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ON ELLIPTIC SYSTEMS IN L^1

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Introduction and basic notation

It is the aim of this paper to study regular elliptic problems in the framework of L^1. We are interested in existence and regularity of solutions and in estimates depending on a parameter leading to results of generation of analytic semigroups.

We start by considering what (in our knowledge) already exists on this subject.

In [2] the authors prove the accretiveness of certain realizations of Dirichlet and Neumann problems with homogeneous boundary conditions for second order elliptic equations in variational form, in cases where a maximum principle is available.

In [1] H. Amann takes advantage of some results of Stewart [14] for elliptic problems in spaces of continuous functions and of certain duality arguments to obtain some results of generation of semigroups in L^1 space again for realizations of second order elliptic problems in variational form. The same basic idea is used in Pazy's short treatment of Dirichlet problem for operators of arbitrary order (see [10]).

In the book [16] H. Tanabe, using ideas of R. Beals and L. Hömander, estimates the kernels of $(A_p-\lambda)^{-1}$ and $\exp(tA_p)$, where A_p is the realization in $L^p(\Omega)$ of a certain elliptic operator with certain homogeneous boundary conditions and $\exp(tA_p)$ is the semigroup generated by it. Then, a semigroup $G(t)$ in $L^1(\Omega)$ is defined by

$$(G(t)f)(x):=\int_\Omega G(t,x,y)f(y)\,dy,$$

where $G(t,x,y)$ is the kernel of $\exp(tA_p)$ which is of course independent of p. Finally, the L^1 realization A_1 of the elliptic operator is defined as the infinitesimal generator of $G(t)$. These results require however the existence of a dual problem of the same type (in other words of a Green's formula) and so a variational formulation or a certain regularity of the coefficients (for a statement of the needed assumptions in a western language see also [11]). Moreover, only the case of equations is considered.

Instead, we are interested in problems for elliptic systems in nonvariational form and in results requiring minimal assumptions of regularity of the coefficients.

We recall also that the Dirichlet problem for second order elliptic systems in nonvariational form is treated in [18], who proves, in this particular case, a result of generation of analytic semigroups under assumptions similar to ours, using again a duality argument. However, the solution is only intended in a so called "ultraweak" sense and no attempt is made to study its properties and regularity more precisely.

We go now to explain the organization of the paper; the first paragraph contains the study of elliptic systems in $\boldsymbol{R}^n$. The main result is contained in 1.7 (generation of analytic semigroups in $L^1(\boldsymbol{R}^n)^N$ by elliptic operators with hölder continuous coefficients). The solution is constructed using the classical method of Levi.

In corollary 1.9 it is stated and proved that, if $u \in B^{2m}_{1,\infty}(\boldsymbol{R}^n)^N$ and $A(x, \partial)u \in L^1(\boldsymbol{R}^n)^N$, all the derivatives of order not overcoming $2m-1$ of u are regular distribution in the variable x_n, and so admit a sectional trace $u(\cdot, x_n)$ for any $x_n \in \boldsymbol{R}$ (we say that $v \in \mathcal{D}'(\boldsymbol{R}^n)$ is regular in the variable x_n if there exist $V \in C(\boldsymbol{R}; \mathcal{D}'(\boldsymbol{R}^{n-1}))$ such that for any $\phi \in \mathcal{D}(\boldsymbol{R}^n)$ $u(\phi) = \int_{\boldsymbol{R}} \langle V(x_n), \phi(\cdot, x_n)\rangle \, dx_n$; it is natural to identify $V(c)$ with the sectional trace of v in the hyperplane $x_n = c$). The paragraph ends with some results (in the constant coefficient case and for equations) showing that the same precise information concerning the regularity of the solution as in case $p > 1$ cannot be expected. For results of analogous type in spaces of continuous functions, even in the case of nonconstant coefficients see [13], where the author promises analogous considerations for L^1. Ours are simply intended as examples and, due to their simplicity, give some insight into the difference between the two cases $p=1$ and $1 < p < +\infty$.

The second paragraph is rather technical in its content and deals with elliptic boundary value problems in half spaces. It is essentially directed to the proof of the key result, contained in the statement of 2.16. Here, too, the basic technique is a variant of Levi's method (see for a discussion the notes following 2.6).

The third paragraph contains the main result of the paper, in 3.3. Essentially, a result of generation of an analytic semigroup by a certain realization of an elliptic problem in $L^1(\Omega)^N$ is given, for nonvariational problems under "minimal regularity" assumptions on the coefficients.

The fourth and final paragraph contains some results concerning Besov spaces which are used here and there in the paper. We refer without comment to this paragraph for the basic definitions and properties.

Now we introduce some notation: let X be an open subset of $\boldsymbol{R}^n$, E a locally convex Hausdorff space; then, $C(X, E)$ is the space of continuous functions from X to E; if $j \in \boldsymbol{N}$, $C^j(X, E)$ is the set of functions from X to E with all partial derivatives of order less than or equal to m continuous; $C^\infty(X, E)$ stands for the intersection of all the spaces $C^j(X, E)$; if $\alpha>0$, $\alpha=j+\beta$, with $j \in \boldsymbol{N}$ or $j=0$ and $0<\beta<1$, $C^\alpha(X, E)$ is the set of elements of $C^j(X, E)$ such that for any multiindex γ of weight j, for any continuous seminorm p in E, there exists $C \geq 0$, depending on j and p such that $p(\partial^\gamma f(x)-\partial^\gamma f(y)) \leq C|x-y|^\beta$ for any $x, y \in X$. $BC^j(X, E)$ will be the subspace of $C^j(X, E)$ whose elements have all the derivatives of order not exceeding j bounded. An analogous meaning will have $BC^\alpha(X, E)$. If E is dropped in these notations, we shall always assume $E=\boldsymbol{C}$. Here and there we shall consider also the case where X is substituted by its topological closure $\bar{X}$. We shall mean the subset of elements of the corresponding space continuously extendable together with their derivatives to $\bar{X}$.

$\mathcal{O}_M$ will indicate the space of functions which are C^∞ in $\boldsymbol{R}^n$ such that for any multiindex α there exists $m(\alpha)$ real such that $\partial^\alpha u=O(|x|^{m(\alpha)})$ $(|x| \rightarrow+\infty)$.

Let $\delta \in \boldsymbol{R}$. We set $BC_\delta(\boldsymbol{R}^n):=\{f \mid |x|^\delta f \in L^\infty(\boldsymbol{R}^n)\}$.

If $A=(a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is an $m \times n$ matrix, $\|A\|=\max_{1 \leq i \leq m, 1 \leq j \leq n}|a_{ij}|$.

If $u \in \mathcal{S}'(\boldsymbol{R}^n)$, $r>0$, $\tau \in \boldsymbol{R}$, we set $(r-\Delta)^\tau u:=F^{-1}((r+|\xi|^2)^\tau Fu)$, where F is the Fourier tranform and F^{-1} is its invevse Fourier tranform.

$(\cdot, \cdot)_{[\theta]}$ and $(\cdot, \cdot)_{\theta, q}$ $(0<\theta<1, 1 \leq q \leq+\infty)$ are the complex and real interpolation functors.

The notations "C" and "const" will mean constants (which may be different in each case) which we are not interested to precise. $C(\alpha, \beta, \gamma, \cdots)$ will mean a constant depending on $\alpha, \beta, \cdots$. If $\alpha \in \boldsymbol{R}$, $[\alpha]$ is the largest integer not larger than α, $[\alpha]^-$ the largest integer strictly less than α, $[\alpha]=\alpha-[\alpha]$, $\{\alpha\}^-=\alpha-[\alpha]^-$.

If X is a Banach space, $\|\cdot\|_X$ will stand for the norm in the space X.

If $R>0$, B_R is the open ball with center 0 and radius R, B_R^+ is the subset of elements of B_R with the last coordinate positive.

$\langle\cdot, \cdot\rangle$ stands for the duality between a certain locally convex space X and its dual space X'. If f is a function of certain distinguished arguments $\partial_j^\alpha f$ will be the derivative with respect to the j-th argument. If $1 \leq r \leq n$, e_r is the r-th element of the canonical basis of $\boldsymbol{R}^n$. If $1 \leq p \leq+\infty$, $\|\cdot\|_p$ will indicate the norm in the space $L^p(\boldsymbol{R}^n)$.

For alternative notations (with the same meaning) in the case of Besov spaces see the fourth section of the paper.

1. Problems in $\boldsymbol{R}^n$

The next proposition will be crucial in the following:

Proposition 1.1. *Let $m \in C^\infty(\boldsymbol{R}^n)$, $s \in \boldsymbol{R}$. Assume that $\forall \alpha \in \boldsymbol{N}_0^n$ there exists $C(\alpha) \geq 0$ such that $|\partial^\alpha m(\xi)| \leq C(\alpha)(1+|\xi|)^{-s-|\alpha|}$. We set $K=F^{-1} m$. Then,*

(a) $K\in B^s_{1,\infty}(\boldsymbol{R}^n)$.
(b) $K|_{\boldsymbol{R}^n\setminus\{0\}}\in C^\infty(\boldsymbol{R}^n\setminus\{0\})$ *and is rapidly decreasing together with all its derivatives at infinity.*
(c) *If* $s\leq n$, *for any* $\varepsilon>0$ *there exists* $C(\zeta)\geq 0$ *such that*

$$|K(x)|\leq C(\varepsilon)\ |x|^{s-n-\varepsilon},\ \forall x\in\boldsymbol{R}^n\setminus\{0\}\ .$$

(d) *If* $s>n$, $K\in C(\boldsymbol{R}^n)$.

Proof. (a) We put $m'(\xi)=(1+|\xi|^2)^{s/2}\,m(\xi)$. Then, by [4], m' is a Fourier multiplier for the space $BC^\alpha(\boldsymbol{R}^n)$ $(0<\alpha<1)$. This implies (see [15] 2, th. 2) that $F^{-1}\,m'\in B^0_{1,\infty}(\boldsymbol{R}^n)$. From 4.5 one has the result.
(b) If $\alpha,\beta\in\boldsymbol{N}^n_0$, with $|\beta|-|\alpha|<s-n$, then $\xi\to\partial^\alpha(\xi^\beta\,m(\xi))\in L^1(\boldsymbol{R}^n)$. This implies that $x\to x^\alpha\,\partial^\beta\,K(x)$ is continuous and bounded in $\boldsymbol{R}^n$, which proves (b).
(c) Assume $n-1<s\leq n$. Then, for $j=1,\cdots,n$, $\partial_j\,m\in L^1(\boldsymbol{R}^n)$. As $\int_{\boldsymbol{R}^n}\partial_j\,m(\xi)\,d\xi=0$, one has, for $0<a\leq 1$:

$$(-ix_j)\,K(x)=(2\pi)^{-n}\int_{\boldsymbol{R}^n}[(\exp(ix\cdot\xi)-1]\ |x\cdot\xi|^{-a}\ |x\cdot\xi|^a\ \partial_j\,m(\xi)\,d\xi$$

which implies, using the inequality $|[(\exp(ix\cdot\xi)-1]|x\cdot\xi|^{-a}|\leq C$,

$$|x_j\,K(x)|\leq C\,|x|^a\int_{\boldsymbol{R}^n}|\xi|^a|\partial_j\,m(\xi)|\ d\xi<+\infty$$

if $a<s+1-n$. So (c) is proved if $n-1<s\leq n$.

If $n-2<s\leq n-1$, for $j=1,\cdots,n$, $-ix_jK(x)=F^{-1}(\partial_j\,m)\,(x)$ satisfies

$$|x_j\,K(x)|\leq C(\varepsilon)|x|^{s+1-n-\varepsilon}$$

$\forall\varepsilon>0$, $x\in\boldsymbol{R}^n\setminus\{0\}$, for $j=1,\cdots,n$. Iterating the method, one obtains the general result.
(d) follows from the fact that $m\in L^1(\boldsymbol{R}^n)$.

REMARK 1.2. All the constants appearing in the statement of 1.1 depend on $\sup_{\xi\in\boldsymbol{R}^n,|\alpha|\leq M}(1+|\xi|)^{s+|\alpha|}|\partial^\alpha\,m(\xi)|$ with M suitably large.

Let $A(x,\partial)=(A_{ij}(x,\partial))_{1\leq i\leq N,1\leq j\leq N}$ be a differential operator valued matrix $(x\in\boldsymbol{R}^n)$. We assume that
(h1) $\forall(i,j)$ *the order of* $A_{ij}(x,\partial)$ *does not exceed* $2m$ $(m\in\boldsymbol{N})$.
(h2) *The coefficients of* $A_{ij}(x,\partial)$ *are of class* $BC^\beta(\boldsymbol{R}^n)$ $(\beta>0)$.
We set $\beta':=\min\{\beta,1\}$.

Next, we indicate with $A^\sharp_{ij}(x,\partial)$ the part of order $2m$ of $A_{ij}(x,\partial)$, $A^\sharp(x,\partial)=(A^\sharp_{ij}(x,\partial))_{1\leq i\leq N,1\leq j\leq N}$. Let $\theta\in[-\pi/2,\pi/2]$. We assume:
(h3) For any $x\in\boldsymbol{R}^n$, *for any* $(\xi,r)\in\boldsymbol{R}^n\times[0,+\infty[\setminus\{(0,0)\}$ *the matrix* $A^\sharp(x,i\xi)-r^{2m}\exp(i\theta)$ *is invertible and* $|\det(A^\sharp(x,i\xi)-r^{2m}\exp(i\theta))|\geq c(|\xi|+r)^{2mN}$, *with c*

positive and independent of x, ξ, r.

Now we set (for a fixed $\theta\in[-\pi/2, \pi/2]$),

$$K(\cdot, y, r) = (K_{ji}(\cdot, y, r))_{1\le j\le N, 1\le i\le N} = F^{-1}((A^{\sharp}(y, i\cdot)-r^{2m}\exp(i\theta))^{-1})$$

$(r>0)$. As $(\xi, r)\to A^{\sharp}(y, i\xi)-r^{2m}\exp(i\theta)$ is positively homogeneous of degree $2m$, we have for $r\ge 1$, $\alpha\in \boldsymbol{N}_0^n$

$$||\partial_\xi^\alpha(A^{\sharp}(y, i\cdot)-r^{2m}\exp(i\theta))^{-1}||\le C(\alpha)(r+|\xi|)^{-2m-|\alpha|}, \tag{1}$$

so that from 1.1 and 1.2 we have easily:

Proposition 1.3. *Under the assumptions* (h1) *and* (h2) *one has for* $1\le i\le N$, $1\le j\le N$:
(a) $\forall r>0$, $\forall y\in \boldsymbol{R}^n$
$y\to K_{ji}(\cdot, y, r)\in BC^\beta(\boldsymbol{R}^n, B^{2m}_{1,\infty}(\boldsymbol{R}^n))$ *and* $||K_{ji}(\cdot, y, r)||_{B^{2m}_{1,\infty}(\boldsymbol{R}^n)}\le C$, independent of $r\ge 1$, $y\in \boldsymbol{R}^n$;
(b) *For any* $\chi\in C^\infty(\boldsymbol{R}^n)$, *such that* $\chi(\xi)=0$ *in a neighbourhood of* 0, $\chi(\xi)=1$ *for* $|\xi|$ *large, for any* $r>0$, $y\to\chi(\cdot)K_{ji}(\cdot, y, r)\in C^\beta(\boldsymbol{R}^n, \mathcal{S}(\boldsymbol{R}^n))$
(c) *If* $\alpha, \gamma\in \boldsymbol{N}_0^n$, $2m-|\alpha|\le n$, $|\gamma|\le\beta$, $\forall\varepsilon>0$ *there exists* $C(\alpha, \varepsilon, r)>0$, *such that*

$$||\partial_x^\alpha\,\partial_y^\gamma K(x, y, r)||\le C(\alpha, \varepsilon, r)\ |x|^{2m-|\alpha|-n-\varepsilon},\ \forall x\in \boldsymbol{R}^n\backslash\{0\}\ ;$$

moreover, if $|\gamma|=[\beta]$,

$||\partial_x^\alpha\,\partial_y^\gamma K(x, y_1, r)-\partial_x^\alpha\,\partial_y^\gamma K(x, y_2, r)||\le C(\alpha, \varepsilon, r)\ |x|^{2m-|\alpha|-n-\varepsilon}|y_1-y_2|^{\{\beta\}}$, $\forall x\in \boldsymbol{R}^n\backslash\{0\}$, $y_1, y_2\in \boldsymbol{R}^n$.
(d) *If* $\alpha, \gamma\in \boldsymbol{N}_0^n$ $2m-|\alpha|>n$, $|\gamma|\le\beta$, *there exists* $C(\alpha, \varepsilon, r)>0$, *such that*
$||\partial_x^\alpha\,\partial_y^\gamma K(x, y, r)||\le C(\alpha, r)$, *for any* $y\in \boldsymbol{R}^n$, $r>0$,
$||\partial_x^\alpha\,\partial_y^\gamma K(x, y_1, r)-\partial_x^\alpha\,\partial_y^\gamma K(x, y_2, r)||\le C(\alpha, \varepsilon, r)\ |y_1-y_2|^{\{\beta\}}$, $\forall x\in \boldsymbol{R}^n\backslash\{0\}$, $y_1, y_2\in \boldsymbol{R}^n$.
(e) *If* $\alpha\in \boldsymbol{N}_0^n$, $r>0$, $\partial_x^\alpha K(x, y, r)=r^{n+|\alpha|-2m}\,\partial_x^\alpha K(rx, y, 1)$.
(f) *If* $|\alpha|\le 2m-1$, $1\le i, j\le N$,
$||\partial_x^\alpha K_{ji}(\cdot, y, r)||_{L^1(\boldsymbol{R}^n)}=r^{|\alpha|-2m}\,||\partial_x^\alpha K_{ji}(\cdot, y, 1)||_{L^1(\boldsymbol{R}^n)}$.

Lemma 1.4. *Let* $\phi\in L^1(\boldsymbol{R}^n)^N$. *If* $r>0$, *we set*

$$T_r\phi(x) = \int_{\boldsymbol{R}^n} K(x-y, y, r)\,\phi(y)\,dy\,.$$

Then,
(a) $T_r\in\mathcal{L}(L^1(\boldsymbol{R}^n)^N; B^{2m}_{1,\infty}(\boldsymbol{R}^n)^N)$.
(b) *If* $0\le j\le 2m-1$, $||T_r||_{\mathcal{L}(L^1(\boldsymbol{R}^n)^N\,;\,W^{j,1}(\boldsymbol{R}^n)^N)}\le Cr^{j-2m}$, *with* $C>0$ *independent of* $r\ge 1$.

(c) *If* $0 \leq s \leq 2m$ $||T_r||_{\mathcal{L}(L^1(\boldsymbol{R}^n)^N\,;\,B^s_{1,\infty}(\boldsymbol{R}^n)^N)} \leq Cr^{s-2m}$.

Proof. From 1.3 (f) one has that $T_r \in \mathcal{L}(L^1(\boldsymbol{R}^n)^N; W^{j,1}(\boldsymbol{R}^n)^N)$ if $j \leq 2m-1$ and it is easily seen that

$$||T_r||_{\mathcal{L}(L^1(\boldsymbol{R}^n)^N\,;\,W^{j,1}(\boldsymbol{R}^n)^N)} \leq C \sup_{y\in\boldsymbol{R}^n, 1\leq j,i\leq N, |\alpha|\leq j} ||\partial_x^\alpha K_{ji}(\cdot, y, r)||_{L^1(\boldsymbol{R}^n)} \leq Cr^{j-2m}.$$

Analogously, using directly the definition of $B^{2m}_{1,\infty}(\boldsymbol{R}^n)^N$ in 4.1, one can show (c) in case $s=2m$. The proof of (*c*) in case $0<s<2m$ follows by interpolation (see 4.8).

Lemma 1.4. *For any* $r>0$, $\phi \in L^1(\boldsymbol{R}^n)^N$

$$(2)\quad [A(x, \partial)-r^{2m}\exp(i\theta)]\, T_r\, \phi(x) = \phi(x) + \int_{\boldsymbol{R}^n} [A(x, \partial_x)-A^\sharp(y, \partial_x)]\, K(x-y, y, r)\, \phi(y)\, dy\,.$$

Proof. First of all, by (h2), 1.4 and 4.6, $[A(\cdot, \partial)-r^{2m}\exp(i\theta)]\, T_r\, \phi$ is well defined and belongs to $B^0_{1,\infty}(\boldsymbol{R}^n)^N$. Assume that the coefficients of the system are of class C^∞. We indicate with $A(x, \partial)^T$ the dual system. One has for $\Psi \in \mathcal{D}(\boldsymbol{R}^n)^N$:

$$\langle[A(x, \partial)-r^{2m}\exp(i\theta)]\, T_r\, \phi, \Psi\rangle = \int_{\boldsymbol{R}^n} \Big(\int_{\boldsymbol{R}^n} K(x-y, y, r)\, \phi(y)\, dy\Big).\ [A(x, \partial)^T - r^{2m}\exp(i\theta)]\, \Psi(x)\, dx =$$

$$\int_{\boldsymbol{R}^n} \Big(\int_{\boldsymbol{R}^n} K(x-y, y, r)\, \phi(y) \cdot [A(x, \partial)^T - r^{2m}\exp(i\theta)]\, \Psi(x)\, dx\Big)\, dy\,.$$

As $[A(x, \partial_x)-r^{2m}\exp(i\theta)]\, K(x-y, y, r) = \delta(x-y)\, I_N + [A(x, \partial_x)-A^\sharp(y, \partial_x)]\, K(x-y, y, r)$,

(I_N is the $N\times N$ identity matrix) the result follows.

If the coefficients are not C^∞, they can be approximated in $BC^{\beta'}(\boldsymbol{R}^n)$ by BC^∞ coefficients for any $\beta'<\beta$ and the result follows from the convergence of the corresponding terms of (2) in the sense of distributions.

Corollary 1.6. (a) *For any* $r>0$, $\forall \phi \in L^1(\boldsymbol{R}^n)^N$

$$[A(\cdot, \partial)-r^{2m}\exp(i\theta)]\, T_r\, \phi \in L^1(\boldsymbol{R}^n)^N$$

Proof. The proof follows immediately from 1.5, (h2) and 1.3(c).

Theorem 1.7. *Assume that the asssumption* (h3) *is satisfied* $\forall \theta \in [-\pi/2, \pi/2]$. *We consider the following operator*:

$$D(A) = \{u \in B^{2m}_{1,\infty}(\boldsymbol{R}^n)^N : A(x, \partial)\, u \in L^1(\boldsymbol{R}^n)^N\},\ Au = A(x, \partial)\, u\ (u \in D(A))\,.$$

Then, A is the infinitesimal generator of an analytic semigroup in $L^1(\boldsymbol{R}^n)^N$.

Proof. By 4.9 $\rho(A)$ contains $\{z \in \boldsymbol{C} \mid \mathrm{Re}\, z \geq 0,\ |z| \geq R\}$ for $R \geq 0$ suitably large. Further, it is clear that $D(A)$ is dense in $L^1(\boldsymbol{R}^n)^N$ because it contains $W^{2m,1}(\boldsymbol{R}^n)^N$. It remains to show that $\forall\theta\in[-\pi/2, \pi/2]$

$$||(A-r^{2m}\exp(i\theta))^{-1}||_{\mathcal{L}(L^1(\boldsymbol{R}^n)^N)} = O(r^{-2m})\ (r\to+\infty)\,.$$

We consider the equation $Au-r^{2m}\exp(i\theta)\,u=f(f\in L^1(\boldsymbol{R}^n)^N)$. We try to write the unique solution $u\in B^{2m}_{1,\infty}(\boldsymbol{R}^n)^N$ of this problem (with r large enough) in the form

$$u(x) = \int_{\boldsymbol{R}^n} K(x-y, y, r)\,\phi(y)\,dy$$

with $\phi\in L^1(\boldsymbol{R}^n)^N$.

From 1.5 one has that

$$(*)\qquad \phi(x)+\int_{\boldsymbol{R}^n}[A(x,\partial_x)-A^\sharp(y,\partial_x)]\,K(x-y,y,r)\,\phi(y)\,dy = f(x)\,.$$

If $\phi\in L^1(\boldsymbol{R}^n)^N$ one has

$$\begin{aligned}&\int_{\boldsymbol{R}^n}\Big|\int_{\boldsymbol{R}^n}[A(x,\partial_x)-A^\sharp(y,\partial_x)]\,K(x-y,y,r)\,\phi(y)\,dy\Big|\,dx\le\\ &\sup_{y\in\boldsymbol{R}^n,1\le i,j,l\le N}||[A_{ij}(\cdot,\partial_x)-A^\sharp_{ij}(y,\partial_x)]\,K_{jl}(\cdot-y,y,r)||_{L^1(\boldsymbol{R}^n)}\,||\phi||_{L^1(\boldsymbol{R}^n)^N}\\ &\quad\le Cr^{-\beta'}\,||\phi||_{L^1(\boldsymbol{R}^n)^N}\end{aligned}$$

owing to (h2) and (c), (e) and (f) in 1.3.

This implies that if r is large enough (*) has a unique solution in $L^1(\boldsymbol{R}^n)^N$ and $||\phi||_{L^1(\boldsymbol{R}^n)^N}\le C||f||_{L^1(\boldsymbol{R}^n)^N}$ with $C>0$ independent of r and f. So the desired estimate follows from 1.4 (b).

In view of the treatment of boundary value problems, we are going to consider the existence of sectional traces on x_n=const of $T_r\phi$ and of some of its derivatives ($\phi\in L^1(\boldsymbol{R}^n)^N$). We start with the following

Lemma 1.8. *Assume* $\beta\in\boldsymbol{N}^n_0$, $1\le j, i\le N$.
(a) *For any* $r>0$, *for any* $\delta>0$, $(x_n, y)\to\partial^\beta_x K_{ji}(\cdot, x_n, y, r)\in C((]-\infty, -\delta]\cup[\delta, +\infty[)\times\boldsymbol{R}^n; \mathcal{S}(\boldsymbol{R}^n))$.
(b) *If* $|\beta|\le 2m-1$, $\sup_{y\in\boldsymbol{R}^n, x_n\in\boldsymbol{R}\setminus\{0\}}||\partial^\beta_x K_{ji}(\cdot, x_n, y, r)||_{B^{2m-|\beta|-1}_{1,\infty}(\boldsymbol{R}^{n-1})}\le C(r)$.

Proof. (a) follows immediately from 1.3(b).
(b) follows from 1.3 (a), 4.5 and 4.13 if $|\beta|\le 2m-2$ or $|\beta|=2m-1$ and $\beta\neq(0,\cdots,2m-1)$. Assume $\beta=(0,\cdots,2m-1)$. One has, for $x_n\in\boldsymbol{R}\setminus\{0\}$:

$$F(\partial^{2m-1}_n K(\cdot, x_n, y, r))\,(\xi') = F^{-1}_{\xi_n}((i\xi_n)^{2m-1}\,[A^\sharp(i\xi', i\xi_n, y, r)-r^{2m}\exp(i\theta)]^{-1})\,(x_n)\,.$$

Each element of the matric $[A^\sharp(i\xi', i\xi_n, y, r)-r^{2m}\exp(i\theta)]^{-1}$ is of the type

$$\frac{\alpha(\xi', \xi_n, y, r)}{\det([A^\sharp(i\xi', i\xi_n, y, r)-r^{2m}\exp(i\theta)])}$$

with $\alpha(\cdot, \cdot, y, \cdot)$ homogeneous polynomial of order $2m(N-1)$, $y\to\alpha(\xi', \xi_n, y, r)\in C^\beta(\boldsymbol{R}^n)$ $\forall(\xi', \xi_n, r)$. Moreover,

$$\det(A^\sharp(i\xi', i\xi_n, y, r)-r^{2m}\exp(i\theta)) = \sum_{l=0}^{2mN} P_l(\xi', y, r)\,\xi_n^l$$

with $P_l(\cdot, y, \cdot)$ homogeneous polynomial of order $2mN-l, y\to P_l(\xi', y, r)\in BC^\beta(\boldsymbol{R}^n)$, for any ξ', r. It follows that, for a fixed $\chi\in C^\infty(\boldsymbol{R})$, such that $\chi(t)=0$ in a neighbourhood of 0, $\chi(t)=1$ for $|t|$ large, each element of the matrix

$$(i\xi_n)^{2m-1}\,[A^\sharp(i\xi', i\xi_n, y, r)-r^{2m}\exp(i\theta)]^{-1}$$

is of the type

$$a(y)\,\chi(\xi_n)\,\xi_n^{-1}+Q(\xi', \xi_n, y, r)$$

with $a\in BC^\beta(\boldsymbol{R}^n), y\to Q(\cdot, \cdot, y, \cdot)\in BC^\beta(\boldsymbol{R}^n, C^\infty(\boldsymbol{R}^n\times[0, +\infty[\setminus\{(0, 0)\}))$, $\partial_{\xi_n}^k Q(\xi', \cdot, y, r)=O(\xi_n^{-2-k})$ $(|\xi_n|\to+\infty)$ $\forall k\in\boldsymbol{N}\cup\{0\}$. Therefore we have from 1.1 that, for any $x_n\in\boldsymbol{R}\setminus\{0\}$, $1\leq i, j\leq N$,

$$F(\partial_n^{2m-1}K_{ji}(\cdot, x_n, y, r))\,(\xi') = H(x_n)\,A(y)+\Phi(\xi', x_n, y, r)$$

(H is the Heaviside function)
with $x_n\to\Phi(\xi', x_n, y, r)\in B^2_{1,\infty}(\boldsymbol{R})$, $(\xi', r)\to\Phi(\xi', \cdot, y, r)\in C^\infty(\boldsymbol{R}^{n-1}\times[0, +\infty[\setminus\{(0, 0)\}; B^2_{1,\infty}(\boldsymbol{R}))$ for any $y\in\boldsymbol{R}^n$, $y\to\Phi(\xi', \cdot, y, r)\in BC^\beta(\boldsymbol{R}^n, B^2_{1,\infty}(\boldsymbol{R}))$ for any $(\xi', r)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}^+$. As $B^2_{1,\infty}(\boldsymbol{R})$ is imbedded in $BC(\boldsymbol{R})$, one has

$$|\Phi(\xi', x_n, y, r)|\leq C(\xi', y, r) \quad \text{for any} \quad x_n\in\boldsymbol{R}\,.$$

Moreover, it is easily seen that

$$\begin{aligned}&F(\partial_n^{2m-1}K(\cdot, x_n, y, r))\,(\xi')\\&\quad = F(\partial_n^{2m-1}K(\cdot, x_n(|\xi'|^2+r^2)^{1/2}, y, r(|\xi'|^2+r^2)^{-1/2}))\,((|\xi'|^2+r^2)^{-1/2}\,\xi')\,,\end{aligned}$$

which implies

$$\Phi(\xi', x_n, y, r) = \Phi((|\xi'|^2+r^2)^{-1/2}\,\xi', x_n(|\xi'|^2+r^2)^{1/2}, y, r(|\xi'|^2+r^2)^{-1/2})\,,$$

from which the estimate

$$\begin{aligned}&||\Phi(\xi', \cdot, y, r)||_\infty = ||\Phi((|\xi'|^2+r^2)^{-1/2}\,\xi', \cdot, y, r(|\xi'|^2+r^2)^{-1/2})||_\infty\\&\quad\leq C \ (\text{if } |\xi'|^2+r^2>0)\end{aligned}$$

follows. So,

$$\begin{aligned}&||F(\partial_n^{2m-1}K(\cdot, x_n, y, r))\,(\xi')||\leq C, \quad \text{independent of} \quad x_n(\neq 0), y, r, \xi'\\&\quad \text{with} \quad |\xi'|^2+r^2>0\,.\end{aligned}$$

If $\alpha' \in N_0^{n-1}$, one has

$$\partial_{\xi'}^{\alpha'} F(\partial_n^{2m-1} K(\cdot, x_n, y, r))(\xi') = (2\pi)^{-1} \int_R \exp(ix_n \xi_n)(i\xi_n)^{2m-1} \partial_{\xi'}^{\alpha'} [A^\sharp(i\xi', i\xi_n, y, r) - r^{2m} \exp(i\theta)]^{-1}) \, d\xi_n$$

from which

$$\|\partial_{\xi'}^{\alpha'} F(\partial_n^{2m-1} K(\cdot, x_n, y, r))(\xi')\| \leq C(\alpha')(1+|\xi'|)^{-|\alpha'|}$$

This, together with 1.1, implies (b).

Corollary 1.9. *For any $\phi \in L^1(R^n)^N$, for any $r>0$, for any $\alpha \in N_0^n$ with $|\alpha| \leq 2m-1$, $\partial^\alpha T_r \phi$ is a distribution which is regular in the variable x_n. If $|\alpha| \leq 2m-2$, $\partial^\alpha T_r \phi(\cdot, 0) \in B_{1,\infty}^{2m-1-|\alpha|}(R^{n-1})^N$; if $|\alpha|=2m-1$, $\partial^\alpha T_r \phi(\cdot, 0) \in \bigcap_{\varepsilon>0} B_{1,\infty}^{-\varepsilon}(R^{n-1})^N$.*

Proof. Owing to 1.8(b) and 4.5, if $|\alpha| \leq 2m-1$, for any $\tau>0$, for any $x_n \in R \setminus \{0\}$, $1 \leq i, j \leq N$,

$$(r^2 - \Delta_{x'})^{-\tau} \partial_x^\alpha K_{ji}(\cdot, x_n, y, r) \in L^1(R^{n-1}) \quad \text{and}$$
$$\|(r^2 - \Delta_{x'})^{-\tau} \partial_x^\alpha K_{ji}(\cdot, x_n, y, r)\|_{L^1(R^{n-1})^N} \leq C(r, \tau)$$

so that it follows easily from Fubini's theorem that for any $\tau>0$ $(r^2 - \Delta_{x'})^{-\tau} \partial_x^\alpha T_r \phi$ is regular in x_n, with traces in $L^1(R^{n-1})^N$. Its trace in $x_n = c$ is of course

$$x' \to \int_{R^n} (r^2 - \Delta_{x'})^{-\tau} \partial_x^\alpha K(x'-y', c-y_n, y, r) \phi(y) \, dy$$

The belonging of $\partial^\alpha T_r \phi(\cdot, 0)$ to $B_{1,\infty}^{2m-1-|\alpha|}(R^{n-1})$ in case $|\alpha| \leq 2m-2$ follows from 4.13.

We conclude this paragraph showing that, if $n \geq 2$, there is no hope to obtain optimal regularity results comparable with those available in case $1<p<+\infty$ (see, for example [3]). For simplicity we shall consider only equations with constant coefficients. We recall that a partial differential operator $A(\partial)$ of order $2m$ is strongly elliptic if $\operatorname{Re} A^\sharp(i\xi)>0$ for any $\xi \in R^n \setminus \{0\}$. We start with the following

Lemma 1.10. *Let $m \in C^\infty(R^n)$. Assume that there exists $m_0 \in C^\infty(R^n \setminus \{0\})$, positively homogeneous of degree 0 and $\delta>0$ such that for any $\alpha \in N_0^n$*

$$\partial^\alpha(m-m_0)(\xi) = O(|\xi|^{-\delta-|\alpha|}) \, (|\xi| \to +\infty).$$

Then m is a Fourier multiplier for $L^1(R^n)$ if and only if m_0 is a constant function.

Proof. First of all, we fix $\chi \in C^\infty(R^n)$, such that $\chi(\xi)=0$ if $|\xi| \leq 1$, $\chi(\xi)=1$ if $|\xi| \geq 2$. We have $m = \chi m + (1-\chi) m$. $(1-\chi) m \in \mathcal{D}(R^n)$ and so m is a

Fourier multiplier if and only if χm is a Fourier multiplier. We have $\chi m=\chi m_0+\chi(m-m_0)$. Owing to 1.1, $F^{-1}(\chi(m-m_0))\in B^{\delta}_{1,\infty}(\boldsymbol{R}^n)$. So, m is a Fourier multiplier if and only if χm_0 is a Fourier multiplier. This happens if and only if $F^{-1}(\chi m_0)$ is a finite Borel measure (see [8] th. 1.4). Let f, f_0, f_1 be, respectively, the restrictions of $F^{-1}(\chi m_0)$, $F^{-1} m_0$, $F^{-1}((\chi-1) m_0)$ to $\boldsymbol{R}^n\setminus\{0\}$. Then, $f=f_0+f_1$. So $f\in C^\infty(\boldsymbol{R}^n\setminus\{0\})$, because f_0 and f_1 are elements of $C^\infty(\boldsymbol{R}^n\setminus\{0\})$ (owing, respectively, to [9] th. 7.1.18 and the fact that $(\chi-1) m_0$ has compact support). Now, if $F^{-1}(\chi m_0)$ is a finite Borel measure, f is necessarily in $L^1(\boldsymbol{R}^n\setminus\{0\})$; in particular $f\in L^1(B_1)$. But of course $f_1\in L^1(B_1)$. This implies that $f_0\in L^1(B_1)$. However, f_0 is homogeneous of degree $-n$ (see ([9] th. 7.1.16) which means that f_0 has to be equal to 0. Therefore, the support of $F^{-1} m_0$ is contained in $\{0\}$ and so m_0 is necessarily a homogeneous polynomial function of degree 0.

So the "only if" part is proved. The "if" part follows easily from 1.1.

In [12] the author constructs a function u whose laplacian is in $L^1(\boldsymbol{R}^n)$, but such that, for any $\alpha\in\boldsymbol{N}_0^n$ with $|\alpha|=2$ $\partial^\alpha u\notin L^1(\boldsymbol{R}^n)$. Here we have, more generally

Proposition 1.11. *Let $n\geq 2$, $A(\partial)$ a strongly elliptic differential operator of order $2m$ in $\boldsymbol{R}^n$ with constant coefficients. Set $D(A)=\{u\in L^1(\boldsymbol{R}^n)\,|\,A(\partial)\,u\in L^1(\boldsymbol{R}^n)\}$, $Au=A(\partial)\,u$. Then $D(A)$ contains properly $W^{2m,1}(\boldsymbol{R}^n)$. More precisely, there exists $u\in D(A)$ such that $\partial^\alpha u\notin L^1(\boldsymbol{R}^n)$ $\forall\alpha\in\boldsymbol{N}_0^n$ such that $|\alpha|=2m$.*

Proof. Fix $\lambda_0\in\boldsymbol{C}$, such that $A(i\xi)\neq\lambda_0$ for any $\xi\in\boldsymbol{R}^n$. It is easily seen that $\lambda_0\in\rho(A)$ and, for any $f\in L^1(\boldsymbol{R}^n)$, $(\lambda_0-A)^{-1}f=F^{-1}((\lambda_0-A(i\xi))^{-1}Ff)$. We start by showing that, if $|\alpha|=2m$, there exists $u\in D(A)$ such that $\partial^\alpha u\notin L^1(\boldsymbol{R}^n)$. Assume, by contradiction, that, for some $\alpha\in\boldsymbol{N}_0^n$, $u\in D(A)$ implies $\partial^\alpha u\in L^1(\boldsymbol{R}^n)$. This implies that $m(\xi)=(i\xi)^\alpha(\lambda_0-A(i\xi))^{-1}$ is a Fourier multiplier for $L^1(\boldsymbol{R}^n)$. However, m satisfies the assumptions of 1.10 with $m_0(\xi)=-\xi^\alpha A^\sharp(\xi)^{-1}$, which cannot be constant if $n\geq 2$. Now, define $X_\alpha:=\{u\in D(A)\,|\,\partial^\alpha u\in L^1(\boldsymbol{R}^n)\}$ and set (for $u\in X_\alpha$) $\|u\|_\alpha:=\|u\|_{D(A)}+\|\partial^\alpha u\|_{L^1(\boldsymbol{R}^n)}$. With this norm X_α is a Banach space continuouly imbedded in $D(A)$ and not coinciding with it. It follows from the open mapping theorem that X_α is of the first category as a subset of $D(A)$. So, also the union of all the X_α with $|\alpha|=2m$ is of the first category in $D(A)$ and this proves the result.

Proposition 1.12. *Let $A(\partial)$, $B(\partial)$ strongly elliptic differential operators of order $2m$ with constant coefficients in $\boldsymbol{R}^n$; put $D(A)=\{u\in L^1(\boldsymbol{R}^n)\,|\,A(\partial)\,u\in L^1(\boldsymbol{R}^n)\}$, $Au=A(\partial)\,u$, $D(B)=\{u\in L^1(\boldsymbol{R}^n)\,|\,B(\partial)u\in L^1(\boldsymbol{R}^n)\}$, $Bu=B(\partial)u$. Then, $D(A)=D(B)$ if and only if there exists $c\in\boldsymbol{C}$ such tha $B^\sharp(\xi)=cA^\sharp(\xi)$ for any $\xi\in\boldsymbol{R}^n$.*

Proof. We show the "if" part. Let $u\in D(A)$. Then, $B(\partial)\,u=cAu+B_1(\partial)\,u$

$-cA_1(\partial)u$, where $A_1(\partial)=A(\partial)-A^\sharp(\partial)$, $B_1(\partial)=B(\partial)-B^\sharp(\partial)$. So, owing to the inclusion $D(A)\subseteq B^{2m}_{1,\infty}(\boldsymbol{R}^n)$, $B(\partial)u\in L^1(\boldsymbol{R}^n)$. The opposite inclusion follows from the fact that, clearly, $c\neq 0$.

On the other hand, assume $D(A)=D(B)$. Fix $\lambda_0\in\boldsymbol{C}$, such that $\lambda_0-A(i\xi)\neq 0\ \forall\xi\in\boldsymbol{R}^n$. Then $\lambda_0\in\rho(A)$ and $B(\lambda_0-A)^{-1}\in\mathcal{L}(L^1(\boldsymbol{R}^n))$, which implies that $m(\xi)=B(i\xi)(\lambda_0-A(i\xi))^{-1}$ is a Fourier miltiplier for $L^1(\boldsymbol{R}^n)$. So we can apply 1.10 with $m_0(\xi)=-B^\sharp(\xi)A^\sharp(\xi)^{-1}$.

REMARK 1.13. In 1.12 we have in fact proved something more than what declared in the statement; more precisely, we have shown that, if $A^\sharp$ and $B^\sharp$ are not proportional, there is no type of inclusion between $D(A)$ and $D(B)$.

2. Boundary value problems in a half-space.

We continue to consider a system $A(x,\partial)$ satisfying the assumptions (h1), (h2), (h3) and we couple to it another system of partial differential operators $B(x,\partial)=(B_{\lambda j}(x,\partial))_{1\le\lambda\le mN,1\le j\le N}$. We assume that the following conditions are satisfied:

(h4) *for any* (λ, j) $B_{\lambda j}(x,\partial)$ *is an operator of order less than or equal to* $\sigma_\lambda\le 2m-1$ *with coefficients in* $BC^{2m-\sigma_\lambda-\beta}(\boldsymbol{R}^n)$;
we indicate with $B^\sharp_{\lambda,j}(x,\partial)$ the part of order σ_λ of $B_{\lambda j}$, $B^\sharp(x,\partial)=(B^\sharp_{\lambda,j}(x,\partial))_{1\le\lambda\le mN,1\le j\le N}$. We assume that the following complementary condition is satisfied:
(h5) *for any* $(x',\xi',r)\in\boldsymbol{R}^{n-1}\times\boldsymbol{R}^{n-1}\times[0,+\infty[$, ξ', r *not both* 0, *the O.D.E. problem*

$$A^\sharp(x',0,i\xi',\partial_t)\,v(t)-r^{2m}\exp(i\theta)\,v(t)=0 \text{ in } \boldsymbol{R},$$
$$B^\sharp(x',0,i\xi',\partial_t)\,v(0)=g,$$
$$v \text{ bounded in } \boldsymbol{R}^+$$

has a unique solution $t\to\Omega(\xi',t,x',r)\,g$ *for any* $g\in\boldsymbol{C}^{mN}$;
finally,
(h6) *if* $|x'|$ *is large enough, the coefficients of* $A^\sharp(x',0,\partial)$ *and* $B^\sharp(x',\partial)$ *depend only on* $x'/|x'|$.

From the uniqueness of the solution one has

$$\Omega(\xi',\rho^{-1}t,x',\rho r)=\Omega(\rho^{-1}\xi',t,x',r)\,S(\rho^{-1}). \tag{3}$$

with $S_{\lambda\mu}(\rho)=\delta_{\lambda\mu}\,\rho^{\sigma_\mu}(1\le\lambda,\ \mu\le mN,\ \rho\in\boldsymbol{R}^+)$. Moreover, from the representation of the solution in [19], suppl., th. 2 and (h2), (h4), (h6), it follows that the mapping $x'\to\Omega_{j\lambda}(\cdot,\cdot,x',\cdot)$ is in

$$BC^\beta(\boldsymbol{R}^{n-1}, C^\infty(\{(\xi',t,r)\in\boldsymbol{R}^{n-1}\times[0,+\infty[\times[0,+\infty[:\ (\xi',r)\neq(0,0)\}))$$

Next, we have:

Lemma 2.1. *There exists $\delta>0$ such that for any $\alpha\in\boldsymbol{N}_0^n$, $(\xi',r)\in\boldsymbol{R}^{n-1}\times[0,+\infty[$, $(\xi',r)\neq(0,0)$, $t>0$, $1\leq j\leq N$, $1\leq\lambda\leq mN$, $\forall x'\in\boldsymbol{R}^{n-1}$,*

$$|(\partial_{\xi'})^{\alpha'}(\partial_t)^{\alpha_n}\Omega_{j\lambda}(\xi',t,x',r)|\leq C(\alpha)\exp(-\delta(|\xi'|+r)t)(|\xi'|+r)^{\alpha_n-|\alpha'|-\sigma_\lambda}$$

Proof. Again by [19] supplement, th. 2, if $|\xi'|+r=1$, $|(\partial_{\xi'})^{\alpha'}(\partial_t)^{\alpha_n}\Omega_{j\lambda}(\xi',t,x',r)|\leq C(\alpha)\exp(-\delta t)$, with $\delta>0$, independnent of x'. So the result follows from (3).

Now, let $\tau\in\boldsymbol{R}$. We put

$$H_{j\lambda}(x',x_n,y',r)=F_{\xi'}^{-1}((r^2+|\xi'|^2)^{-\tau/2}\,\Omega_{j\lambda}(\xi',x_n,y',r))(x')\,.$$

Lemma 2.2. *Let $1\leq\lambda\leq mN$, $1\leq j\leq N$, $\tau\in\boldsymbol{R}$. Then,*

(a) $y'\to H_{j\lambda\tau}(\cdot,\cdot,y',r)\in BC^\beta(\boldsymbol{R}^{n-1};C^\infty(\boldsymbol{R}_+^n))\ \forall r>0$;

(b) *Let $\phi\in C^\infty(\boldsymbol{R})$, such that $\phi(t)=0$ if $-\infty<t\leq\delta$, $\phi(t)=1$ if $t\geq2\delta$, $\delta>0$. Then $y'\to((x',x_n)\to\phi(x_n)\,H_{j\lambda\tau}(x',x_n,y',r))\in BC^\beta(\boldsymbol{R}^{n-1};\mathcal{S}(\boldsymbol{R}^n))$.*

(c) *If $\alpha\in\boldsymbol{N}_0^n$ and $|\alpha|\geq\sigma_\lambda+\tau+1-n$, for any $\varepsilon>0$, there esists $C(\varepsilon,r)\geq0$ such that $|\partial_x^\alpha H_{j\lambda\tau}(x,y',r)|\leq C(\varepsilon,r)\,|x|^{1-n+\sigma_\lambda+\tau-|\alpha|-\varepsilon}$;*

(d) *If $\alpha\in\boldsymbol{N}_0^n$ and $|\alpha|<\sigma_\lambda+\tau+1-n$, $|\partial_x^\alpha H_{j\lambda\tau}(x,y',r)|\leq C(\alpha,r)$;*

(e) $\forall x_n\geq0\ H_{j\lambda\tau}(\cdot,x_n,y',r)\in B_{1,\infty}^{\sigma_\lambda+\tau}(\boldsymbol{R}^{n-1})$ *and* $\sup_{x_n\geq0,y\in\boldsymbol{R}^{n-1}}\|H_{j\lambda\tau}(\cdot,x_n,y',r)\|_{B_{1,\infty}^{\sigma_\lambda+\tau}(\boldsymbol{R}^{n-1})}\leq C(r)$;

(f) if $\sigma_\lambda+\tau>-1$, $\forall\varepsilon>0$, $H_{j\lambda\tau}(\cdot,y',r)\in B_{1,\infty}^{\sigma_\lambda+\tau+1-\varepsilon}(\boldsymbol{R}_+^n)$;

(g) $\forall r>0\ H_{j\lambda\tau}(x,y',r)=r^{-\tau-\sigma_\lambda+n-1}\,H_{j\lambda\tau}(rx,y',1)$;

(h) *if $\sigma_\lambda+\tau>-1$, $j\in\boldsymbol{N}_0$, $j<\sigma_\lambda+\tau+1$, $\|H_{j\lambda\tau}(\cdot,y',r)\|_{W^{j,1}(\boldsymbol{R}_+^n)}\leq Cr^{-\tau-\sigma_\lambda-1+j}$, $\|H_{j\lambda\tau}(\cdot,y',r)\|_{1,\infty,\boldsymbol{R}_+^n}^{\sigma_\lambda+\tau+1-\varepsilon}\leq Cr^{-\varepsilon}\ \forall\varepsilon>0$.*

Proof. (a) and (b) are direct consequences of 2.1. For what concerns (c), one has for $\alpha\in\boldsymbol{N}_0^n$, $\alpha=(\alpha',\alpha_n)$, $x_n>0$

$$\partial_x^\alpha H_{j\lambda\tau}(x,y',r)=F_{\xi'}^{-1}((i\xi')^{\alpha'}(r^2+|\xi'|^2)^{-\tau/2}\,\partial_{x_n}^{\alpha_n}\Omega_{j\lambda}(\xi',x_n,y',r))(x')\,.$$

One has for any $\beta\in\boldsymbol{N}_0^{n-1}$

$$\begin{aligned}&|\partial_{\xi'}^\beta((i\xi')^{\alpha'}(r^2+|\xi'|^2)^{-\tau/2}\,\partial_{x_n}^{\alpha_n}\Omega_{j\lambda}(\xi',x_n,y',r)|\\&\leq C(\alpha,\tau,\beta)\exp(-\delta'(|\xi'|+r)x_n)(|\xi'|+r)^{|\alpha|-\sigma_\lambda-\tau-|\beta|}\,.\end{aligned}$$

So, by 1.1,

$$|\partial_x^\alpha H_{j\lambda\tau}(x,y',r)|\leq C(\varepsilon,r)\,|x'|^{1-n+\sigma_\lambda+\tau-|\alpha|-\varepsilon}\,.$$

Moreover,

$$|\partial_x^\alpha H_{j\lambda\tau}(x,y',r)|\leq C\int_{\boldsymbol{R}^{n-1}}(|\xi'|+r)^{|\alpha|-\sigma_\lambda-\tau}\exp(-\delta(|\xi'|+r_n)\,d\xi'\,.$$

We distinguish three cases: if $|\alpha|-\sigma_\lambda-\tau\geq 0$, $(|\xi'|+r)^{|\alpha|-\sigma_\lambda-\tau}\leq C(|\xi'|^{|\alpha|-\sigma_\lambda-\tau}+r^{|\alpha|-\sigma_\nu-\tau})$; if $1-n<|\alpha|-\tau-\sigma_\lambda<0$, $(|\xi'|+r)^{|\alpha|-\sigma_\lambda-\tau}\leq|\xi'|^{|\alpha|-\sigma_\lambda-\tau}$.
In each case we obtain

$$|\partial_x^\alpha H_{j\lambda\tau}(x,y',r)|\leq C(\varepsilon,r)\,|x_n|^{1-n+\sigma_\lambda+\tau-|\alpha|}$$

If $|\alpha|-\tau-\sigma_\lambda=1-n$, from $(|\xi'|+r)^{|\alpha|-\sigma_\lambda-\tau}\leq C(r,\varepsilon)\,(|\xi'|+r)^{|\alpha|-\sigma_\lambda-\tau+\varepsilon}$ for any positive σ, one is reduced again to the second case.

The proof of (d) is similar to the proof of (c). (e) follows immediately from 2.1 and 1.1.

We prove (f); assume $\sigma_\lambda+\tau>-1$. From 2.1 and (c) one has that $H_{j\lambda\tau}(\cdot,y',r)\in L^1(\boldsymbol{R}^n_+)$. Derivating, it is also easily seen that, if $\sigma_\lambda+\tau>m(m\in\boldsymbol{N}_0)$, $H_{j\lambda\tau}(\cdot,y',r)\in W^{m+1,1}(\boldsymbol{R}^n_+)$. These facts remain true if we substitute to τ any complex number with real part equal to τ. We consider, just for simplicity, the case $-1<\sigma_\lambda+\tau<0$. Fix τ_0 such that $-1<\tau_0+\sigma_\lambda<\tau+\sigma_\lambda$ and set $\tau_1=\tau_0+1$,

$$F(z)\,(x',x_n)=F_{\xi'}^{-1}((r^2+|\xi'|^2)^{[(z-1)\tau_0-z\tau_1]/2}\,\Omega_{j\lambda}(\xi',x_n,y',r))\,(x')\;(0\leq\operatorname{Re}z\leq 1)\,.$$

By complex interpolation (see 4.8 and 4.12) it follows $H_{j\lambda\tau}(\cdot,y',r)\in(L^1(\boldsymbol{R}^n_+), W^{1,1}(\boldsymbol{R}^n_+))_{[\tau-\tau_0]}\subseteq(B^{-\varepsilon}_{1,1}(\boldsymbol{R}^n_+), B^{1-\varepsilon}_{1,1}(\boldsymbol{R}^n_+))_{[\tau-\tau_0]}=B^{\tau-\tau_0-\varepsilon}_{1,1}(\boldsymbol{R}^n_+)$ for any $\varepsilon>0$. From the arbitrarity of τ_0 and ε the result follows.
(g) is an immediate consequence of (3).
(h) Extend $H_{j\lambda\tau}(\cdot,y',r)$ to $\boldsymbol{R}^n$ with the reflexion method described in [17] 2.9.2 (step 3). Then apply directly the definition of $B^{\sigma_\lambda+\tau+1-\varepsilon}_{1,\infty}(\boldsymbol{R}^n)$ given in 4.1 (f) and (g).

In the following lemmas we shall study some properties of

$$u(x)=\int_{\boldsymbol{R}^{n-1}}H_{j\lambda\tau}(x'-y',x_n,y',r)\,\psi(y')\,dy' \tag{4}$$

with $\psi\in L^1(\boldsymbol{R}^{n-1})$, $\tau\in\boldsymbol{R}$, $\sigma_\lambda+\tau>-1$.

Lemma 2.3. *$u\in C^\infty(\boldsymbol{R}^n_+)$ and $\forall\alpha\in\boldsymbol{N}^n_0$*

$$\partial^\alpha u(x)=\int_{\boldsymbol{R}^{n-1}}\partial_{x'}^{\alpha'}\partial_{x_n}^{\alpha_n}H_{j\lambda\tau}(x'-y',x_n,y',r)\,\psi(y')\,dy'\,.$$

Proof. It is an almost immediate consequence of 2.2 (b).

Lemma 2.4. *Assume $\sigma_\lambda+\tau>-1$. Then $u\in\bigcap_{\varepsilon>0}B^{\sigma_\lambda+\tau+1-\varepsilon}_{1,\infty}(\boldsymbol{R}^n_+)$. Moreover, if $j\in\boldsymbol{N}_0$, $j<\sigma_\lambda+\tau+1$,*

$$\|u\|_{W^{j,1}(\boldsymbol{R}^n_+)}\leq Cr^{-\tau-\sigma_\lambda-1+j}\,\|\psi\|_{L^1(\boldsymbol{R}^{n-1})}$$
$$\|u\|^{\sigma_\lambda+\tau+1-\varepsilon}_{1,\infty,\boldsymbol{R}^n_+}\leq Cr^{-\varepsilon}\,\|\psi\|_{L^1(\boldsymbol{R}^{n-1})}\;\forall\varepsilon>0\,.$$

Proof. If $\sigma_\lambda+\tau>-1$, one has

$$\|u\|_{L^1(\mathbf{R}^n_+)} \leq \sup_{y' \in \mathbf{R}^{n-1}} \|H_{j\lambda\tau}(\cdot, y', r)\|_{L^1(\mathbf{R}^n_+)} \|\psi\|_{L^1(\mathbf{R}^{n-1})}$$
$$\leq C r^{-\tau-\sigma_\lambda-1} \|\psi\|_{L^1(\mathbf{R}^{n-1})} .$$

by 2.2 (h). Analogously one can treat the case of $j<\sigma_\lambda+\tau+1$. Extending (for example) $H_{i\lambda\tau}$ to $\mathbf{R}^n$ with the reflection method of [17] 2.9.2, step 3 and using directly the definition in 4.1, one has

$$\|u\|_{1,\infty,\mathbf{R}^n_+}^{\sigma_\lambda+\tau+1-\varepsilon} \leq C \sup_{y'} \|H_{j\lambda\tau}(\cdot, y', r)\|_{1,\infty,\mathbf{R}^n_+}^{\sigma_\lambda+\tau+1-\varepsilon} \|\psi\|_{L^1(\mathbf{R}^{n-1})}$$

and the result follows from 2.2(h).

Lemma 2.5. *Assume $\sigma_\lambda+\tau>-1$, the coefficients of $A(x, \partial)$ and $B(x, \partial)$ in $BC^2(\mathbf{R}^n)$. Then, $\forall \theta \in]0, 1[$ if $\psi \in B^\theta_{1,\infty}(\mathbf{R}^{n-1})$, $u \in \bigcap_{\varepsilon>0} B^{\sigma_\lambda+\tau+1+\theta-\varepsilon}_{1,\infty}(\mathbf{R}^n_+)$.*

Proof. Assume $\psi \in W^{1,1}(\mathbf{R}^{n-1})$. From 2.2 (a) and (b) one has, for x', y' in $\mathbf{R}^{n-1}$ $H_{j\lambda\tau}(x'-y', x_n, y', r) = H_{j\lambda\tau}(x'-y', x_n, x', r) + \sum_{r=1}^{n-1} \partial_3^{\varepsilon_r} H_{j\lambda\tau}(x'-y', x_n, x', r)$ $(y_r-x_r) + R(x'-y', x_n, y', x', r)$.
One has, for $\delta \in [0, 1[$,

$$v(x) = \int_{\mathbf{R}^{n-1}} H_{j\lambda\tau}(x'-y', x_n, x', r)\, \psi(y')\, dy' = \int_{\mathbf{R}^{n-1}} H_{j\lambda\tau+\delta}(x'-y', x_n, x', r)$$
$$(r^2-\Delta)^{\delta/2}\, \psi(y')\, dy'$$

With the same method of 2.4, applying 4.3, one can show that

$$\|v\|_{\infty,\mathbf{R}^n_+}^{\sigma_\lambda+\tau+\delta+1-\varepsilon} \leq C(\varepsilon, r)\, \|(r^2-\Delta)^{\delta/2}\, \psi\|_{L^1(\mathbf{R}^{n-1})} \leq C(\varepsilon, r)\, \|\psi\|_{W^{1,1}(\mathbf{R}^{n-1})} .$$

Moreover, if $r \in \{1, \cdots, n-1\}$

$$-y_r\, \partial_3^{\varepsilon_r} H_{j\lambda\tau}(y', x_n, x', r) = -i\, \partial_3^{\varepsilon_r} F^{-1}_{\xi'}(\partial_{\xi_r}((r^2+|\xi'|^2)^{-\tau/2}\, \Omega_{j\lambda}(\xi', x_n, x', r)))\, (y') .$$

From 2.1 and 1.1, if we set

$$w(x) = \int_{\mathbf{R}^{n-1}} \sum_{r=1}^{n-1} \partial_3^{\varepsilon_r} H_{j\lambda\tau}(x'-y', x_n, x', r)\, (y_r-x_r)\, \psi(y')\, dy' ,$$

we obtain

$$\|w\|_{1,\infty,\mathbf{R}^n_+}^{\sigma_\lambda+\tau+\delta+1-\varepsilon} \leq C(\varepsilon, r)\, \|\psi\|_{L^1(\mathbf{R}^{n-1})} .$$

Finally, setting

$$z(x) := \int_{\mathbf{R}^{n-1}} R(x'-y', x_n, y', x', r)\, \psi(y')\, dy' ,$$

using the fact that, for example,

$$\operatorname{Re} R(x'-y', x_n, y', x', r) = -\operatorname{Re} \sum_{|\gamma|=2} (\gamma!)^{-1} F_{\xi'}^{-1}(\partial_{\xi'}^{\gamma} \partial_3^{\gamma} \Omega_{j\lambda\tau}(\cdot, x_n+\theta(y'-x'), r))$$
$$(x'-y')\ (0<\theta<1),$$

in an analogous way one can show that $z \in \bigcap_{\varepsilon>0} B_{1,\infty}^{\sigma_\lambda+\tau+2-\varepsilon}(\boldsymbol{R}_+^n)$. Therefore the result follows by interpolation.

Lemma 2.6. *Assume $\psi \in L^1(\boldsymbol{R}^{n-1})$. Then $\forall \alpha \in \boldsymbol{N}_0^n$ $\partial^\alpha u$ has a sectional trace on $\partial \boldsymbol{R}_+^n$ belong to $\bigcap_{\varepsilon>0} B_{1,\infty}^{\sigma_\lambda+\tau-|\alpha|-\varepsilon}(\boldsymbol{R}^{n-1})$.*

Proof. By lemma 2.4, if $\sigma_\lambda+\tau>0$ u has a sectional trace belonging to $\bigcap_{\varepsilon>0} B_{1,\infty}^{\sigma_\lambda+\tau-\varepsilon}(\boldsymbol{R}_+^n)$ and

$$u(x', 0) = \int_{\boldsymbol{R}^{n-1}} H_{j\lambda\tau}(x'-y', 0, y', r)\, \psi(y')\, dy'$$

(the integral has a meaning owing to 2.2 (c)). Assume $\sigma_\lambda+\tau\leq 0$. Then, for any $\delta>0$, $x_n>0$,

$$u(\cdot, x_n) = (r^2-\Delta_{x'})^{\delta/2}\, (r^2-\Delta_{x'})^{-\delta/2}\, u(\cdot, x_n)\,.$$

However,

$$(r^2-\Delta_{x'})^{-\delta/2}\, u(\cdot, x_n)\,(x') = \int_{\boldsymbol{R}^{n-1}} H_{j\lambda\tau+\delta}(x'-y', x_n, y', r)\, \psi(y')\, dy'\,.$$

This implies what we want if $\alpha=0$ (using 4.3).

Analogous arguments give the result for general α.

After these preliminaries we pass to construct a solution of the problem

$$(5)\qquad \begin{aligned} &A(x, \partial)\, u - r^{2m} \exp(i\theta)\, u = f \text{ in } \boldsymbol{R}_+^n\,,\\ &\gamma(B(\cdot, \partial)\, u) = g \text{ on } \partial\boldsymbol{R}_+^n\,. \end{aligned}$$

with $f\in L^1(\boldsymbol{R}_+^n)^N$, g in a certain subspace of $\mathcal{S}'(\boldsymbol{R}^{n-1})^{mN}$ that we shall make precise and γ the trace operator on $x_n=0$.

To this aim, we introduce the following notation: again we put $\beta' := \min\{\beta, 1\}$, fix $\mu \in]2m-1-\beta', 2m-1]$ and set, for $\lambda=1, \cdots, mN$, $\tau_\lambda=\mu-\sigma_\lambda$. Next, we put

$$H(x', x_n, y', r) = (H_{j\lambda\tau_\lambda}(x', x_n, y', r))_{1\leq j\leq N, 1\leq\lambda\leq mN}$$

and look for a solution of the form

$$(6)\qquad \begin{aligned} u(x) &= \int_{\boldsymbol{R}_+^n} K(x-y, y, r)\, \phi(y)\, dy + \int_{\boldsymbol{R}^{n-1}} H(x'-y', x_n, y', r)\, \psi(y')\, dy' \\ &= v(x)+w(x) \end{aligned}$$

with $\phi\in L^1(\boldsymbol{R}_+^n)^N$, $\psi \in L^1(\boldsymbol{R}^{n-1})^{mN}$.

Owing to 2.3, one has (if we indicate with z the set of the two first arguments of H)

$$\begin{aligned}[A(x, \partial)-r^{2m}\exp(i\theta)] &\int_{R^{n-1}} H(x'-y', x_n, y', r)\, \psi(y')\, dy' \\ &= \int_{R^{n-1}} [A(x, \partial_z)-r^{2m}\exp(i\theta)]\, H(x'-y', x_n, y', r)\, \psi(y')\, dy' \\ &= \int_{R^{n-1}} [A(x, \partial_z)-A^{\sharp}(y', 0, \partial_z)]\, H(x'-y', x_n, y', r)\, \psi(y')\, dy',\end{aligned}$$

as $[A^{\sharp}(y', 0, \partial_z)-r^{2m}\exp(i\theta)]\, H(x'-y', x_n, y', r)=0$ for $x_n>0$. Moreover, by 1.9, if $|\alpha|\leq 2m-1$, $\partial^{\alpha}v$ has sectional traces in the space $(\bigcap_{\varepsilon>0} B_{1,\infty}^{2m-|\alpha|-1-\varepsilon}(R^{n-1}))^N$, so that by 4.6 $\gamma(B(\cdot, \partial)\, v)$ is well defined. For what concerns $\gamma(B(\cdot, \partial)\, w)$ we start by introducing the following notation: let $\rho\in R^+$; we put $T_{\lambda\mu}(\rho)=\rho^{\tau_\lambda}\delta_{\lambda\mu}$ $(1\leq\lambda,\ \mu\leq mN)$, $T(\rho)=(T_{\lambda\mu}(\rho))_{1\leq\lambda\leq mN, 1\leq\mu\geq mN}$.

One has the following

Lemma 2.7. $\gamma(B(\cdot, \partial)\, w)\,(x') = T((r^2\exp(i\theta)-\Delta_{x'})^{-1/2})\, \psi(x')$

$$+ \int_{R^{n-1}} [B(x', 0, \partial_z)-B^{\sharp}(y', 0, \partial_z)]\, H(x'-y', 0, y', r)\, \psi(y')\, dy'.$$

Proof. First of all, by 2.2 (c), the integral in the statement of 2.7 has a meaning. Next, by 2.3, if $x_n>0$,

$$\begin{aligned}B(x, \partial)\, w(x', x_n) &= \int_{R^{n-1}} B(x, \partial_z)\, H(x'-y', x_n, y', r)\, \psi(y')\, dy' \\ &= \int_{R^{n-1}} [B(x, \partial_z)-B^{\sharp}(y', 0, \partial_z)]\, H(x'-y', x_n, y', r)\, \psi(y')\, dy' \\ &+ \int_{R^{n-1}} B^{\sharp}(y', 0, \partial_z)\, H(x'-y', x_n, y', r)\, \psi(y')\, dy' = \zeta_1(x_n)\,(x')+\zeta_2(x_n)\,(x').\end{aligned}$$

$(x=(x', x_n))$

It is easily seen, using the regularity of the coefficients and 2.2(c), that $\zeta_1(x_n)$ tends to $\int_{R^{n-1}} [B(x', 0, \partial_z)-B^{\sharp}(y', 0, \partial_z)]\, H(x'-y', 0, y', r)\, \psi(y')\, dy'$ in $L^1(R^{n-1})^{mN}$.

Next, fix $\delta>0$ sufficiently large, in such a way that each term of the matrix

$$\begin{aligned}(r^2-\Delta_{x'})^{-\delta}\, &B^{\sharp}(y', 0, \partial_z)\, H(\cdot-y', x_n, y', r) \\ &= B^{\sharp}(y', 0, \partial_z)\, (r^2-\Delta_{x'})^{-\delta}\, H(\cdot-y, x_n, y', r)\end{aligned}$$

converges to the corresponding term of $B^{\sharp}(y', 0, \partial_z)\, (r^2-\Delta_{x'})^{-\delta}\, H(\cdot-y', 0, y', r)$ in $L^1(R^{n-1})$ (the existence of such a δ follows from 2.2(e)). However,

$$B^{\sharp}(y', 0, \partial_z)(r^2-\Delta_{x'})^{-\delta} H(\cdot-y', 0, y', r) = (r^2-\Delta_{x'})^{-\delta} F_{\xi'}^{-1} T((r^2+|\xi'|^2)^{-1/2})(\cdot-y')$$

so that $\zeta_2(x_n)$ converges to $T((r^2 \exp(i\theta)-\Delta_{x'})^{-1/2})\psi)$ in $\mathcal{S}'(\boldsymbol{R}^{n-1})^{mN}$.

Now we make precise what kind of data g we shall consider. Owing to the type of solutions we have in mind, a natural choice is the following: if $r>0$, we set

$$Z_r := \{g\in\mathcal{S}'(\boldsymbol{R}^{n-1})^{mN}\colon T((r^2 \exp(i\theta)-\Delta_{x'})^{1/2})g\in L^1(\boldsymbol{R}^{n-1})^{mN}\},$$

with its natural norm.

Now we impose that u (of the form (6)) is a solution of (5); so, owing to 1.5 and 2.7, it should be

$$\begin{aligned}
&\phi(x)+\int_{\boldsymbol{R}^n_+}[A(x, \partial_1)-A^{\sharp}(y, \partial_1)]\, K(x-y, y, r)\,\phi(y)\,dy \\
&\quad+\int_{\boldsymbol{R}^{n-1}}[A(x, \partial_z)-A^{\sharp}(y', 0, \partial_z)]\, H(x'-y', x_n, y', r)\,\psi(y')\,dy' = f(x),\ x\in\boldsymbol{R}^n_+,\\
(7)\quad &\psi(x')+T((r^2 \exp(i\theta)-\Delta_{x'})^{1/2})\Big(\int_{\boldsymbol{R}^n_+}B(x', 0, \partial_z)\, K(x'-y', -y_n, y, r)\,\phi(y)\,dy\Big)\\
&\quad+T((r^2 \exp(i\theta)-\Delta_{x'})^{1/2})\int_{\boldsymbol{R}^{n-1}}[B(x', 0, \partial_z)-B^{\sharp}(y', 0, \partial_z)]\, H(x'-y', 0, y', r)\\
&\quad\psi(y')\,dy' = T((r^2 \exp(i\theta)-\Delta_{x'})^{1/2})g,\ x'\in\boldsymbol{R}^{n-1}.
\end{aligned}$$

With $\int_{\boldsymbol{R}^n_+} B(x', 0, \partial_z)\, K(x'-y', -y_n, y, r)\,\phi(y)\,dy$ we mean of course the trace of

$$x \to \int_{\boldsymbol{R}^n_+} B(x, \partial_z)\, K(x'-y', x_n-y_n, y, r)\,\phi(y)\,dy.$$

In the following we shall study system (7). For convenience, we set $X := L^1(\boldsymbol{R}^n_+)^N$, $Y := L^1(\boldsymbol{R}^{n-1})^{mN}$.

We start by putting, for $\phi\in X$, $r>0$

$$T_{11}(r)\,\phi(x) := \int_{\boldsymbol{R}^n_+}[A(x, \partial_1)-A^{\sharp}(y, \partial_1)]\, K(x-y, y, r)\,\phi(y)\,dy.$$

From the proof of 1.7 we have the following

Lemma 2.8. *Let $\beta' := \min\{\beta, 1\}$. For any $r>0$ $T_{11}(r)\in\mathcal{L}(X)$ and, for $r\geq 1$, $\|T_{11}(r)\|_{\mathcal{L}(X)}\leq Cr^{-\beta'}$.*

But we have also:

Lemma 2.9. *Assume the coefficients of $A(x, \partial)$ of class $BC^2(\boldsymbol{R}^n)$. Then,*

$\forall r>0\ T_{11}(r)\in\mathcal{L}(X, B^1_{1,\infty}(\boldsymbol{R}^n_+)^N)$.

Proof. We have

$$T_{11}(r)\,\phi(x)=\int_{\boldsymbol{R}^n_+}[A^\sharp(x,\partial_1)-A^\sharp(y,\partial_1)]\,K(x-y,y,r)\,\phi(y)\,dy$$
$$+\int_{\boldsymbol{R}^n_+}[A(x,\partial_1)-A^\sharp(x,\partial_1)]\,K(x-y,y,r)\,\phi(y)\,dy\,,$$

Owing to 1.7, the second addend belongs to $B^1_{1,\infty}(\boldsymbol{R}^n_+)^N$.

Next, we have

$$A^\sharp(x,\partial)-A^\sharp(y,\partial)=\sum_{j=1}^n(x_j-y_j)\,\partial_jA^\sharp(x,\partial)+R(x,y,\partial)\,,$$

where the coefficients of R are $O(|x-y|^2)$ $(y\to x)$ uniformly for $x\in\boldsymbol{R}^n$. If $j\in\{1,\cdots,n\}$, $|\alpha|=2m$, one has that

$$x_j\partial_1^\alpha K(\cdot,y,r)=\text{const}\ F^{-1}(\xi^\alpha\,\partial_{\xi_j}(A^\sharp((y,i\xi)-r^{-2m}\exp(i\theta))^{-1})\,,$$

so that, by 1.1, if $1\leq j, i\leq N$,

$$\sup_{y\in\boldsymbol{R}^n,r\geq1}||x_j\,\partial_1^\alpha\,K(\cdot,y,r)||_{B^1_{1,\infty}(\boldsymbol{R}^n)}<+\infty\,.$$

Therefore, it follows that

$$||\int_{\boldsymbol{R}^n_+}(x_j-y_j)\,\partial_1^\alpha\,K(x-y,y,r)\,\phi(y)\,dy||_{B^1_{1,\infty}(\boldsymbol{R}^n_+)^N}\leq C(r)\,||\phi||_X\,.$$

Finally, it is easily seen that, in the sense of distributions,

$$\partial_{x_j}(\int_{\boldsymbol{R}^n_+}R(x,y,\partial_1)\,K(x-y,y,r)\,\phi(y)\,dy)=\int_{\boldsymbol{R}^n_+}\partial_{x_j}R(x,y,\partial_1)\,K(x-y,y,r)\,\phi(y)\,dy$$
$$+\int_{\boldsymbol{R}^n_+}R(x,y,\partial_1)\,\partial_{x_j}\,K(x-y,y,r)\,\phi(y)\,dy\in X\,,$$

so that the result follows from the inclusion $W^{1,1}(\boldsymbol{R}^n_+)^N\subseteq B^1_{1,\infty}(\boldsymbol{R}^n_+)^N$.

Next, if $r>0$, $\psi\in Y$, we put

$$T_{12}(r)\,\psi(x):=\int_{\boldsymbol{R}^{n-1}}[A(x,\partial_z)-A^\sharp(y',0,\partial_z)]\,H(x'-y',x_n,y',r)\,\psi(y')\,dy'\,.$$

We have

Lemma 2.10. *For any* $r>0$ $T_{12}(r)\in\mathcal{L}(Y,X)$ *and there exists* $C>0$ *such that* $\forall r\geq1$ $||T_{12}(r)||_{\mathcal{L}(Y,X)}\leq Cr^{2m-1-\mu-\beta'}$.

Proof. It is easily seen that

$$\|T_{12}(r)\|_{\mathcal{L}(Y,X)} \leq \sup_{y'\in \boldsymbol{R}^{n-1}} \int_{\boldsymbol{R}^n_+} \|[A(x,\partial_z)-A^\sharp(y',0,\partial_z)]\, H(x'-y',x_n,y',r)\|\, dx$$

$$\leq \sup_{y'\in \boldsymbol{R}^{n-1}} \int_{\boldsymbol{R}^n_+} \|[A(x,\partial_z)-A^\sharp(x,\partial_z)]\, H(x'-y',x_n,y',r)\|\, dx$$

$$+\sup_{y'\in \boldsymbol{R}^{n-1}} \int_{\boldsymbol{R}^n_+} \|[A^\sharp(x,\partial_z)-A^\sharp(y',0,\partial_z)]\, H(x'-y',x_n,y',r)\|\, dx\,.$$

For any $y'\in \boldsymbol{R}^{n-1}$ one has

$$\int_{\boldsymbol{R}^n_+} \|[A(x,\partial_z)-A^\sharp(x,\partial_z)]\, H(x'-y',x_n,y',r)\|\, dx$$

$$\leq C\sum_{|\alpha|\leq 2m-1} \int_{\boldsymbol{R}^n_+} \|\partial_z^\alpha H(x'-y',x_n,y',r)\|\, dx \leq Cr^{2m-\mu-2}\,,$$

as a consequence of 2.2 (f).

Analogously, one has, for any $y'\in \boldsymbol{R}^{n-1}$,

$$\int_{\boldsymbol{R}^n_+} \|[A^\sharp(x,\partial_z)-A^\sharp(y',0,\partial_z)]\, H(x'-y',x_n,y',r)\|\, dx$$

$$\leq C\sum_{|\alpha|=2m} \int_{\boldsymbol{R}^n_+} (|x-y'|+x_n)^{\beta'}\, \|\partial_z^\alpha H(x'-y',x_n,y',r)\|\, dx$$

$$\leq Cr^{2m-\mu-1-\beta'} \quad \text{(applying 2.2(c) and (g))}.$$

Lemma 2.11. *Assume that the coefficients of $A(x,\partial)$ are of class $BC^2(\boldsymbol{R}^n)$ and the coefficients of $B_{\lambda j}$ are of class $BC^{2m-\sigma_\lambda+2}(\boldsymbol{R}^{n-1})$; then if $r>0$, $T_{12}(r)\in \mathcal{L}(Y, \cap_{\varepsilon>0} B^{\mu+2-2m-\varepsilon}_{1,\infty}(\boldsymbol{R}^n_+)^N)$.*

Proof. One has

$$T_{12}(r)\,\psi(x) = \int_{\boldsymbol{R}^n_+} [A^\sharp(x,\partial_z)-A^\sharp(y',0,\partial_z)]\, H(x'-y',x_n,y'\,r)\,\psi(y')\, dy'$$

$$+\int_{\boldsymbol{R}^n_+} [A(x,\partial_z)-A^\sharp(x,\partial_z)]\, H(x'-y',x_n,y',r)\,\psi(y')\, dy' = v_1(x)+v_2(x)\,.$$

$$\|v_2\|_{B^{\mu+2-2m-\varepsilon}_{1,\infty}(\boldsymbol{R}^n_+)^N}$$

$$\leq C\|\Psi\|_Y \max_{1\leq j\leq N, 1\leq\lambda\leq mN} \sup_{y'\in\boldsymbol{R}^{n-1}} \|H_{j\lambda\tau_\lambda}(\cdot,y',r)\|_{B^{\mu+1-\varepsilon}_{1,\infty}(\boldsymbol{R}^n_+)^N} \leq C(r)$$

by 2.2(f). Moreover, if $|\alpha|=2m$, we have

$$A_\alpha(x)-A_\alpha(y',0) = \sum_{k=1}^{n-1}(x_k-y_k)\,\partial_k A_\alpha(y',0)+x_n\,\partial_n A_\alpha(y',0)+R_\alpha(x,y')\,,$$

with $\|R_\alpha(x,y')\|\leq C(|x'-y'|+x_n)^2$ and, for $k=1,\cdots,n$, $\|\partial_{x_k} R_\alpha(x,y')\| \leq C(|x'-y'|+x_n)$.

If $1\leq k\leq n-1$, $1\leq j\leq N$, $1\leq\lambda\leq mN$, $|\alpha|=2m$ one has

$$x_k\,\partial^\alpha H_{j\lambda\tau\lambda}(x,y',r)=c(\alpha)\,F_{\xi'}^{-1}(\partial_{\xi_k}[\xi^\alpha(r^2+|\xi'|^2)^{-\tau_\lambda/2}\,\partial_n^{\alpha_n}\,\Omega_{j\lambda}(\xi',x_n,y',r))])\,(x')$$
$$\partial^\alpha\,H_{j\lambda\tau\lambda}(x,y',r)=c(\alpha)\,F_{\xi'}^{-1}(\xi^\alpha(r^2+|\xi'|^2)^{-\tau_\lambda/2}\,x_n\,\partial_n^{\alpha_n}\,\Omega_{j\lambda}(\xi',x_n,y',r))])\,(x')$$

and

$$x_n\,\partial^\alpha\,H_{j\lambda\tau\lambda}(x,y',r)=c(\alpha)\,F_{\xi'}^{-1}(\xi^\alpha(r^2+|\xi'|^2)^{-\tau_\lambda/2}\,x_n\,\partial_n^{\alpha_n}\,\Omega_{j\lambda}(\xi',x_n,y',r))])\,(x')$$

The same argument of 2.2(f) implies that, for $1\leq k\leq n$,

$$\sup_{y'\in\mathbf{R}^{n-1}}||x_k\,\partial^\alpha\,H_{j\lambda\tau\lambda}(\cdot,y',r)||_{B_{1,\infty}^{\mu+2-2m-\varepsilon}(\mathbf{R}_+^n)^N}\leq C(r)\,,$$

so that

$$||\int_{\mathbf{R}_+^n}\sum_{|\alpha|=2m}\sum_{k=1}^{n-1}[(x_k-y_k)\,\partial_k\,A_\alpha(y',0)+x_n\,\partial_n\,A_\alpha(y',0)]$$
$$\partial^\alpha H(x'-y,x_n,y',r)\,\psi(y')\,dy')||_{B_{1,\infty}^{\mu+2-2m-\varepsilon}(\mathbf{R}_+^n)^N}$$
$$\leq C(\sum_{|\alpha|=2m,1\leq k\leq n,1\leq j\leq N,1\leq\lambda\leq mN}\sup_{y'\in\mathbf{R}^{n-1}}||x_k\,\partial^\alpha\,H_{j\lambda\tau\lambda}(\cdot,y',r)||_{B_{1,\infty}^{\mu+2-2m-\varepsilon}(\mathbf{R}_+^n)})\,||\psi||_Y$$

Finally, the terms of $R_\alpha(\cdot,y)\,\partial^\alpha H(\cdot,y',r)$ are bounded in $W^{1,1}(\mathbf{R}_+^n)$, which implies our result.

Let $\phi\in X$, $r>0$. We set

$$T_{21}(r)\,\phi(x'):=T((r^2-\Delta_{x'})^{1/2})\int_{\mathbf{R}_+^n}B(\cdot,0,\partial_x)\,K(\cdot-y',-y_n,y',r)\,\phi(y)\,dy)$$

One has:

Lemma 2.12. *For any $r>0$ $T_{21}(r)\in\mathcal{L}(X,Y)$ and, for $r\geq 1$, $||T_{21}(r)||_{\mathcal{L}(X,Y)}\leq Cr^{1+\mu-2m}$.*

Proof. One has

$$||T_{21}||_{\mathcal{L}(X,Y)}\leq C\,\sup_{y\in\mathbf{R}_+^n,1\leq\lambda\leq mN,1\leq j,i\leq N}$$
$$||(r^2-\Delta_{x'})^{\tau_\lambda/2}[B_{\lambda j}(\cdot,0,\partial_z)\,K_{ji}(\cdot-y',-y_n,y,r)]||_{L^1(\mathbf{R}^{n-1})}\,.$$

So we have to estimate $||(r^2-\Delta_{x'})^{\tau_\lambda/2}[B_{\lambda j}(\cdot,0,\partial_z)\,K_{ji}(\cdot-y',-y_n,y,r)]||_{L^1(\mathbf{R}^{n-1})}$. We remark, first of all, that, owing to 1.3(a), 4.13, 4.3 one has that

$$(r^2-\Delta_{x'})^{\tau_\lambda/2}[B_{\lambda j}(\cdot,0,\partial_z)\,K_{ji}(\cdot-y',-y_n,y,r)]\in B_{1,\infty}^{2m-\mu-1}(\mathbf{R}^{n-1})\subseteq L^1(\mathbf{R}^{n-1}).$$

Moreover,

$$||(r^2-\Delta_{x'})^{\tau_\lambda/2}[B_{\lambda j}(\cdot,0,\partial_z)\,K_{ji}(\cdot-y',-y_n,y,r)]||_{L^1(\mathbf{R}^{n-1})}$$
$$\leq||(r^2-\Delta_{x'})^{\tau_\lambda/2}\{[B_{\lambda j}(\cdot,0,\partial_z)-B_{\lambda j}^\sharp(y',0,\partial_z)]\,K_{ji}(\cdot-y',-y_n,y,r)]\}||_{L^1(\mathbf{R}^{n-1})}$$
$$+||(r^2-\Delta_{x'})^{\tau_\lambda/2}\,B_{\lambda j}^\sharp(y',0,\partial_z)\,K_{ji}(\cdot-y',-y_n,y,r))||_{L^1(\mathbf{R}^{n-1})}\,.$$

Now observe that

$$(r^2-\Delta_{x'})^{\tau_\lambda/2} B^\sharp_{\lambda j}(y', 0, \partial_z) K_{ji}(\cdot-y', -y_n, y, r)$$
$$= r^{\mu+n-2m} B^\sharp_{\lambda j}(y', 0, \partial_z) [(1-\Delta_{x'})^{\tau_\lambda/2} K_{ji}] (r(\cdot-y'), -ry_n, y, 1),$$

which implies

$$\|(r^2-\Delta_{x'})^{\tau_\lambda/2} B^\sharp_{\lambda j}(y', 0, \partial_z) K_{ji}(\cdot-y', -y_n, y, r)\|_{L^1(\mathbf{R}^{n-1})} \leq Cr^{\mu+1-2m}.$$

To estimate $\|(r^2-\Delta_{x'})^{\tau_\lambda/2} \{[B_{\lambda j}(\cdot, 0, \partial_z)-B^\sharp_{\lambda j}(y', 0, \partial_z)] K_{ji}(\cdot-y', -y_n, y, r)]\}\|_{L^1(\mathbf{R}^{n-1})}$, we shall distinguish the two cases $\sigma_\lambda=2m-1$ and $\sigma_\lambda\leq 2m-2$.

Assume $\sigma_\lambda=2m-1$. Then $\tau_\lambda<0$. For $g\in L^1(\mathbf{R}^{n-1})$ sufficiently regular

$$\|(r^2-\Delta_{x'})^{\tau_\lambda/2} g\|_{L^1(\mathbf{R}^{n-1})} \leq \|F^{-1}_{\xi'}(r^2+|\xi'|^2)^{\tau_\lambda/2}\|_{L^1(\mathbf{R}^{n-1})} \|g\|_{L^1(\mathbf{R}^{n-1})} \leq Cr^{\tau_\lambda} \|g\|_{L^1(\mathbf{R}^{n-1})},$$

as $F^{-1}_{\xi'}((r^2+|\xi'|^2)^{\tau_\lambda/2})(x')=r^{n-1+\tau_\lambda} F^{-1}_{\xi'}((1+|\xi'|^{2\tau_\lambda/2})(rx')$. So,

$$\|(r^2-\Delta_{x'})^{\tau_\lambda/2}\{[B_{\lambda j}(\cdot, 0, \partial_z)-B^\sharp_{\lambda j}(y', 0, \partial_z)] K_{ji}(\cdot-y', -y_n, y, r)\}\|_{L^1(\mathbf{R}^{n-1})}$$
$$\leq Cr^{\tau_\lambda} \|[B_{\lambda j}(\cdot, 0, \partial_z)-B^\sharp_{\lambda j}(y', 0, \partial_z)] K_{ji}(\cdot-y', -y_n, y, r))\|_{L^1(\mathbf{R}^{n-1})}$$
$$\leq Cr^{\tau_\lambda}\Big[\sum_{|\alpha|=2m-1} \int_{\mathbf{R}^{n-1}} |x'-y'|\,|\partial^\alpha_z K_{ji}(x'-y', -y_n, y', r)|\, dx'$$
$$+\sum_{|\alpha|\leq 2m-2} \int_{\mathbf{R}^{n-1}} |\partial^\alpha_z K_{ji}(x'-y', -y_n, y', r)|\, dx'\Big] \leq Cr^{\tau_\lambda-1}.$$

owing to 1.3(e).

Next, assume $\sigma_\lambda\leq 2m-2$, which implies $\tau_\lambda>0$. Let $g\in L^1(\mathbf{R}^{n-1})$. Then, one has for $\varepsilon>0$ and g sufficiently regular

$$\|(r^2-\Delta_{x'})^{\tau_\lambda/2} g\|_{L^1(\mathbf{R}^{n-1})} \leq C\, \|(r^2-\Delta_{x'})^{\tau_\lambda/2} g\|_{B^\varepsilon_{1,\infty}(\mathbf{R}^{n-1})}.$$

As, for any $\alpha\in \mathbf{N}^{n-1}_0$

$$|\partial^\alpha_\xi((r^2+|\xi'|^2)^{\tau_\lambda/2}(r^{\tau_\lambda}+(1+|\xi'|^2)^{\tau_\lambda/2})^{-1})| \leq C(\alpha)$$

(independent of $\xi'\in \mathbf{R}^{n-1}$, $r\geq 1$), from 4.5 it follows

$$\|(r^2-\Delta_{x'})^{\tau_\lambda/2} g\|_{B^\varepsilon_{1,\infty}(\mathbf{R}^{n-1})} \leq C[r^{\tau_\lambda} \|g\|_{B^\varepsilon_{1,\infty}(\mathbf{R}^{n-1})}+\|g\|_{B^{\varepsilon+\tau_\lambda}_{1,\infty}(\mathbf{R}^{n-1})}].$$

So, we have to estimate

$$\|[B_{\lambda j}(\cdot, 0, \partial_z)-B^\sharp_{\lambda j}(y', 0, \partial_z)] K_{ji}(\cdot-y', -y_n, y, r)\|_{B^\varepsilon_{1,\infty}(\mathbf{R}^{n-1})}.$$

Let $\varepsilon\in]0, 1[$. By interpolation

$$\|[B_{\lambda j}(\cdot, 0, \partial_z)-B^\sharp_{\lambda j}(y', 0, \partial_z)] K_{ji}(\cdot-y', -y_n, y, r)\|_{B^\varepsilon_{1,\infty}} \leq C\{\|[B_{\lambda j}(\cdot, 0, \partial_z)$$
$$-B^\sharp_{\lambda j}(y', 0, \partial_z)] K_{ji}(\cdot-y', -y_n, y, r)\|_{L^1(\mathbf{R}^{n-1})}\}^{1-\varepsilon} \{\|[B_{\lambda j}(\cdot, 0, \partial_z)$$
$$-B^\sharp_{\lambda j}(y', 0, \partial_z)] K_{ji}(\cdot-y', -y_n, y, r)\|_{W^{1,1}(\mathbf{R}^{n-1})}\}^\varepsilon.$$

One has

$$\|[B_{\lambda j}(\cdot, 0, \partial_z)-B^{\sharp}_{\lambda j}(y', 0, \partial_z)]\, K_{ji}(\cdot-y', -y_n, y, r)\}\|_{L^1(R^{n-1})}$$
$$\leq C\Big(\sum_{|\alpha|=\sigma_\lambda} \int_{R^{n-1}} |x'-y'|\; |\partial_z^\alpha K_{ji}(x'-y', -y_n, y', r)|\; dx'$$
$$+\sum_{|\alpha|\leq\sigma_\lambda-1} \int_{R^{n-1}} |\partial_z^\alpha K_{ji}(x'-y', -y_n, y', r)|\; dx'\Big)\leq Cr^{\sigma_\lambda-2m}.$$

Differentiating under the integral, one can analogously show that

$$\|[B_{\lambda j}(\cdot, 0, \partial_z)-B^{\sharp}_{\lambda j}(y', 0, \partial_z)]\, K_{ji}(\cdot-y', -y_n, y, r)\|_{W^{1,1}(R^{n-1})}\leq Cr^{\sigma_\lambda+1-2m}.$$

So we have

$$\|[B_{\lambda j}(\cdot, 0, \partial_z)-B^{\sharp}_{\lambda j}(y', 0, \partial_z)]\, K_{ji}(\cdot-y', -y_n, y, r)\|_{B^{\varepsilon}_{1,\infty}(R^{n-1})}\leq Cr^{\sigma_\lambda+\varepsilon-2m}.$$

With the same arguments, if $0<\varepsilon<1$, one has

$$\|[B_{\lambda j}(\cdot, 0, \partial_z)-B^{\sharp}_{\lambda j}(y', 0, \partial_z)]\, K_{ji}(\cdot-y', -y_n, y, r)\|_{B^{\varepsilon+\tau_\lambda}_{1,\infty}(R^{n-1})}\leq Cr^{\mu+\varepsilon-2m}.$$

With this the statement is completely proved.

Lemma 2.13. *For any* $r>0$, $\forall\varepsilon>0$ $T_{21}(r)\in\mathcal{L}(X, B^{2m-\mu-1-\varepsilon}_{1,\infty}(R^{n-1})^{mN})$. *Moreover, if* $\phi\in B^{\delta}_{1,\infty}(R^n_+)$, *for some* $\delta\in]0, \beta'[$, $T_{21}(r)\,\phi\in B^{2m-\mu-1+\delta}_{1,\infty}(R^{n-1})^{mN}$.

Proof. The first statement can be proved simply remarking that, if $\delta<2m-\mu-1$,

$$\|(r^2-\Delta_{x'})^{(\tau_\lambda+\delta)/2}[B^{\sharp}_{\lambda j}(\cdot, 0, \partial_z)\, K_{ji}(\cdot-y, -y_n, y, r)]\|_{L^1(R^{n-1})}\leq C(r).$$

and using 4.3. For what concerns the second statement, it is a simple consequence of 4.9.

We set, if $r>0$, $\psi\in Y$:

$$T_{22}(r)\,\psi := T((r^2-\Delta_{x'})^{1/2})\,\Big\{\int_{R^{n-1}} [B(\cdot, 0, \partial_z)-B^{\sharp}(y', 0, \partial_z)]\, H(\cdot-y', 0, y', r)\,\psi(y')\, dy'\Big\}.$$

We have:

Lemma 2.14. *For any* $r>0$ $T_{22}(r)\in\mathcal{L}(Y)$ *and there exists* $C>0$ *such that* $\forall r\geq 1$

$$\|T_{22}(r)\|_{\mathcal{L}(Y)}\leq Cr^{-1}.$$

Proof. Analogous to the proof of 2.12, using 2.2.

Lemma 2.15. *For any* $r>0$ $T_{22}(r)\in\mathcal{L}(Y, B^{1-\varepsilon}_{1,\infty}(R^{n-1})^{mN})$ *for any* $\varepsilon>0$.

Proof. Analogous to 1.13.

So we have the following

Proposition 2.16. *Let $f\in L^1(\boldsymbol{R}^n_+)^N$, $g\in Z_r$. Let $\theta\in[-\pi/2,\pi/2]$ and assume that the assumptions* (h1)-(h6) *are satisfied. Then problem* (5) *has, for $r>0$ and sufficiently large, a unique solution of the form*

$$u(x)=\int_{\boldsymbol{R}^n_+} K(x-y,y,r)\,\phi(y)\,dy+\int_{\boldsymbol{R}^{n-1}} H(x'-y',x_n,y',r)\,\psi(y')\,dy'\,,$$

with $\phi\in L^1(\boldsymbol{R}^n_+)^N$, $\psi\in L^1(\boldsymbol{R}^{n-1})^{mN}$. Moreover,

$$\|u\|_{L^1(\boldsymbol{R}^n_+)^N}\leq C(r^{-2m}\|f\|_{L^1(\boldsymbol{R}^n_+)^N}+r^{-\mu-1}\|g\|_{Z_r}) \tag{8}$$

and $u\in\bigcap_{\varepsilon>0}B^{\mu+1-\varepsilon}_{1,\infty}(\boldsymbol{R}^n_+)^N$.

Finally, for $j\in\boldsymbol{N}$, $j<\mu+1$,

$$\|u\|_{W^{j,1}(\boldsymbol{R}^n_+)^N}\leq Cr^j(r^{-2m}\|f\|_{L^1(\boldsymbol{R}^n_+)^N}+r^{-\mu-1}\|g\|_{Z_r})$$

and for any $\varepsilon>0$

$$\|u\|_{B^{\mu+1-\varepsilon}_{1,\infty}(\boldsymbol{R}^n_+)^N}\leq Cr^{\mu+1-\varepsilon}(r^{-2m}\|f\|_{L^1(\boldsymbol{R}^n_+)^N}+r^{-\mu-1}\|g\|_{Z_r})\,.$$

Proof. The existence and the unicity of a unique solution of the form declared follows from 2.8, 2.10, 2.12, 2.14 and the contraction mapping principle. Next, remark that there exists $C>0$ such that for any r large enough,

$$\|\phi\|_X\leq\|f\|_X+C(r^{\beta'}\|\phi\|_X+r^{2m-1-\mu-\beta'}\|\psi\|_Y)\,,$$
$$\|\psi\|_Y\leq\|g\|_{Z_r}+C(r^{1+\mu-2m}\|\phi\|_X+r^{-1}\|\psi\|_Y)$$

so that, if r is large enough,

$$\|\phi\|_X\leq C(\|f\|_X+r^{2m-1-\mu-\beta'}\|g\|_{Z_r}),\quad \|\psi\|_Y\leq C(r^{1+\mu-2m}\|f\|_X+\|g\|_{Z_r})\,.$$

So, (8) follows from 1.4 (b) and 2.4.

The final statement is again a consequence of 2.4.

Lemma 2.17. *Assume that* (h1)-(h6) *are satisfied. There exists $p>1$ such that, if $f\in L^p(\boldsymbol{R}^n_+)^N\cap L^1(\boldsymbol{R}^n_+)^N$ and $g\in\prod_{\lambda=1}^{mN}W^{2m-\sigma_\lambda-p^{-1},p}(\boldsymbol{R}^{n-1})\cap Z_r$, $u\in W^{2m,p}(\boldsymbol{R}^n_+)^N$ and is the only solution in this space of problem* (5).

Proof. Owing to Agmon's estimates (see [3]) problem (3) has, for any $p\in\,]1,+\infty[$, for any $r>0$ sufficiently large, at most one solution u in the space $W^{2m,p}(\boldsymbol{R}^n_+)^N$ for any $f\in L^p(\boldsymbol{R}^n_+)^N$, $g\in\prod_{\lambda=1}^{mN}W^{2m-\sigma_\lambda-p^{-1},p}(\boldsymbol{R}^{n-1})$.

Now assume first that the coefficients of $A(x,\partial)$ are of class $BC^2(\boldsymbol{R}^n)$, the coefficients of $B(x', \partial)$ are of class $BC^{2m-\sigma_\lambda+2}$, $f \in B^{\delta}_{1,\infty}(\boldsymbol{R}^n_+)^N$, with $0<\delta<\beta'$, $g \in \prod_{\lambda=1}^{mN} B^{2m+\delta-\sigma_\lambda-1}_{1,\infty}(\boldsymbol{R}^{n-1})$. Then, by 4.5, $T((r^2-\Delta_{x'})^{1/2}) g \in B^{2m+\delta-\mu-1}_{1,\infty}(\boldsymbol{R}^{n-1})^{mN}$ and so $g \in Z_r$. Then, by 2.9 and 2.11, $\phi = T_{11}(r)\phi + T_{12}(r)\Psi + f \in B^{\delta'}_{1,\infty}(\boldsymbol{R}^n_+)^N$, for some $\delta'>0$, which implies, by 4.12 and 4.9, that $x \to \int_{\boldsymbol{R}^n_+} K(x-y, y, r)\phi(y)\, dy \in B^{2m+\delta}_{1,\infty}(\boldsymbol{R}^n_+)^N$. Moreover, by 2.13 and 2.15, $T_{21}(r)\phi \in \bigcap_{\varepsilon>0} B^{2m-\mu-1+\delta'-\varepsilon}_{1,\infty}(\boldsymbol{R}^{n-1})^{mN}$ and $T_{22}(r)\psi \in \bigcap_{\varepsilon>0} B^{\beta'-\varepsilon}_{1,\infty}(\boldsymbol{R}^{n-1})^{mN}$, so that, by 2.5,

$$x \to \int_{\boldsymbol{R}^{n-1}} H(x'-y', x_n, y', r)\psi(y')\, dy' \in \bigcap_{\varepsilon>0} B^{2m+\delta-\varepsilon}_{1,\infty}(\boldsymbol{R}^n_+)^N$$

if δ' is sufficiently small.

Now one has that, by Sobolev theorem (see [17] 2.7) $B^{\delta}_{1,\infty}(\boldsymbol{R}^n_+)^N \subseteq L^p(\boldsymbol{R}^n_+)^N$ if $1 \le p < n(n-\delta')^{-1}$ so that $u \in W^{2m,p}(\boldsymbol{R}^n_+)^N$ if $1 \le p < n(n-\delta')^{-1}$. Now take f_ν $(\nu \in \boldsymbol{N}) \in B^{\delta'}_{1,\infty}(\boldsymbol{R}^n_+)^N$ such that $f_\nu \to f$ in $L^p(\boldsymbol{R}^n_+)^N \cap L^1(\boldsymbol{R}^n_+)^N$.

Let $g \in \prod_{\lambda=1}^{mN} W^{2m-\sigma_\lambda-p^{-1},p}(\boldsymbol{R}^{n-1}) \cap Z_r$. If $g_\nu = g * \omega_\nu$ (ω_ν is the usual mollifier), $g_\nu \to g$ in $\prod_{\lambda=1}^{mN} W^{2m-\sigma_\lambda-p^{-1},p}(\boldsymbol{R}^{n-1}) \cap Z_r$. Indicate with u_ν the solution of (5) with data f_ν, g_ν; then $u_\nu \to u$ in $X \cap W^{2m,p}(\boldsymbol{R}^n_+)^N$ and so the result is proved if the coefficients are regular.

The general case follows by approximation.

REMARK 2.18. From 2.17 one draws the fact that the solution of (3) in form (4) does not depend on the choice of μ, at least if g is sufficiently regular.

3. Boundary value problems in a domain

In the following Ω will be a fixed bounded open subset of $\boldsymbol{R}^n$ with the boundary $\partial\Omega$ which is a submanifold of $\boldsymbol{R}^n$ of dimension $n-1$ and class $C^{2m+\beta}$ ($\beta>0$) and Ω lying on one side of $\partial\Omega$.

We want to study the following problem:

$$\text{(9)} \qquad \begin{aligned} & r^{2m}\exp(i\theta)\, u - A(x, \partial)\, u = f \quad \text{in } \Omega\,, \\ & \gamma(B(x, \partial)\, u) = 0\,, \end{aligned}$$

(γ is the trace operator on $\partial\Omega$), with $r>0$, $-\pi/2 \le \theta \le \pi/2$ under the following assumptions:

(I1) *for any* $x \in \bar{\Omega}$ $A(x, \partial) = (A_{ij}(x, \partial))_{1 \le i \le N, 1 \le j \le N}$ *with coefficients in* $C^{\beta}(\bar{\Omega})$ *satisfies* (h1), (h3) ($\beta>0$);

(I2) $B(x, \partial) = (B_{\lambda j}(x, \partial))_{1 \le \lambda \le mN, 1 \le j \le N}$, *with order of* $B_{\lambda j}$ *not exceeding* σ_λ ($0 \le \sigma_\lambda \le 2m-1$) *and coefficients in* $C^{2m-\sigma_\lambda+\beta}(\bar{\Omega})$: *we indicate with* $B^{\sharp}_{\lambda,j}$ *the part of order* σ_λ *of* $B_{\lambda j}$, $B^{\sharp} = (B^{\sharp}_{\lambda,j})_{1 \le \lambda \le mN, 1 \le j \le N}$

(I3) (complementing condition) *for any* $x'\in\partial\Omega$, *for any* $r\geq 0$, *for any* $\xi'\in T_{x'}(\partial\Omega)$, *r and* ξ' *not both* 0, *for any* $\theta\in[-\pi/2, \pi/2]$ *the O.D.E. problem*

$$\begin{aligned}&[r^{2m}e^{i\theta}-A^{\sharp}(x', i\xi'+\nu(x')\partial_t)]\,w(t)=0 \quad \textit{in } \boldsymbol{R},\\ &B^{\sharp}(x', i\xi'+\nu(x')\partial_t)\,w(0)=g\\ &w \textit{ bounded in } \boldsymbol{R}^+\end{aligned}$$

has a unique solution for any $g\in\boldsymbol{C}^{mN}$ (have $\nu(x')$ is the inward normal unit vector to $\partial\Omega$ in $x'\in\partial\Omega$).

We start with the following technical lemma:

Lemma 3.1. *Let* $R>0$, $A(x,\partial)=(A_{ij}(x,\partial))_{1\leq i\leq N, 1\leq j\leq N}$ *with coefficients in* $C^{\beta}(\bar{B}_R^+)$, *satisfying* (h1) *and* (h3); *moreover, let* $B(x,\partial)=(B_{\lambda j}(x,\partial))_{1\leq\lambda\leq mN, 1\leq j\leq N}$, *with order of* $B_{\lambda j}$ *not overcoming* $\sigma_\lambda (0\leq\sigma_\lambda\leq 2m-1)$ *and coefficients in* $C^{2m-\sigma_\lambda+\beta}(\bar{B}_R^+)$; *we indicate with* $B^{\sharp}_{\lambda,j}$ *the part of order* σ_λ *of* $B_{\lambda j}$, $B^{\sharp}=(B^{\sharp}_{\lambda,j})_{1\leq\lambda\leq mN, 1\leq j\leq N}$ *and assume that* (I3) *is satisfied in any point* $(x', 0)$ *with* $|x'|\leq R$.

Then, there exist differential operators $A^{\wedge}(x,\partial)$, $B^{\wedge}(x,\partial)$ *satisfying* (I1)-(I3) *whose restrictions to* $\bar{B}_R^+$ *are* $A(x,\partial)$ *and* $B(x,\partial)$.

Proof. It is easily seen that $A(x,\partial)$ and $B(x,\partial)$ are extensible to operators $A'(x,\partial)$ and $B'(x,\partial)$ defined on $\bar{B}^+_{R+\varepsilon}$ and preserving the properties of $A(x,\partial)$, $B(x,\partial)$. Let $\psi\in C^\infty([0,+\infty[)$, such that $\psi(r)=1$ if $0\leq r\leq R$, $\psi(r)=0$ if $r\geq R+\varepsilon$ and $0\leq\psi(r)\leq 1$ $\forall r\in[0,+\infty[$. Set $\phi(r)=\int_0^r\psi(s)\,ds$. Then $\phi\in C^\infty([0,+\infty[)$, $\phi(r)=r$ if $0\leq r\leq R$, $\phi(r)\leq r$ $\forall r\in[0,+\infty[$, $\phi(r)=C$ with $C\leq R+\varepsilon$ if $r\geq R+\varepsilon$. Set $A^{\wedge}(x,\partial)=A(\phi(|x|)|x|^{-1}x,\partial)$, $B^{\wedge}(x,\partial)=B(\phi(|x|)|x|^{-1}x,\partial)$ $(x\in\boldsymbol{R}^n)$.

Lemma 3.2. *Assume* (I1)-(I3) *are satisfied. Let* $p>1$ *and* $u\in W^{2m,p}(\Omega)^N$, *such that* $\gamma(B(\cdot,\partial)u)=0$. *Then, there exists* $C>0$ *such that, if* $r\geq C$,

$$||u||_{L^1(\Omega)^N}\leq Cr^{-2m}\,||(r^{2m}\exp(i\theta)-A(\cdot,\partial))\,u||_{L^1(\Omega)^N},$$

and, for $0<\sigma<2m$,

$$||u||_{B^{\sigma}_{1,\infty}(\Omega)^N}\leq Cr^{\sigma-2m}\,||(r^{2m}\exp(i\theta)-A(\cdot,\partial))\,u||_{L^1(\Omega)^N}$$

Proof. Let $x'\in\partial\Omega$. Then, there exist U neighbourhood of x' in $\boldsymbol{R}^n$, $R>0$ and Φ: $U\to B_R$ diffeomorphism of class $C^{2m+\beta}$ such that $\Phi(U\cap\Omega)=B_R^+$ and such that $\Phi(U\cap\partial\Omega)=\{y\in B_R\,|\,y_n=0\}$. We set $A_\Phi v=A(\cdot,\partial)(v\circ\Phi)\circ\Phi^{-1}$, $B_\Phi v=B(\cdot,\partial)(v\circ\Phi)\circ^{-1}$. We assume that the $B(\cdot,\partial)$ is defined in a neighbourhood of $\partial\Omega$ with the same regularity of the coefficients. It is well known that Φ may be chosen in such a way that A_Ω and B_Ω satisfy the assumptions of 3.1, so that they can be extended to operators $A^{\wedge}(y,\partial)$, $B^{\wedge}(y,\partial)$ in the way described in 3.1. Let $\Omega\subseteq\bigcup_{1\le s\le S} U_s$, with U_s domain of Φ_s with the properties described or $U_s\subseteq\Omega$. Let $\{\phi_s \mid 1\le s\le S\}$ be a partition of unity subordinated to this covering of Ω. Assume that U_s is not contained in Ω. Then $v_s=(\phi_s u)\circ\Phi_s^{-1}$ satisfies

$$r^{2m} e^{i\theta} v_s - A_{\Phi_s} v_s = (\phi_s f)\circ\Phi_s^{-1} + (A' u)\circ\Phi_s^{-1}\,,$$
$$\gamma(B_{\Omega_s} v_s) = \gamma((B' u)\circ\Phi_s^{-1})\,,$$

with $A'=(A'_{ij}(x,\partial))_{1\le i\le N, 1\le j\le N}$, the order of $A'_{ij}(x,\partial)$ less than or equal to $2m-1$ and the coefficients of class $C^{\beta}(\overline{\Omega})$ vanishing out of U_s, $B'=(B'_{\lambda j}(x,\partial))_{1\le\lambda\le mN, 1\le j\le N}$, the order of $B'_{\lambda j}(x,\partial)$ less than or equal to $\sigma_\lambda-1(B'_{ij}(x,\partial)=0$ if $\sigma_\lambda=0)$ and coefficients of class $C^{2m-\sigma_\lambda}(\overline{\Omega})$ vanishing out of U_s.

Now we think of $\gamma((B'u)\circ\Phi_s^{-1})$ as extended with 0 to the whole $\boldsymbol{R}^{n-1}$. If $1\le\lambda\le mN$, one has $\gamma((B'_{\lambda j}(\cdot,\partial)\, u\circ\Phi_s^{-1})\in W^{2m+1-\sigma_\lambda-p^{-1},p}(\boldsymbol{R}^{n-1})$, so that, using the fact that if has a compact support, one has $\gamma((B'_{\lambda j}(\cdot,\partial)\, u\circ\Phi_s^{-1})\in B^{2m-\sigma_\lambda}_{1,\infty}(\boldsymbol{R}^{n-1})$, so that, if $\tau<2m-\sigma_\lambda$, by 4.5 $(r^2-\Delta_{x'})^{\tau/2}\,\gamma((B'_{\lambda j}(\cdot,\partial)\, u\circ\Phi_s^{-1})\in L^1(\boldsymbol{R}^{n-1})$.

Using 3.1, we can extend A_{Ω_s} and B_{Ω_s} in such a way to be able of applying the machinery of the second section (whose notations we are going to use). By 2.16, we have

$$\|v_s\|_X\le C(r^{-2m}\,\|(\phi_s f)\circ\Phi_s^{-1}\|_X + r^{-2m}\,\|(A'u)\circ\Phi_s^{-1}\|_X + r^{-\mu-1}\,\|\gamma((B'u)\circ\Phi_s^{-1})\|_{Z_r})\,.$$

Now we estimate $\|\gamma((B'u)\circ\Phi_s^{-1})\|_{Z_r}$.

If $\sigma_\lambda=2m-1$, with the same method employed in 2.12 we have

$$\|(r^2-\Delta_{x'})^{\tau_\lambda/2}\,\gamma(B'_{\lambda j}(\cdot,\partial)\, u_j\circ\Phi_s^{-1})\|_{L^1(\boldsymbol{R}^{n-1})}\le Cr^{\tau_\lambda}\,\|\gamma(B'_{\lambda j}(\cdot,\partial)\, u_j\circ\Phi_s^{-1})\|_{L^1(\boldsymbol{R}^{n-1})}$$
$$\le Cr^{\tau_\lambda}\,\|\gamma(B'_{\lambda j}(\cdot,\partial)\, u_j\circ\Phi_s^{-1})\|_{B^{\varepsilon}_{1,\infty}(\boldsymbol{R}^{n-1})}(\varepsilon>0)\le Cr^{\tau_\lambda}\,\|B'_{\lambda j}(\cdot,\partial)\, u_j\circ\Phi_s^{-1}\|_{B^{1+\varepsilon}_{1,\infty}(\boldsymbol{R}^n_+)}$$

(by 4.13)

$$\le Cr^{\tau_\lambda}\,\|B'_{\lambda j}(\cdot,\partial)\, u_j\|_{B^{1+\varepsilon}_{1,\infty}(\Omega)}$$
$$\le Cr^{\tau_\lambda}\,\|u_j\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)}\,.$$

Analogously, one has

$$\|(r^2-\Delta_{x'})^{\tau_\lambda/2}\,\gamma(B'_{\lambda j}(\cdot,\partial)\, u_j\circ\Phi_s^{-1})\|_{L^1(\boldsymbol{R}^{n-1})}\le C(r^{\tau_\lambda}\,\|u_j\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)}+\|u_j\|_{B^{\mu+\varepsilon}_{1,\infty}(\Omega)})$$
$$(0<\varepsilon\le 2m-\mu)\ \text{if}\ 0\le\sigma_\lambda\le 2m-2\,.$$

So, we have, coming back to Ω and summing up in s:

$$\|u\|_{L^1(\Omega)^N}\le C[r^{-2m}\,\|f\|_{L^1(\Omega)^N}+r^{-2m}\,\|u\|_{W^{2m-1}(\Omega)^N}+r^{-\mu-1}\,\|u\|_{B^{\mu+\varepsilon}_{1,\infty}(\Omega)^N}$$
$$+\sum_{\lambda=1}^{mN} r^{-\sigma_\lambda-1}\,\|u\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)^N}]\,. \tag{10}$$

With the same method, applying the last statement of 2.16,

$$\|u\|_{W^{2m-1}(\Omega)^N} \leq C r^{2m-1} [r^{-2m} \|f\|_{L^1(\Omega)^N} + r^{-2m} \|u\|_{W^{2m-1}(\Omega)^N} + r^{-\mu-1} \|u\|_{B^{\mu+\varepsilon}_{1,\infty}(\Omega)^N}$$
$$+ \sum_{\lambda=1}^{mN} r^{\sigma_\lambda - 1} \|u\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)^N}] ,$$

and, if $0<\sigma<\mu+1$,

$$\|u\|_{B^{\sigma}_{1,\infty}(\Omega)^N} \leq C r^{\sigma} [r^{-2m} \|f\|_{L^1(\Omega)^N} + r^{-2m} \|u\|_{W^{2m-1}(\Omega)^N} + r^{-\mu-1} \|u\|_{B^{\mu+\varepsilon}_{1,\infty}(\Omega)^N}$$
$$+ \sum_{\lambda=1}^{mN} r^{-\sigma_\lambda - 1} \|u\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)^N}] .$$

Assume, for example, $\max_\lambda \sigma_\lambda = 2m-1$. For $\sigma=2m-1+\varepsilon$ we obtain, if r is large,

$$\|u\|_{B^{2m-1+\varepsilon}_{1,\infty}(\Omega)^N}$$
$$\leq C [r^{\varepsilon-1} \|f\|_{L^1(\Omega)^N} + r^{\varepsilon-1} \|u\|_{W^{2m-1}(\Omega)^N} + r^{2m+\varepsilon-\mu-2} \|u\|_{B^{\mu+\varepsilon}_{1,\infty}(\Omega)^N}$$
$$+ \sum_{\sigma_\lambda \leq 2m-2} r^{2m+\varepsilon-\sigma_\lambda-2} \|u\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)^N}] .$$

Substituting this estimate in (10) we obtain easily

$$\|u\|_{L^1(\Omega)^N} \leq C [r^{-2m} \|f\|_{L^1(\Omega)^N} + r^{-2m} \|u\|_{W^{2m-1}(\Omega)^N} + r^{-\mu-1} \|u\|_{B^{\mu+\varepsilon}_{1,\infty}(\Omega)^N}$$
$$+ \sum_{\sigma_\lambda \leq 2m-2} r^{-\sigma_\lambda-1} \|u\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)^N}] ,$$
$$\|u\|_{W^{2m-1}(\Omega)^N} \leq C r^{2m-1} [r^{-2m} \|f\|_{L^1(\Omega)^N} + r^{-2m} \|u\|_{W^{2m-1}(\Omega)^N} + r^{-\mu-1} \|u\|_{B^{\mu+\varepsilon}_{1,\infty}(\Omega)^N}$$
$$+ \sum_{\sigma_\lambda \leq 2m-2} r^{-\sigma_\lambda-1} \|u\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)^N}]$$

and, if $0<\sigma<\mu+1$,

$$\|u\|_{B^{\sigma}_{1,\infty}(\Omega)^N} \leq C r^{\sigma} [r^{-2m} \|f\|_{L^1(\Omega)^N} + r^{-2m} \|u\|_{W^{2m-1}(\Omega)^N} + r^{-\mu-1} \|u\|_{B^{\mu+\varepsilon}_{1,\infty}(\Omega)^N}$$
$$+ \sum_{\sigma_\lambda \leq 2m-2} r^{-\sigma_\lambda-1} \|u\|_{B^{\sigma_\lambda+\varepsilon}_{1,\infty}(\Omega)^N}] .$$

Iterating the method (eliminating the strongest norms first) after a finite number of passages one obtains the desired estimate, taking into account the fact that $\mu+1$ can be chosen arbitrarily near $2m$ (owing to 2.18).

Thoerem 3.3. *Assume* (I1)-(I3) *are satisfied. If* $1<p<+\infty$, *set*

$$D(A_p) = \{u \in W^{2m,p}(\Omega)^N \mid \gamma(B(\cdot, \partial)\, u) = 0\} ,$$
$$A_p u = A(x, \partial)\, u .$$

Then:

(a) *A_p is closable in $L^1(\Omega)^N$;*

(b) *the closure of A_p does not depend on p;*

if A_1 is this closure,

(c) $D(A_1) \subseteq \bigcap_{\varepsilon>0} B^{2m-\varepsilon}_{1,\infty}(\Omega)^N$;

(d) *if $u \in D(A_1)$, $\lambda \in \boldsymbol{C}$,* Re $\lambda \geq 0$ *and* $|\lambda|$ *is sufficiently large,* $f=(\lambda-A_1)\, u$,

$$\|u\|_{L_1(\Omega)^N} \leq C|\lambda|^{-1}\|f\|_{L^1(\Omega)^N},$$
$$\|u\|_{B^{\sigma}_{1,\infty}(\Omega)^N} \leq C|\lambda|^{\sigma/(2m)-1}\|f\|_{L^1(\Omega)^N} \quad if \quad 0<\sigma<2m\,;$$

(e) *if* $\max_\lambda \sigma_\lambda \leq 2m-2$,

$$D(A_1) = \{u \in B^{2m}_{1,\infty}(\Omega)^N \mid \gamma(B(\cdot,\partial)u) = 0,\ A(\cdot,\partial)u \in L^1(\Omega)^N\}\,;$$

(f) A_1 *is the infinitesimal generator of an analytic semigroup in* $L^1(\Omega)^N$.

Proof. From [3] one has that for any $p>1$ A_p is the infinitesimal generator of an analytic semigroup in $L^p(\Omega)^N$ and it is easily seen that, if $1<p<p'<+\infty$, $A_{p'}$ is closable in $L^p(\Omega)^N$ and its closure is A_p. This implies that the closure of the graph of A_p in $L^1(\Omega)^N \times L^1(\Omega)^N$ is the same for any $p>1$.

Now, fix $p>1$. Then, by 3.2, if Re $\mu \geq 0$ and $|\mu|$ is sufficiently large, $(\mu-A_p)^{-1}$ is extensible to a linear bounded operator $R_1(\mu) \in \mathcal{L}(L^1(\Omega)^N, B^{\sigma}_{1,\infty}(\Omega)^N)$ for any $\sigma<2m$ and

(II) $\|R_1(\mu)\|_{\mathcal{L}(L^1(\Omega)^N)} \leq C|\mu|^{-1}$, $\|R_1(\mu)\|_{\mathcal{L}(L^1(\Omega)^N, B^{\sigma}_{1,\infty}(\Omega)^N)} \leq C|\mu|^{\sigma/(2m)-1}$.

It is immediately seen that $\{R_1(\mu)/\text{Re}\ \mu \geq 0,\ |\mu| \geq C\}$ is a pseudoresolvent and from the first estimate in (11) and [20] VIII.4, lemma 1', one has that Ker $(R_1(\mu)) \cap \overline{R(R_1(\mu))} = \{0\}$. However, it is clear that $R(R_1(\mu))$ is dense in $L^1(\Omega)^N$ and this implies that $R_1(\mu)$ is injective and there exists a linear operator A_1 such that $R_1(\mu) = (\mu - A_1)^{-1}$. From this (a)-(d) and (f) follow easily. (e) follows from 4.14.

REMARK 3.4. In case $\max_\lambda \sigma_\lambda = 2m-1$, if $u \in D(A_1)$, the boundary condition $\gamma(\sum_{j=1}^{N} B_{\lambda j} u_j) = 0$ may be intended in the following sense: fix a point $x' \in \partial\Omega$ and a local change of variable Φ of class $C^{2m+\beta}$ such that, if U is the domain of Φ, $\Phi(U)$ is of the form $V \times]-T, T[$ $(T>0)$, $\Phi(U \cap \Omega) = \{y \in \Phi(U) \mid y_n > 0\}$, $\Phi(U \cap \partial\Omega) = \{y \in \Phi(U) \mid y_n = 0\}$ and the transformed operators A_Φ and B_Φ satisfy the assumptions of 3.1. Then, $B_\Phi(u \circ \Phi^{-1})$ has a sectional trace in $V \times \{0\}$ equal to 0.

REMARK 3.5. The method employed can be used to study boundary value problems for elliptic systems in spaces which are different from L^1. For example with an analogous method one can study problems in spaces of Borel measures (taking into account that the space of finite Borel measures $M(\boldsymbol{R}^n)$ is included in the space $B^0_{1,\infty}(\boldsymbol{R}^n)$ (see [15] 2) and in $L^\infty(\boldsymbol{R}^n)$ (using spaces $B^s_{\infty,\infty}$). In each case one can prove results of generation of semigroups which are not strongly continuous in 0, because the domains of the infinitesimal generators are not dense.

4. Appendix: Besov spaces $B^{\alpha}_{1,q}$

In this appendix we collect for convenience some results concerning Besov spaces $B^{\alpha}_{1,q}(1\leq q\leq+\infty)$ which were used in the previous paragraphs. Most of these results are only stated without proof, as they are already present in the literature.

DEFINITION 4.1. Let $\alpha\in\boldsymbol{R}$, $0<\alpha\leq 1$, $1\leq q\leq+\infty$. We set

$$B^{\alpha}_{1,\infty}(\boldsymbol{R}^n):=\{u\in L^1(\boldsymbol{R}^n)\,|\,\sup_{h\in\boldsymbol{R}^n\setminus\{0\}}|h|^{-\alpha}\|u(\cdot+h)-2u+u(\cdot-h)\|_{L^1(\boldsymbol{R}^n)}<+\infty\},$$

$$B^{\alpha}_{1,q}(\boldsymbol{R}^n):=\{u\in L^1(\boldsymbol{R}^n)\,|\int_{\boldsymbol{R}^n}(\|u(\cdot+h)-2u+u(\cdot-h)\|_{L^1(\boldsymbol{R}^n)})^q|h|^{-n-\alpha q}\,dh<+\infty\}$$

Let $\alpha>1$.

$$B^{\alpha}_{1,q}(\boldsymbol{R}^n):=\{u\in W^{[\alpha]^-,1}(\boldsymbol{R}^n)\,|\,\forall\beta\in\boldsymbol{N}^n_0\quad\text{with}\quad|\beta|=[\alpha]^-\ \partial^\beta u\in B^{\{\alpha\}^-}_{1,p}(\boldsymbol{R}^n)\}.$$

These spaces will be always considered with their natural norms $\|\cdot\|^{\alpha}_{1,q}$:

$$\|u\|^{\alpha}_{1,\infty}:=\|u\|_{W^{[\alpha]^-,1}(\boldsymbol{R}^n)}$$
$$+\sum_{|\beta|-[\alpha]^-}\sup_{h\in\boldsymbol{R}^n\setminus\{0\}})\,|h|^{-\alpha}\|\partial^\beta u(\cdot+h)-2\partial^\beta u+\partial^\beta u(\cdot-h)\|_{L^1(\boldsymbol{R}^n)},$$
$$\|u\|^{\alpha}_{1,q}:=\|u\|_{W^{[\alpha]^-,1}(\boldsymbol{R}^n)}$$
$$+\sum_{|\beta|-[\alpha]^-}(\int_{\boldsymbol{R}^n}(\|\partial^\beta u(\cdot+h)-2\partial^\beta u+\partial^\beta u(\cdot-h)\|_{L^1(\boldsymbol{R}^n)})^q|h|^{-n-\alpha q}\,dh)^{1/q}$$

In case $0<\{\alpha\}^-<1$ equivalent norms can be obtained replacing $\partial^\beta u(\cdot+h)-2\partial^\beta u+\partial^\beta u(\cdot-h)$ with $\partial^\beta u(\cdot+h)-\partial^\beta u$ (see [17] 2.5.12).

Thoerem 4.2. *For any $\alpha>0$ there exist $C>0$, $N\in\boldsymbol{N}$ such that, for any $m\in\mathcal{O}_M$, for any $f\in B^{\alpha}_{1,q}(\boldsymbol{R}^n)$,*

$$\|F^{-1}(mFf)\|^{\alpha}_{1,q}\leq C\sup_{|\beta|\leq N,x\in\boldsymbol{R}^n}(1+|x|^2)^{|\beta|/2}\,|\partial^\beta m(x)|\,\|f\|^{\alpha}_{1,q}\,.$$

Proof. It is a particular case of [4].

Theorem 4.3. *Let $\alpha,\beta\in\boldsymbol{R}$, $\alpha>0$, $\alpha+\beta>0$; then, $u\to(1-\Delta)^{-\beta/2}u$ is a linear and topological isomorphism between $B^{\alpha}_{1,q}(\boldsymbol{R}^n)$ and $B^{\alpha+\beta}_{1,q}(\boldsymbol{R}^n)$.*

Proof. See [15] 1, th. 8.

DEFINITION 4.4. Let $\alpha\in\boldsymbol{R}$, $\alpha\leq 0$, $\alpha=\beta-\gamma$, with $\beta>0$. We set

$$B^{\alpha}_{1,q}(\boldsymbol{R}^n):=\{u=(1-\Delta)^{\gamma/2}v\,|\,v\in B^{\gamma}_{1,q}(\boldsymbol{R}^n)\}.$$

Owing to 4.3 this definition is independent of $\beta>0$. So, if $\alpha\leq 0$, we can set

$$\|u\|^{\alpha}_{1,q}:=\|(1-\Delta)^{(\alpha-1)/2}u\|^{1}_{1,q}\,.$$

Proposition 4.5. *Let α, $\beta\in\boldsymbol{R}$, $m\in\mathcal{O}_M$. There exist $C>0$, $N\in\boldsymbol{N}$ such that $\forall f\in B^{\alpha}_{1,q}(\boldsymbol{R}^n)$*

$$\|F^{-1}(mFf)\|^{\beta}_{1,q} \le C \sup_{\xi\in\boldsymbol{R}^n, |\gamma|\le N}(1+|\xi|)^{\beta-\alpha+|\gamma|}|\partial^\gamma m(\xi)| \ \|f\|^{\alpha}_{1,q}\,.$$

A consequence of 4.5 is that $\forall\alpha\in\boldsymbol{R}$, $\beta\in\boldsymbol{N}_0^n$ $u\to\partial^\beta u$ is a linear bounded operator from $B^{\alpha}_{1,q}(\boldsymbol{R}^n)$ to $B^{\alpha-|\beta|}_{1,q}(\boldsymbol{R}^n)$.

For what concerns pointwise multipliers, we have the following result:

Proposition 4.6. (see [17] 2.8.2) *Let $\alpha, \beta\in\boldsymbol{R}$, $\beta>|\alpha|$. Then $BC^\beta(\boldsymbol{R}^n)$ is a space of pointwise multipliers for $B^{\alpha}_{1,q}(\boldsymbol{R}^n)$.*

Lemma 4.7. *Let $a\in BC^\beta(\boldsymbol{R}^n)$, with $\beta>0$, $a(0)=0$. If $|\alpha|<\beta$, $1\le q\le +\infty$, $u\in B^{\alpha}_{1,q}(\boldsymbol{R}^n)$, $u(x)=0$ if $|x|>R$,*

$$\|au\|^{\alpha}_{1,q}\le\omega(R)\ \|u\|^{\alpha}_{1,q}+\eta(R)\ \|u\|^{\alpha-1}_{1,q}\,,$$

with $\omega(R)$ and $\eta(R)$ indepedent of u and $\omega(R)\to 0$ $(R\to 0)$.

Proof. See [5] prop. 4.2.

Proposition 4.8. *Assume $\alpha_0, \alpha_1\in\boldsymbol{R}$, $1\le q, q_0, q_1\le+\infty$.*
If $\alpha_0<\alpha_1$, $B^{\alpha_1}_{1,q_1}(\boldsymbol{R}^n)\subseteq B^{\alpha_0}_{1,q_0}(\boldsymbol{R}^n)$.
If $q_0\le q_1$, $B^{\alpha_0}_{1,q_0}(\boldsymbol{R}^n)\subseteq B^{\alpha_0}_{1,q_1}(\boldsymbol{R}^n)$.
If $\alpha_0\neq\alpha_1$, $\theta\in]0,1[$,
$(B^{\alpha_0}_{1,q_0}(\boldsymbol{R}^n), B^{\alpha_1}_{1,q_1}(\boldsymbol{R}^n))_{\theta,q}=B^{\alpha}_{1,q}(\boldsymbol{R}^n)$, with $\alpha=(1-\theta)\,\alpha_0+\theta\alpha_1$;
if $q_0, q_1<+\infty$, $(B^{\alpha_0}_{1,q_0}(\boldsymbol{R}^n), B^{\alpha_1}_{1,q_1}(\boldsymbol{R}^n))_{[\theta]}=B^{\alpha}_{1,q}(\boldsymbol{R}^n)$, with $\alpha=(1-\theta)\,\alpha_0+\theta\alpha_1$, $q^{-1}=(1-\theta)\,q_0^{-1}+\theta q_1^{-1}$.
Finally, $\forall N\in\boldsymbol{N}_0$, $B^{N}_{1,1}(\boldsymbol{R}^n)\subseteq W^{N,1}(\boldsymbol{R}^n)\subseteq B^{N}_{1,\infty}(\boldsymbol{R}^n)$

Proof. See [17] 2.3.2, 2.4.2, 2.4.7, 2.5.7.

Proposition 4.9. *Let $A(x,\partial)=(A_{ij}(x,\partial))_{1\le i,j\le N}$ be a differential operator valued matrix with coefficients in $BC^\beta(\boldsymbol{R}^n)$ $(\beta>0)$. Assume further that* (h1), (h3) *are satisfied. Consider the problem*

$$\lambda u-A(x,\partial)\,u=f\ \text{in}\ \boldsymbol{R}^n\,, \tag{12}$$

with $\lambda\in\boldsymbol{C}$, $\operatorname{Re}\lambda\ge 0$, $f\in B^{\alpha}_{1,q}(\boldsymbol{R}^n)^N$, with $|\alpha|<\beta$. Then, there exists $R\ge 0$ such that, if $|\lambda|\ge R$, the problem has a unique solution $u\in B^{2m+\alpha}_{1,q}(\boldsymbol{R}^n)^N$.

Proof. We start from the constant coefficient case; assume first that $A(\partial)=A^\sharp(\partial)$ and $\theta\in[-\pi/2,\pi/2]$. Then, for any $\xi\in\boldsymbol{R}^n$ and any $r\ge 0$, with $(r,\xi)\neq(0,0)$, the matrix $r^{2m}\,e^{i\theta}-A(i\xi)$ is invertible. We set $(r^{2m}\,e^{i\theta}-A(i\xi))^{-1}=(\alpha_{ij}(r,\xi))_{1\le i,j\le N}$. It is easily seen that α_{ji} is homogeneous of order $-2m$ in the variables (r,ξ). Therefore, problem (12) has a unique solution u in $\mathcal{S}'(\boldsymbol{R}^n)^N$ and such a solution, owing to 4.5, belongs to $B^{2m+\alpha}_{1,q}(\boldsymbol{R}^n)^N$. Moreover, if $r\ge 1$, we have the estimate

$$r^{2m} \|u\|_{B^{\alpha}_{1,q}(\boldsymbol{R}^n)^N} + \|u\|_{B^{2m+\alpha}_{1,q}(\boldsymbol{R}^n)^N} \leq C \|f\|_{B^{\alpha}_{1,q}(\boldsymbol{R}^n)^N}, \tag{13}$$

which can be obtained by arguments similar to [6] proposition 2.3.

A simple perturbation argument gives the same result also in case $A(\partial) \neq A^{\sharp}(\partial)$.

Now we consider the case of variable coefficients. Using 4.7 and the same method of (for example) [6] (lemma 2.4 and prop. 2.5) one obtains estimate (9) in case $q=1$ and the existence of a solution of (8) in the case of BC^{∞} coefficients. A standard approximation argument allows to extend these results to the case of coefficients in BC^{β}. By interpolation the case $q \neq 1$ follows.

DEFINITION 4.10. Let Ω be an open subset of $\boldsymbol{R}^n$, $\alpha \in \boldsymbol{R}$, $1 \leq q \leq +\infty$. We set

$$B^{\alpha}_{1,q}(\Omega) = \{u|_{\Omega} \mid u \in B^{\alpha}_{1,q}(\boldsymbol{R}^n)\} .$$

$B^{\alpha}_{1,q}(\Omega)$ is a Banach space with its natural topology of quotient space. We have:

Proposition 4.11. *Let Ω, O be open subsets of $\boldsymbol{R}^n$, Φ: $O \to \Omega$ a diffeomorphism of class $C^{\rho}(\rho \geq 1)$. Let $u \in B^{\alpha}_{1,q}(\Omega)$, with $0 < \alpha < \rho$ or $\alpha \leq 0$, $|\alpha| + 1 < \rho$ and* supp(u) *compact in Ω. Then, $u \circ \Phi$ is a well defined element of $B^{\alpha}_{1,q}(O)$.*

Proof. See [5] prop. 4.3.

Proposition 4.12. *Let $N \in \boldsymbol{N}$. There exists a linear operator P_N belonging to $\mathcal{L}(B^{\alpha}_{1,q}(\boldsymbol{R}^n_+), B^{\alpha}_{1,q}(\boldsymbol{R}^n))$ for any $\alpha \in]-N, N[$, such that $P_N u|_{\boldsymbol{R}^n_+} = u$. Moreover, if $-1 < \alpha < 1$, given any element $u \in B^{\alpha}_{1,q}(\boldsymbol{R}^n_+)$, there exists a unique element $u_0 \in B^{\alpha}_{1,q}(\boldsymbol{R}^n)$ such that u is the restriction of u_0 to $\boldsymbol{R}^n_+$ and the support of u_0 is contained in the closure of $\boldsymbol{R}^n_+$.*

Proof. See [17] 3.3.4, 2.8.7. The uniqueness of u_0 can be obtained like in [5] prop. 4.4.

With the help of 4.11 and 4.12 it is now easy to obtain natural extensions of 4.6, 4.8 and 4.12 itself to the case $\Omega = \boldsymbol{R}^n_+$ or a bounded open subset of $\boldsymbol{R}^n$, with boundary $\partial\Omega$ of class C^{ρ} and Ω lying on one side of $\partial\Omega$. Moreover, using 4.11, it is possible to define, by local charts, $B^{\alpha}_{1,q}(\partial\Omega)$ for any $\alpha \in \boldsymbol{R}$, $1-\rho < \alpha < \rho$. One has:

Proposition 4.13. *Let $\alpha \in \boldsymbol{R}$, $1 < \alpha < \rho$. If $u \in B^{\alpha}_{1,q}(\Omega)$, u has a trace on $\partial\Omega$. Such a trace belongs to the space $B^{\alpha-1}_{1,q}(\partial\Omega)$.*

Proof. See [17] 3.3.3.

Now, (just with obvious modifications of the proof) one has the following variant of theorem 3.1 in [7]:

Proposition 4.14. *Assume the assumptions* (I1)-(I3) *are satisfied. Let* $u \in B_{1,q}^{2m}(\Omega)^N$ *such that* $\gamma(B(x', \partial)\,u)=0$ $(1 \leq q \leq +\infty)$. *Let* $\lambda \in \boldsymbol{C}$, $\lambda u - A(x, \partial)\,u = f$ *in* Ω. *Then there exists* $C(\lambda, q) > 0$ *independent of* f *such that*

$$\|u\|_{B_{1,q}^{2m}(\boldsymbol{R}^n)^N} \leq C(\lambda, q)\,(\|f\|_{B_{1,q}^{0}(\boldsymbol{R}^n)^N} + \|u\|_{B_{1,q}^{0}(\boldsymbol{R}^n)^N})\,.$$

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