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ON THE DECAY RATE OF LOCAL ENERGY FOR THE ELASTIC WAVE EQUATION

Dedicated to Professor Hiroki Tanabe on his sixtieth birthday

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0. Introduction

Let Ω be an exterior domain in $\boldsymbol{R}^n$ ($n\geq 3$) with smooth and compact boundary. We set

$$A(\partial_x)\,u=\sum_{i,j=1}^{n}\partial_{x_i}(a_{ij}\,\partial_{x_j}u)\,,\quad u={}^t(u_1,u_2,\cdots,u_n)\,,$$

where $a_{ij}=(a_{ipjq}\,|\,{}^{p\downarrow 1,\cdots,n}_{q\rightarrow 1,\cdots,n})$ are $n\times n$ matrices and each a_{ipjq} is constant. We consider the elastic wave equation with the Dirichlet or the Neumann boundary condition

$$\text{(0.1)}\qquad \begin{cases}(\partial_t^2-A(\partial_x))\,u(t,x)=0 & \text{in}\quad \boldsymbol{R}\times\Omega\,,\\ B(\partial_x)\,u(t,x)=0 & \text{on}\quad \boldsymbol{R}\times\partial\Omega\,,\\ u(0,x)=f_1(x)\,,\quad \partial_t u(0,x)=f_2(x) & \text{on}\quad \Omega\,.\end{cases}$$

Here the boundary operator is of the form

$$B(\partial_x)\,u=\begin{cases}u|_{\partial\Omega}, & \text{(the Dirichlet condition)},\\ \sum_{i,j=1}^{n}\nu_i(x)\,a_{ij}\,\partial_{x_j}u|_{\partial\Omega}, & \text{(the Neumann condition)},\end{cases}$$

where $\nu(x)={}^t(\nu_1(x),\nu_2(x),\cdots,\nu_n(x))$ is the unit outer normal to Ω at $x\in\partial\Omega$. The purpose of this paper is to show that in the case of the even $n\geq 4$, if there is any rate of local-energy decay, then we can know the explicit order of the decay rate.

We assume that

$$\text{(A.1)}\qquad a_{ipjq}=a_{pijq}=a_{jqip}\,,$$

$$\text{(A.2)}\qquad \sum_{i,p,j,q=1}^{n}a_{ipjq}\,\varepsilon_{jq}\,\bar{\varepsilon}_{ip}\geq\delta_1\sum_{i,p=1}^{n}|\varepsilon_{ip}|^2\,,$$

(A.3) $\quad A(\xi)=\sum_{i,j=1}^{n}a_{ij}\,\xi_i\,\xi_j$ has d characteristic roots of constant multiplicity for any $\xi\in\boldsymbol{R}^n\backslash 0$, where (ε_{jq}) is any $n\times n$ symmetric matrix and δ_1 is some positive constant independent of (ε_{jq}).

Under the assumptions (A.1)$\sim$(A.3), Shibata and Soga [13] formulate the scattering theory which is analogous to the theory of Lax and Phillips [9]. Hence, the same argument as in Lax, Morawetz and Phillips [8] implies that in the case of the odd $n\geq 3$, if there is any rate of local-energy decay, then there is an exponential rate of decay. We can also prove the same statement stated above by using the argument in Morawetz [11]. Thus, in the odd dimensional case we can find the explicite order of the decay rate.

In the even dimensional case, however, we can not apply their methods since the Cauchy problem for the operator $\partial_t^2-A(\partial_x)$ has Huygens's Principle if and only if the space dimension n is odd and $n\geq 3$, and Huygens's Principle is indispensable to carry out the arguments in [8] and [11]. In the following, our interest is to investigate the similar problem in the even dimensional case.

For a domain $D\subset \boldsymbol{R}^n$, we define the local energy $||f||_{H(D)}$ of the data $f={}^t(f_1, f_2)$ in D as

$$||f||^2_{H(D)}=\frac{1}{2}\int_D \{\sum_{i,p,j,q=1}^n a_{ipjq}\,\partial_{x_j}f_{1q}(x)\,\overline{\partial_{x_i}f_{1p}(x)}+|f_2(x)|^2\}\,dx\,.$$

Shibata and Soga [13] introduce the Hilbert space of the data $f={}^t(f_1, f_2)$ defined as the completion of $\{f\in C_0^\infty(\overline{\Omega})\,|\,B(\partial_x)f_1=0 \text{ on } \partial\Omega\}$ in the norm $||f||_H=||f||_{H(\Omega)}$. Furthermore, they show that the mapping $f\mapsto {}^t(u(t,\cdot), \partial_t u(t,\cdot))$ becomes a group of unitary operators $\{U(t)\}_{t\in\boldsymbol{R}}$ on H.

In the following, we fix a constant $\rho>0$ with $\partial\Omega\subset \boldsymbol{B}_\rho=\{x\in\boldsymbol{R}^n\,|\,|x|<\rho\}$ and assume that

(D) there exist a function $p\in C([0,\infty))$ and a constant $\tilde{\rho}>0$ satisfying $\lim_{t\to\infty} p(t)=0$ and
$$||U(t)f||_{H(\Omega_{\rho+\tilde{\rho}})}\leq p(t)\,||f||_H$$
for any $t\geq 0$ and $f\in H^\rho$.

In the above $H^\rho=\{f\in H\,|\,\text{supp}\, f\subset\overline{\Omega}\cap B_\rho\}$ and $\Omega_{\rho+\tilde{\rho}}=\Omega\cap B_{\rho+\tilde{\rho}}$. Then our main theorem in the present paper is

Theorem 0.1. *If we assume* (D), *then there exists a constant* $C=C(\rho)>0$ *such that the following estimates hold:*

$$||U(t)f||_{H(\Omega_\rho)}\leq C(1+t)^{-(n-1)}\,||f||_H\,, \tag{0.2}$$

$$||[U(t)f]_2||_{L^2(\Omega_\rho)}\leq C(1+t)^{-n}\,||f||_H \tag{0.3}$$

for any $t\geq 0$ *and* $f\in H^\rho$,

where $[U(t)f]_2$ *means the second component of* $U(t)f$.

Next, in the case of the isotropic elastic wave equation (i.e. $a_{ipjq}=\lambda\delta_{ip}\delta_{jq}+\mu(\delta_{ij}\delta_{pq}+\delta_{iq}\delta_{jp})$), we give some applications of Theorem 0.1. We assume that the Lamé constants λ and μ are independent of the variables t and x, and satisfy $\lambda+\frac{2}{n}\mu>0$, $\mu>0$. Note that the assumptions (A.1)$\sim$(A.3) hold under these assumptions of λ and μ. We start with the case of the Neumann boundary condition. In this case, it is well known that there is a wave called the Rayleigh surface wave which seems to propagate along the boundary. Hence, we can not expect that there is a uniform rate of local-energy decay. In fact, we have the following result as a corollary of Theorem 0.1 in the persent paper and Theorem 0.2 in Kawashita [7].

Corollary 0.2. *In the case of the isotropic elastic wave equation with the Neumann boundary condition, the problem* (0.1) *does not have the uniform local-energy decay property.*

In the above, we say the problem (0.1) has the uniform local-energy decay property when for any bounded domain D and $D_0\subset \boldsymbol{R}^n$ there exists a function $p\in C([0,\infty))$ such that $\lim_{t\to\infty} p(t)=0$ and $\|U(t)f\|_{H(\Omega\cap D)}\leq p(t)\|f\|_H$ holds for any $t\geq 0$ and $f\in H$ with $\operatorname{supp} f\subset\overline{\Omega}\cap D_0$.

Note that for the odd $n\geq 3$ Corollary 0.2 is already obtained as Corollary 0.3 in [7]. In the case of the even $n\geq 4$ if we assume that (0.1) has the uniform local-energy decay property, then from Theorem 0.1 it follows that the problem (0.1) has the uniform local-energy decay property of the strong type (cf. [7]). In the case of the Neumann boundary condition, however, the above statement contradicts Theorem 0.2 in [7], which means the correctness of Corollary 0.2.

The second application is to decide the uniform decay rate for the isotropic elastic wave equation with the Dirichlet condition. In this case, B. V. Kapitonov [6] shows that we have the uniform decay rate with $p(t)=C(1+t)^{-1/2}$ if the obstacle $\boldsymbol{R}^n\backslash\Omega$ is star-shaped. Combining the result of Kapitonov [6] with Theorem 0.1, we obtain the following corollary.

Corollary 0.3. *In the case of the isotropic elastic wave equation with the Dirichlet boundary condition, if the obstacle $\boldsymbol{R}^n\backslash\Omega$ is star-shaped, then we have the estimates* (0.2) *and* (0.3).

The proof of Theorem 0.1 is based on some local-energy decay estimates for the free space solution (cf. §2) and the decay estimate stated in Proposition 4.1 in §4, that is,

$$\|P^{\rho}_{+,0}EU(t)f\|_{H(\boldsymbol{R}^n)}\leq C(1+t)^{-n/2}\|f\|_H \quad \text{for any } t\geq 0 \text{ and } f\in H^{\rho}\,, \tag{0.4}$$

where E is an extension operator from Ω to $\boldsymbol{R}^n$, and $P^{\rho}_{+,0}$ is the projection operator into the orthogonal complement of the outgoing subspace in the Hilbert space of the data for the free space problem. Using these estimates we prove Theorem 0.1 in §5.

R.B. Melrose [10] obtains the estimate (0.4) in the case of the scalar valued wave equation under the assumption that Ω is non-trapping. In [10], it seems that he intends to decompose $P^{\rho}_{+,0}EU(t)f$ into the energy escaping part and the part behaving like free space solution. Using this decomposition he obtains the estimate of the time $T_p \geq 0$ satisfying $\|P^{\rho}_{+,0}EU(t)f\|_{H(\boldsymbol{R}^n)} \leq 2^{-p}\|E\|\,\|f\|_H$ for any $t \geq T_p$ and $f \in H^{\rho}$. On the procedure in [10], the non-trapping condition is used to show the existense of the energy escaping part. Hence, if we can show the existence of good decomposition like the above mension we can expect that the idea getting the estimate of T_p implies the estimate (0.4). However, it does not seem easy to use the same argument as in [10] since the decomposition of $P^{\rho}_{+,0}EU(t)f$ in [10] is very complicated. Hence, we introduce a different decomposition to prove Proposition 4.1.

In §1, we introduce a class of the data $V_{\alpha}(C)$ which is indispensable to get Proposition 4.1, and refer to some propositions in [10] which are used later. Note that the class $V_{\alpha}(C)$ is originally defined by Melrose [10].

In §3, we introduce some operator which reflects the fact that the remaining energy tends to zero uniformly as $t \to \infty$ under the assumptions in Theorem 0.1 (cf. Proposition 3.1 in §3). Proposition 3.1 corresponds to the existence of the energy escaping part in [10], and the proof of Proposition 3.1 is the main part of the proof of Proposition 4.1.

In §4 we prove Proposition 4.1 by means of the decay estimates in §2 and Proposition 3.1. An essential idea is the same as in Melrose [10] stated above. However, there are some differences between our argument and that of [10]. One difference is the way of the decomposition of $P^{\rho}_{+,0}EU(t)f$. The form of the energy escaping part is also different. The differences give us not only simplicity in the argument to obtain Proposition 4.1 but also the clear reason why the order of the estimate (0.4) should be $-n/2$.

Finally, we note a result related with Theorem 0.1. Iwashita and Shibata [5] and Iwashita [4] show that the estimates (0.2) and (0.3) hold under the assumptions (A.1)$\sim$(A.3) and the assumption that Ω is non-trapping in the sense; for any $a>0$ with $\partial\Omega \subset B_a$ there exists a constant $T=T(a,\Omega)>0$ such that $U(t)f(x) \in C^{\infty}([T,\infty)\times\bar{\Omega}_a)$ for any $f={}^t(0,f_2)\in H$ with $\operatorname{supp} f \subset \bar{\Omega}\cap B_a$. Thus, they have a sharper result than that of Melrose [10]. To prove the estimates (0.2) and (0.3) they intend to get an estimate of resolvent. From this estimate they obtain (0.2) and (0.3) by using the Laplace transformation. Hence, to prove Theorem 0.1 we can not use the methods in [5] and [4] since it does not seem easy to deduce the estimate of resolvent under the assumption (D). This is the reason why we use the idea in Melrose [10] to prove Theorem 0.1 though the result in [10] is not better than that in Iwashita and Shibata [5] and Iwashita [4].

1. A class of the data $V_\alpha(C)$

In this section, first we review the properties of the translation representation obtained by Shibata and Soga [13], which are used later. Second, we define a class of the data $V_\alpha(C)$ which plays an important role in this paper.

Let H_0 be the Hilbert space of the data for the free space problem (i.e. in the case that $\Omega=\boldsymbol{R}^n$). We set $\{U_0(t)\}_{t\in\boldsymbol{R}}$ is a group of unitary operators on H_0 which is a solution operator of the free space problem. By the assumptions (A.1)$\sim$(A.3), we can enumerate the characteristic roots $\lambda_j(\xi)$ $(j=1, 2, \cdots, d)$ of $A(\xi)$ in the following way: $0<\lambda_1(\xi)<\lambda_2(\xi)<\cdots<\lambda_d(\xi)$ for any $\xi\in\boldsymbol{R}^n\backslash 0$. We set $P_j(\xi)$ as the eigenprojector for the eigenvalue $\lambda_j(\xi)$ for each $j=1, \cdots, d$. The free space translation representation $T_0^+\colon H_0\to L^2(\boldsymbol{R}\times S^{n-1})$ has the following representation:

$$T_0^+ f(s, \omega) = \sum_{j=1}^{d} \lambda_j(\omega)^{1/4} P_j(\omega) (J_+\mathcal{R}_j f) (\lambda_j(\omega)^{1/2} s, \omega) \qquad \text{for any } f\in C_0^\infty(\boldsymbol{R}^n)\,,$$

where

$$\mathcal{R}_j f(s, \omega) = -\lambda_j(\omega)^{1/2}\, \partial_s \tilde{f}_1(s, \omega)+\tilde{f}_2(s, \omega) \quad (j = 1, 2, \cdots, d)\,,$$

$$\tilde{f}_j(s, \omega) = \int_{x\cdot\omega=s} f_j(x)\, dS_x \quad (j = 1, 2) \quad \text{(Radon transform)},$$

and $J_+=(-\partial_s)^{(n/2)-1}\,\lambda_+(D_s)$ with

$$\lambda_+(\sigma) = \begin{cases} \dfrac{1-i}{\sqrt{2}}\,\sigma^{1/2} & (\text{for } \sigma\geq 0)\,, \\[2ex] \dfrac{1+i}{\sqrt{2}}\,|\sigma|^{1/2} & (\text{for } \sigma<0)\,. \end{cases}$$

For $\rho>0$ (stated in §0), we define the outgoing subspace D_+^ρ as

$$D_+^\rho = U_0(C_{\min}^{-1}\,\rho)\, D_+^0$$

where $D_+^0=\{f\in H_0\,|\, T_0^+ f(s, \omega)=0 \text{ in } s<0\}=\{f\in H_0\,|\, U_0(t) f=0 \text{ in } |x|<C_{\min}\, t\}$ and $C_{\min}=\min\limits_{j=1,\cdots,d}\ \inf\limits_{\omega\in S^{n-1}} \{\lambda_j(\omega)^{1/2}\}>0$. The outgoing subspace D_+^ρ is the closed subspace in H_0 and H. We denote by $P_{+,0}^\rho$ and P_+^ρ the orthogonal projectors of the closed subspaces $(D_+^\rho)^\perp$ in H_0 and H, respectively. By using the method of Seeley [12] and the estimate

$$(1.1) \qquad \|v\|_{L^2(\Omega_\rho)}\leq C_\rho\, \|\nabla v\|_{L^2(\Omega)} \quad \text{for any } v\in C_0^\infty(\overline{\Omega})$$

(cf. [13]), we can construct the linear extension operator $E: H \rightarrow H_0$ satisfying the following conditions:

(1.2) $$Ef = f \text{ in } \Omega \quad \text{for any } f \in H.$$

(1.3) For any integer $l \geq 0$, $E: D(A^l) \rightarrow D(A_0^l)$ is bounded with respect to the graph norms, where A and A_0 are the generators of $\{U(t)\}_{t\in \boldsymbol{R}}$ and $\{U_0(t)\}_{t\in \boldsymbol{R}}$ respectively.

(1.4) There exists a constant ρ_0 $(0<\rho_0<\rho)$ such that $\partial\Omega \subset B_{\rho_0}$ and for $\rho' \geq \rho_0$, we have $Ef=0$ in $|x|<\rho'$ for any $f\in H$ satisfying $f=0$ in $|x|<\rho'$.

(1.5) $$\|Ef\|_{H(B_\rho)} \leq C\,\{\|f\|_{H(\Omega_\rho)} + \|f_1\|_{L^2(\Omega_\rho)}\} \quad \text{for any } f \in H.$$

Furthermore, we have

(1.6) $$E P_+^\rho = P_{+,0}^\rho E.$$

Proof of (1.6). We note the fact that $g(x)=0$ in $|x|<\rho$ for any $g\in D_+^\rho$. From (1.2) and the above fact it follows that $((I-P_{+,0}^\rho)\,E\,P_+^\rho f, g)_{H_0} = (E\,P_+^\rho f, (I-P_{+,0}^\rho)\,g)_{H_0} = (P_+^\rho f, (I-P_{+,0}^\rho)\,g)_H = 0$, where $(\ ,\ )_{H_0}$ and $(\ ,\ )_H$ mean the inner products of H_0 and H, respectively. Thus, we have $(I-P_{+,0}^\rho)\,E\,P_+^\rho = 0$. Since $(I-P_+^\rho)f=0$ in $|x|<\rho$, by (1.2) and (1.4) we have $E(I-P_+^\rho)f = (I-P_+^\rho)f$ in H_0, which implies that $P_{+,0}^\rho\,E(I-P_+^\rho)=0$. This completes the proof of (1.6).

Now, we define a class of the data $V_\alpha(C)$, which is originally introduced by Melrose [10].

Definition 1.1. *For any $\alpha>0$ and $C>0$, we say that $f\in V_\alpha(C)$ if and only if the element $f\in H$ satisfies*

$$(T_0^+\,EU(t)f)\,(s,\omega) \in C^\infty((-\infty, -C_{\min}^{-1}(\rho+2)), L^2(S^{n-1})) \quad \textit{for any } t\geq 0$$

and the following estimate

$$\|(\partial_s^l\,T_0^+\,EU(t)f)\,(s,\cdot)\|_{L^2(S^{n-1})} \leq C\,|s|^{-\alpha-l}\,\|f\|_H$$

$$\textit{for any } t\geq 0,\ s< -C_{\min}^{-1}(\rho+2) \quad \textit{and} \quad l=0,1,\cdots,\frac{n}{2}+1.$$

Remark. Our definition of $V_\alpha(C)$ is slightly different from the one in Melrose [10].

The following estimate obtained by Melrose [10] is a key result to know the uniform decay rate.

Lemma 1.2. *For any $f\in H$ and $t\geq 0$, we have*

$$(T_0^+(EU(t)-U_0(t)E)f)(s,\cdot)\in C^\infty((-\infty,-C_{\min}^{-1}(\rho+2)),L^2(S^{n-1})),$$

and for any integer $l\geq 0$ there exists a constant $C_l>0$ such that

$$\begin{aligned}&\|(\partial_s^l T_0^+(EU(t)-U_0(t)E)f)(s,\cdot)\|_{L^2(S^{n-1})}\\&\leq C_l\int_{-2C_{\max}^{-1}}^{t-C_{\max}^{-1}}|t-s-r|^{-(n+1)/2-l}e_f(r)^{1/2}dr\\&\quad\text{for any } f\in H,\ t\geq 0 \text{ and } s<-C_{\min}^{-1}(\rho+2),\end{aligned}$$

where $C_{\max}=\max_{j=1,2,\cdots,d}\sup_{\omega\in S^{n-1}}\{\lambda_j(\omega)^{1/2}\}>0$ is the propagation speed of (0.1) *and $e_f(r)=\|P_{+,0}^\rho EU(r)f\|_H^2$ for $r\geq 0$, $e_f(r)=e_f(0)$ for $r<0$.*

We can prove Lemma 1.2 by using the argument in Lemma 4.5 of Melrose [10]. Hence, we omit the proof of Lemma 1.2.

For a data $f\in H^\rho$, it follows that supp $\mathcal{R}_j Ef\subset(-\rho,\rho)\times S^{n-1}$ and $\|\mathcal{R}_j Ef\|_{L^2(\boldsymbol{R}\times S^{n-1})}\leq C\rho^{(n-1)/2}\|f\|_H$, where $C>0$ is a fixed constant. Thus, noting that in the region $s<-(\rho+2)$ the mapping J_+ is a convolution operator with a homogeneous kernel of order $-(n+1)/2$, we have

$$\begin{aligned}&\|(\partial_s^l T_0^+U_0(t)Ef)(s,\cdot)\|_{L^2(S^{n-1})}\leq C_l|s-t|^{-(n+1)/2-l}\|f\|_H\\&\quad\text{for any } t\geq 0,\ s<-C_{\min}^{-1}(\rho+2) \text{ and } f\in H^\rho,\end{aligned}\tag{1.7}$$

where $C_l=C_l(\rho)>0$ is a fixed constant. Combining (1.7) with Lemma 1.2, we get the following corollary.

Corollary 1.3.

(1.8) *There is a constant $C=C(\rho)>0$ such that $H^\rho\subset V_{(n-1)/2}(C)$.*

(1.9) *For any $\beta>2$ and $C_0>0$, there exists a constant* $\boldsymbol{C}=\boldsymbol{C}(\rho,\beta,C_0)>0$ *such that*

$$W_\beta(C_0)\cap H^\rho\subset V_{(n+1)/2}(C),$$

where $W_\beta(C_0)=\{f\in H\,|\,e_f(t)\leq C_0(1+t)^{-\beta}\|f\|_H^2 \quad \text{for any } t\geq 0\}$.

(1.10) *For any constant $C_0>0$, there exists a constant*

$$C=C(\rho,C_0)>0 \text{ such that } W_2(C_0)\cap H^\rho\subset V_{n/2}(C).$$

(cf. see Melrose [10]).

2. Uniform decay estimates for the free space solutions

In this section, we give some uniform decay estimates for the free space solutions, which are some parts of the basic tools to prove the main theorem.

To start with, we set up a class of mappings to unify treatments for the deduction of the decay estimates used in the following sections.

Definition 2.1. For any $C_0>0$, $\alpha>0$ and a subset $G\subset H$, we say that a mapping F: $G\to H_0$ belongs to $\Lambda(C_0, \alpha; G)$ if and only if the following conditions are satisfied:

$$(2.1)\qquad ||Ff||_{H_0}\leq C_0||f||_H \quad \textit{for any } f\in G,$$

$$(2.2)\qquad (T_0^+ Ff)(s, \cdot)\in C^\infty((-\infty, -C_{\min}^{-1}(\rho+2)), L^2(S^{n-1})) \quad \text{for any } f\in G,$$

$$(2.3)\qquad ||(\partial_s^l T_0^+ Ff)(s, \cdot)||_{L^2(S^{n-1})}\leq C_0|s|^{-\alpha-l}||f||_H$$
$$\text{for any } f\in G, s<-C_{\min}^{-1}(\rho+2) \text{ and } l=0, 1, \cdots, \frac{n}{2}+1.$$

We have the following

Proposition 2.2. *For any $C_0>0$, $\alpha>1/2$ and any subset $G\subset H$, there exists a constant $C=C(\rho, \alpha, C_0, G)>0$ such that*

$$||P_{+,0}^\rho U_0(t) Ff||_{H_0}\leq C(1+t)^{-(\alpha-1/2)}||f||_H$$

holds for any $t\geq 0$, $F\in\Lambda(C_0, \alpha; G)$ and $f\in G$.

Proof of Proposition 2.2. Noting that the definition of $P_{+,0}^\rho$ and $||T_0^+ g||_{L^2(\mathbf{R}\times S^{n-1})}^2=4(2\pi)^{n-1}||g||_{H_0}^2$ (cf. [13]), we have

$$||P_{+,0}^\rho U_0(t) Ff||_{H_0}^2 = 4^{-1}(2\pi)^{-(n-1)}\int_{-\infty}^{C_{\min}^{-1}\rho-1}||(T_0^+ Ff)(s, \cdot)||_{L^2(S^{n-1})}^2 ds.$$

Thus, combining the above equality with (2.3), we get

$$||P_{+,0}^\rho U_0(t) Ff||_{H_0}^2\leq 4^{-1}(2\pi)^{-(n-1)}\frac{C_0^2}{2\alpha-1}(t-C_{\min}^{-1}\rho)^{-(2\alpha-1)}||f||_H^2$$
$$\text{for any } t>2C_{\min}^{-1}(\rho+1) \text{ and } f\in G,$$

which implies Proposition 2.2 if we note that by the estimate (2.1) $||P_{+,0}^\rho U_0(t) Ff||_{H_0}\leq C_0||f||_H$ holds for all $t\geq 0$ and $f\in G$.

Furthermore, we need the following estimate:

Proposition 2.3. *For any $C_0>0$, $\alpha>1/2$, $\rho'\geq\rho$ and any subset $G\subset H$, there exists a constant $C=C(\rho', \alpha, C_0, G)>0$ such that*

$$(2.4)\qquad ||U_0(t) Ff||_{H(B_{\rho'})}\leq C(1+t)^{-(\alpha-1/2)-n/2}||f||_H,$$

$$\|[U_0(t)Ff]_1\|_{L^2(B_{\rho'})}\le C(1+t)^{-(\alpha-3/2)-n/2}\|f\|_H \quad (2.5)$$
for any $t\ge 0$, $F\in\Lambda(C_0,\alpha;G)$ *and* $f\in G$,

where $[U_0(t)Ff]_1$ *means the first component of* $U_0(t)Ff$.

Proof of Proposition 2.3. Noting that for any $g\in D(A_0^\infty)$, we have

$$[U_0(t)g]_1(x) = -2^{-1}(2\pi)^{1-n}$$
$$\int_{S^{n-1}}\sum_{j=1}^d \lambda_j(\omega)^{-n/4}P_j(\omega)(\partial_s^{-1}J_+^* T_0^+ g)(\lambda_j(\omega)^{-1/2}x\cdot\omega-t,\omega)\,d\omega,$$
$$[U_0(t)g]_2(x) = 2^{-1}(2\pi)^{1-n}$$
$$\int_{S^{n-1}}\sum_{j=1}^d \lambda_j(\omega)^{-n/4}P_j(\omega)(J_+^* T_0^+ g)(\lambda_j(\omega)^{-1/2}x\cdot\omega-t,\omega)\,d\omega$$

(cf. Theorem 2.1 of Shibata and Soga [13]). These imply that there is a constant $C_1>0$ such that

$$\|U_0(t)g\|_{H(B_{\rho'})}\le C_1\rho'^{(n-1)/2}\{\int_{-C_{\min}^{-1}\rho'-t}^{C_{\min}^{-1}\rho'-t}\|(J_+^* T_0^+ g)(s,\cdot)\|_{L^2(S^{n-1})}^2\,ds\}^{1/2}$$
for any $g\in D(A_0^\infty)$.

Take $\varphi\in C_0^\infty(\boldsymbol{R})$ satisfying supp $\varphi\subset(-1,1)$ and $\int_{-\infty}^{\infty}\varphi(s)\,ds=1$. For any $\varepsilon>0$, $F\in\Lambda(C_0,\alpha;G)$, and $f\in G$, we set

$$g_\varepsilon=(T_0^+)^{-1}[\varphi_\varepsilon * T_0^+ Ff]\in D(A_0^\infty),$$

where $\varphi_\varepsilon(s)=\varepsilon^{-1}\varphi(\varepsilon^{-1}s)$. Then, for $s<-C_{\min}^{-1}(\rho+2)-\varepsilon$ and $l=0,1,\cdots,\frac{n}{2}+1$, we have

$$\begin{aligned}&\|(\partial_s^l T_0^+ g_\varepsilon)(s,\cdot)\|_{L^2(S^{n-1})}\\ &=\{\int_{S^{n-1}}\left|\int_{-\infty}^{\infty}\varphi_\varepsilon(s')(\partial_s^l T_0^+ Ff)(s-s',\omega)\,ds'\right|^2 d\omega\}^{1/2}\\ &\le\int_{-\infty}^{\infty}\varphi_\varepsilon(s')\|(\partial_s^l T_0^+ Ff)(s-s',\cdot)\|_{L^2(S^{n-1})}\,ds',\end{aligned}$$

which implies

$$\|(\partial_s^l T_0^+ g_\varepsilon)(s,\cdot)\|_{L^2(S^{n-1})}\le C_0|s+\varepsilon|^{-\alpha-l}\|f\|_H \quad (2.6)$$
for $l=0,1,\cdots,\frac{n}{2}+1$, $s<-C_{\min}^{-1}(\rho+2)-\varepsilon$ and $f\in G$.

Since $\overline{J_+(\sigma)}=(i\sigma)^{n/2}\hat{\chi}_+(\sigma)$, where $\chi_+(\tau)=\tau^{-1/2}$ for $\tau\ge 0$, $\chi_+(\tau)=0$ for $\tau<0$ (cf. §1 in Soga [14]), for $k\in C_0^\infty(\boldsymbol{R}\times S^{n-1})$ $J_+^* k$ is represented as

$$J_+^* k(s,\omega)=\frac{1}{\sqrt{\pi}}\int_0^\infty s'^{-1/2}(\partial_s^{n/2}k)(s-s',\omega)\,ds',$$

which yields

$$(2.7)\qquad \|J_+^* k(s,\cdot)\|_{L^2(S^{n-1})}\leq \frac{1}{\sqrt{\pi}}\int_0^\infty s'^{-1/2}\,\|(\partial_s^{n/2}k)(s-s',\cdot)\|_{L^2(S^{n-1})}\,ds'$$

for all $k\in C_0^\infty(\boldsymbol{R}, L^2(S^{n-1}))$ by using the density argument.

Now, we take $\Phi\in C_0^\infty(\boldsymbol{R})$ satisfying $0\leq\Phi\leq 1$ in $\boldsymbol{R}$, $\Phi(s)=1$ in $|s|<1$, and $\Phi(s)=0$ in $|s|>2$. For any $R>0$, we set

$$g_{\varepsilon,R}=(T_0^+)^{-1}[\tilde{\Phi}_R(s)\,(\varphi_\varepsilon * T_0^+ Ff)(s,\omega)]\in H_0\,,$$

where $\tilde{\Phi}_R(s)=\Phi(R^{-1}s)$. Since $T_0^+ g_{\varepsilon,R}\in C_0^\infty(\boldsymbol{R}, L^2(S^{n-1}))$, from (2.6) and (2.7) we have

$$\|(J_+^* T_0^+ g_{\varepsilon,R})(s,\cdot)\|_{L^2(S^{n-1})}\leq \frac{C_0}{\sqrt{\pi}}\|f\|_H\sum_{l=0}^{n/2}M_l R^{-(n/2)+l}I_l(s)$$
$$\text{for all } s<-C_{\min}^{-1}(\rho+2)-\varepsilon \text{ and } R>0\,,$$

where $M_l=\binom{n/2}{l}\max_{s\in\boldsymbol{R}}|(\partial_s^{(n/2)-l}\Phi)(s)|>0$ and

$$I_l(s)=\int_0^\infty s'^{-1/2}\,|s-s'+\varepsilon|^{-\alpha-l}\,ds'\,.$$

Note that for $s<-C_{\min}^{-1}(\rho+2)-\varepsilon$ and $l=0, 1, \cdots, n/2$,

$$\begin{aligned}I_l(s)&=\int_0^{-s} s'^{-1/2}\,|s-s'+\varepsilon|^{-\alpha-l}\,ds'+\int_{-s}^\infty s'^{-1/2}\,|s-s'+\varepsilon|^{-\alpha-l}\,ds'\\&\leq 2|s|^{1/2}\max_{0\leq s'\leq -s}|s-s'+\varepsilon|^{-\alpha-l}+\int_{-s}^\infty s'^{-(1/2)-\alpha-l}\,ds'\\&\leq\frac{4\alpha}{2\alpha-1}(1+\varepsilon\,C_2(\alpha,\rho))\,|s+\varepsilon|^{-\alpha+1/2-l}\,,\end{aligned}$$

where $C_2(\alpha,\rho)>0$ is independent of $\varepsilon>0$. Hence, it follows that

$$\begin{aligned}\|U_0(t)\,g_{\varepsilon,R}\|_{H(B_{\rho'})}&\leq\frac{4C_0\,C_1\,\alpha\rho'^{(n-1)/2}}{\sqrt{\pi}(2\alpha-1)}(1+\varepsilon\,C_2(\alpha,\rho))\sqrt{2C_{\min}^{-1}\rho'}\,\|f\|_H\\&\qquad\sum_{l=0}^{n/2}M_l R^{-(n/2)+l}(t-C_{\min}^{-1}\rho'-\varepsilon)^{-(\alpha-1/2)-l}\end{aligned}$$
$$\text{for any } t>C_{\min}^{-1}(\rho'+\rho+2)+\varepsilon,\ \varepsilon>0 \text{ and } R>0\,.$$

Then, we get (2.4) if we note that $U_0(t)\,g_{\varepsilon,R}\to U_0(t)\,g_\varepsilon$ as $R\to\infty$ in H_0, $U_0(t)\,g_\varepsilon\to U_0(t)\,Ff$ as $\varepsilon\to 0$ in H_0 and $\|U_0(t)\,Ff\|_{H_0}\leq C_0\,\|f\|_H$ for any $t\geq 0$ and $f\in G$.

As for (2.5), we note that

$$\begin{aligned}&\|[U_0(t)\,g]_1\|_{L^2(B_{\rho'})}\\&\quad\leq C_1'\,\rho'^{(n-1)/2}\Big\{\int_{-C_{\min}^{-1}\rho'-t}^{C_{\min}^{-1}\rho'-t}\|(\partial_s^{-1}J_+^* T_0^+ g)(s,\cdot)\|_{L^2(S^{n-1})}^2\,ds\Big\}^{1/2}\end{aligned}$$

for any $g\in D(A_0^\infty)$, where $C_1'>0$ is a fixed constant. From the estimate (1.1) and the assumption (A.2), we have $\lim_{j\to\infty}\|g^{(j)}\|_{L^2(B\rho')}=\|g\|_{L^2(B\rho')}$ for any sequence $g^{(j)}\in H_0$ with $\lim_{j\to\infty} g^{(j)}=g$ in H_0. Hence, the same argument as for (2.4) yields the estimate (2.5), which completes the proof of Proposition 2.3.

Now, we give some applications of Propositions 2.2 and 2.3, which are used in the following sections.

Corollary 2.4. *For any $C_0>0$ and $\alpha>1/2$, there exists a constant $C=C(\rho, \alpha, C_0)>0$ such that*

$$\|P_{+,0}^{\rho}\, U_0(t)\, EU(t')f\|_{H_0}\leq C(1+t)^{-(\alpha-1/2)}\,\|f\|_H\,,$$
$$\|U_0(t)\, EU(t')f\|_{H(B\rho)}\leq C(1+t)^{-(\alpha-1/2)-n/2}\,\|f\|_H\,,$$
$$\|[U_0(t)\, EU(t')f]_1\|_{L^2(B\rho)}\leq C(1+t)^{-(\alpha-3/2)-n/2}\,\|f\|_H$$
$$\text{for any } t'\geq 0,\ t\geq 0 \text{ and } f\in V_\alpha(C_0)\,.$$

Proof of Corollary 2.4. By the definition of $V_\alpha(C_0)$, we have

$$EU(t')\in\Lambda(C_0', \alpha;\, V_\alpha(C_0)) \quad \text{for any } t'\geq 0\,, \tag{2.8}$$

where $C_0'=\max\{C_0, \|E\|\}$. Thus, from Propositions 2.2 and 2.3, we have Corollary 2.4.

Corollary 2.5. *For any $C_0>0$ and $\alpha>1/2$, there exists a constant $C=C(\rho, \alpha, C_0)>0$ satisfying*

$$\|(\frac{d}{dt}\, U_0)\,(t)\, EU(t')f\|_{H(B\rho)}\leq C(1+t)^{-(\alpha+1/2)-n/2}\,\|f\|_H\,,$$
$$\|[(\frac{d}{dt}\, U_0)\,(t)\, EU(t')f]_1\|_{L^2(B\rho)}\leq C(1+t)^{-(\alpha-1/2)-n/2}\,\|f\|_H$$
$$\text{for any } t'\geq 0,\ t>2C_{\min}^{-1}(\rho+1) \text{ and } f\in V_\alpha(C_0)\cap D(A)\,.$$

Proof of Corollary 2.5. The representation of $U_0(t)\,g$ with $g\in D(A_0^\infty)$ yields

$$\|(\frac{d}{dt}\, U_0)\,(t)\, g\|_{H(B\rho)}$$
$$\leq C_3\,\rho^{(n-1)/2}\{\int_{-C_{\min}^{-1}\rho-t}^{C_{\min}^{-1}\rho-t}\|(\partial_s J_+^*\, T_0^+\, g)\,(s, \cdot)\|_{L^2(S^{n-1})}^2\, ds\}^{1/2}\,,$$
$$\|[(\frac{d}{dt}\, U_0)\,(t)\, g]_1\|_{L^2(B\rho)}$$
$$\leq C_3\,\rho^{(n-1)/2}\{\int_{-C_{\min}^{-1}\rho-t}^{C_{\min}^{-1}\rho-t}\|(J_+^*\, T_0^+\, g)\,(s, \cdot)\|_{L^2(S^{n-1})}^2\, ds\}^{1/2}\,,$$

where $C_3>0$ is independnent of $g\in D(A_0^\infty)$ and $t\geq 0$. Hence, noting (2.8), we obtain Corollary 2.5 by the same argument as for Proposition 2.3.

3. The operator $X(t, t')$

In the case of the odd $n\geq 3$, Lax, Morawetz and Phillips [8] introduce the semigroup $\{Z(t)\}$ defined as $Z(t)=P_+^\rho U(t) P_-^\rho$. In [8], the property $\lim_{t\to\infty} \|Z(t)\|_{B(H,H)}=0$ derived by the assumption (D) plays a crucial role to show that there is an exponential rate of the local-energy decay. We can say that the decaying property of $\{Z(t)\}$ ensures the existence of the energy escaping part. In the even dimensional case, however, the operator $Z(t)$ is not connected with the rate of the local-energy decay. Thus, we have to make another operator $X(t, t')$ which show the existence of the energy escaping part in the case of the even $n\geq 4$. This is the perpose in this section.

First, we fix a constant $0<\rho_0'<\rho_0$ with $\partial\Omega\subset B_{\rho_0'}$ and take $\psi\in C^\infty(\boldsymbol{R}^n)$ satisfying $0\leq\psi\leq 1$ in $\boldsymbol{R}^n$, $\psi=1$ in $|x|>\rho_0$, $\psi=0$ in $|x|<\rho_0'$, where $\rho_0>0$ is in (1.4). For any $t, t'\geq 0$, we define the operator $X(t, t')$: $H\to H_0$ as

$$X(t, t') = P_{+,0}^\rho EU(t)\,(U(t')-\psi U_0(t')\,E)\,.$$

Then we have the main result of this section which is a basic tool to prove Theorem 0.1.

Proposition 3.1. *Under the assumptions that are in Theorem 0.1, we have*

$$\lim_{t\to\infty} \|X(t, t')\|_{B(H,H_0)} = 0 \qquad \textit{uniformly in } t'\geq 0\,.$$

To prove Proposition 3.1, we need the following extension operator.

Lemma 3.2. *There exists a bounded linear operator $\tilde{E}$: $H\to H_0$ satisfying*

(3.1) $$\tilde{E}f=f \qquad in \quad |x|>\rho_0\,,$$

(3.2) *there is a constant $C_1>0$ such that*

$$\|\tilde{E}f\|_{H(B_{\rho'})}\leq C_1\, \|f\|_{H(\Omega_{\rho'})} \quad \textit{for all } f\in H,\ \rho'\geq\rho_0,\ \textit{where } \Omega_{\rho'} = \Omega\cap B_{\rho'}\,,$$

(3.3) $\tilde{E}: D(A^l) \to D(A_0^l)$ *is bounded with respect to the graph norm,*

(3.4) $$\tilde{E}\,P_+^\rho = P_{+,0}^\rho\,\tilde{E}\,.$$

Proof of Lemma 3.2. Note that it is sufficient to construct a linear operator $E^{(1)}$: $C_0^\infty(\bar{\Omega})\to C_0^\infty(\boldsymbol{R}^n)$ satisfying

(3.5) $$E^{(1)}\,u = u \qquad \text{in} \quad |x|>\rho_0\,,$$

(3.6) $$\|\partial_x^\alpha\, E^{(1)}\, u\|_{L^2(\boldsymbol{R}^n)}\leq C \sum_{1\leq|\beta|\leq|\alpha|} \|\partial_x^\beta\, u\|_{L^2(\Omega)} \quad \text{for any } u\in C_0^\infty(\bar{\Omega}) \text{ and } |\alpha|\geq 1\,,$$

$$(3.7)\qquad |||E^{(1)}u|||_{B\rho'}\leq C\,|||u|||_{\Omega\rho'} \quad \text{for any } u\in C_0^\infty(\bar{\Omega}),\ \rho'\geq\rho_0\,,$$

where $C>0$ is independnet of $\rho'\geq\rho_0$ and

$$|||u|||_D^2=\frac{1}{2}\int_D\sum_{i,p,j,q=1}^n a_{ipjq}\,\partial_{x_j}u_q(x)\,\overline{\partial_{x_i}u_p(x)}\,dx\,.$$

Because, employing the extension operator $E^{(2)}$: $C_0^\infty(\bar{\Omega})\to C_0^\infty(\boldsymbol{R}^n)$ obtained by Seeley [12], we define $\tilde{E}$ as $\tilde{E}f={}^t(E^{(1)}f_1, E^{(2)}f_2)$, and then we can check that $\tilde{E}$ posesses the properties (3.1)$\sim$(3.3). The equality (3.4) is guaranteed by the same argument as for (1.6).

Now, we construct $E^{(1)}$ by the same methods as in Ito [3]. For $u\in C_0^\infty(\bar{\Omega})$, we define $b_{ij}(u)$ and $a_i(u)$ as

$$b_{ij}(u)=\frac{1}{2|\Omega_{\rho_0}|}\int_{\Omega\rho_0}\{\partial_{x_j}u_i(x)-\partial_{x_i}u_j(x)\}\,dx\,,$$

$$a_i(u)=\frac{1}{|\Omega_{\rho_0}|}\int_{\Omega\rho_0}(u_i(x)-\sum_{j=1}^n b_{ij}(u)\,x_j)\,dx\,,$$

where $|\Omega_{\rho_0}|$ is the volume of Ω_{ρ_0}. Set

$$(E^{(1)}u(x))_i=\psi(x)\,v_i(x)+(a_i(u)+\sum_{j=1}^n b_{ij}(u)\,x_j)\quad(i=1,2,\cdots,n)\,,$$

where $v_i(x)=u_i(x)-(a_i(u)+\sum_{j=1}^n b_{ij}(u)\,x_j)$. Then $E^{(1)}$ is linear, and (3.5) and (3.6) are obvious.

Noting that $b_{ij}(u)$ is skew symmetric and the assumption (A.1), we have

$$(3.8)\qquad |||E^{(1)}u|||_{B\rho'}=|||\psi v|||_{B\rho'}\,,$$

where $v={}^t(v_1,v_2,\cdots,v_n)$. Hence, the assumption (A.1) implies that

$$|||E^{(1)}u|||_{B\rho'}^2=\frac{1}{2}\int_{B\rho'}\langle\mathcal{E}(x)+\tilde{\mathcal{E}}(x),\mathcal{E}(x)+\tilde{\mathcal{E}}(x)\rangle\,dx\,,$$

where $\mathcal{E}(x)=(\mathcal{E}_{ij}(x))$, $\mathcal{E}_{ij}(x)=\psi(x)\,(\partial_{x_i}v_j(x)+\partial_{x_j}v_i(x))/2$, $\tilde{\mathcal{E}}(x)=(\tilde{\mathcal{E}}_{ij}(x))$, $\tilde{\mathcal{E}}_{ij}(x)=((\partial_{x_i}\psi)(x)\,v_j(x)+(\partial_{x_j}\psi)(x)\,v_i(x))/2$, and

$$\langle\mathcal{E},\tilde{\mathcal{E}}\rangle=\sum_{i,p,j,q=1}^n a_{ipjq}\,\tilde{\mathcal{E}}_{jq}\,\overline{\mathcal{E}_{ip}}\,.$$

Since the assumptions (A.1) and (A.2) yield that $\langle\ ,\ \rangle$ is an inner product in the space of the symmetric $n\times n$-matrix, we obtian

$$(3.9)\qquad |||E^{(1)}u|||_{B\rho'}^2\leq\int_{B\rho'}\{\langle\mathcal{E}(x),\mathcal{E}(x)\rangle+\langle\tilde{\mathcal{E}}(x),\tilde{\mathcal{E}}(x)\rangle\}\,dx \leq|||v|||_{\Omega\rho'}^2+C_2\,||v||_{L^2(\Omega\rho_0)}^2\,,$$

where $C_2=\max\{\sum_{i,p,j,q=1}^n a_{ipjp}\,\partial_{x_j}\psi(x)\,\partial_{x_i}\psi(x)\,\xi_q\,\bar{\xi}_p\,|\,x\in\boldsymbol{R}^n,\ \xi\in\boldsymbol{C}^n$ with $|\xi|=$

1}. Since we have $\int_{\Omega\rho_0} v(x)\,dx=0$ and $\int_{\Omega\rho_0} \{\partial_{x_i} v_j(x)-\partial_{x_j} v_i(x)\}\,dx=0$ for $i, j=1, 2, \cdots, n$, Poincare's inequality, Korn's second inequality, (cf. Ito [3] or Duvant and Lions [2]) and the assumption (A.2) imply that

$$||v||_{L^2(\Omega\rho_0)}\leq C(\rho_0)\,||\nabla_x v||_{L^2(\Omega\rho_0)}\leq C'(\rho_0)\,|||v|||_{\Omega\rho'}\,.$$

Combining the above estimate with (3.9), we have

$$|||E^{(1)} u|||^2_{B\rho'}\leq(1+C_2\,C'(\rho_0))\,|||v|||_{\Omega\rho'}\,.$$

Thus the estimate (3.7) follows from $|||v|||_{\Omega\rho'}=|||u|||_{\Omega\rho'}$ which is guaranteed by the same reason as for (3.8). This completes the proof of Lemma 3.2.

Next, we give the following estimate which is indispensable to prove Proposition 3.1.

$$\begin{aligned}(3.10)\qquad ||X(t, t')f||_{H_0}\leq C\sum_{j=1}^{N}||V(t-\frac{rj}{C_{\min}}, t')f||_{H(\Omega\rho+\tilde{\rho})}\\ +||P^{\rho}_{+,0}\,EU_0\left(\frac{rN}{C_{\min}}\right)\tilde{E}V\left(t-\frac{rN}{C_{\min}}, t'\right)f||_{H_0}\end{aligned}$$

for any $t\geq rN/C_{\min}$, $t'\geq0$, $f\in H$, and $N\geq1$ integer, where

$$V(t, t')=U(t)\,(U(t')-\psi_r U_0(t')\,E)\,.$$

In the above, we set $r=2^{-1}\,C_{\min}\,C^{-1}_{\max}\,\tilde{\rho}$, where $\tilde{\rho}>0$ is the same constant as in the assumption (**D**).

Proof of (3.10). We denote by $\tilde{H}$ the Hilbert space of the data with the Neumann boundary condition and define an operator $M\colon H\to\tilde{H}$ as

$$M=U\left(\frac{r}{C_{\min}}\right)-U_0\left(\frac{r}{C_{\min}}\right)\tilde{E}\,.$$

The operator M is well-defined because H is a closed subspace of $\tilde{H}$ and the restriction operator $H_0\ni f\mapsto f|_{\Omega}\in\tilde{H}$ is well-defined and bounded. Since the propagation speed is less than $C_{\max}$, by the estimate (3.2) we obtain $Mf=0$ in $|x|>\rho+(\tilde{\rho}/2)$ for any $f\in H$ and

$$(3.11)\qquad ||Mf||_{\tilde{H}}\leq C_3\,||f||_{H(\Omega\rho+\tilde{\rho})}\quad\text{for any } f\in H\,.$$

Now, by induction we start to prove that the estimate (3.10) holds with the constant $C=||E||\max\{C_3, C_1(C_1+C_3+1)\}>0$, where $C_1>0$ (resp. $C_3>0$) is the same constant as in (3.2) (resp. (3.11)).

For the case of $N=1$, we decompose the operator $X(t, t')$ in the following way:

$$X(t, t') = P^{\rho}_{+,0}\, EMV\left(t-\frac{r}{C_{\min}}, t\right) + P^{\rho}_{+,0}\, EU_0\left(\frac{r}{C_{\min}}\right)\tilde{E}V\left(t-\frac{r}{C_{\min}}, t\right),$$

where we note that E is able to be defined on $\tilde{H}$ and has the same properties as (1.2), (1.4) and (1.5). Combining the above equality with the estimate (3.11), we obtain (3.10) for $N=1$ with the constant $C=||E||\max\{C_3, C_1(C_1+C_3+1)\}$.

Next, assuming that (3.10) is true for an integer $N\geq 1$, we show the estimate (3.10) for $N+1$. First, we decompose $\tilde{E}V(t-rN/C_{\min}, t')$ as

$$\begin{aligned}\tilde{E}V\left(t-\frac{rN}{C_{\min}}, t'\right) &= \tilde{E}MV\left(t-\frac{r(N+1)}{C_{\min}}, t'\right)\\ &+\left(\tilde{E}U_0\left(\frac{r}{C_{\min}}\right)-U_0\left(\frac{r}{C_{\min}}\right)\right)\tilde{E}V\left(t-\frac{r(N+1)}{C_{\min}}, t'\right)\\ &+U_0\left(\frac{r}{C_{\min}}\right)\tilde{E}V\left(t-\frac{r(N+1)}{C_{\min}}, t'\right).\end{aligned} \tag{3.12}$$

From the property (3.1) and the estimate (3.2), it follows that

$$||\left(\tilde{E}U_0\left(\frac{r}{C_{\min}}\right)-U_0\left(\frac{r}{C_{\min}}\right)\right)\tilde{E}g||_{H_0}\leq(C_1+1)\,||U_0\left(\frac{r}{C_{\min}}\right)\tilde{E}g||_{H(B\rho)},$$

which yields

$$||\left(\tilde{E}U_0\left(\frac{r}{C_{\min}}\right)-U_0\left(\frac{r}{C_{\min}}\right)\right)\tilde{E}g||_{H_0}\leq C_1(C_1+1)\,||g||_{H(\Omega\rho+\tilde{\rho}/2)}, \tag{3.13}$$

because of (3.2) and the fact that the propagation speed is less than $C_{\max}$. Hence, the estimates (3.11), (3.13) and the decomposition (3.12) imply that

$$\begin{aligned}&||P^{\rho}_{+,0}\, EU_0\left(\frac{rN}{C_{\min}}\right)\tilde{E}V\left(t-\frac{rN}{C_{\min}}, t'\right)f||_{H_0}\\ &\leq||E||\,\{C_1 C_3+C_1(C_1+1)\}\,||V\left(t-\frac{r(N+1)}{C_{\min}}, t'\right)f||_{H(\Omega\rho+\tilde{\rho})}\\ &+||P^{\rho}_{+,0}\, EU_0\left(\frac{r(N+1)}{C_{\min}}\right)\tilde{E}V\left(t-\frac{r(N+1)}{C_{\min}}, t'\right)f||_{H_0}\\ &\qquad\text{for any } t\geq\frac{r(N+1)}{C_{\min}},\ t'\geq 0 \text{ and } f\in H.\end{aligned}$$

Combining the above estimate with the assumption of the induction yields (3.10) for $N+1$ with the constant $C=||E||\max\{C_3, C_1(C_1+C_3+1)\}>0$. This completes the proof of (3.10) for each integer $N\geq 1$.

Now, we prove Proposition 3.1. First, note that we have the following estimate.

Lemma 3.3. *Under the same assumption that in Propotition* 3.1, *there is a function* $q\in C([0,\infty))$ *such that* $\lim_{t\to\infty} q(t)=0$ *and*

$$\|V(t,t')f\|_{H(\Omega_\rho+\tilde{\rho})}\leq q(t)\,\|f\|_H \quad \text{for any } t,t'\geq 0 \text{ and } f\in H.$$

We postpone the proof of Lemma 3.3, which will be given in the last part of this section.

By means of Lemma 3.3 and (3.10), we can get Proposition 3.1 if we show that there is a constant $C>0$ such that

$$\|P^{\rho}_{+,0}\,EU_0(s)\,\tilde{E}V(t,t')\|_{B(H,H_0)}\leq C(1+s)^{-(n/2)+1} \quad \text{for any } t,t',s\geq 0. \tag{3.14}$$

Noting that $P^{\rho}_{+,0}\,E(I-P^{\rho}_{+,0})\,g=0$ for any $g\in H_0$, we have $P^{\rho}_{+,0}\,EU_0(s)\,\tilde{E}V(t,t')=P^{\rho}_{+,0}\,EP^{\rho}_{+,0}\,U_0(s)\,\tilde{E}V(t,t')$. Thus, by means of Proposition 2.2, to obtain (3.14) it is sufficient to prove that there exists a constant $C_0>0$ satisfying

$$\tilde{E}V(t,t')\in\Lambda\left(C_0,\frac{n-1}{2};H\right) \quad \text{for any } t,t'\geq 0. \tag{3.15}$$

First, we write $\tilde{E}V(t,t')f$ as follows:

$$\tilde{E}V(t,t')f=k_1(t,t';f)+k_2(t,t';f), \tag{3.16}$$

where

$$\begin{aligned} k_1(t,t';f) &= (\tilde{E}-E)\,U(t)\,(U(t')-\psi_\rho U_0(t')\,E)f \\ &\quad +U_0(t)\,(U_0(t')-E\,\psi_\rho U_0(t'))\,Ef \\ k_2(t,t';f) &= (EU(t+t')-U_0(t+t')\,E)f \\ &\quad -(EU(t)-U_0(t)\,E)\,\psi_\rho U_0(t')\,Ef. \end{aligned}$$

Since $\operatorname{supp}(g-E\psi_\rho g)\subset B_\rho$ for any $g\in H_0$ and $\operatorname{supp}((\tilde{E}-E)g)\subset B_\rho$ for any $g\in H$, the same argument as for the estimate (1.7) implies that

$$\begin{aligned} &\|(\partial_s^l\,T_0^+\,k_1(t,t';f))\,(s,\cdot)\|_{L^2(S^{n-1})}\leq C\,|s|^{-(n+1)/2-l}\,\|f\|_H \\ &\text{for any } s<-C_{\min}^{-1}(\rho+2),\ t,t'\geq 0,\ f\in H, \\ &\text{and } l=0,1,\cdots,\frac{n}{2}+1. \end{aligned} \tag{3.17}$$

Combining the estimate $e_f(t)=\|P^{\rho}_{+,0}\,EU(t)f\|^2_{H_0}\leq\|E\|^2\,\|f\|^2_H$ with the estimate in Lemma 1.2, we have

$$\begin{aligned} &\|(\partial_s^l\,T_0^+\,k_2(t,t';f))\,(s,\cdot)\|_{L^2(S^{n-1})}\leq C\,|s|^{-(n-1)/2-l}\,\|f\|_H \\ &\text{for any } s<-C_{\min}^{-1}(\rho+2),\ t,t'\geq 0,\ f\in H, \text{ and } l=0,1,\cdots,\frac{n}{2}+1. \end{aligned} \tag{3.18}$$

From (3.16), (3.17) and (3.18), it follows that

$$\|(\partial_s^l T_0^+ \tilde{E}V(t,t')f)(s,\cdot)\|_{L^2(S^{n-1})} \leq C\,|s|^{-(n-1)/2-l}\,\|f\|_H$$
$$\text{for any } s < -C_{\min}^{-1}(\rho+2),\ t, t' \geq 0,\ f \in H,$$
$$\text{and } l = 0, 1, \cdots, \frac{n}{2}+1.$$

Hence, combining the above estimate with the fact that $\tilde{E}V(t,t')\colon H \to H_0$ are uniformly bounded with respect to $t, t' \geq 0$, we get (3.15). This completes the proof of Proposition 3.1.

The rest in this section is devoted to proof of Lemma 3.3. To start with, we decompose $V(t,t')f$ as

$$V(t,t')f = (U(t)-\psi U_0(t)E)(U(t')-\psi U_0(t')E)f + \psi U_0(t)E(U(t')-\psi U_0(t')E)f. \tag{3.19}$$

By the same reason as for (3.15) with $t=0$ and $t' \geq 0$, it follows that there is a constant $C_0 > 0$ such that

$$E(U(t')-\psi U_0(t')E) \in \Lambda\left(C_0, \frac{n-1}{2}; H\right) \quad \text{for any } t' \geq 0. \tag{3.20}$$

Thus, Proposition 2.3 implies that

$$\|\psi U_0(t)E(U(t')-\psi U_0(t')E)f\|_{H(\Omega_{\rho+\tilde{\rho}})} \leq C(1+t)^{-n+2}\|f\|_H \quad \text{for all } t, t' \geq 0 \text{ and } f \in H, \tag{3.21}$$

since

$$\|\psi g\|_{H(\Omega_{\rho'})} \leq C\{\|g\|_{L^2(\Omega_{\rho'})} + \|g_1\|_{L^2(\Omega_{\rho'})}\} \quad \text{for any } \rho' \geq \rho \text{ and } g \in H_0. \tag{3.22}$$

For any $g \in D(A)$, we set $h(t) = (U(t)-\psi U_0(t)E)g$. Note that $h(t) \in D(A)$ (cf. (1.3)). Since $Af = {}^t(f_2, A(\partial_x)f_1)$ for $f \in D(A)$ (cf. §2 of [13]), we have

$$\begin{cases} \dfrac{d}{dt}h(t) = Ah(t) + QU_0(t)Eg & \text{for } t \in \boldsymbol{R}, \\ h(0) = (I-\psi E)g, \end{cases}$$

where $Q\colon H_0 \to H$ is the bounded operator defined as

$$Qf = {}^t(0, A(\partial_x)(\psi f_1) - \psi A(\partial_x)f_1). \tag{3.23}$$

Thus Duhamel's principle and the fact that $D(A)$ is dense in H yield

$$(U(t)-\psi U_0(t)E)g = U(t)(I-\psi E)g + \int_0^t U(t-s)QU_0(s)Eg\,ds \quad \text{for any } t \in \boldsymbol{R} \text{ and } g \in H.$$

Hence, we have

$$(3.24)\qquad \|(U(t)-\psi U_0(t)\,E)\,g_{t'}\|_{H(\Omega_{\rho+\tilde{\rho}})}\leq\|U(t)\,(I-\psi E)\,g_{t'}\|_{H(\Omega_{\rho+\tilde{\rho}})} +\left\|\int_0^t U(t-s)\,QU_0(s)\,Eg_{t'}\,ds\right\|_{H(\Omega_{\rho+\tilde{\rho}})},$$

where we set $g_{t'}=(U(t')-\psi U_0(t')\,E)f$. Since $\mathrm{supp}\,(I-\psi E)\,g_{t'}\subset\bar{\Omega}\cap B_\rho$ and $\|(I-\psi E)\,g_{t'}\|_H\leq C\|f\|_H$ for a constant C independent of $t'\geq 0$ and $f\in H$, the assumption (D) implies that

$$(3.25)\qquad \|U(t)\,(I-\psi E)\,g_{t'}\|_{H(\Omega_{\rho+\tilde{\rho}})}\leq Cp(t)\,\|f\|_H \quad \text{for any } t,\, t'\geq 0 \text{ and } f\in H\,.$$

Noting that $\mathrm{supp}\,Qg\subset\bar{\Omega}\cap B_\rho$ for any $g\in H_0$, we obtain

$$(3.26)\qquad \|U(t-s)\,QU_0(s)\,Eg_{t'}\|_{H(\Omega_{\rho+\tilde{\rho}})}\leq p(t-s)\,\|QU_0(s)\,Eg_{t'}\|_H \quad \text{for any } t\geq s\geq 0,\, t'\geq 0,\, f\in H\,,$$

by the assumption (D). Since the definition of Q (cf. (3.23)) implies that

$$(3.27)\qquad \|Qg\|_H\leq C\,\{\|g\|_{H(\Omega_\rho)}+\|g_1\|_{L^2(\Omega_\rho)}\} \quad \text{for any } g\in H_0\,,$$

Proposition 2.3 and the fact (3.20) yield that there exists a constant $C>0$ such that

$$(3.28)\qquad \|QU_0(s)\,Eg_{t'}\|_H\leq C\,(1+s)^{-(n-2)}\,\|f\|_H \quad \text{for any } s,\, t'\geq 0 \text{ and } f\in H\,.$$

From the estimates (3.26) and (3.28), it follows that

$$\|U(t-s)\,QU_0(s)\,Eg_{t'}\|_{H(\Omega_{\rho+\tilde{\rho}})}\leq Cp(t-s)\,(1+s)^{-(n-2)}\,\|f\|_H \quad \text{for any } t\geq s\geq 0,\, t'\geq 0 \text{ and } f\in H\,,$$

which yields

$$(3.29)\qquad \left\|\int_0^t U(t-s)\,QU_0(s)\,Eg_{t'}\,ds\right\|_{H(\Omega_{\rho+\tilde{\rho}})}\leq CJ(t)\,\|f\|_H \quad \text{for any } t,\, t'\geq 0 \text{ and } f\in H\,.$$

In (3.29), we set $J(t)=\int_0^t p(t-s)\,(1+s)^{-(n-2)}\,ds=\int_0^\infty \chi(t,s)\,p(t-s)\,(1+s)^{-(n-2)}\,ds$, where $\chi(t,s)=1$ for $s\leq t$ and $\chi(t,s)=0$ for $s>t$. Noting that $n\geq 4$ and $\lim_{t\to\infty}\chi(t,s)\,p(t-s)=0$ for all fixed $s\geq 0$, we have $\lim_{t\to\infty}J(t)=0$ by the Lebesgue convergence theorem. Hence, from (3.19), (3.21), (3.24), (3.25), and (3.29), it follows that there exists a function $q(t)\in C([0,\infty))$ satisfying $\lim_{t\to\infty}q(t)=0$ and

$$\|V(t, t')f\|_{H(\Omega_\rho+\tilde{\rho})} \leq q(t)\,\|f\|_H \quad \text{for any } t, t' \geq 0 \text{ and } f \in H,$$

where $q(t)$ is of the form $q(t)=C\{(1+t)^{-(n-2)}+p(t)+J(t)\}$. This completes the proof of Lemma 3.3.

4. The uniform decay rate of $\|P^\rho_{+,0}\, EU(t)\, f\|_{H_0}$

Our goal in this section is to prove the following property which plays an important role in the proof of Theorem 0.1.

Proposition 4.1. *Under the same assumption that in Theorem* 0.1, *there exists a constant* $C=C(\rho)>0$ *such that*

$$\|P^\rho_{+,0}\, EU(t)f\|_{H_0} \leq C(1+t)^{-n/2}\,\|f\|_H \quad \textit{for any } t \geq 0 \textit{ and } f \in H^\rho.$$

Proposition 4.1 is a direct consquence from Corollary 1.3 and the following proposition.

Proposition 4.2. *Under the same assumption that in Theorem* 0.1, *for any* $C_0>0$ *and* $\alpha>1/2$, *there exists a constant* $C=C(\rho, \alpha, C_0)>0$ *such that*

$$\|P^\rho_{+,0}\, EU(t)f\|_{H_0} \leq C(1+t)^{-(\alpha-1/2)}\,\|f\|_H \quad \textit{for any } t \geq 0 \textit{ and } f \in V_\alpha(C_0).$$

In fact, (1.8), Proposition 4.2 and (1.10) imply that $H^\rho \subset V_{n/2}(C'_0)$ for some fixed constant $C'_0=C'_0(\rho)>0$. Hence, from Proposition 4.2 and (1.9) it follows that there is a constant $C_0=C_0(\rho)>0$ such that

$$H^\rho \subset V_{(n+1)/2}(C_0), \tag{4.1}$$

which yields Proposition 4.1. This procedure is the same as in Melrose [10].

In the following of this section, we prove Proposition 4.2. The original idea of the proof is given by Melrose [10]. However, his proof is very complicated. Hence, we give the different proof of Proposition 4.2. We set

$$T_p = \inf\{t \geq 0 \mid \|P^\rho_{+,0}\, EU(t')f\|_{H_0} \leq 2^{-p}\,\|Ef\|_{H_0} \quad \text{for any } t' \geq t \text{ and } f \in V_\alpha(C_0)\},$$

for each non-negative integer p. Note that $T_0 \geq 0$ exists. By induction we show that $T_p \geq 0$ exists really for every $p \geq 1$. Noting that $P^\rho_+\, U(t)\,(I-P^\rho_+)=0$ for all $t \geq 0$, we have

$$\begin{aligned} P^\rho_+\, U(t+t'+T_p)f &= P^\rho_+\, U(t)\, U(t')\, P^\rho_+\, U(T_p)f \\ &= P^\rho_+\, U(t)\,(U(t')-\psi U_0(t')\, E)\, P^\rho_+\, U(T_p)f \end{aligned}$$

$$+P_+^\rho U(t) P_+^\rho \psi U_0(t') E P_+^\rho U(T_p) f$$
for any $t, t' \geq 0$ and $f \in H$.

Thus, the property (1.6) implies that

$$\text{(4.2)} \quad \begin{aligned} P_{+,0}^\rho EU(t+t'+T_p) f = {} & X(t, t') P_+^\rho U(T_p) f \\ & + P_{+,0}^\rho EU(t) P_+^\rho \psi U_0(t') P_{+,0}^\rho EU(T_p) f \\ & \text{for any } t, t' \geq 0 \text{ and } f \in H . \end{aligned}$$

Proposition 3.1 yields that there exists a constant $T>0$ such that

$$\|X(t, t')\|_{B(H,H_0)} \leq 1/4 \quad \text{for any } t \geq T \text{ and } t' \geq 0 ,$$

which imples that

$$\text{(4.3)} \quad \begin{aligned} & \|X(t, t') P_+^\rho U(T_p) f\|_{H_0} \leq 2^{-(p+2)} \|Ef\|_{H_0} \\ & \quad \text{for any } t \geq T, t' \geq 0 \text{ and } f \in V_\alpha(C_0) . \end{aligned}$$

In (4.3), we use the definition of T_p and the estimate

$$\|P_+^\rho U(T_p) f\|_H \leq \|EP_+^\rho U(T_p) f\|_{H_0} = \|P_{+,0}^\rho EU(T_p) f\|_{H_0} .$$

Noting that $\psi P_{+,0}^\rho g \in (D_+^\rho)^\perp$ (cf. the argument for (1.6)) and $(\psi-1)(I-P_{+,0}^\rho) g=0$ for any $g \in H_0$ we have

$$\text{(4.4)} \quad P_+^\rho(\psi g) = \psi P_{+,0}^\rho g \quad \text{for any } g \in H_0 ,$$

which yields

$$\text{(4.5)} \quad \begin{aligned} & P_{+,0}^\rho EU(t) P_+^\rho \psi U_0(t') P_{+,0}^\rho EU(T_p) f \\ & \quad = P_{+,0}^\rho EU(t) \psi P_{+,0}^\rho U_0(t') EU(T_p) f \\ & \quad \text{for any } t, t' \geq 0 \text{ and } f \in H , \end{aligned}$$

where we use the fact that $P_{+,0}^\rho U_0(t)(I-P_{+,0}^\rho)=0$ for all $t \geq 0$. Since the operator $P_{+,0}^\rho EU(t)\psi$: $H_0 \to H_0$ is uniformly bounded for any $t \in \boldsymbol{R}$, Corollary 2.4 and (4.5) imply that there is a constant $C_1=C_1(\alpha, C_0)>0$ satisfying

$$\begin{aligned} & \|P_{+,0}^\rho EU(t) P_+^\rho \psi U_0(t') P_{+,0}^\rho EU(T_p) f\|_{H_0} \leq C_1(1+t')^{-(\alpha-1/2)} \|Ef\|_{H_0} \\ & \quad \text{for any } t, t' \geq 0 \text{ and } f \in V_\alpha(C_0) . \end{aligned}$$

Now, we take t_p' as $C_1 t_p'^{-(\alpha-1/2)}=2^{-(p+2)}$, that is, $t_p'=C_2 2^{2(p+2)/(2\alpha-1)}$ where $C_2=C_2^{2/(2\alpha-1)}$. Then it follows that

$$\begin{aligned} & \|P_{+,0}^\rho EU(t) P_+^\rho \psi U_0(t_p') P_{+,0}^\rho EU(T_p) f\|_{H_0} \leq 2^{-(p+2)} \|Ef\|_{H_0} \\ & \quad \text{for any } t \geq 0 \text{ and } f \in V_\alpha(C_0) . \end{aligned}$$

Combining the above estimate with (4.3) and (4.2), we get

$$(4.6)\qquad \|P^{\rho}_{+,0}\, EU(t+t'_p+T_p)f\|_{H_0}\le 2^{-(p+1)}\,\|Ef\|_{H_0} \quad \text{for any } t\ge T \text{ and } f\in V_\alpha(C_0)\,.$$

By (4.6), existence of T_{p+1} is obvious and furthermore we obtain that $T_{p+1}\le C_3\, 2^{2(p+2)/(2\alpha-1)}+T_p$ for any positive integer p, where the constant $C_3=C_2+T$ is independent of p. Now, we define $\{\widetilde{T}_p\}_{p=0,1,\cdots}$ as $\widetilde{T}_{p+1}=\widetilde{T}_p+C_3\, 2^{2(p+2)/(2\alpha-1)}$, $\widetilde{T}_0=0$. Then it follows that $\widetilde{T}_p\ge T_p$ for any $p=0, 1, \cdots$, and there exist constants $C_4>0$ and $C_5>0$ satisfying $\widetilde{T}_p\le C_4\, 2^{2p/(2\alpha-1)}$ and $\widetilde{T}_{p+1}\le C_5\,\widetilde{T}_p$ for any $p=1, 2, \cdots$. Thus, for any $t\ge 0$ with $\widetilde{T}_p\le t\le \widetilde{T}_{p+1}$ and $p\ge 1$, we have

$$(4.7)\qquad \begin{aligned}\|P^{\rho}_{+,0}\, EU(t)f\|_{H_0}&\le 2^{-p}\,\|Ef\|_{H_0}\\ &\le (C_4\, C_5)^{(\alpha-1/2)}\,\|E\|\, t^{-(\alpha-1/2)}\,\|f\|_H\\ &\quad \text{for any } f\in V_\alpha(C_0)\,,\end{aligned}$$

where we use $2^{-p}\le C_4^{(\alpha-1/2)}\,\widetilde{T}_p^{-(\alpha-1/2)}\le (C_4\, C_5)^{(\alpha-1/2)}\,\widetilde{T}_{p+1}^{-(\alpha-1/2)}$ and $\widetilde{T}_{p+1}^{-1}\le t^{-1}$. Combining the estimate (4.7) with the fact that $\|P^{\rho}_{+,0}\, EU(t)f\|_{H_0}\le \|E\|\,\|f\|_H$ for any $t\ge 0$ and $f\in H$, we obtain Proposition 4.2.

5. Proof of Theorem 0.1

Now, we begin to prove (0.2). To start with, for any $t, t'\ge 0$ and $f\in H$ we have

$$(5.1)\qquad \begin{aligned}\|U(2t)f\|_{H(\Omega\rho)}&\le \|(U(t)-\psi_\rho U_0(t)\, E)\, U(t)f\|_{H(\Omega\rho)}\\ &\quad +\|\psi_\rho U_0(t)\, EU(t)f\|_{H(\Omega\rho)}\,.\end{aligned}$$

From Corollary 2.4, (4.1) and the estimate (3.22), it follows that there exists a constant $C_1>0$ satisfying

$$(5.2)\qquad \|\psi_\rho U_0(t)\, EU(t)f\|_{H(\Omega\rho)}\le C_1(1+t)^{-n+1}\,\|f\|_H \quad \text{for any } t\ge 0 \text{ and } f\in H^\rho\,.$$

In the same way as (3.24), we have

$$(5.3)\qquad \|(U(t)-\psi_\rho U_0(t)\, E)\, U(t)f\|_{H(\Omega\rho)}\le \sum_{j=1}^{3} I_j(t;f)\,,$$

where

$$\begin{aligned}I_1(t;f)&=\|U(t)\,(I-\psi_\rho E)\, U(t)f\|_{H(\Omega\rho)}\,,\\ I_2(t;f)&=\Big\|\int_0^{t/2} U(t-s)\, QU_0(s)\, EU(t)f\, ds\Big\|_{H(\Omega\rho)}\,,\\ I_3(t;f)&=\Big\|\int_{t/2}^{t} U(t-s)\, QU_0(s)\, EU(t)f\, ds\Big\|_{H(\Omega\rho)}\,.\end{aligned}$$

Since $P_+^\rho g=g$ in Ω_ρ, from Proposition 4.1, (1.2) and (1.6) it follows that

$$\|U(t)g\|_{H(\Omega_\rho)}\leq C_2(1+t)^{-n/2}\|g\|_H \quad \text{for any } t\geq 0 \text{ and } g\in H^\rho. \tag{5.4}$$

Noting that $P_+^\rho U(t)(I-P_+^\rho)=0$ for any $t\geq 0$ and $P_+^\rho(\psi_\rho g)=\psi_\rho P_+^\rho g$ for any $g\in H$ which is proved by the same argument to deduce (4.4), we have

$$\begin{aligned} I_1(t;f) &= \|U(t)(1-\psi_\rho E)P_+^\rho U(t)f\|_{H(\Omega_\rho)} \\ &\leq C_2(1+t)^{-n/2}\|(1-\psi_\rho E)P_+^\rho U(t)f\|_H, \end{aligned}$$

where we use (5.4) and supp $(1-\psi_\rho E)g\subset B_\rho$. Hence, Proposition 4.1 implies

$$I_1(t;f)\leq C_3(1+t)^{-n}\|f\|_H \quad \text{for any } t\geq 0 \text{ and } f\in H^\rho. \tag{5.5}$$

From the estimate (3.27) and the fact that $P_{+,0}^\rho g=g$ in B_ρ, it follows that $\|Qg\|_H\leq C\{\|P_{+,0}^\rho g\|_{H(\Omega_\rho)}+\|[P_{+,0}^\rho g]_1\|_{L^2(\Omega_\rho)}\}$, which yields

$$\|QU_0(s)EU(t)f\|_H\leq C_4\|P_{+,0}^\rho U_0(s)EU(t)f\|_{H_0} \quad \text{for any } s, t\geq 0 \text{ and } f\in H^\rho, \tag{5.6}$$

if we note the estimate (1.1), Korn's inequality (cf. [3] or [11]) and the assumption (A.2). Since we have $P_{+,0}^\rho U_0(s)(I-P_{+,0}^\rho)=0$ for all $s\geq 0$, Proposition 4.1 implies that

$$\|QU_0(s)EU(t)f\|_H\leq C_5(1+t)^{-n/2}\|f\|_H \quad \text{for any } s, t\geq 0 \text{ and } f\in H^\rho.$$

From the above inequality and (5.4), it follows that

$$\begin{aligned} &\|U(t-s)QU_0(s)EU(t)f\|_{H(\Omega_\rho)} \\ &\leq C_2C_5(1+t-s)^{-n/2}(1+t)^{-n/2}\|f\|_H \quad \text{for any } t\geq s\geq 0 \text{ and } f\in H^\rho, \end{aligned}$$

where we use the fact that supp $Qg\subset\bar{\Omega}\cap B_\rho$ for all $g\in H_0$ (cf. (3.23)). Hence, we obtain

$$I_2(t;f)\leq\frac{2C_2C_5}{n-2}\left(1+\frac{t}{2}\right)^{-n+1}\|f\|_H \quad \text{for any } t\geq 0 \text{ and } f\in H^\rho. \tag{5.7}$$

For $I_3(t;f)$, by means of the estimate (3.27), the same argument as for (5.2) yields that

$$\|QU_0(s)EU(t)f\|_H\leq C_6(1+s)^{-n+1}\|f\|_H \quad \text{for any } s, t\geq 0 \text{ and } f\in H^\rho. \tag{5.8}$$

From the estimates (5.4) and (5.8) it follows that

$$\|U(t-s)\,QU_0(s)\,EU(t)f\|_{H(\Omega_\rho)}$$
$$\le C_2\,C_6(1+t-s)^{-n/2}(1+s)^{-n+1}\,\|f\|_H$$
$$\text{for any } t\ge s\ge 0 \text{ and } f\in H^\rho\,.$$

Hence, we have

$$I_3(t;f)\le\frac{2C_2\,C_6}{n-2}\left(1+\frac{t}{2}\right)^{-n+1}\|f\|_H \tag{5.9}$$
$$\text{for any } t\ge 0 \text{ and } f\in H^\rho\,.$$

From (5.1), (5.2), (5.3), (5.5), (5.7) and (5.9) it follows that

$$\|U(2t)f\|_H\le C_7\left(1+\frac{t}{2}\right)^{-n+1}\|f\|_H$$
$$\text{for any } t\ge 0 \text{ and } f\in H^\rho\,,$$

which completes the proof of (0.2).

In the last, we prove (0.3). In the argument for (0.2), we replace $\|U(2t)f\|_{H(\Omega_\rho)}$ by $\|[U(2t)f]_2\|_{L^2(\Omega_\rho)}$. Then from the estimate (0.2) it follows that the estimate (0.3) holds if we obtain the following estimate:

$$I(t;f)\le C\left(1+\frac{t}{2}\right)^{-n}\|f\|_H \tag{5.10}$$
$$\text{for any } t\ge 0 \text{ and } f\in H^\rho\cap D(A)\,,$$

where

$$I(t;f)=\|\int_{t/2}^{t}[U(t-s)\,QU_0(s)\,EU(t)f]_2\,ds\|_{L^2(\Omega_\rho)}\,.$$

For any data $g\in D(A_0)$, we have

$$[U(t-s)\,QU_0(s)\,g]_2=[AU(t-s)\,QU_0(s)\,g]_1=[\left(\frac{d}{dt}\,U\right)(t-s)\,QU_0(s)\,g]_1\,,$$

where we note that $Qg\in D(A)$ if $g\in D(A_0)$, Since $[Qg]_1=0$ for any $g\in H_0$ (cf. (3.23)), the integration by parts yields

$$I(t;f)\le\int_{t/2}^{t}\|[U(t-s)\,Q\left(\frac{d}{ds}\,U_0\right)(s)\,EU(t)f]_1\|_{L^2(\Omega_\rho)}\,ds \tag{5.11}$$
$$+\|[U\left(\frac{t}{2}\right)QU_0\left(\frac{t}{2}\right)EU(t)f]_1\|_{L^2(\Omega_\rho)}$$
$$\text{for any } f\in H^\rho\cap D(A)\,.$$

By Proposition 4.1 we have

$$(5.12) \qquad \|[U(t-s)g]_1\|_{L^2(\Omega_\rho)} \leq C_8(1+t-s)^{-n/2}\|g\|_H \quad \text{for any } t\geq s\geq 0 \text{ and } g\in H,$$

where we use the same argument as for (5.6). From Corollary 2.5 and the estimate (3.27) it follows that

$$\|Q\left(\frac{d}{ds}U_0\right)(s)\,EU(t)f\|_H \leq C_9(1+s)^{-n}\|f\|_H \quad \text{for any } t\geq 0,\ s>2C_{\min}^{-1}(\rho+1) \text{ and } f\in H^\rho\cap D(A).$$

By the above estimate, (5.12) and (5.11) we have

$$I(t;f)\leq \frac{2C_8C_9}{n-2}\left(1+\frac{t}{2}\right)^{-n}\|f\|_H + \|[U\left(\frac{t}{2}\right)QU_0\left(\frac{t}{2}\right)EU(t)f]_1\|_{L^2(\Omega_\rho)}. \quad \text{for any } t>4C_{\min}^{-1}(\rho+1) \text{ and } f\in H^\rho\cap D(A).$$

The above estimate, (5.12) and (5.8) imply that

$$I(t;f)\leq\left(\frac{2C_8C_9}{n-2}+C_6C_8\right)\left(1+\frac{t}{2}\right)^{-n}\|f\|_H \quad \text{for any } t>4C_{\min}^{-1}(\rho+1) \text{ and } f\in H^\rho\cap D(A).$$

Since we have $I(t;f)\leq I_3(t;f)$ for any $t\geq 0$ and $f\in H$, from the above estimate and (5.9) it follows that the estimate (5.10) holds. This completes the proof of Theorem 0.1.

References

[1] P.G. Ciarlet: "Mathematical elasticity volume 1: three-dimensional elasticity," North-Holland, New York, 1988.

[2] G. Duvant and J.L. Lions: "Inequalities in Mechanics and Physics," Springer-Verlag, Berlin Heidelberg New York, 1976.

[3] H. Ito: *Extended Korn's inequalities and the associated best possible constants*, J. Elasticity **24** (1990), 43–78.

[4] H. Iwashita: *A remark on the analyticity of spectral functions for some exterior boundray value problem*, Sci. Rep. of Niigata Univ. **24** (1988), 25–31.

[5] H. Iwashita and Y. Shibata: *On the analyticity of spectral functions for some exterior boundray value problems*, Glasnik Matematicki **23 (43)** (1988), 291–313.

[6] B.V. Kapitonov: *On the behaviour as $t\to\infty$ of the solutions of the exterior boundary value problem for a hyperbolic system*, Siberian Math. J. **28** (1987), 444–457.

[7] M. Kawashita: *On the local energy decay property for the elastic wave equation with the Neumann boundary condition*, Duke Math. J. **67** (1992), 333–351.

[8] P.D. Lax, C.S. Morawetz and R.S. Phillips: *Exponential decay of solutions of*

the wave equation in the exterior of a star-shaped obstacle, Comm. Pure Appl. Math. **16** (1963), 477–486.
[9] P.D. Lax and R.S. Phillips: "Scattering theory revised edition," Academic Press, New York, 1989.
[10] R.B. Melrose: *Singularities and energy decay in acoustical scattering*, Duke Math. J. **46** (1979), 43–59.
[11] C.S. Morawetz: *Exponential decay of solutions of the wave equation*, Comm. Pure Appl. Math. **19** (1966), 439–444.
[12] R. Seeley: *Extension of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. **15** (1964), 625–626.
[13] Y. Shibata and H. Soga: *Scattering theory for the elastic wave equation*, Publ. RIMS Kyoto Univ. **25** (1989), 861–887.
[14] H. Soga: *Singularities of the scattering kernel for convex obstacles*, J. Math. Kyoto Univ. **22** (1983), 729–765.

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