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SOME EXAMPLES OF HYPOELLIPTIC OPERATORS OF INFINITELY DEGENERATE TYPE

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0. Introduction

The object of the present paper is to study some examples of the operators of the form

$$P = D_x^2 + a(x)D_y^2 + b(x)D_y\,, \tag{1}$$

on $\boldsymbol{R}^2$ where $D_x = -i\dfrac{\partial}{\partial x}$, $D_y = -i\dfrac{\partial}{\partial y}$, $a(x)$ and $b(x)$ are functions satisfying:

(2) (i) $a(x), b(x) \in C^\infty(\boldsymbol{R})$,
(ii) $a(x) > 0$ for $x \neq 0$, $\partial^\alpha a(0) = \partial^\alpha b(0) = 0$ for any α.

We consider here C^∞-hypoellipticity of the operator P on $x=0$. In general it is hypoelliptic if $b(x)$ is small compared with $a(x)$, and conversely, not hypoelliptic if $b(x)$ is big. Such conditions for the hypoellipticity were investigated in the previous paper [5]. But the examples considered here cannot be explained by the method of [5] (we cannot regard $b(x)$ small nor big in what follows). They are analogous to the one which A. Menikoff considered in [6], i.e., the finitely degenerate case where $a(x) = x^{2k}$ and $b(x) = bx^{k-1}$. We prove the following theorems.

Theorem 1. *Let* $a(x) = |x|^{-4}\exp(-2|x|^{-1})$ *and* $b(x) = b\cdot|x|^{-4}\exp(-|x|^{-1})$ *with b being a complex constant. Then the operator P is hypoelliptic if and only if b is not odd integer.*

Theorem 2. *Let* $a(x) = |x|^{-4}\exp(-2|x|^{-1})$ *and* $b(x) = b\cdot\operatorname{sgn} x\cdot|x|^{-4}\exp(-|x|^{-1})$ *with b being a complex constant. Then the operator P is hypoelliptic.*

Remark 1: By the similar argument of the proof of theorem 1 in T. Morioka [8], we can conclude that P is micro-hypoelliptic when P is hypoelliptic.

The hypoellipticity of P is closely connected to the branching of singularities of solutions for the weakly hyperbolic operator $Q = -D_x^2 + a(x)D_y^2 + b(x)D_y$.

G.R. Aleksandryan [1] dealt with the one for Q which corresponds the cases in Theorem 1 and Theorem 2. In Section 1, we shall prove the non-hypoellipticity part of Theorem 1, by using the observation of Aleksandryan. Section 2 is devoted to the proof of hypoellipticity parts of Theorem 1 and Theorem 2. We shall show them by constructing the parametrix of P explicitly.

The author is grateful to Professor K. Kajitani for introducing him the article [1] of Aleksandryan.

1. Proof of non-hypoellipticity

In this section we prove that P is not hypoelliptic if $a(x)$ and $b(x)$ are those in Theorem 1, and b satisfies $b=2n+1$ for some $n\in \boldsymbol{Z}$. Also we shall explain the reason why Theorem 2 is free from such a condition. Here we adopt the notations from Aleksandryan [1].

At first, let us set $\Lambda(x)=\exp(-|x|^{-1})$ and $\mu(x)=\Lambda'(x)$ $(=\operatorname{sgn} x\cdot|x|^{-2}\exp(-|x|^{-1}))$. Then the partial Fourier transform of the equation $Pu=0$ with respect to y can be written in the following form:

$$-\hat{u}_{xx}+\left(\mu(x)^2\eta^2+b\frac{\mu(x)^2}{\Lambda(x)}\eta\right)\hat{u}=0\,. \tag{3}$$

Furthermore making a change in such a way that $\hat{u}(x,\eta)=xw(\tau)$, $\tau=\Lambda(x)\eta$, it becomes

$$-w_{\tau\tau}-\frac{w_\tau}{\tau}+\left(1+\frac{b}{\tau}\right)w=0. \tag{4}$$

Set now $z=2\tau$ and $f(z)=e^{z/2}w\left(\frac{z}{2}\right)$. Then (4) turns into Kummer's equation

$$zf''(z)+(1-z)f'(z)-\alpha f(z)=0\,, \tag{5}$$

where $\alpha=\dfrac{1+b}{2}$. Hence we have the following

Proposition 1. (i) *Suppose* $\eta>0$. *Then there exist solutions* $\hat{u}_1(x,\eta)$ *and* $\hat{u}_2(x,\eta)$ *of* (3) *which have the following expressions:*

$$\hat{u}_1(x,\eta)=xe^{-\Lambda(x)\eta}\,\Psi(\alpha,1;2\Lambda(x)\eta)\qquad for\quad x>0\,,$$

and

$$\hat{u}_2(x,\eta)=-xe^{-\Lambda(x)\eta}\,\Psi(\alpha,1;2\Lambda(x)\eta)\qquad for\quad x<0\,,$$

where $\Psi(\alpha,1;z)$ *is a solution of* (5) *for* $z>0$ *defined in A. Erdelyi et al.* [2, *page* 255–256].

(ii) *Suppose* $\eta<0$. *Then there exist solutions* $\hat{u}_1(x,\eta)$ *and* $\hat{u}_2(x,\eta)$ *of* (3) *which have the following expressions:*

$$\hat{u}_1(x, \eta) = xe^{\Lambda(x)\eta}\, \Psi(1-\alpha, 1; -2\Lambda(x)\eta) \qquad \textit{for} \quad x>0\,,$$

and

$$\hat{u}_2(x, \eta) = -xe^{\Lambda(x)\eta}\, \Psi(1-\alpha, 1; -2\Lambda(x)\eta) \qquad \textit{for} \quad x<0\,.$$

REMARK 2: It holds that $\hat{u}_2(x, \eta)=\hat{u}_1(-x, \eta)$ for $x<0$. Generally, it does not hold that $\hat{u}_1(x, \eta)=-\hat{u}_2(x, \eta)$ (they are linearly independent in generic case), because $\Psi(\alpha, \gamma; z)$ is many-valued holomorphic function of z and its principal branch can be at most defined in the plane cut along negative real axis (see page 257 of [2]).

REMARK 3: Since $\Psi(\alpha, \gamma; z)=O(z^{-\alpha})$ as positive number z tends to infinity (see [2, page 278]), $\hat{u}_1(x, \eta)$ is uniformly bounded for $(\log 2|\eta|)^{-1}\leq x\leq 1$ and also $\hat{u}_2(x, \eta)$ is uniformly bounded for $-1\leq x\leq -(\log 2|\eta|)^{-1}$.

Proof. (i) We can see the result concerning $\hat{u}_1(x, \eta)$ since $z=\Lambda(x)\eta>0$ for $x>0$ rnd $\eta>0$ (recall that $\Psi(\alpha, 1; z)$ satisfies (5) for $z>0$). In order to obtain the result concerning $\hat{u}_2(x, \eta)$, we make the change of variable $\tilde{x}=-x$ in the equation (3) (notice that $\tilde{x}>0$ for $x<0$). Then (3) becomes the same equation with respect to the variable $\tilde{x}$ since $\Lambda(\tilde{x})=\Lambda(x)$ and $\mu(\tilde{x})^2=\mu(x)^2$. Thus we can see that there is a solution of (3) which have the expression: $\hat{u}_2(x, \eta)= \tilde{x}e^{-\Lambda(x)\eta}\, \Psi(\alpha, 1; 2\Lambda(\tilde{x})\eta)$ for $\tilde{x}>0$. This implies the result.

(ii) To obtain the result concerning $\hat{u}_1(x, \eta)$, we set $z=-2\tau$ and $f(z)= e^{z/2}w\left(-\frac{z}{2}\right)$ in the equation (4) (notice that $z=-2\Lambda(x)\eta>0$ for $x>0$, $\eta<0$). Then (4) becomes

$$zf''(z)+(1-z)f'(z)-(1-\alpha)f(z) = 0\,, \tag{5'}$$

and this implies the result. The argument to obtain the one concerning $\hat{u}_2(x, \eta)$ is also similar. ■

Next we investigate the Wronskian of $\hat{u}_1$ and $\hat{u}_2$, namely,

$$W(\eta) = \hat{u}_1(0, \eta)\hat{u}_2'(0, \eta)-\hat{u}_1'(0, \eta)\hat{u}_2(0, \eta)\,.$$

We can compute the value of $W(\eta)$ which is essential to the proof of Theorem 1.

Proposition 2. (i) *For $\eta>0$, it holds that*

$$W(\eta) = \frac{2}{\Gamma(\alpha)^2}\{\log 2\eta+\psi(\alpha)-2\psi(1)\}\,, \tag{6}$$

where $\Gamma(\alpha)$ is Euler's Gamma function and $\psi(\alpha)=\Gamma'(\alpha)/\Gamma(\alpha)$.

(ii) *For $\eta<0$, it holds that*

$$W(\eta)=\frac{2}{\Gamma(1-\alpha)^2}\{\log(-2\eta)+\psi(1-\alpha)-2\psi(1)\}. \tag{7}$$

Proof. Here we prove the case (i). The argument for the proof of (ii) is completely parallel if α and η are respectively replaced by $1-\alpha$ and $-\eta$.

At first, let us recall that $\Psi(\alpha, n+1; z)$ $(n=0, 1, \cdots)$ has the following asymptotic behavior as $z\downarrow 0$ (see page 261 of [2]):

$$\begin{aligned}\Psi(\alpha, n+1; z) &= \frac{(-1)^{n-1}}{n!\,\Gamma(\alpha-n)}\{\Phi(\alpha, n+1; z)\log z \\ &+\sum_{r=0}^{\infty}\frac{(\alpha)_r}{(n+1)_r}[\psi(\alpha+r)-\psi(1+r)-\psi(1+n-r)]\frac{z^r}{r!}\} \\ &+\frac{(n-1)!}{\Gamma(\alpha)}\sum_{r=0}^{n-1}\frac{(\alpha-n)_r}{(1-n)_r}\cdot\frac{z^{r-n}}{r!},\end{aligned} \tag{8}$$

where $(\alpha)_r=\alpha(\alpha+1)\cdots(\alpha+r-1)$ and

$$\Phi(\alpha, \gamma; z)=\sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\gamma)_r}\cdot\frac{z^r}{r!}.$$

Hence we can conclude that

$$\begin{aligned}\hat{u}_1(0, \eta) &= \lim_{x\downarrow 0} xe^{-\Lambda(x)\eta}\,\Psi(\alpha, 1; 2\Lambda(x)\eta) \\ &= \frac{1}{\Gamma(\alpha)}.\end{aligned} \tag{9}$$

Next let us recall the following relation (see page 258 of [2]):

$$\frac{d}{dz}\Psi(\alpha, \gamma; z)=-\alpha\Psi(\alpha+1, \gamma+1; z). \tag{10}$$

This implies that

$$\begin{aligned}\hat{u}_1'(x, \eta) &= e^{-\Lambda(x)\eta}\,\Psi(\alpha, 1; 2\Lambda(x)\eta) \\ &\quad -x\Lambda'(x)\eta e^{-\Lambda(x)\eta}\,\Psi(\alpha, 1; 2\Lambda(x)\eta) \\ &\quad -2\alpha x\Lambda'(x)\eta e^{-\Lambda(x)\eta}\,\Psi(\alpha+1, 2; 2\Lambda(x)\eta),\end{aligned} \tag{11}$$

for $x>0$. Take now the limit of the equation (11) as $x\downarrow 0$, keeping (8) in mind. Then the cancelation will occur between the terms of order $O(x^{-1})$. Thus we get

$$\hat{u}_1'(0, \eta)=-\frac{1}{\Gamma(\alpha)}\{\log 2\eta+\psi(\alpha)-2\psi(1)\}, \tag{12}$$

Similarly, from the expression of $\hat{u}_2(x, \eta)$ for $x<0$, we obtain

$$(13) \qquad \hat{u}_2(0, \eta) = \frac{1}{\Gamma(\alpha)},$$

$$\hat{u}_2'(0, \eta) = \frac{1}{\Gamma(\alpha)}\{\log 2\eta + \psi(\alpha) - 2\psi(1)\}.$$

The equations (9), (12) and (13) immediately give our assertion.

REMARK 4. From Proposition 2, we can see that $\hat{u}_1(x, \eta)$ and $\hat{u}_2(x, \eta)$ are linearly dependent for $\eta>0$ if $b=-1, -3, -5, \cdots$ and for $\eta<0$ if $b=1, 3, 5, \cdots$ (recall that $\alpha=\frac{1+b}{2}$ and $\frac{1}{\Gamma(-n)}=0$ for $n=0, 1, 2, \cdots$). Also it is clear that, for sufficiently large $|\eta|$, $\hat{u}_1(x, \eta)$ and $\hat{u}_2(x, \eta)$ are linearly independent if b is not odd integer.

Next we investigate Theorem 2. In the case of Theorem 2 we consider the equation (3) with $\Lambda(x)$ and $\mu(x)$ being respectively replaced by $\tilde{\Lambda}(x)=\operatorname{sgn} x \cdot \exp(-|x|^{-1})$ and $\tilde{\mu}(x)=\tilde{\Lambda}'(x)$. The similar argument as above gives us the folloiwng

Proposition 3. (i) *For $\eta>0$, there exist solutions which have the following expressions:*

$$\hat{u}_1(x, \eta) = xe^{-\tilde{\Lambda}(x)\eta}\, \Psi(\alpha, 1; 2\tilde{\Lambda}(x)\eta) \qquad \text{for} \quad x>0,$$

and

$$\hat{u}_2(x, \eta) = -xe^{-\tilde{\Lambda}(x)\eta}\, \Psi(1-\alpha, 1; -2\tilde{\Lambda}(x)\eta) \qquad \text{for} \quad x<0.$$

Moreover the Wronskian of them is

$$W(\eta) = \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\{2\log 2\eta - 4\psi(1) + \psi(\alpha) + \psi(1-\alpha)\}.$$

(ii) *For $\eta<0$, there exist solutions which have the following expressions:*

$$\hat{u}_1(x, \eta) = xe^{\tilde{\Lambda}(x)\eta}\, \Psi(1-\alpha, 1; -2\tilde{\Lambda}(x)\eta) \qquad \text{for} \quad x>0,$$

and

$$\hat{u}_2(x, \eta) = -xe^{-\tilde{\Lambda}(x)\eta}\, \Psi(\alpha, 1, 2\tilde{\Lambda}(x)\eta) \qquad \text{for} \quad x<0.$$

Moreover the Wronskian of them is

$$W(\eta) = \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\{2\log(-2\eta) - 4\psi(1) + \psi(\alpha) + \psi(1-\alpha)\}.$$

REMARK 5. As in the case of Theorem 1, we can conclude from Proposition 3 that, for sufficiently large $|\eta|$, $\hat{u}_1(x, \eta)$ and $\hat{u}_2(x, \eta)$ are linearly independent if b is not odd integer (i.e., $\alpha \in \boldsymbol{Z}$). Moreover, even if $\alpha \in \boldsymbol{Z}$,

$$W(\eta)=\frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\{\psi(\alpha)+\psi(1-\alpha)\}$$
$$=\begin{cases} 1 \text{ if } |2\alpha-1|\equiv 3 \bmod 4\,, \\ -1 \text{ if } |2\alpha-1|\equiv 1 \bmod 4\,. \end{cases}$$

(See page 15 of [2].) Thus we see that $\hat{u}_2(x,\eta)$ and $\hat{u}_2(x,\eta)$ are linearly independent for such α. This is the reason why Theorem 2 is free from such an assumption as in Theorem 1.
Now w turn to prove the non-hypoellipticity part of Theorem 1.

Proof of non-hypoellipticity in Theorem 1. First let us observe thrt, if P is hypoelliptic, we get the following inequality from the argument of Banach's closed graph theorem.

For any positive number l and for any pair of open sets Ω and Ω' satisfying $\bar{\Omega}'\subset\Omega$, there exist a positive integer m and a constant C such that

$$(14)\qquad ||D_y^l u||_{L^2(\Omega')}\leq C\Big\{\sum_{m_1+m_2\leq m}||D_x^{m_1}D_y^{m_2}Pu||_{L^2(\Omega)}+||u||_{L^2(\Omega)}\Big\}\,,$$
$$\forall u\in C^\infty(\bar{\Omega})\,.$$

We are now going to show that the inequality (14) never holds provided b is odd integer. Let us set $\Omega=(-\delta,\delta)\times(-\delta,\delta)$ and $\Omega'=(-\delta',\delta')\times(-\delta',\delta')$ with δ and δ' satisfying $0<\delta'<\delta<1$. Moreover set

$$(15)\qquad u_\eta(x,y)=e^{iy\eta}\,\hat{u}_1(x,\eta)$$
$$=-e^{iy\eta}\,\hat{u}_2(x,\eta)\,,$$

with $\eta>0$ if $b=-1,-3,\cdots$ and with $\eta<0$ if $b=1,3,\cdots$ (Observe that $\hat{u}_1(x,\eta)=-\hat{u}_2(x,\eta)$ provided b is odd integer. To see this, compare (9), (12) and (13).) Let us substitute $u_\eta(x,y)$ into (14) and compare the asymptotic behavior of the both hand sides as $|\eta|\to\infty$. Clearly, in the right hand side, it holds that $Pu_\eta=0$.

Observe now that there exists a constant C (independent of η) such that

$$|\hat{u}_1(x,\eta)|\leq C \qquad \text{for} \quad 0\leq x\leq 1\,,$$

and

$$|\hat{u}_2(x,\eta)|\leq C \qquad \text{for} \quad -1\leq x\leq 0\,.$$

This can be seen from the remark after the statement of Proposition 1 and the asymptotic behaviors of $\Psi(\alpha,1;z)$ and $\Psi(1-\alpha,1;z)$ as $z\downarrow 0$. Indeed, for example, it follows from (8) that

$$|\hat{u}_1(x,\eta)|=|x|e^{-\Lambda(x)\eta}|\Psi(\alpha,1;2\Lambda(x)\eta)|$$
$$\leq C_1|x|(1+|\log 2\Lambda(x)\eta|)$$

$$\leq C_1(|x|+|x|\log 2+|x\log\Lambda(x)|+|x\log\eta|)$$
$$\leq C_2,$$

for $0<x<(\log 2\eta)^{-1}$ and $\eta\geq e/2$. Hence, if we substitute u_η into (14), the right hand side is not larger than

$$\|u_\eta\|_{L^2(\Omega)}\leq 4\delta^2\cdot C. \tag{16}$$

On the other hand, in the left hand side of (14), it is clear that

$$\|D_y^l u_\eta\|_{L^2(\Omega')}=|\eta|^l\cdot 2\delta'\cdot\left(\int_{-\delta'}^{\delta'}|\hat{u}_1(x,\eta)|^2\,dx\right)^{1/2}.$$

Moreover, from the asymptotic behavior of $\Psi(\alpha,1;z)$ as $z\to\infty$, it follows that there exist positive constants ε and M such that

$$|\hat{u}_1(x,\eta)|\geq\varepsilon|x|\qquad\text{for}\quad M\leq 2\Lambda(x)|\eta|\leq 2M.$$

Hence we obtain that

$$\begin{aligned}\|D_y^l u_\eta\|_{L^2(\Omega')}&\\ \geq|\eta|^l\cdot 2\delta'\cdot&\left(\int_{M\leq 2\Lambda(x)|\eta|\leq 2M}|\hat{u}_1(x,\eta)|^2\,dx\right)^{1/2}\\ \geq|\eta|^l\cdot 2\delta'\cdot&\varepsilon\cdot 3^{-1/2}\left\{\left(\log\frac{|\eta|}{M}\right)^{-3}-\left(\log\frac{2|\eta|}{M}\right)^{-3}\right\}^{1/2}.\end{aligned}\tag{17}$$

Finally taking $l\geq 1$ immediately implies the contradiction among (14), (16) and (17). ■

2. Proof of hypoellipticity

In the present section, we assume that $W(\eta)\neq 0$ for $|\eta|\geq C$, and denote by $Q(x,x';\eta)$ the Green function of (3) (in the case of Theorem 2, $\Lambda(x)$ and $\mu(x)$ being replaced respectively by $\tilde{\Lambda}(x)$ and $\tilde{\mu}(x)$), i.e.,

$$Q(x,x';\eta)=\begin{cases}\dfrac{\hat{u}_2(x,\eta)\,\hat{u}_1(x',\eta)}{W(\eta)} & (x<x'),\\[2ex] \dfrac{\hat{u}_2(x',\eta)\,\hat{u}_1(x,\eta)}{W(\eta)} & (x'<x).\end{cases}$$

Then we have the following

Proposition 4. *For any non-negative integer m, there exists a constant C_m such that the following inequalities hold:*

$$\int_{-1}^{1}|\partial_\eta^m Q(x,x';\eta)|\,dx'\leq C_m|\eta|^{-m}\ \textit{for}\ -1\leq x\leq 1\ \textit{and}\ |\eta|\geq\max\{C,e\}, \tag{18}$$

(19) $\int_{-1}^{1} |\partial_\eta^m Q(x, x'; \eta)| dx \le C_m |\eta|^{-m}$ *for* $-1 \le x' \le 1$ *and* $|\eta| \ge \max\{C, e\}$.

Proof. Here we prove the proposition in the case of Theorem 1. First we shall verify (18) when $m=0$. Observe now the following inequality:

(20)
$$\int_{-1}^{1} |Q(x, x; \eta)| dx'$$
$$\le \left\{ |\hat{u}_2(x, \eta)| \int_x^1 |\hat{u}_1(x', \eta)| dx' + |\hat{u}_1(x, \eta)| \int_{-1}^{x} |\hat{u}_2(x', \eta)| dx' \right\} \cdot |W(\eta)|^{-1}.$$

Let us set $x_\eta = (\log|\eta|)^{-1}$ (then $\Lambda(x_\eta)|\eta| = 1$). We are going to estimate the right hand side of (20). Here we assume $\eta > 0$. In the case of $\eta < 0$, the argument is completely parallel if α is replaced by $1-\alpha$.

(I) Now we are going to show that the value of $\int_{-1}^{1} |Q(x, x'; \eta)| dx'$ is uniformly bounded for $x_\eta \le x \le 1$ and $\eta \ge \max\{C, e\}$. Concerning the first term on the right hand side of (20), we can use the expression of $\hat{u}_1(x', \eta)$ for $x' > 0$ and the asymptotic behavior $\Psi(\alpha, \gamma, z) = O(z^{-\alpha})$ as $z \to \infty$. Hence we have

$$|\hat{u}_1(x', \eta)| \le Ce^{-\Lambda(x')\eta}(\Lambda(x')\eta)^{-\alpha} \qquad \text{for} \quad x_\eta \le x' \le 1.$$

We cannot use the expression of $\hat{u}_2(x, \eta)$ for $x > 0$. So let us express it by linear combination of $\hat{u}_1(x, \eta)$ and

$$\hat{u}_3(x, \eta) = xe^{-\Lambda(x)\eta}\, \Phi(\alpha, 1; 2\Lambda(x)\eta) \qquad \text{for} \quad x > 0$$

(concerning the definition of $\Phi(\alpha, \gamma; z)$, see page 248 of [2]). From the facts that $\hat{u}_3(0, \eta) = 0$ and $\hat{u}_3'(0, \eta) = 1$, it follows

(21)
$$\hat{u}_2(x, \eta) = A\hat{u}_1(x, \eta) + B\hat{u}_3(x, \eta),$$

where

$$A = 1,$$
$$B = \frac{2}{\Gamma(\alpha)} \{\log 2\eta + \psi(\alpha) - \psi(1)\}.$$

Hence we obtain

(22)
$$|\hat{u}_2(x, \eta)| \int_x^1 |\hat{u}_1(x', \eta)| dx' \cdot |W(\eta)|^{-1}$$
$$\le \{|W(\eta)|^{-1} \cdot |\hat{u}_1(x, \eta)| + C \cdot |\hat{u}_3(x, \eta)|\} \int_x^1 |\hat{u}_1(x', \eta)| dx'.$$

Now recall that

$$\Phi(\alpha, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z z^{\alpha-\gamma}(1 + O(|z|^{-1})) \text{ as } z \to +\infty$$

(see page 278 of [2]), and $\Lambda'(x')\eta=\mu(x')\eta\geq(x_\eta)^{-2}$ for $x_\eta\leq x'\leq 1$. Hence, concerning the second term on the right of (22), we have furthermore

$$
\begin{aligned}
&|\hat{u}_3(x,\eta)|\int_x^1|\hat{u}_1(x',\eta)|\,dx'\\
&\leq C_1\cdot x_\eta^2\cdot e^{\Lambda(x)\eta}(\Lambda(v)\eta)^{\alpha-1}\int_x^1(\Lambda(x')\eta)^{-\alpha}\,e^{-\Lambda(x')\eta}\,\Lambda'(x')\eta\,dx'\\
&\leq C_1\cdot x_\eta^2\cdot e^{\Lambda(x)\eta}(\Lambda(x)\eta)^{\alpha-1}\int_{\Lambda(x)\eta}^{\infty}t^{-\alpha}\,e^{-t}\,dt\\
&\leq C_2\cdot x_\eta^2\cdot(\Lambda(x)\eta)^{-1}\leq C_3\,.
\end{aligned}
$$

Here we have used the fact that $\int_s^\infty t^{-\alpha}e^{-t}dt=O(s^{-\alpha}e^{-s})$ as $s\to+\infty$. The similar argument is applicable for estimating the first term on the right of (22). Consequently, the first term on the right of (20) is uniformly bounded for $x_\eta\leq x\leq 1$ and $\eta\geq\max\{C, e\}$.

Concerning the second term on the right of (20), let us decompose it in the following way:

$$
\begin{aligned}
|\hat{u}_1(x,\eta)|\int_{-1}^{x}|\hat{u}_2(x',\eta)|\,dx'\cdot|W(\eta)|^{-1}&=|\hat{u}_1(x,\eta)\cdot W(\eta)^{-1}|\\
\times\left\{\int_{-1}^{-x_\eta}|\hat{u}_2(x',\eta)|\,dx'+\int_{-x_\eta}^{x_\eta}|\hat{u}_2(x',\eta)|\,dx'\right.&\left.+\int_{x_\eta}^{x}|\hat{u}_2(x',\eta)|\,dx'\right\}.
\end{aligned}
$$

For $x'\leq x_\eta$, the expression of $\hat{u}_2(x',\eta)$ can be applied, and also for $x_\eta\leq x'$, $\hat{u}_2(x',\eta)$ can be decomposed as (21). Thus, by using the asymptotic behaviors of $\Psi(\alpha, 1; z)$ and $\Phi(\alpha, 1; z)$ as $z\to\infty$, we see that the first and the third terms are uniformly bounded. Concerning the integral with $-x_\eta\leq x'<0$, the expression of $\hat{u}_2(x',\eta)$ and the asymptotic behavior of $\Psi(\alpha, 1; z)$ as $z\downarrow 0$ can be applied (see page 262 of [2]). Hence it holds that

$$
\begin{aligned}
&|\hat{u}_1(x,\eta)\cdot W(\eta)^{-1}|\cdot\int_{-x_\eta}^{0}|\hat{u}_2(x',\eta)|\,dx'\\
&\leq C_1\,e^{-\Lambda(x)\eta}\,(\Lambda(x)\eta)^{-\alpha}\cdot|W(\eta)|^{-1}\int_{-x_\eta}^{0}|x'\log 2\Lambda(x')\eta|\,dx'\\
&\leq C_2\cdot|W(\eta)|^{-1}\cdot\left\{\int_{-x_\eta}^{0}dx'+(\log 2\eta)\int_{-x_\eta}^{0}|x'|\,dx'\right\}\\
&\leq C_3\,.
\end{aligned}
$$

Concerning the integral with $0\leq x'\leq x_\eta$, we can estimate in the similar way, by using the fact (21). Thus we see that the second term on the right of (20) is also uniformly bounded for $x_\eta\leq x\leq 1$ and $\eta\geq\max\{C, e\}$.

(II) For $-1\leq x\leq -x_\eta$, the argument for the estimate is completely parallel if we interchange the roles of $\hat{u}_1(x,\eta)$ and $\hat{u}_2(x,\eta)$. Also for $-x_\eta\leq x\leq x_\eta$, the argument is similar if we rewrite (20) as

$$\int_{-1}^{1} |Q(x, x'; \eta)| dx'$$
$$\leq |W(\eta)^{-1}\cdot\hat{u}_2(x, \eta)| \cdot \left\{\int_{x_\eta}^{1} |\hat{u}_1(x', \eta)| dx' + \int_{x}^{x_\eta} |\hat{u}_1(x', \eta)| dx'\right\}$$
$$+ |W(\eta)^{-1}\cdot\hat{u}_1(x, \eta)| \cdot \left\{\int_{-1}^{-x_\eta} |\hat{u}_2(x', \eta)| dx' + \int_{-x_\eta}^{x} |\hat{u}_2(x', \eta)| dx'\right\},$$

and estimate the each term on the right hand side. Consequently, we see that the value of $\int_{-1}^{1}|Q(x, x'; \eta)|dx'$ is uniformly bounded for $-1\leq x\leq 1$ and $\eta\geq \max\{C, e\}$.

The argument to show (18) for $m>0$ is similar to the above if we notice the fact (10) and

$$\frac{d}{dz}\Phi(\alpha, \gamma; z) = \frac{\alpha}{\gamma}\Phi(\alpha+1, \gamma+1; z)$$

(see page 254 of [2]). Thus the proof of Proposition 4 is clear. ■

Now we are in position to verify the hypoellipticity parts of Theorems.

Proof of hypoellipticity: First let us notice that the operator P is elliptic except $x=0$. Hence we can restrict our consideration at $(0, y_0)$. Moreover, since P is non-characteristic with respect to the variable x, the smoothness of the solution w.r.t. the variable x follows from the one w.r.t. the variable y. To be more precise, let $H^{k,l}$ be the space of distributions u satisfying $(1+\xi^2)^{k/2}(1+\eta^2)^{l/2}\hat{u}(\xi, \eta) \in L^2(\boldsymbol{R}^2)$ (ξ and η are the dual variables of x and y respectively). Then $u\in H^{k,l}$ and $Pu\in C^\infty$ at $(0, y_0)$ implies that $u\in \bigcap_{m=1}^{\infty} H^{k+2m,l-2m}$ at $(0, y_0)$. Thus $u\in H^{0,\infty}$ and $Pu\in C^\infty$ at $(0, y_0)$ implies that $u\in C^\infty$ at $(0, y_0)$. So it suffices to prove that $u\in H^{0,\infty}$ at $(0, y_0)$ when $Pu\in C^\infty$ at $(0, y_0)$.

Secondly we can assume that the support of the solution is contained in a small neighborhood of $(0, y_0)$. To observe this, let us take a function $\chi(x, y)\in C_0^\infty$ satisfying $\chi(x, y)\equiv 1$ for $|x|+|y-y_0|\leq\delta/2$ and $\chi(x, y)\equiv 0$ for $|x|+|y-y_0| \geq\delta$. Then the second term on the right of

$$Pu = P\chi u + P(1-\chi)u$$

is equal to 0 in a neighborhood of $(0, y_0)$. So it suffices to show that χu is smooth at $(0, y_0)$ provided $P\chi u$ is smooth there.

Now take a function $\phi(\eta)\in C^\infty$ such that $\phi(\eta)\equiv 0$ for $|\eta|\leq\max\{C, e\}$ and $\phi(\eta)\equiv 1$ for $|\eta|\geq 2\max\{C, e\}$, and set

$$Qu(x, y) = \frac{1}{2\pi}\iiint e^{i(y-y')\eta}\, Q(x, x'; \eta)\, \phi(\eta)\, u(x', y')\, dx'\, dy'\, d\eta\,. \tag{23}$$

Then it follows from (18) and (19) with $m=0$ that Q is a bounded operator $H^{0,l}((-1, 1)\times\boldsymbol{R})$ for all $l\in\boldsymbol{R}$. Moreover since $Q(x, x'; \eta)$ is the Green function of (3), it holds that $PQ=I+K_1$, where K_1 is an operator with symbol $1-\phi(\eta)$, in particular, it is regularizing operator w.r.t. y, i.e., the one from $H^{0,l}((-1,1)\times \boldsymbol{R})$ into $H^{0,\infty}(-1,1)\times \boldsymbol{R})$. Now let $Q_1(x,x';\eta)$ be the Green function of (3) with b being replaced by $\bar{b}$, and let R be the adjoint operator of (23) with $Q(x,x';\eta)$ being replaced by $Q_1(x,x';\eta)$. Then it holds that

$$RP=I+K,$$

where K is an operator from $H^{0,l}((-1,1)\times \boldsymbol{R})$ into $H^{0,\infty}((-1,1)\times \boldsymbol{R})$. Furthermore R has pseudo-local property w.r.t. y mod $H^{0,\infty}$. To be more precise, let $\chi_1(y)$ be a function of class C_0^∞ satisyfing $\chi_1(y)\equiv 1$ for $|y-y_0|\leq \varepsilon$. Then the second term of the right of

$$Rf=R\chi_1 f+R(1-\chi_1)f$$

belongs to $H^{0,\infty}$ at $(0,y_0)$ for any $f\in H^{0,l}$. It is a consequence of (18) and (19). Indeed, $R(1-\chi_1)f$ is expressed as

$$\frac{1}{2\pi}\iint e^{i(y'-y)\eta}\,dy'd\eta\int Q_2(y,y';x,x';\eta)f(x,'y')\,dx',$$

where

$$Q_2(y,y';x,x';\eta)=\frac{1-\chi_1(y')}{(y'-y)^m}D_\eta^m\{\bar{Q}_1(x',x;\eta)\phi(\eta)\},$$

for arbitrary positive integer m, and the values of

$$|\eta|^{m+\gamma}\int|\partial_y^\alpha\partial_{y'}^\beta\partial_\eta^\gamma Q_2(y,y';x,x';\eta)|\,dx'$$

and

$$|\eta|^{m+\gamma}\int|\partial_y^\alpha\partial_{y'}^\beta\partial_\eta^\gamma Q_2(y,y';x,x';\eta)|\,dx$$

are uniformly bounded for $\eta\in\boldsymbol{R}$, $|y-y_0|\leq\varepsilon$ and $y'\in\boldsymbol{R}$. Hence $Q_2(y,y';\cdot,\cdot;\eta)$ is an operator valued symbol of class $S^{-\infty}((y_0-\varepsilon,y_0+\varepsilon)\times\boldsymbol{R}_{y'}\times\boldsymbol{R}_\eta;L^2(-1,1))$.

Thus, if $P\,u$ is of class $H^{0,\infty}$ at $(0,y_0)$ and the supports of χ and χ_1 are taken properly small, then all terms on the right hand side of the equation

$$\begin{aligned}\chi u&=RP\chi u-K\chi u\\&=R\chi_1 P\chi u+R(1-\chi_1)P\chi u-K\chi u\end{aligned}$$

are of class $H^{0,\infty}$ at $(0,y_0)$, since χu becomes of class $H^{0,l}$ for some l and the singular support of χu becomes contained in $\{(x,y)\,|\,x=0\}\cap\operatorname{supp}\chi$. This completes the proof. ■

References

[1] G.R. Aleksandryan: *The parametrix and propagation of the wave front of the solution of the Cauchy problem for a model equation,* Soviet Journal of Contemporary Mathematical Analysis **19** (1984), 33–46.

[2] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi: "Higher Transcendental Functions, vol. 1," McGraw Hill, New York, 1953.

[3] V.S. Fedii: *On a criterion for hypoellipticity,* Math. USSR Sb. **14** (1971), 15–45.

[4] A. Gilioli and F. Treves: *An example in the solvability theory of linear PDE's,* Amer. J. Math. **96** (1974), 367–385.

[5] T. Hoshiro: *On Levi-type conditions for hypoellipticity of certain differential operators,* Comm. PDE **17** (1992), 905–922.

[6] A. Menikoff: *Some examples of hypoelliptic partial differential equations,* Math. Ann. **221** (1976), 167–181.

[7] Y. Morimoto: *On the hypoellipticity for infinitely degenerate semi-elliptic operators,* J. Maht. Soc. Japan **30** (1978), 327–358.

[8] T. Morioka: *Some remarks on micro-hypoelliptic operators of infinitely degenerate type,* Osaka J. Math. (to appear).

[9] M. Suzuki: *Hypoellipticity for a class of degenerate elliptic operators of second order,* Tsukuba Math. J. **16** (1992), 217–234.

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