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UNLINKING TWO COMPONENT LINKS

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1. Introduction

We define a *knot* to be a piecewise linear embedding of a circle, S^1, in either Euclidean 3-space, $\boldsymbol{R}^3$, or the 3-sphere, S^3. A *link* is defined to be the disjoint union of circles in $\boldsymbol{R}^3$ or S^3. A natural question to ask about a knot or link is: *how can this be untied?* Here by "untying" we mean, how many crossings need to be changed to transform our knot or link into a collection (with one element in the case of a knot) of trivial circles. Formally, we define the ***unknotting number***, $u(K)$, for a knot K (or the ***unlinking number*** for a link) to be the minimal number of crossing changes necessary to convert the diagram of K into a diagram of a trivial knot (link). This minimum is taken over all diagrams of the knot or link.

A list of unknotting numbers has been complied by Y. Nakanishi [8] for prime knots having 9 or fewer crossings. Of the 84 knots listed, the unknotting numbers of nearly one quarter were unknown in 1981. In the last decade, due to techniques by Lickorish [6], Kanenobu and Murakami [4] and others, the number of these small knots with unknown unknotting numbers has been reduced to about a half dozen. In this paper we provide a list of unlinking numbers for the "small" classical two component links. These are the prime, nonsplit links which have diagrams with 9 or fewer crossings.

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2. Four methods for determining $u(L)$

We begin with a link, L, with components A_L and B_L. Individually the components of L may be unknotted. In general, however, $u(A_L)\geq 0$ and $u(B_L)\geq 0$. We shall use $lk(A_L, B_L)$ to denote the ***linking number*** of A_L and B_L.

To determine the unlinking number of a link we need both upper and lower bounds. The upper bound is found experimentally by examining diagrams of the link. Generally, in our small links, we will see that a sharp upper bound can be found in a ***minimal*** diagram of the link. This is a diagram with the minimal number of crossings, where again this minimum is taken over all diagiams of the link.

Many of the smaller knots and links are two-bridge. Therefore we introduce the following notation. We let $S(p, q)$ denote the two-bridge knot or link for which the lens space $L(p, q)$ is the two-fold branced cover of S^3. The following theorem [5] classifies all two-bridge links with unlinking number one.

Theorem 1. *The two-bridge link L has $u(L)=1$ if and only if L is equivalent to $S(2n^2, 2mn\pm 1)$, where m and n are relatively prime.*

This theorem will provide a basis for one of several methods we shall use to determine unlinking numbers.

Before listing these methods we note that if a single crossing change in some diagram of a non-trivial link, L, unlinks L, then clearly $u(L)=1$. A corollary of Theorem 1 states that for a two-bridge link with $u(L)=1$, this crossing will be found in a minimal diagram of L.

METHOD 1.

A simple lower bound for the unlinking number is given by:

$$u(L)\geq u(A_L)+u(B_L)+|lk(A_L, B_L)|. \tag{1}$$

This inequality is sharp. Another corollary of Theorem 1 states that for a two-bridge link with $u(L)=1, lk(A_L, B_L)=0$ when n is even and $|lk(A_L, B_L)|=1$ when n is odd. In every two-bridge link each component is individually unknotted. Thus for this class of two component links, we have strict inequality in (1) when n is even and equality when n is odd.

METHOD 2.

Any two-bridge link with unlinking number one can be detected using Theorem 1. For any other two-bridge link, L, if two crossings can be found in some diagram of L which unlink L, we may conclude that $u(L)=2$.

Before stating Proposition 2, the basis for our next method, we introduce some additional notation. Let $N(K)$ be the regular neighborhood of the knot K in a closed orientable 3-manifold M, with μ a meridian of $N(K)$. Let X be the exterior of K in M, that is,

$$X = M - \text{int}\, N(K).$$

Now let $X(r)$ denote the closed manifold obtained by attaching a solid torus T to X so that a curve of slope r bounds a disk in T. By slope, we mean the isotopy class of a non-trivial simple closed curve in ∂X. We say that $X(r)$ is the result of *r-surgery* on K in M. For two slopes r and s in ∂X, let $\Delta(r, s)$ be their minimal geometric inersection number.

Finally, every simple closed curve, γ, in $S^1\times S^2$ defines a homology class $[\gamma]\in H_1(S^1\times S^2)\cong\boldsymbol{Z}$. We define the *winding number* of K, $n\geq 0$, to be the absolute value of $[K]$ in $\boldsymbol{Z}$ under this mapping.

Proposition 2. *Let K be a knot in $S^1\times S^2$ with meridian μ and winding number n. Let $Y=S^1\times S^2-\operatorname{int}N(K)$ and $\Delta(r,\mu)=t$. Then*

$$|H_1(Y(r))| = \begin{cases} tn^2 & \text{if } n>0 \\ \infty & \text{if } n=0 \end{cases}$$

(*where $|G|$ denotes the order of the group G*).

Proof. If $n=0$, then $K\sim O$ (is homologous to the unknot). The order of $H_1(Y(r))$ will be the same as the order of the manifold obtained by performing r-surgery on O in S^3 and then taking the *connected sum* of this manifold with $S^1\times S^2$. This group will be a direct sum $A\oplus\boldsymbol{Z}$. Thus its order is infinite.

Now, let V and W be solid tori, $S^1\times D^2$. Let $\mu,\lambda\in H_1(\partial V)=H_1(S^1\times\partial D^2)$ be given by $\mu=[*\times\partial D^2]$, $*\in S^1$ and $\lambda=[S^1\times *]$, $*\in\partial D^2$. Then $S^1\times S^2=V\cup_\partial W$, such that the meridians of the two solid tori are identified. By an isotopy, we may regard $K\subset V$, so that $Y\cong(V-\operatorname{int}N(K))\cup W$. We choose a zero-framing for K and let α,β be the usual meridian-longitude classes for $H_1(\partial N(K))$. We note that $H_1(V-\operatorname{int}N(K))\cong\boldsymbol{Z}\langle\lambda\rangle\oplus\boldsymbol{Z}\langle\alpha\rangle$. The identification which attaches ∂W to ∂V has the effect on $H_1(V-\operatorname{int}N(K))$ of setting $\mu=n\alpha=0$, where n is the winding number.

In terms of the α,β basis for $H_1(\partial N(K))$ we may express r as $r=\frac{s}{t}\in\boldsymbol{Q}\cup\{\infty\}$. Hence, at this boundary component, the r-surgery has the effect of setting $t\beta+s\alpha=0$ in $H_1(V-\operatorname{int}N(K))$. Since $\beta=n\lambda$, we have $tn\lambda+s\alpha=0$. Thus,

$$H_1(Y(r))\cong(\boldsymbol{Z}\langle\lambda\rangle\oplus\boldsymbol{Z}\langle\alpha\rangle)/(n\alpha,\, tn\lambda+s\alpha)\,.$$

The order of this group is simply the absolute value of the determinant of the matrix,

$$\begin{pmatrix} 0 & n \\ tn & s \end{pmatrix},$$

which is tn^2. ∎

We now let M_L denote the two-fold covering of S^3 branched along the link L.

Corollary 3. *Let L be a two component link with $u(L)=1$. The order of $H_1(M_L)$ is either infinite or equal to $2n^2$ for some integer n.*

Proof. Since $u(L)=1$, M_L is obtained by r-surgery on some knot K in

$S^1 \times S^2$, where $\Delta(r, \mu)=2$. (See Lemma 1 [6] or Lemma 3 [5]). Let n be the winding number of K in $S^1 \times S^2$. The result follows from Proposition 2. ∎

As in Method 2, this corollary enables us to conclude that $u(L)=2$ for certain links. Here, however, we are not restricted by the bridge index.

METHOD 3.

If the order of the homology of M_L of a link L is finite and not equal to $2n^2$ for any integer n, we may conclude that $u(L) \geq 2$. Thus, if 2 crossings can be found in a diagram of L which would unlink L, we again may conclude that $u(L)=2$.

METHOD 4.

Another lower bound for $u(L)$ [7] is given by

$$0 \leq \frac{1}{2} |\sigma(L)| \leq u(L), \tag{2}$$

where $\sigma(L)$ is the signature of the link L. For a link, the signature is a function of the orientations assigned to the components. If both of the orientations are reversed, the signature does not change, but reversing the orientation of one component may result in a different signature. In this case we are interested in maximizing $|\sigma(L)|$, thus we choose an appropriate orientation.

The signature enables us to determine the unlinking number for several of the links in our list for which the previous methods fail. As an example we consider the link 9^2_{13} (Figure 1). This link is not two-bridge, one component is unknotted, the other component is a (5, 2)-torus knot and $|lk(A_{9^2_{13}}, B_{9^2_{13}})|=0$. Since the unknotting number of a (5, 2)-torus knot is 2, and the order of $H_1(M_{9^2_{13}})$ is 24, all we may conclude is that $u(9^2_{13}) \geq 2$. But $|\sigma(9^2_{13})|=5$, so by (2), $u(9^2_{13}) \geq 3$. In fact, 9^2_{13} may be unlinked by changing the three indicated crossings of Figure 1.

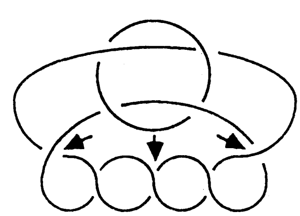

Figure 1: The three indicated crossings would unlink 9^2_{13}.

These four methods are practical and effective for determining the unlinking numbers we wish to calculate. Each of the constituent invariants is relatively easy to compute. The linking number may be evaluated by looking at the intersections of one component with a Seifert surface of the other component [9]. Unknotting numbers are known for all knots with up to 8 crossings ([8], [6], [4]). Thus computing a lower bound for $u(L)$ using Method 1 is simple. Similarly, we know which of the links have bridge index 2. Applying Theorem 1 is also straightforward. Both the order of $H_1(M_L)$ and the signature can be computed from a *Goeritz matrix* of L ([1], [3]), thus Methods 3 and 4 are easy to use.

3. Another method

Of the 91 links in our list, the unlinking numbers of only 5 cannot be determined using one of these methods. For these links an upper bound for $u(L)$ has been found to be 3, but unfortunately, the lower bounds predicted by these methods is at most 2. We now discuss one more method which can be used to compute $u(L)$. It is not *needed* for any of the links in our table, although it could have been used for several. The primary reason for introducing it here is that with a variation of this argument we may predict the "types" of crossing changes required for unlinking 4 of the 5 remaining unlinking numbers.

METHOD 5.

Again L is a two component link. Method 5 will apply only to certain links with $lk(A_L, B_L)=0$. It will be convenient to define two new link invariants. The *unlinking number of L restricted to A*, denoted $u_A(L)$, is the minimal number of crossing changes necessary to convert L into a trivial link, where this minimum is taken over all diagrams of L and the crossing change operations involve *only* arcs of component A. We similarly define $u_B(L)$, the *unlinking number* of L *restricted to B*. We remark that if one of the components is knotted, B say, then it will not be possible to unlink L using component A alone. In this case we set $u_A(L)=\infty$. In general $u_A(L)$ and $u_B(L)$ are independent.

We shall explain Mehtod 5 while using it to show that $u(9^2_{25})=2$. See Figure 2. Let $A_{9^2_{25}}$ be the unknotted component of this link. Since $B_{9^2_{25}}$ is knotted $u_A(9^2_{25})=\infty$. Assume that $u(9^2_{25})=1$. Since $lk(A_{9^2_{25}}, B_{9^2_{25}})=0$ and a crossing change between different components of any link will change $|lk(A_L, B_L)|$ by 1, it follows that $u_B(9^2_{25})=1$. We pass to the double cover of S^3 branched along $A_{9^2_{25}}$. See Figure 3. In this cover $B_{9^2_{25}}$ lifts to a two component link $\tilde{B}$. It may be seen that each component of $\tilde{B}$ is unknotted, but the absolute value of the linking number of the components of $\tilde{B}$ is 4. From this we may conclude that $u(\tilde{B})\geq 4$. This implies that $u_B(9^2_{25})\geq 2$, which is a contradiction. Thus $u(9^2_{25})=2$.

This method can also be used to show that some links with *two* unknotted components and $lk(A_L, B_L)=0$ have $u(L)=2$. In these cases we would need to look at the double cover of S^3 branched along A_L and also the cover branched along B_L.

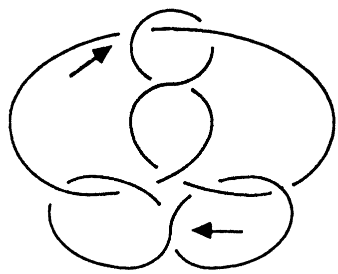

Figure 2: The two indicated crossings would unlink 9^2_{25}.

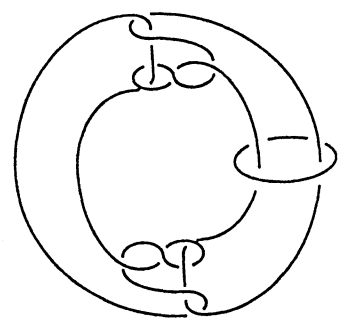

Figure 3: In the double cover of S^3 branched along the unknotted component of 9^2_{25}, the lift of $B_{9^2_{25}}$ is the two component link $\tilde{B}$.

4. A discussion of $u(L)$ for four particular links

We now let $\mathcal{L}=\{9^2_{15}, 9^2_{27}, 9^2_{31}, 9^2_{36}\}$. Each of the four links $L\in\mathcal{L}$ has $lk(A_L, B_L)=0$. None of these links has a double branched cover with order $2n^2$ for any integer n. Thus, by Corollary 3, we know that $u(L)\geq 2$. The diagrams in Figure 4 show that for each of these links $u(L)\leq 3$. We prove the following:

If $u(L)=2$, for $L\in\mathcal{L}$, then one of the crossing changes is between strands of one component and the other is between strands of the other component.

When a link has $u(L)=2$ and $lk(A_L, B_L)=0$, there are four possibilities:

(1) $u_A(L)=2$
(2) $u_B(L)=2$
(3) Both crossing changes involve both components
(4) One crossing change involves one component and the other crossing change involves the other component.

We show that for $L\in\mathcal{L}$, only the last case is possible. For each $L\in\mathcal{L}$, one component is knotted and the other component is unknotted. Let A_L be the unknotted component. Since B_L is knotted, $u_A(L)=\infty$. In addition, if two crossings changes involving *both* components are made, B_L will still be knotted. Thus cases (1) and (3) cannot occur here. We eliminate case (2) with an argument similar to that used in Method 5. In the double cover of S^3 branched along A_L, $\tilde{B}$, the lift of B_L has $u(\tilde{B})\geq 6$ for each $L\in\mathcal{L}$. Thus, $u_B(L)\geq 3$. Figure 5 shows $\tilde{B}$ for 9^2_{15}.

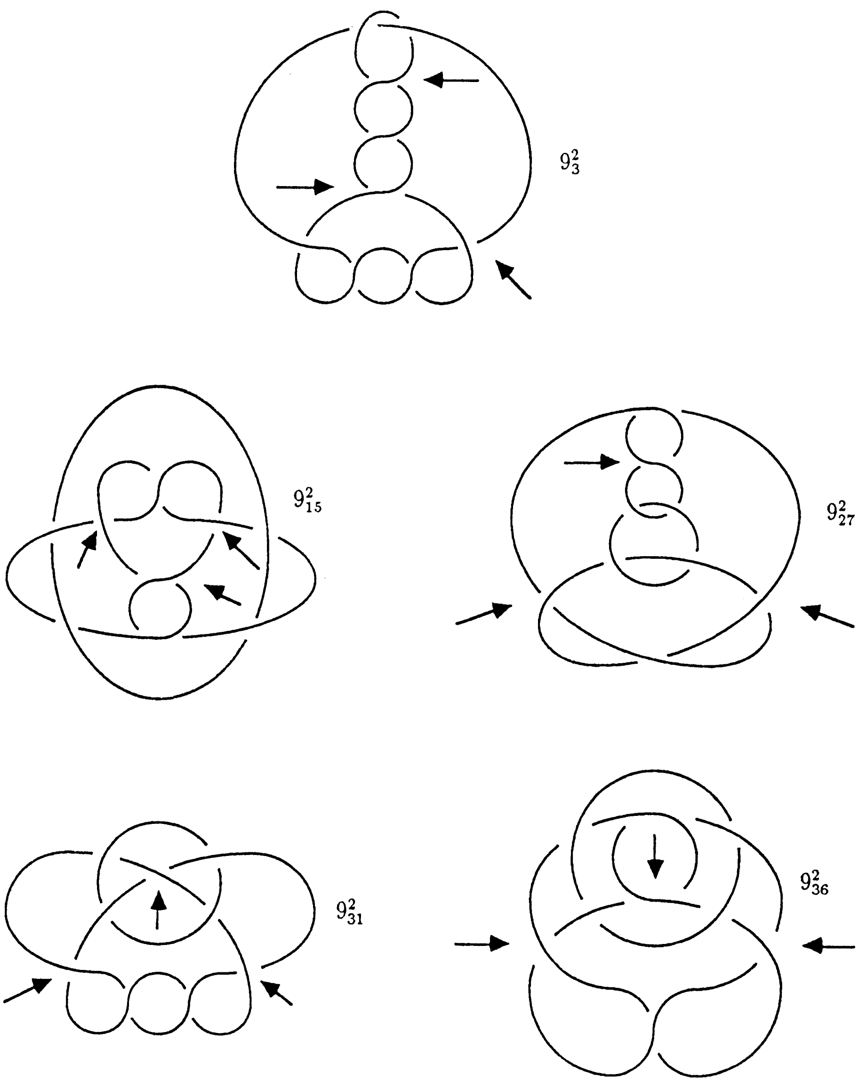

Figure 4: The links 9^2_3, 9^2_{15}, 9^2_{27}, 9^2_{31}, 9^2_{36}. The linking number $|lk(A_{9^2_3}, B_{9^2_3})|=1$. For each of the others, $lk(A_L, B_L)=0$. When changed, the indicated crossings would unlink these links.

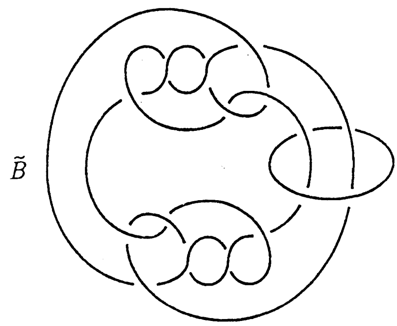

Figure 5: The double cover of S^3 branched along $A_{9^2_{15}}$.
The unlinking number of $\tilde{B}$ is 6.

5. A table of unlinking numbers

For the links in the accompanying table we include the following information, $|lk(A_L, B_L)|$, $|\sigma|$, $|H_1(M_L)|$, $u(L)$, and which of the methods described above was used to compute $u(L)$. When a method is listed for a link L, that means that a lower bound for $u(L)$ was found using this method *and* then a diagram was found in which exactly that number of crossing changes would unlink L. Note that for each link in the table, with the exception of 9^2_4 this was always a minimal diagram of the link. The two-bridge link 9^2_4 has $u(9^2_4)=2$, but no combination of two crossings in a minimal diagram will unlink 9^2_4. Two crossings will unlink 9^2_4 if we use the second diagram of Figure 6.

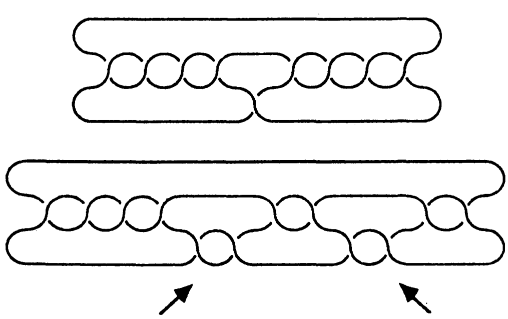

Figure 6: A minimal diagram of the link 9^2_4 is on top. In the second diagram the two indicated crossings, when changed, would unlink 9^2_4.

It should not be inferred that if a method for unlinking a particular link is *not* listed, it could not be used to compute a sharp lower bound for $u(L)$. For example, although Method 3 is listed for showing that $u(9^2_{25})=2$, Method 5 could also be used, as shown above.

Finally, for several of the links we were unable to determine an exact number for $u(L)$. For these 5 links, the entry 2, 3? means the unlinking number is either 2 or 3. As discussed above, for the four links in $\mathcal{L}$, if $u(L)$ is, in fact, 2, the crossing changes required would need to be of the form described.

Table

L	$\lvert \ell k(A_L, B_L)\rvert$	$\lvert\sigma\rvert$	$\lvert H_1(M_L)\rvert$	$u(L)$	Method
2^2_1	1	1	2	1	2
4^2_1	2	3	4	2	2
5^2_1	0	1	8	1	2
6^2_1	3	5	6	3	1
6^2_2	3	3	10	3	1
6^2_3	2	3	12	2	2
7^2_1	1	3	14	2	2
7^2_2	1	1	18	1	2
7^2_3	0	1	16	2	2
7^2_4	0	3	16	2	3
7^2_5	2	3	20	3	1
7^2_6	0	1	24	2	3
7^2_7	2	5	4	3	1
7^2_8	0	1	8	1	1
8^2_1	4	7	8	4	1
8^2_2	4	5	16	4	1
8^2_3	3	5	22	3	1
8^2_4	4	5	24	4	1
8^2_5	3	3	26	3	1
8^2_6	2	1	20	2	2
8^2_7	1	3	30	2	2
8^2_8	1	1	34	2	2
8^2_9	2	3	28	3	2
8^2_{10}	0	1	32	1	1
8^2_{11}	2	5	28	3	1
8^2_{12}	0	1	32	1	1
8^2_{13}	0	1	40	2	1
8^2_{14}	2	3	36	3	3
8^2_{15}	0	1	8	1	1
8^2_{16}	2	3	12	3	1

L	$\lvert \ell k(A_L, B_L)\rvert$	$\lvert\sigma\rvert$	$\lvert H_1(M_L)\rvert$	$u(L)$	Method
9^2_1	2	5	20	3	4
9^2_2	2	3	28	2	2
9^2_3	1	3	30	2, 3?	
9^2_4	0	3	24	2	2
9^2_5	0	1	32	1	2
9^2_6	2	3	36	2	2
9^2_7	2	3	44	2	2
9^2_8	1	1	34	2	2
9^2_9	0	1	40	2	2
9^2_{10}	0	1	24	2	2
9^2_{11}	1	3	46	2	2
9^2_{12}	1	1	50	1	2
9^2_{13}	0	5	24	3	4
9^2_{14}	2	5	36	4	1
9^2_{15}	0	3	40	2, 3?	
9^2_{16}	2	3	44	3	1
9^2_{17}	2	3	36	3	1
9^2_{18}	0	3	48	2	3
9^2_{19}	1	5	26	3	4
9^2_{20}	3	3	34	4	1
9^2_{21}	1	3	38	2	1
9^2_{22}	3	5	46	4	1
9^2_{23}	2	3	36	2	1
9^2_{24}	3	5	54	3	1
9^2_{25}	0	1	48	2	3
9^2_{26}	2	3	52	3	1
9^2_{27}	0	3	40	2, 3?	
9^2_{28}	2	5	44	3	1
9^2_{29}	2	5	44	4	1
9^2_{30}	2	5	52	3	1

L	$\lvert \ell k(A_L, B_L)\rvert$	$\lvert\sigma\rvert$	$\lvert H_1(M_L)\rvert$	$u(L)$	Method
9^2_{31}	0	3	40	2, 3?	
9^2_{32}	0	1	56	2	3
9^2_{33}	0	1	56	2	3
9^2_{34}	1	1	50	1	1
9^2_{35}	1	3	46	2	3
9^2_{36}	0	1	48	2, 3?	
9^2_{37}	0	1	48	2	3
9^2_{38}	2	3	60	2	1
9^2_{39}	1	3	54	2	1
9^2_{40}	3	3	50	4	1
9^2_{41}	0	1	56	2	3
9^2_{42}	1	1	66	2	3.
9^2_{43}	2	7	4	4	1
9^2_{44}	0	3	16	2	1
9^2_{45}	2	5	12	3	1
9^2_{46}	0	1	16	2	3
9^2_{47}	0	1	18	1	1
9^2_{48}	2	5	20	3	1
9^2_{49}	3	7	2	4	1
9^2_{50}	1	1	10	2	1
9^2_{51}	3	5	14	4	1
9^2_{52}	1	3	22	2	1
9^2_{53}	4	6	∞	4	1
9^2_{54}	1	1	18	1	1
9^2_{55}	0	3	24	2	1
9^2_{56}	0	3	24	2	3
9^2_{57}	2	3	12	2	1
9^2_{58}	2	3	28	2	1
9^2_{59}	2	5	4	4	1
9^2_{60}	2	5	4	3	1
9^2_{61}	4	4	∞	4	1

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