p_n/q_n| \gg q_n^{-(c+\varepsilon)},$$

which together with (20) yields (14). The transcendency follows from Lemma 6 with (16), (18), and

$$\log q_n'/\log q_n \ll \lambda_n/\mu_n \to 0 \quad (n\to\infty).$$

This completes the proof of Theorem 2.

REMARK 4. If $c=\varlimsup_{n\to\infty} \mu_{n+1}/\mu_n>2$ in Theorem 2, x is transcendental by Roth's Theorem and (19), without the condition $\varliminf_{n\to\infty} \mu_{n+1}/\mu_n>1$. In particular, $c>2$ if $a\geqq 3$.

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