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J-GROUPS OF SUSPENSIONS OF STUNTED LENS SPACES MOD 8

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1. Introduction

Let $L^n(q)=S^{2n+1}/\boldsymbol{Z}_q$ be the $(2n+1)$-dimensional standard lens space mod q. As defined in [7], we set

$$(1.1)\qquad \begin{aligned} L_q^{2n+1} &= L^n(q)\,, \\ L_q^{2n} &= \{[z_0, \cdots, z_n]\in L^n(q)\,|\, z_n \text{ is real and } z_n\geqq 0\}\,. \end{aligned}$$

In the previous paper [10], we determined the J-groups $\tilde{J}(S^j(L_q^m/L_q^n))$ of the suspensions of the stunted lens spaces L_q^m/L_q^n for $q=4$ and for $j\equiv 1 \pmod 2$. The purpose of this paper is to determine the KO- and J-groups of suspensions of stunted lens spaces mod 8.

This paper is organized as follows. In section 2 we state the main theorems: the structures of $\widetilde{KO}(S^j(L_8^m/L_8^n))$ and $\tilde{J}(S^j(L_8^m/L_8^n))$ for $j\equiv 0 \pmod 2$ are given in Theorems 1 and 2 respectively. In section 3 we prepare some lemmas and recall known results in [8], [9] and [11]. By virtue of the results in [8], the proofs of Theorems 1 and 2 for the case $j\equiv 0 \pmod 4$ are given in section 4. Applying the method used in the corresponding parts of [10], we prove Theorems 1 and 2 for the case $j\equiv 2 \pmod 4$ in the final section.

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2. Statement of results

We prepare functions $h_1, h_2, h_3, h_4, a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 defined by

$$(2.1)\qquad \begin{cases} h_1(n) = [n/4]+[(n+7)/8]+[(n+4)/8] \\ h_2(n) = [n/4]+[(n+7)/8]+[n/8]+1 \\ h_3(m, n) = [m/4]-[n/4] \\ h_4(m, n) = [m/8]-[n/8]\,. \end{cases}$$

$$
(2.2)\qquad \begin{cases} a_1(m, n) = h_1(m)-[(n+1)/4]-[(n+1)/8]-[(n+6)/8]+1 \\ a_2(m, n) = h_3(m, n+1) \\ a_3(m, n) = h_4(m-2, n+5) \\ a_4(m, n) = h_4(m, n+7) \\ a_5(m, n) = a_1(m, n)-[(m+4)/8]+[m/8] \\ a_6(m, n) = [(m+4)/8]+[(m-2)/8]-[(n+1)/4] \\ a_7(m, n) = 2[(m+4)/8]-[(n+5)/4]\,. \end{cases}
$$

Let $\boldsymbol{Z}/k$ denote the cyclic group $\boldsymbol{Z}/k\boldsymbol{Z}$ of order k. For an integer n, $G(n)$ denotes the group defined by

$$
(2.3)\qquad G(n) = \begin{cases} \boldsymbol{Z}/2\oplus\boldsymbol{Z}/2 & (n\equiv 1 \pmod 8) \\ \boldsymbol{Z}/2 & (n\equiv 0 \text{ or } 2 \pmod 8) \\ 0 & (\text{otherwise})\,. \end{cases}
$$

Theorem 1. *Let j, m and n be non-negative integers with $j\equiv 0$ (mod 2) and $m>n$.*

(1) *Suppose* $j\equiv 0$ (mod 4).

i) *If $n\not\equiv 3$ (mod 4) and $m\geqq 4\,[(n+j+15)/8]+2[(n-j)/4]$, then we have*

$$
\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \oplus_{i=1}^4 \boldsymbol{Z}/2^{a_i(m+j,n+j)}\,.
$$

ii) *If $n\not\equiv 3$ (mod 4) and $4\,[(n+j+15)/8]+2[(n-j)/4]>m>n$, then we have*

$$
\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/2^{a_1(m+j,n+j)} & (m\geqq 4\,[n/4]+4) \\ \boldsymbol{Z}/2\oplus\boldsymbol{Z}/2 & (n+j\equiv 0 \pmod 8) \text{ and } n+4>m\geqq n+2) \\ \boldsymbol{Z}/2 & (h_4(n+j+6,\, n+j) = [m/2]-[(n+1)/2] = 0) \\ 0 & (\textit{otherwise})\,. \end{cases}
$$

iii) *If $n\equiv 3$ (mod 4), then we have*

$$
\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}\oplus\widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & (m\geqq n+2) \\ \boldsymbol{Z} & (m = n+1)\,. \end{cases}
$$

(2) *Suppose* $j\equiv 2$ (mod 4).

i) *If $m\geqq 8\,[(n+j+15)/8]-j+2$, then we have*

$$
\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \boldsymbol{Z}/2^{h_3(m+j,n+j-3)}\oplus(\oplus_{i=0}^1 \boldsymbol{Z}/2^{h_4(m+j-4i,\,n+j-4i+5)}) \oplus G(m+j)\oplus H(n+j)\,,
$$

where $G(m)$ is the group defined by (2.3) *and $H(n)$ is the group defined by*

$$
H(n) = \begin{cases} \boldsymbol{Z} & (n\equiv 3 \pmod 4) \\ G(n) & (\textit{otherwise})\,. \end{cases}
$$

ii) *If* $8[(n+j+15)/8]-j+2>m\geqq 6[(n+j+7)/8]+2[(n+j+1)/8]-j+4$, *then we have*

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \boldsymbol{Z}/2^{h_3(m+j,n+j-3)}\oplus\boldsymbol{Z}/2^{h_4(m+j-4,n+j+1)}$$
$$\oplus G_1(m+j)\oplus H_1(n+j)\,,$$

where $G_1(m)$ *is the group defined by*

$$(2.4)\qquad G_1(m)=\begin{cases}\boldsymbol{Z}/4\oplus\boldsymbol{Z}/2 & (m\equiv 1 \pmod 8)\\ \boldsymbol{Z}/4 & (m\equiv 0 \pmod 8)\\ G(m) & (\textit{otherwise})\end{cases}$$

and $H_1(n)$ *is the group defined by*

$$H_1(n)=\begin{cases}\boldsymbol{Z} & (n\equiv 3 \pmod 4)\\ G_1(2-n) & (\textit{otherwise})\,.\end{cases}$$

iii) *If* $6[(n+j+7)/8]+2[(n+j+1)/8]-j+4>m\geqq n+3$, *then we have*

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong G_2(m+j)\oplus H_2(n+j)\,,$$

where $G_2(m)$ *is the group defined by*

$$G_2(m)=\begin{cases}\boldsymbol{Z}/8\oplus\boldsymbol{Z}/2 & (m\equiv 1 \pmod 8)\\ \boldsymbol{Z}/8 & (m\equiv 0 \pmod 8)\\ G(m) & (\textit{otherwise})\,,\end{cases}$$

and $H_2(n)$ *is the group defined by*

$$H_2(n)=\begin{cases}\boldsymbol{Z} & (n\equiv 3 \pmod 4)\\ G_2(2-n) & (\textit{otherwise})\end{cases}$$

iv) *If* $n+3>m>n$, *then we have* $\widetilde{KO}(S^j(L_8^m/L_8^n))\cong\widetilde{KO}(L_8^{m+j}/L_8^{n+j})$.

REMARK. (1) Combining this theorem with [13, Theorem 2], we obtain the complete results for the groups $\widetilde{KO}(S^j(L_8^m/L_8^n))$.
(2) The partial results for the case $n=0$ of this theorem have been obtained in [8].

Let $\nu_p(s)$ denote the exponent of the prime p in the prime power decomposition of s, and $\mathfrak{m}(s)$ the function defined on positive integers as follows (cf. [3]):

$$\nu_p(\mathfrak{m}(s))=\begin{cases}0 & (p\neq 2 \text{ and } s\not\equiv 0 \pmod{(p-1)})\\ 1+\nu_p(s) & (p\neq 2 \text{ and } s\equiv 0 \pmod{(p-1)})\\ 1 & (p=2 \text{ and } s\not\equiv 0 \pmod 2)\\ 2+\nu_2(s) & (p=2 \text{ and } s\equiv 0 \pmod 2)\,.\end{cases}$$

In order to state the next theorem, we set

$$(2.5)\quad \begin{cases} b_1(j,m,n)=\begin{cases}\min\{\nu+h_4(n+9,n-3),\ a_1(m+j,n+j)\} & (j\equiv 4\ (\mathrm{mod}\ 8))\\ \min\{\nu+h_4(n+9,n-2),\ a_5(m,n)\} & (\text{otherwise})\end{cases}\\ b_2(j,m,n)=\begin{cases}\min\{\nu+h_4(n+3,n-7),\ a_7(m,n)\} & (j\equiv 4\ (\mathrm{mod}\ 8))\\ \min\{\nu+h_4(n+5,n-7),\ a_6(m,n)\} & (\text{otherwise})\end{cases}\\ b_3(j,m,n)=\begin{cases}\min\{\nu+1,\ a_3(m+j,n+j)\} & (j\equiv 4\ (\mathrm{mod}\ 8))\\ \min\{\nu+1,\ a_4(m,n)\} & (\text{otherwise}),\end{cases}\end{cases}$$

where ν is the integer defined by

$$\nu=\begin{cases}\nu_2(j) & (j\neq 0)\\ m & (j=0).\end{cases}$$

Main result is the following theorem.

Theorem 2. *Let j, m and n be non-negative integers with $j\equiv 0$ (mod 2) and $m>n$.*

(1) *Suppose $j\equiv 0$ (mod 4) and $n\not\equiv 3$ (mod 4).*

i) *If $m\geqq 2[n/4]+4[n/8]+6+h_4(j,j-4)(2h_4(n-2,n+4)-4a_4(n,n))$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n))\cong\begin{cases}(\oplus_{i=1}^3 \boldsymbol{Z}/2^{b_i(j,m,n)})\oplus\boldsymbol{Z}/2 & (n\equiv 2+2h_4(j+4,j)\ (\mathrm{mod}\ 8))\\ \oplus_{i=1}^3\boldsymbol{Z}/2^{b_i(j,m,n)} & (\textit{otherwise}).\end{cases}$$

ii) *If $2[n/4]+4[n/8]+6+h_4(j,j-4)(2h_4(n-2,n+4)-4a_4(n,n))>m>n$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n))\cong\begin{cases}\boldsymbol{Z}/2^{b_1(j,m,n)}\oplus\boldsymbol{Z}/4 & (h_4(n+j+5,n+j-2)=h_3(m,n+6)=0)\\ \boldsymbol{Z}/8 & (h_4(n+j+6,n+j-1)=h_3(m,n+3)=0)\\ \widetilde{KO}(S^j(L_8^m/L_8^n)) & (\textit{otherwise}).\end{cases}$$

(2) *Suppose $j\equiv n+1\equiv 0$ (mod 4). Set $M=\mathfrak{m}((n+j+1)/2)$ and $b_i=b_i(j,m,n)$ $(1\leqq i\leqq 3)$.*

i) *If $m\geqq n+2h_4(j+4,j)h_4(n+1,n)+5$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n))\cong\begin{cases}\boldsymbol{Z}/2^{c_1}M\oplus\boldsymbol{Z}/2^{c_2+i_1}\oplus\boldsymbol{Z}/2^{b_3}\oplus\boldsymbol{Z}/2 & (j(n+1)\equiv 4\ (\mathrm{mod}\ 8))\\ \boldsymbol{Z}/2^{c_1}M\oplus\boldsymbol{Z}/2^{c_2+i_1}\oplus\boldsymbol{Z}/2^{c_3+i_2}\oplus\boldsymbol{Z}/2^{i_3} & (\textit{otherwise}),\end{cases}$$

where i_1, i_2, i_3, c_1, c_2 and c_3 are integers defined by

$$\begin{cases} i_1=\begin{cases}\min\{b_1,\ \nu_2(n+1)-1\} & (n+j\equiv 7\ (\mathrm{mod}\ 8))\\ \min\{b_1,\ \nu_2(n+1)\} & (n+j\equiv 3\ (\mathrm{mod}\ 8))\end{cases}\\ i_2=\begin{cases}\min\{b_2,\ \nu_2(n+1)-2\} & (n+j\equiv 7\ (\mathrm{mod}\ 8))\\ \min\{b_2,\ \nu_2(n+1)-1\} & (n+j\equiv 3\ (\mathrm{mod}\ 8))\end{cases}\end{cases}$$

(2.6)
$$\begin{cases} i_3 = \begin{cases} \min\{b_3, \nu_2(n+1)-2\} & (j \equiv n-3 \equiv 4 \pmod 8) \\ b_3 & (\textit{otherwise}) \end{cases} \\ c_1 = \max\{b_k - i_k \mid 1 \leqq k \leqq 3\} \\ c_3 = \min\{b_k - i_k \mid 1 \leqq k \leqq 3\} \\ c_2 = (\Sigma_{k=1}^3 b_k - i_k) - c_1 - c_3 \,. \end{cases}$$

ii) *If* $n+2h_4(j+4, j)\, h_4(n+1, n)+5>m>n$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/2^{b-i} M \oplus \boldsymbol{Z}/2^i & (m \geqq n+5) \\ \boldsymbol{Z}/M \oplus \widetilde{KO}(S^j(L_8^m/L_8^{n-1})) & (n+5>m>n+1) \\ \boldsymbol{Z}/M & (m = n+1)\,, \end{cases}$$

where $b=b_1$ *and* $i=\min\{b, \nu_2(n+1)\}$.

(3) *Suppose* $j \equiv 2 \pmod 4$ *and* $n \not\equiv 1 \pmod 4$.

i) *If* $m \geqq 8[(n+17)/8]+h_4(j+4, j)\,(6h_4(n-2, n+1)+2h_4(n-6, n+9))$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/2^b \oplus \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/2 \oplus G(n+j) \oplus G(m+j) \\ \qquad\qquad (h_4(n+5-2h_4(j, j-4), n-2) = 0) \\ \boldsymbol{Z}/2^b \oplus \boldsymbol{Z}/8 \oplus G(n+j) \oplus G(m+j) \quad (\textit{otherwise})\,, \end{cases}$$

where $b=b_3(j, m+2, n)$ *and* $G(m)$ *is the group defined by* (2.3).

ii) *If* $8[(n+17)/8]+h_4(j+4, j)\,(6h_4(n-2, n+1)+2h_4(n-6, n+9))>m\geqq 8[(n+2)/8]+10$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/2 \oplus G_1(m+j) & (h_4(n+5-2h_4(j, j-4), n-2) = 0) \\ \boldsymbol{Z}/8 \oplus G_1(h_4(j+4, j)\,(2n-1)-n) \oplus G_1(m+j) & (\textit{otherwise})\,, \end{cases}$$

where $G_1(m)$ *is the group defined by* (2.4).

iii) *If* $8[(n+2)/8]+10>m>n$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(S^j(L_8^m/L_8^n))\,.$$

(4) *Suppose* $j \equiv n+1 \equiv 2 \pmod 4$. *Set* $M=\mathfrak{m}((n+j+1)/2)$.

i) *If* $m \geqq 8[n/8]+2h_4(j, j-4)+14$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/2^b M \oplus \boldsymbol{Z}/4 \oplus \boldsymbol{Z}/2 \oplus G(m+j) & (n \equiv 1 \pmod 8) \\ \boldsymbol{Z}/4M \oplus \boldsymbol{Z}/2^b \oplus \boldsymbol{Z}/2 \oplus G(m+j) & (n \equiv 5 \pmod 8))\,, \end{cases}$$

where $b=b_3(j, m+2, n)$ *and* $G(m)$ *is the group defined by* (2.3).

ii) *If* $8[n/8]+14+2h_4(j, j-4)>m\geqq 4h_4(j+4, j)\, h_4(n-4, n)+8[n/8]+10$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/2M\oplus\boldsymbol{Z}/4\oplus G_1(m+j) & (j\equiv n+1\equiv 2 \pmod 8) \\ \boldsymbol{Z}/2^{c+1}M\oplus\boldsymbol{Z}/2\oplus G_1(m+j) & (j\equiv n-3\equiv 2 \pmod 8) \\ \boldsymbol{Z}/2^cM\oplus\boldsymbol{Z}/4\oplus G(m+j) & (j\equiv n+5\equiv 6 \pmod 8) \\ \boldsymbol{Z}/2M\oplus\boldsymbol{Z}/2^d\oplus G_1(m+j) & (j\equiv n+1\equiv 6 \pmod 8)\,, \end{cases}$$

where $c=[(m-n-1)/8]$, $d=h_4(m-4,n)$ *and* $G_1(m)$ *is the group defined by* (2.4).

iii) *If* $4h_4(j+4,j)\,h_4(n-4,n)+8\,[n/8]+10>m>n$, *then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \boldsymbol{Z}/M\oplus\widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & (m\geqq 4\,[n/4]+6) \\ \boldsymbol{Z}/M\oplus\boldsymbol{Z}/2 & (n+j\equiv 7 \pmod 8) \text{ and } n+4>m\geqq n+2) \\ \boldsymbol{Z}/M & (\text{otherwise})\,. \end{cases}$$

REMARK. (1) Combining this theorem with [10, Theorem 1], we obtain the complete results for the groups $\tilde{J}(S^j(L_8^m/L_8^n))$.
(2) The partial results for the case $j=n=0$ of this theorem have been obtained in [9].

3. Preliminaries

In this section we prepare some lemmas and recall known results which are needed to prove Theorems 1 and 2.

Lemma 3.1. *Let* j, k, l *and* s *be integers with* $j>0$, $\nu=\nu_2(j)\geqq s\geqq 1$, $l\geqq 2$ *and* $k\equiv\pm 1 \pmod{2^l}$. *Then we have*

(1) $$k^j-1\equiv(k^{2^\nu}-1)\,(j/2^\nu) \pmod{2^{2\nu+2l}}\,.$$

(2) $$k^j\equiv 1 \pmod{2^{\nu+l}}\,.$$

(3) $$k^j-1\equiv(k^{2^s}-1)\,(j/2^s) \pmod{2^{\nu+s+2l-1}}\,.$$

Proof. Since $k^2\equiv 1 \pmod{2^{l+1}}$, by making use of the method used in the proof of [10, Lemma 3.1] we can show that

$$k^j-1\equiv(k^2-1)\,(j/2) \pmod{2^{\nu+2l}}\,.$$

This implies (2). In particular, we have

$$k^{2^\nu}\equiv 1 \pmod{2^{\nu+l}}\,.$$

Then, the rest of the proof is similar to that of [10, Lemma 3.1]. q.e.d.

Considering the $\boldsymbol{Z}/8$-action on $S^{2n+1}\times\boldsymbol{C}$ given by

$$\exp(2\pi\sqrt{-1}/8)\,(z,u)=(z\cdot\exp(2\pi\sqrt{-1}/8),\,u\cdot\exp(2\pi\sqrt{-1}/8))$$

for $(z,u)\in S^{2n+1}\times\boldsymbol{C}$, we have a complex line bundle

$$\eta\colon (S^{2n+1}\times \boldsymbol{C})/(\boldsymbol{Z}/8) \to L_8^{2n+1}\,.$$

Then we have the following elements

$$\text{(3.2)}\qquad \sigma(i) = \eta^{2^i}-1\in \tilde{K}(L_8^{2n+1})\subset K(L_8^{2n+1}) \quad (0\leqq i\leqq 2)\,.$$

We denote the restriction of $\sigma(i)$ in $\tilde{K}(L_8^{2n})$ by the same symbol, and $\sigma(0)$ by σ. The following proposition is well known.

Proposition 3.3 (Mahammed [11]). *The ring $K(L_8^m)$ is isomorphic to the truncated polynomial ring $\boldsymbol{Z}[\sigma]/(\sigma^{[m/2]+1}, (\sigma+1)^8-1)$, where $(\sigma^{[m/2]+1}, (\sigma+1)^8-1)$ means the ideal of $\boldsymbol{Z}[\sigma]$ generated by $\sigma^{[m/2]+1}$ and $(\sigma+1)^8-1$.*

In order to state the next lemma, we set

$$\text{(3.4)}\quad (1)\quad \begin{cases} \sigma_{2^i} = \sigma(i) & (0\leqq i\leqq 2) \\ \sigma_6 = \sigma_4\,\sigma_2 & \\ \sigma_{2i+1} = \sigma_{2i}\,\sigma_1 & (1\leqq i\leqq 3)\,. \end{cases}$$

$$(2)\quad \begin{cases} b(n) = ((-68-48\sqrt{2})^n+(-68+48\sqrt{2})^n)/2 \\ c(n) = ((-68-48\sqrt{2})^n-(-68+48\sqrt{2})^n)/2\sqrt{2}\,. \end{cases}$$

Then we have

$$\text{(3.5)}\quad (1)\quad \begin{cases} b(0) = 1,\ b(n+1) = -68b(n)-96c(n)\,, \\ c(0) = 0,\ c(n+1) = -48b(n)-68c(n)\,. \end{cases}$$

$$(2)\quad \begin{cases} b(n) \equiv (-4)^n & (\mathrm{mod}\ 2^{2n+4}) \\ c(n) = 0 & (\mathrm{mod}\ 2^{2n+2})\,. \end{cases}$$

The following lemma is obtained by Proposition 3.3 and (3.5) (1).

Lemma 3.6. *Let u be a positive integer. Then, in $K(L_8^m)$,*

$$\sigma^u = \Sigma_{i=1}^7\, a_{u,i}\,\sigma_i\,,$$

where $a_{u,i}$ $(1\leqq i\leqq 7)$ are integers defined by $a_{u,1} = (-2)^{u-1}$,

$$\begin{aligned}
a_{u,2} &= (1/5)\,(-4)^{[u/4]+1}+(2/5)\,(-4)^{[(u-1)/4]}-(1/5)\,(-4)^{[(u+2)/4]} \\
&\qquad +(3/5)\,(-4)^{[(u+1)/4]}\,, \\
a_{u,3} &= -(-2)^{u-2}-(1/2)\,a_{u+1,2}\,, \\
a_{u,4} &= -(1/2)\,h_4(u, u-1)\,b([u/8])+h_4(u-1, u-2)\,c([u/8]) \\
&\qquad -h_4(u-2, u-3)\,c([u/8])+h_4(u+4, u+3)\,(b([u/8])+2c([u/8])) \\
&\qquad -h_4(u+3, u+2)\,(4b([u/8])+6c([u/8])) \\
&\qquad +h_4(u+2, u+1)\,(10b([u/8])+14c([u/8])) \\
&\qquad -h_4(u+1, u)\,(20b([u/8])+28c([u/8]))\,, \\
a_{u,5} &= -(-2)^{u-2}-a_{u+1,4}-a_{u+2,4}-(1/2)\,a_{u+3,4}\,, \\
a_{u,6} &= (1/2)\,a_{u,2}-a_{u+1,4}-(1/2)\,a_{u+2,4}
\end{aligned}$$

and $a_{u,7}=(-2)^{u-3}-(1/4)\,a_{u+1,2}-(1/2)\,a_{u+1,4}$.

Proof. By making use of the relation $(\sigma+1)^8=1$, we obtain equalities $a_{u+1,1}=-2a_{u,1}$, $a_{u+1,2}=a_{u,1}-2a_{u,3}$, $a_{u+1,3}=a_{u,2}-2a_{u,3}$, $a_{u+1,4}=a_{u,3}-2a_{u,7}$, $a_{u+1,5}=a_{u,4}-2a_{u,5}$, $a_{u+1,6}=a_{u,5}-2a_{u,7}$ and $a_{u+1,7}=a_{u,6}-2a_{u,7}$, where $a_{1,1}=1$ and $a_{1,i}=0$ $(2\leqq i\leqq 7)$. Thus the lemma is proved by the induction with respect to u. q.e.d.

In order to state the next proposition, we set

(3.7) (1) Let $F(x)$ denote the free abelian group generated by $\{x_i\,|\,1\leqq i\leqq 7\}$. Then X_i and $X_i(n)$ $(7\geqq i\geqq 1,\ n\geqq 0)$ denote the elements of $F(x)$ defined by $X_1=4x_1+2x_3+2x_5+x_7$, $X_2=2x_2+x_6$, $X_3=2x_3+x_7$, $X_6=x_6+x_7$, $X_i=x_i$ $(i=4,5$ or $7)$, $X_1(n)=2^{[n/2]}X_1$,

$$\begin{aligned}
X_2(n) &= 2^{[n/4]}X_2-2^{2[n/4]}X_1\,,\\
X_3(n) &= 2^{[(n-2)/4]}X_3+2^{2[n/4]-1}h_3(n,n-2)\,X_1\,,\\
X_4(n) &= 2^{[n/8]}X_4+2^{2[n/8]}h_4(n+4,n)\,X_2+2^{2[n/8]+[n/4]}X_1\,,\\
X_5(n) &= 2^{[(n-2)/8]}X_5+2^{2[(n-2)/8]}h_4(n+2,n-2)\,X_3\\
&\quad -2^{[(n+2)/4]+2[(n-2)/8]}X_1\,,\\
X_6(n) &= 2^{[(n-4)/8]}X_6+2^{[n/4]-1}h_4(n,n-4)\,X_2\\
&\quad -2^{[n/4]+2[(n-4)/8]+1}X_1
\end{aligned}$$

and

$$\begin{aligned}
X_7(n) &= 2^{[(n-6)/8]}X_7-2^{[(n-6)/4]}h_4(n+6,n+2)\,(X_3-2X_2)\\
&\quad +2^{[(n-2)/4]+2[(n+2)/8]}X_1\,.
\end{aligned}$$

(2) Let $\varphi\colon F(x)\to\tilde{K}(L_8^m)$ be the homomorphism defined by setting $\varphi(x_i)=\sigma_i$ $(1\leqq i\leqq 7)$.

Proposition 3.8 (Kobayashi and Sugawara [9]). *The homomorphism* φ *is an epimorphism, and the kernel of* φ *coincides with the subgroup of* $F(x)$ *generated by* $\{X_i(m)\,|\,1\leqq i\leqq 7\}$.

According to [1], we have the following lemma.

Lemma 3.9. *The Adams operations are given by the following formulae, where* $s_i=\varphi(X_i)$ $(1\leqq i\leqq 7)$.

$$(1)\qquad \psi^k(s_1)=\begin{cases} s_1 & (k\equiv 1 \pmod 2)\\ 2s_2 & (k\equiv 2 \pmod 4)\\ 4s_4 & (k\equiv 4 \pmod 8)\\ 0 & (k\equiv 0 \pmod 8)\,.\end{cases}$$

$$(2)\qquad \psi^k(s_2)=\begin{cases} s_2 & (k\equiv 1 \pmod 2)\\ 2s_4 & (k\equiv 2 \pmod 4)\\ 0 & (k\equiv 0 \pmod 4)\,.\end{cases}$$

$$(3)\qquad \psi^k(s_3)=\begin{cases} s_3-2h_3(k+1,k)(s_2+s_3) & (k\equiv 1 \pmod 2)\\ 2s_6-2s_7 & (k\equiv 2 \pmod 8)\\ -4s_4-2s_6+2s_7 & (k\equiv 6 \pmod 8)\\ 0 & (k\equiv 0 \pmod 4).\end{cases}$$

$$(4)\qquad \psi^k(s_4)=\begin{cases} s_4 & (k\equiv 1 \pmod 2)\\ 0 & (k\equiv 0 \pmod 2).\end{cases}$$

$$(5)\qquad \psi^k(s_5)=\begin{cases} s_5-2h_4(k+4,k)(s_4+s_5) & (k\equiv 1 \pmod 4)\\ s_5+s_6-2h_4(k+4,k)(s_4+s_6+s_5) & (k\equiv 3 \pmod 4)\\ 0 & (k\equiv 0 \pmod 2).\end{cases}$$

$$(6)\qquad \psi^k(s_6)=\begin{cases} (1-2h_4(k+5,k+1))\,s_6 & (k\equiv 1 \pmod 2)\\ 0 & (k\equiv 0 \pmod 2).\end{cases}$$

$$(7)\qquad \psi^k(s_7)=\begin{cases} s_7-2h_4(k+4,k)\,s_6 & (k\equiv 1 \pmod 4)\\ 2s_4-s_7+2h_4(k+4,k)\,s_6 & (k\equiv 3 \pmod 4)\\ 0 & (k\equiv 0 \pmod 2).\end{cases}$$

For each integer n with $0\leqq n<m$, we denote the inclusion map of L_8^n into L_8^m by i_n^m, and denote the kernel of the homomorphism

$$(i_n^m)^!\colon \tilde{K}(L_8^m)\to \tilde{K}(L_8^n)$$

by V_n. Then by Proposition 3.8, Lemma 3.6 and (3.5) (2), we obtain the following lemma.

Lemma 3.10. (1) *The group V_n is the subgroup of $\tilde{K}(L_8^m)$ generated by $\{\varphi(X_i(n))\mid 1\leqq i\leqq 7\}$.*
(2) *Let u be a positive integer with $2u<m$. Then we have*

$$\sigma^u\equiv\begin{cases} -\varphi(X_4(2u-2)) & (u\equiv 0 \pmod 4)\\ -\varphi(X_5(2u-2)) & (u\equiv 1 \pmod 4)\\ -\varphi(X_6(2u-2)+X_1(2u-2)) & (u\equiv 2 \pmod 4)\\ \varphi(X_7(2u-2)) & (u\equiv 3 \pmod 4)\end{cases}$$

modulo the subgroup V_{2u}.

Considering the $\boldsymbol{Z}/8$-action on $S^{2n+1}\times\boldsymbol{R}$ given by

$$\exp(2\pi\sqrt{-1}/8)(z,v)=(z\cdot\exp(2\pi\sqrt{-1}/8),-v)$$

for $(z,v)\in S^{2n+1}\times\boldsymbol{R}$, we have a real line bundle

$$\nu\colon (S^{2n+1}\times\boldsymbol{R})/(\boldsymbol{Z}/8)\to L_8^{2n+1}.$$

We set $\kappa=\nu-1\in\widetilde{KO}(L_8^{2n+1})$. It is easy to see that

$$(3.11)\qquad \begin{cases} c(\kappa) = \sigma(2) \\ r(\sigma(2)) = 2\kappa\,, \end{cases}$$

where c: $KO \to K$ is the complexification and r: $K \to KO$ is the real restriction.

Let I: $\tilde{K}(X) \to \tilde{K}(S^2X)$ (resp. I_R: $\widetilde{KO}(X) \to \widetilde{KO}(S^8X)$) be the Bott periodicity isomorphisms for K(resp. KO)-theory. In order to state the next proposition, we set

(3.12) Let j be a non-negative integer with $j \equiv 0 \pmod 4$.

$$(1)\qquad \begin{cases} \tau_i = r(I^{j/2}(\sigma_i)) & (1 \leqq i \leqq 2) \\ \tau_3 = r(I^{j/2}(\sigma_5)) & \\ \tau_4 = \begin{cases} I_R^{j/8}(\kappa) & (j \equiv 0 \pmod 8) \\ r(I^{j/2}(\sigma_4)) & (j \equiv 4 \pmod 8)\,. \end{cases} \end{cases}$$

(2) Let $F(y)$ denote the free abelian group generated by y_1, y_2, y_3 and y_4. Then X_i^j, Y_i^j, $X_i^j(n)$ and $Y_i^j(n)$ $(1 \leqq i \leqq 4,\ n \geqq 0)$ denote the elements of $F(y)$ defined by $Y_4^j = y_4$,

$$\begin{aligned}
Y_1^j &= (-1)^{(j/4)} X_1^j = h_4(j+12, j)\,(2y_1 - y_2 + y_3) + y_4\,, \\
Y_2^j &= X_2^j = h_4(j+12, j)\, y_2 - y_4\,, \\
Y_3^j &= -y_3 - h_4(j, j-12)\, y_4\,, \\
X_i^j &= Y^j_{i+h_4(j+4,j)(7-2i)}\ (3 \leqq i \leqq 4)\,, \\
X_1^j(n) &= (-1)^{(j/4)}\, Y_1^j(n) = 2^{h_1(n+j)-h_1(j)}\, X_1^j\,, \\
X_2^j(n) &= Y_2^j(n) = 2^{[n/4]-h_4(j+4,j)}\, (X_2^j - (-2)^{[n/4]}\, h_4(n+j+7, n+j)\, Y_1^j)\,, \\
X_3^j(n) &= 2^{h_4(n+j-2,j+4)}\, X_3^j + 2^{[n/4]-h_4(j+12,j)}\, h_4(n+j, n+j-2)\, X_2^j \\
&\qquad - 2^{2h_4(n+j+4,j+4)+[n/4]-1}\, h_4(n+j+4, n+j-2)\, X_1^j\,, \\
X_4^j(n) &= 2^{h_4(n+j,j+4)}\, X_4^j + 2^{2[(n-4)/8]}\, h_4(n+j+4, n+j)\, X_2^j \\
&\qquad + 2^{2h_4(n+j,j+4)+[n/4]}\, h_4(n+j+7, n+j)\, X_1^j
\end{aligned}$$

and $Y_i^j(n) = X^j_{i+h_4(j+4,j)(7-2i)}(n)\quad (3 \leqq i \leqq 4)$.

(3) Let μ_j: $F(y) \to \widetilde{KO}(S^jL_8^m)$ be the homomorphism defined by setting $\mu_j(y_i) = \tau_i$ $(1 \leqq i \leqq 4)$.

Proposition 3.13 (Kobayashi [8]). *Let j be a non-negative integer with $j \equiv 0 \pmod 4$. Then the homomorphism μ_j is an epimorphism, and the kernel of μ_j coincides with the subgroup of $F(y)$ generated by $\{Y_i^j(m) \mid 1 \leqq i \leqq 4\}$.*

According to [1] and [4], we have the following lemma.

Lemma 3.14. *Let j be a non-negative integer with $j \equiv 0 \pmod 4$. Then the Adams operations are given by the following formulae, where $t_i = \mu_j(Y_i^j)$*

$(1\leqq i\leqq 4)$ *and* $k\equiv 1 \pmod 2$.

(1) $\psi^k(t_i)=k^{j/2}t_i$ $(i=1, 2$ *or* $4)$.

(2) $\psi^k(t_3)=k^{j/2}(1-2h_4(k+5, k+1))\,t_3$.

By Lemma 3.9 and (3.11), we have the following Lemma.

Lemma 3.15. *Let j be a non-negative integer with $j\equiv 0 \pmod 4$. Then homomorphisms c: $KO\to K$ and r: $K\to KO$ are given by the following formurae, where $s_i=I^{j/2}(\varphi(X_i))$ $(1\leqq i\leqq 7)$ and $t_i=\mu_j(Y_i^j)$ $(1\leqq i\leqq 4)$.*

(1) $r(s_i)=h_4(j, j-12)\,t_i$ $(i=1, 2$ *or* $4)$.

(2) $r(s_3)=-r(s_2)$.

(3) $r(s_5)=-t_3-h_4(j, j-12)\,t_4$.

(4) $r(s_6)=2t_3$.

(5) $r(s_7)=2t_3+h_4(j, j-12)\,t_4$.

(6) $c(t_i)=h_4(j+12, j)\,s_i$ $(i=1, 2$ *or* $4)$.

(7) $c(t_3)=s_6$.

4. Proof for the case $j\equiv 0 \pmod 4$

In this section we prove the parts of the case $j\equiv 0 \pmod 4$ of Theorems 1 and 2. Throughout this section, j denotes a non-negative integer with $j\equiv 0$ (mod 4), and ν the integer defined by

$$\nu=\begin{cases}\nu_2(j) & (j>0)\\ m & (j=0)\,.\end{cases} \tag{4.1}$$

For each integer n with $0\leqq n<m$, we denote the kernel of the homomorphism

$$(i_n^m)^!\colon \widetilde{KO}(S^jL_8^m)\to\widetilde{KO}(S^jL_8^n)$$

by $VO_{m,n}^j$. It follows from Proposition 3.13 that we have

(4.2) *The group $VO_{m,n}^j$ is the subgroup of $\widetilde{KO}(S^jL_8^m)$ generated by*

$$\{\mu_j(Y_i^j(n))\,|\,1\leqq i\leqq 4\}\;,$$

where μ_j: $F(y)\to\widetilde{KO}(S^jL_8^m)$ is the homomorphism defined in (3.12).

By Lemma 3.14, we have the following lemma.

Lemma 4.3. *The Adams operations are given by the following formulae, where $T_i=\mu_j(Y_i^j(n))$ $(1\leqq i\leqq 4)$ and $k\equiv 1 \pmod 2$.*

(1) $\psi^k(T_i)=k^{j/2}\,T_i$ $(i=1, 2$ *or* $4)$.

(2) $\psi^k(T_3) = k^{j/2}(T_3+h_4(k+5, k+1)(-2T_3+\alpha(0,j,n)T_2-\alpha(1,j,n)T_1))$, *where* $\alpha(l,j,n)=h_4(n+h_4(j,j-28l), n+(-2)^{1-l}h_4(j+12, j+16l))$ $(0\leqq l\leqq 1)$.

We set

$$UO^j_{m,n} = \sum_k \left(\bigcap_e k^e(\psi^k-1)VO^j_{m,n}\right), \tag{4.4}$$

where the intersection runs over all non-negative integers e. Since the order of $VO^j_{m,n}$ is equal to a power of 2, we have

$$UO^j_{m,n} = \sum_{k\,:\,\mathrm{odd}}(\psi^k-1)VO^j_{m,n}\,.$$

It follows from Lemmas 3.1 and 4.3 that we have

(4.5) *The group* $UO^j_{m,n}$ *is the subgroup of* $VO^j_{m,n}$ *generated by*

$$\{2^{\nu+1}T_i \mid i=1, 2 \text{ or } 4\} \cup \{R\}\,,$$

where $T_i=\mu_j(Y^j_i(n))$ $(1\leqq i\leqq 4)$,

$$R = 2(2^\nu-1)T_3+\alpha(0,j,n)T_2-\alpha(1,j,n)T_1$$

and $\alpha(l,j,n)$ *is the integer defined in Lemma* 4.3 $(0\leqq l\leqq 1)$.

4.1. Proof for the case $n\not\equiv 3 \pmod 4$. Suppose that $n\not\equiv 3 \pmod 4$. According to [13], we have the exact sequence

$$0 \to \widetilde{KO}(S^j(L^m_8/L^n_8)) \to \widetilde{KO}(S^jL^m_8) \xrightarrow{(i^m_n)^!} \widetilde{KO}(S^jL^n_8) \to 0\,.$$

Hence we have

$$\widetilde{KO}(S^j(L^m_8/L^n_8)) \cong VO^j_{m,n}\,.$$

Since the order of $VO^j_{m,n}$ is finite, we have

$$\tilde{J}(S^j(L^m_8/L^n_8)) \cong VO^j_{m,n}/UO^j_{m,n}\,.$$

It follows from Proposition 3.13 that we have

$$VO^j_{m,n} \cong \langle\{w_i \mid 1\leqq i\leqq 4\}\rangle/\langle\{R_i \mid 1\leqq i\leqq 4\}\rangle\,,$$

where $w_i = X^j_i(n)$ $(1\leqq i\leqq 4)$, $R_1 = 2^{h_1(m+j)-h_1(n+j)}w_1$,

$R_2 = 2^{h_3(m,n)+a_4(n+j,n+j)}(\tilde{w}_2+2^{h_3(m,n)}w_1)$,

$R_3 = 2^{a_3(m+j,n+j)}(\tilde{w}_3+2^{a_4(m+j+4,n+j)}(\tilde{w}_2-2^{a_7(m+j,n+j)+1}w_1))$,

$R_4 = 2^{a_4(m+j,n+j)}(\tilde{w}_4+2^{h_4(m+j,n+j+4)}(\tilde{w}_2+2^{a_7(m+j-4,n+j)+2}w_1))$,

$\tilde{w}_2 = h_4(n+j+15, n+j)w_2-(-1)^{[(n+j)/4]}a_4(n+j, n+j)w_1$,

$\tilde{w}_3 = h_4(n+j+12, n+j-2)w_3-h_4(n+j, n+j-2)w_2+a_4(n+j+12, n+j)w_1$

and $\tilde{w}_4 = h_4(n+j+15, n+j)w_4-h_4(n+j+4, n+j)w_2+a_4(n+j+4, n+j)w_1$.

Suppose that $m \geqq 4\,[(n+j+7)/8]+4\,[(n-j+12)/8]$, and set $A_i = R_i\ (3 \leqq i \leqq 4)$,

$$\begin{aligned}
A_1 &= R_1-(-1)^{[(n+j)/4]}\, 2^{h_1(m+j)-2[(n+j)/4]-h_3(m,n)}\, R_2 \\
&\quad -h_4(n+j, n+j-2)\, 2^{a_1(m+j,n+j)}\, (2^{h_4(n+j,m+j+6)}\, R_3-2^{h_4(n+j,m+j)-1}\, R_2) \\
&\quad +h_4(n+j+4, n+j)\, 2^{a_1(m+j,n+j)}\, (2^{h_4(n+j,m+j)}\, R_4-2^{h_4(n+j,m+j+4)}\, R_2), \\
A_2 &= R_2+h_4(n+j-1, n+j-4)\, 2^{h_4(m+j+4,n+j)}\, R_4 \\
&\quad +h_4(n+j+4, n+j)\, 2^{h_3(m+j,n+j)-h_4(m+j-2,n+j)}\, R_3, \\
u_1 &= -(-1)^{[(n+j)/4]}\, 2^{h_3(m,n)+a_4(n+j,n+j)-1}\, w_1 \\
&\quad -h_4(n+j-2, n+j-4)\, (w_2+2^{h_3(m,n)-2}\, w_1) \\
&\quad -h_4(n+j, n+j-2)\, (w_3-2^{a_4(m+j-4,n+j)}\, (2^{h_3(m,n)}+2^{a_7(m+j,n+j)+1})\, w_1) \\
&\quad +h_4(n+j+4, n+j)\, (w_4-2^{h_4(m+j,n+j)-2}\, (2^{h_3(m,n)}-2^{2h_4(m+j,n+j)})\, w_1), \\
u_2 &= w_2+2^{h_3(m,n)+a_4(n+j,n+j)}\, w_1 \\
&\quad +h_4(n+j-1, n+j-4)\, (w_4+2^{a_4(m+j,n+j)}\, (\tilde{w}_2+2^{2h_4(m+j,n+j)+1}\, w_1)) \\
&\quad +h_4(n+j+4, n+j)\, (w_3+2^{a_4(m+j-4,n+j)}\, (\tilde{w}_2-2^{a_7(m+j,n+j)+1}\, w_1)), \\
u_3 &= \tilde{w}_3+2^{a_4(m+j+4,n+j)}\, (\tilde{w}_2-2^{a_7(m+j,n+j)+1}\, w_1)
\end{aligned}$$

and $u_4 = \tilde{w}_4+2^{h_4(m+j,n+j+4)}\, (\tilde{w}_2+2^{a_7(m+j-4,n+j)+2}\, w_1)$. Then we have

$$\begin{aligned}
A_i &= 2^{a_i(m+j,n+j)}\, u_i\ \ (1 \leqq i \leqq 4), \\
w_1 &= h_4(n+j, n+j-2)\, h_4(n+j+15, n+j)\, (2u_1+u_2+u_3-2^{h_4(m+j+4,n+j)}\, u_2) \\
&\quad +h_4(n+j-1, n+j-2)\, (2^{h_4(m+j+4,n+j)}-1)\, u_4 \\
&\quad +h_4(n+j-2, n+j-4)\, (2u_1+2u_2-u_4) \\
&\quad +h_4(n+j+4, n+j)\, (4u_1-2u_2+u_3-2u_4+2^{h_4(m+j,n+j)}(2u_2-u_3))
\end{aligned}$$

and

$$\begin{aligned}
w_2 = &-2^{h_3(m,n)+a_4(n+j,n+j)}\, w_1-h_4(n+j-2, n+j-4)\, u_1+h_4(n+j, n+j-1)\, u_2 \\
&-h_4(n+j-1, n+j-2)\, (2u_1+u_3-2^{h_4(m+j-4,n+j)}\, (2u_2-u_4)) \\
&+h_4(n+j+4, n+j)\, (2u_1-u_4+2^{a_4(m+j,n+j)}\, (2u_2-u_3)).
\end{aligned}$$

This implies that $\langle \{w_i \mid 1 \leqq i \leqq 4\} \rangle = \langle \{u_i \mid 1 \leqq i \leqq 4\} \rangle$ and

$$VO^j_{m,n} \cong \langle \{u_i \mid 1 \leqq i \leqq 4\} \rangle / \langle \{A_i \mid 1 \leqq i \leqq 4\} \rangle \cong \oplus_{i=1}^4 \boldsymbol{Z}/2^{a_i(m+j,n+j)}\,.$$

Suppose that $4[(n+j+7)/8]+4[(n-j+12)/8] > m \geqq h_1(n+j)+2[(n-j+4)/8]+2[(n-j+6)/8]+1$, and set $A_i = R_i\ (2 \leqq i \leqq 4)$, $u_{2i} = R_{2i}\ (1 \leqq i \leqq 2)$,

$$A_1 = \begin{cases} 2^{h_3(m+3,n)}(2R_4-R_2)-R_1 & (h_4(n+j, n+j-4)=0) \\ R_1-4R_2+8R_3 & (n+j \equiv 1 \pmod 8 \text{ and } m \geqq n+3) \\ R_1+4R_2 & (n+j \equiv 2 \pmod 8)) \\ R_1 & (h_3(m, n)=0)\,, \end{cases}$$

$$u_1 = \begin{cases} w_4+w_1 & (h_4(n+j, n+j-4)=0) \\ w_3-2w_1 & (n+j \equiv 1 \pmod 8 \text{ and } m \geqq n+3) \\ w_2+2w_1 & (n+j \equiv 2 \pmod 8)) \\ w_1 & (h_3(m, n)=0) \end{cases}$$

and

$$u_3=\begin{cases} w_3+w_2-w_1 & (h_4(n+j,n+j-4)=0) \\ 2w_3+w_2-3w_1 & (n+j\equiv 1 \pmod 8 \text{ and } m\geqq n+3) \\ w_3+2w_2-3w_1 & (n+j\equiv 2 \pmod 8) \\ w_3 & (h_3(m,n)=0)\,. \end{cases}$$

Then we have $A_i=u_i$ ($i=2$ or 4),

$$A_1=\begin{cases} 2^{a_1(m+j,n+j)}\,u_1 & (m\geqq 4\,[n/4]+4) \\ 2u_1 & (n+j\equiv 0 \pmod 8) \\ u_1 & (n+j\equiv 1 \pmod 8 \text{ and } n+3>m>n)\,, \end{cases}$$

$$A_3=\begin{cases} 2u_3 & (h_4(n+j-2,n+j-4)=[(m+j)/2]-4\,[(n+j+4)/8]-1=0) \\ u_3 & (\text{otherwise})\,, \end{cases}$$

$$w_1=\begin{cases} 2u_4-u_2-4u_1 & (h_4(n+j,n+j-4)=0) \\ 4u_1+u_2-2u_3 & (n+j\equiv 1 \pmod 8 \text{ and } m\geqq n+3) \\ 2u_1-u_2 & (n+j\equiv 2 \pmod 8) \\ u_1 & (h_3(m,n)=0) \end{cases}$$

and

$$w_2=\begin{cases} 2u_1+u_2-u_4 & (h_4(n+j,n+j-4)=0) \\ -6u_1-u_2+3u_3 & (n+j\equiv 1 \pmod 8 \text{ and } m\geqq n+3) \\ 2u_2-3u_1 & (n+j\equiv 2 \pmod 8) \\ u_2-u_1 & (h_3(m,n)=0)\,. \end{cases}$$

This implies that $\langle\{w_i\,|\,1\leqq i\leqq 4\}\rangle=\langle\{u_i\,|\,1\leqq i\leqq 4\}\rangle$ and

$$VO^j_{m,n}\cong\begin{cases} \boldsymbol{Z}/16\oplus\boldsymbol{Z}/2 & (h_4(n+j,n+j-4)=[m/2]-2\,[n/4]-3=0) \\ \boldsymbol{Z}/2^{a_1(m+j,n+j)} & (4\,[(n+j+15)/8]+2\,[(n-j)/4]>m\geqq 4\,[n/4]+4) \\ \boldsymbol{Z}/2\oplus\boldsymbol{Z}/2 & (n+j\equiv 0 \pmod 8 \text{ and } [m/2]=(n/2)+1) \\ \boldsymbol{Z}/2 & (h_4(n+j+6,n+j)=[m/2]-[(n+1)/2]=0)\,. \end{cases}$$

If $h_1(n+j)+2\,[(n-j+4)/8]+2\,[(n-j+6)/8]\geqq m>n$, then we have $VO^j_{m,n}\cong 0$. Thus the proof for the case $j\equiv 0 \pmod 4$ and $n\not\equiv 3 \pmod 4$ of Theorem 1 is completed.

Consider the case $j\equiv 0 \pmod 8$. It follows from (4.5) that we have

$$\mu_j^{-1}(UO^j_{m,n})=\langle\{2^{\nu+h_3(i+5,i)}\,w_i\,|\,1\leqq i\leqq 4\}\cup\{R_i\,|\,0\leqq i\leqq 4\}\rangle\,,$$

where ν is the integer defined by (4.1) and

$$R_0=2\,(2^\nu-1)\,w_3+h_4(n,n-2)\,w_2-h_4(n+4,n-1)\,w_1\,.$$

Suppose that $m\geqq 4\,[(n+15)/8]+2\,[(n+6)/8]+2\,[n/8]$, and set

$$\begin{aligned} v_1 &= h_4(n-2, n-4)\, w_2 + h_4(n, n-2)\, w_3 + h_4(n+4, n)\, w_4\,, \\ v_2 &= h_4(n-1, n-2)\,(w_4 + 2w_3 + 2^{h_3(m,n)+1}(1-2^{h_4(m+6,n)})\, w_3 + 2^{\nu}\, w_3) \\ &\quad + h_4(n-2, n-4)\,(w_4 + w_2 + 2^{h_3(m,n)}\, w_2) + h_4(n+4, n)\,(w_3 + 2w_4 - 2^{h_3(m,n)+1}\, w_4) \\ &\quad + h_4(n, n-1)\,(w_2 + 2^{h_3(m,n)+1}\, w_3 + 2^{\nu}\, w_3)\,, \\ v_3 &= h_4(n, n-1)\,(w_4 + 2^{h_4(m,n)}\, w_2 - 2^{3h_4(m,n)+2}\, w_3) \\ &\quad + h_4(n-1, n-2)\,(2w_4 - w_1 + 2^{h_4(m,n)+1}(w_4 + 2w_3) - 2^{3h_4(m,n)+3}\, w_3) \\ &\quad + h_4(n-2, n-4)\,(2w_4 - w_1 + 2^{h_4(m,n)+1}(w_4 + w_2) - 2^{3h_4(m,n)+2}\, w_2) \\ &\quad + h_4(n+4, n)\,(2w_4 - w_2 + 2^{h_4(m,n)}(w_3 + 2w_4) + 2^{3h_4(m,n)+1}\, w_4) \end{aligned}$$

and $v_4 = R_0 + h_4(n-2, n-3)\,(w_3 - R_0)$. Then we have $\langle\{w_i \mid 1 \leqq i \leqq 4\}\rangle = \langle\{v_i \mid 1 \leqq i \leqq 4\}\rangle$ and

$$VO^j_{m,n}/UO^j_{m,n} \cong \begin{cases} (\oplus_{i=1}^3 \boldsymbol{Z}/2^{b_i(j,m,n)}) \oplus \boldsymbol{Z}/2 & (n \equiv 2 \pmod 8) \\ \oplus_{i=1}^3 \boldsymbol{Z}/2^{b_i(j,m,n)} & (\text{otherwise})\,. \end{cases}$$

Suppose that $4\,[(n+15)/8] + 2\,[(n+6)/8] + 2\,[n/8] > m \geqq 4\,[(n+14)/8] + 4\,[n/8]$, and set

$$\begin{aligned} v_1 &= h_4(n-2, n-3)\, w_2 + h_4(n-1, n-2)\, w_3 + h_4(n+4, n)\, w_4\,, \\ v_2 &= h_4(n-2, n-3)\,(w_4 + 5w_2) + h_4(n-1, n-2)\,(w_4 - 2w_3) \\ &\quad + h_4(n+4, n)\,(w_3 - 2w_4)\,, \\ v_3 &= h_4(n-2, n-3)\,(2w_4 - w_1 - 16w_2) + h_4(n-1, n-2)\,(w_1 - 4w_3) \\ &\quad + h_4(n+4, n)\,(w_2 + 6w_4) \end{aligned}$$

and $v_4 = R_0 + h_4(n-2, n-3)\,(w_3 + 2w_2 - 3w_1 - R_0)$. Then we have $\langle\{w_i \mid 1 \leqq i \leqq 4\}\rangle = \langle\{v_i \mid 1 \leqq i \leqq 4\}\rangle$ and

$$VO^j_{m,n}/UO^j_{m,n} \cong \begin{cases} \boldsymbol{Z}/2^{b_1(j,m,n)} \oplus \boldsymbol{Z}/4 & (n \equiv 2 \pmod 8) \\ \boldsymbol{Z}/8 & (n \equiv 1 \pmod 8) \\ VO^j_{m,n} & (h_4(n, n-4) = 0)\,. \end{cases}$$

If $4\,[(n+14)/8] + 4\,[n/8] > m > n$, then we have $UO^j_{m,n} \cong 0$. Thus the proof for the case $j \equiv 0 \pmod 8$ and $n \not\equiv 3 \pmod 4$ of Theorem 2 is completed.

Consider the case $j \equiv 4 \pmod 8$. It follows from (4.5) that we have

$$\mu_j^{-1}(UO^j_{m,n}) = \langle\{2^{3+h_3(i,i-1)}\, w_i \mid 1 \leqq i \leqq 4\} \cup \{R_i \mid 0 \leqq i \leqq 4\}\rangle\,,$$

where $R_0 = 2w_4 - h_4(n+3, n)\,(8w_4 + w_1) - h_4(n, n-4)\,(8w_4 + w_2)$. Suppose that $m \geqq 2\,[n/4] + 4\,[(n+2)/8] + 6$, and set

$$\begin{aligned} v_1 &= h_4(n, n-4)\, w_4 + h_4(n+4, n+2)\, w_3 + h_4(n+2, n)\, w_2\,, \\ v_2 &= h_4(n, n-4)\,(w_3 - 2w_4) + h_4(n+4, n+3)\,(w_2 + w_4 - 2^{h_3(m,n)+1}\, w_3) \\ &\quad + h_4(n+3, n+2)\,(w_4 - 2w_3) + h_4(n+2, n)\,(w_4 + w_2)\,, \end{aligned}$$

$$
\begin{aligned}
v_3 = {} & h_4(n, n-4)\,(2w_3+w_1+2^{h_4(m,n)+1}(w_3-2w_4)) \\
& +h_4(n+4, n+3)\,(2w_3-w_2+w_1+2^{h_4(m,n)}\,w_2) \\
& +h_4(n+3, n+2)\,(2w_3-w_2+2^{h_4(m,n)}(w_4-2w_3)) \\
& +h_4(n+2, n)\,(w_3+2^{h_4(m,n)}(w_4+w_2))
\end{aligned}
$$

and $v_4=R_0+h_4(n+4, n+3)\,(w_4-R_0)$. Then we have $\langle\{w_i \mid 1\leqq i\leqq 4\}\rangle=\langle\{v_i \mid 1\leqq i\leqq 4\}\rangle$ and

$$
VO^j_{m,n}/UO^j_{m,n} \cong \begin{cases} (\oplus_{i=1}^3 \boldsymbol{Z}/2^{b_i(j,m,n)})\oplus\boldsymbol{Z}/2 & (n\equiv 4 \pmod 8) \\ \oplus_{i=1}^3 \boldsymbol{Z}/2^{b_i(j,m,n)} & (\text{otherwise}). \end{cases}
$$

If $2\,[n/4]+4\,[(n+2)/8]+6>m>n$, then we have $UO^j_{m,n}\cong 0$. Thus the proof for the case $j\equiv 4 \pmod 8$ and $n\not\equiv 3 \pmod 4$ of Theorem 2 is completed.

4.2. Proof for the case $n\equiv 3$ (mod 4). Now, we turn to the case $n\equiv 3 \pmod 4$. It follows from [13] that we have the following commutative diagram, in which rows are exact.

$$
\begin{array}{ccccccccc}
0 \to & VO^j_{m,n+1} & \xrightarrow{f_1} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_2} & \widetilde{KO}(S^{j+n+1}) & \to 0 \\
& \| & & \downarrow f_3 & & \downarrow & \\
0 \to & VO^j_{m,n+1} & \to & \widetilde{KO}(S^jL_8^m) & \xrightarrow{f_4} & \widetilde{KO}(S^jL_8^{n+1}) & \to 0\,.
\end{array} \tag{4.6}
$$

Since $\widetilde{KO}(S^{j+n+1})$ is isomorphic to $\boldsymbol{Z}$, the upper row of (4.6) splits. Choose $y\in\widetilde{KO}(S^j(L_8^m/L_8^n))$ such that $\beta=f_2(y)$ generates the group $\widetilde{KO}(S^{j+n+1})$. Then we have an isomorphism

$$
f\colon VO^j_{m,n+1}\oplus\widetilde{KO}(S^{j+n+1}) \to \widetilde{KO}(S^j(L_8^m/L_8^n))
$$

defined by $f(x, k\beta)=f_1(x)+ky$ for every $(x, k)\in VO^j_{m,n+1}\oplus\boldsymbol{Z}$. This proves the case $j\equiv n+1\equiv 0 \pmod 4$ of Theorem 1.

Lemma 4.7. *If $j\equiv n+1\equiv 0$ (mod 4), then there is an element $y\in \widetilde{KO}(S^j(L_8^m/L_8^n))$ which satisfies the following conditions:*

(1) *$\beta=f_2(y)$ generates the group $\widetilde{KO}(S^{j+n+1})$,*

(2) $$f_3(y) = \begin{cases} \mu_j(Y_4^j(n)) & (j\equiv n+1\equiv 0 \pmod 8) \\ \mu_j(Y_3^j(n)+Y_1^j(n)) & (j\equiv n+1\equiv 4 \pmod 8) \\ \mu_j(Y_3^j(n)+Y_2^j(n)+2Y_1^j(n)) & (j\equiv n-3\equiv 0 \pmod 8) \\ \mu_j(Y_4^j(n)+Y_2^j(n)) & (j\equiv n-3\equiv 4 \pmod 8)\,. \end{cases}$$

Proof. Consider the following commutative diagram, in which rows are exact:

$$\begin{array}{ccccccccc} 0 & \to & V^j_{m,n+1} & \xrightarrow{f_{C,1}} & \tilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_{C,2}} & \tilde{K}(S^{n+j+1}) & \to & 0 \\ & & \| & & \downarrow f_{C,3} & & \downarrow & & \\ 0 & \to & V^j_{m,n+1} & \longrightarrow & \tilde{K}(S^jL_8^m) & \xrightarrow{f_{C,4}} & \tilde{K}(S^jL_8^{n+1}) & \to & 0\,. \end{array}$$

According to Lemma 3.10, there is an element $x\in\tilde{K}(S^j(L_8^m/L_8^n))$ such that $f_{C,2}(x)$ generates the group $\tilde{K}(S^{n+j+1})$ and

$$f_{C,3}(x)=\begin{cases} I^{j/2}(\varphi(X_4(n))) & (n\equiv 7 \ (\mathrm{mod}\ 8)) \\ I^{j/2}(\varphi(X_6(n)+X_1(n))) & (n\equiv 3 \ (\mathrm{mod}\ 8))\,. \end{cases}$$

If $n+j\equiv 3$ (mod 8), then $r\colon \tilde{K}(S^{n+j+1})\to\widetilde{KO}(S^{n+j+1})$ is an isomorphism. It follows from Lemma 3.15 that $y=r(x)$ satisfies the conditions (1) and (2). If $n+j\equiv 7$ (mod 8), then $c\colon \widetilde{KO}(S^{n+j+1})\to\tilde{K}(S^{n+j+1})$ is an isomorphism and $c\colon \widetilde{KO}(S^jL_8^{n+1})\to\tilde{K}(S^jL_8^{n+1})$ is a monomorphism. There is an element $\bar{y}\in\widetilde{KO}(S^j(L_8^m/L_8^n))$ such that $f_2(\bar{y})$ generates the group $\widetilde{KO}(S^{n+j+1})$ and $f_{C,4}(c(f_3(\bar{y})))=f_{C,4}(f_{C,3}(x))$. It follows from Lemma 3.15 that we have $f_{C,3}(x)=c(Y)$, where

$$Y=\begin{cases} \mu_j(Y_4^j(n)) & (n+1\equiv j\equiv 0 \ (\mathrm{mod}\ 8)) \\ \mu_j(Y_3^j(n)+Y_1^j(n)) & (n+1\equiv j\equiv 4 \ (\mathrm{mod}\ 8))\,. \end{cases}$$

This implies that $f_4(f_3(\bar{y}))=f_4(Y)$ and $y=\bar{y}+f_1(Y-f_3(\bar{y}))$ satisfies the conditions (1) and (2). q.e.d.

In the rest of this section, we fix an element $y\in\widetilde{KO}(S^j(L_8^m/L_8^n))$ which satisfies the conditions of Lemma 4.7, and set

(4.8) $$T_i=\mu_j(Y_i^j(n+1)) \quad (1\leqq i\leqq 4)\,.$$

Lemma 4.9. *If k is an odd integer, then the Adams operation ψ^k is given by*

$$\psi^k(y)=k^{(n+j+1)/2}y+((k^{j/2}-k^{(n+j+1)/2})/8)f_1(8f_3(y))+k^{j/2}f_1(w)\,,$$

where $w=h_4(k+5,k+1)(W-8f_3(y))$ and

(4.10) $$W=8f_3(y)-h_4(n,n-4)(T_3+h_4(j,j-4)(T_3+T_1))\,.$$

Proof. We necessarily have

$$\psi^k(y)=uy+f_1(x)$$

for some integer u and an element $x\in VO^j_{m,n+1}$. By using the ψ-map f_2, we see that $u=k^{(n+j+1)/2}$. Under f_3, $f_1(x)$ maps into x and y maps into $f_3(y)$, and we see that

$$\psi^k(f_3(y)) = k^{(n+j+1)/2} f_3(y) + x .$$

It follows from Lemma 4.3 that

$$k^{j/2}(f_3(y)+w) = k^{(n+j+1)/2} f_3(y) + x .$$

This implies that

$$x = ((k^{j/2} - k^{(n+j+1)/2})/8)\,(8f_3(y)) + k^{j/2}\,w . \qquad \text{q.e.d.}$$

We now recall some definition in [3]. Set $Y = \widetilde{KO}(S^j(L_8^m/L_8^n))$ and let f be a function which assigns to each integer k a non-negative integer $f(k)$. Given such a function f, we define Y_f to be the subgroup of Y generated by

$$\{k^{f(k)}(\psi^k - 1)\,(y) \mid k \in \boldsymbol{Z}, y \in Y\} ;$$

that is,

$$Y_f = \langle \{k^{f(k)}(\psi^k - 1)\,(y) \mid k \in \boldsymbol{Z}, y \in Y\} \rangle .$$

Then the kernel of the homomorphism $J''\colon Y \to J''(Y)$ coincides with $\bigcap_f Y_f$, where the intersection runs over all functions f.

Suppose that f satisfies

$$(4.11) \qquad f(k) \geqq m + \max\{\nu_p(\mathfrak{m}((n+j+1)/2)) \mid p \text{ is a prime divisor of } k\}$$

for every $k \in \boldsymbol{Z}$. For each odd integer i, $N(i)$ denotes the integer chosen to satisfy the property

$$(4.12) \qquad iN(i) \equiv 1 \pmod{2^m} .$$

In the following calculation we put $(n+j+1)/2 = u$ for the sake of simplicity. From Lemma 3.1, (4.5) and Lemma 4.9, we have

$$\begin{aligned}
& k^{f(k)}(\psi^k - 1)\,(y) \\
&\quad = k^{f(k)}(k^u - 1)\,y + (k^{f(k)}(k^{j/2} - k^u)/8)\,f_1(8f_3(y)) + k^{f(k)+(j/2)} f_1(w) \\
&\quad \equiv k^{f(k)}(k^u - 1)\,y + (k^{f(k)}(k^{j/2} - k^u)/8)\,f_1(W) \pmod{f_1(UO^j_{m,n+1})} \\
&\quad = k^{f(k)}(k^u - 1)\,y + (k^{f(k)}\,N(u/2^{\nu_2(u)})\,(u(k^{j/2} - 1) - u\,(k^u - 1))/2^{\nu_2(8u)})\,f_1(W) \\
&\quad \equiv k^{f(k)}(k^u - 1)\,y \\
&\qquad + (k^{f(k)}\,N(u/2^{\nu_2(u)})\,((j/2)\,(k^u - 1) - u(k^u - 1))/2^{\nu_2(8u)})\,f_1(W)\,(2^\nu + 1) \\
&\qquad\qquad \pmod{f_1(UO^j_{m,n+1})} \\
&\quad = (k^{f(k)}(k^u - 1)/2^{\nu_2(4u)})\,(2^{\nu_2(4u)}\,y - N(u/2^{\nu_2(u)})\,((n+1)/4)\,f_1(W)\,(2^\nu + 1)) .
\end{aligned}$$

By virtue of [3; II, Theorem (2.7) and Lemma (2.12)], we have

$$\begin{aligned}
Y_f &= \langle f_1(UO^j_{m,n+1}) \cup \{k^{f(k)}(\psi^k - 1)\,(y) \mid k \in \boldsymbol{Z}\} \rangle \\
&= \langle f_1(UO^j_{m,n+1}) \cup \{\mathfrak{m}(u)\,y - Mf_1(W)\} \rangle ,
\end{aligned}$$

where $M=(\mathfrak{m}(u)/2^{\nu_2(u)+2})\,N(u/2^{\nu_2(u)})\,((n+1)/4)\,(2^\nu+1)$. Since this is true for every function f which satisfies (4.11), we have

$$\text{(4.13)}\qquad J''(Y) \cong Y/\langle f_1(UO^j_{m,n+1}) \cup \{\mathfrak{m}((n+j+1)/2)\,y - Mf_1(W)\}\rangle,$$

where $\nu_2(M)=\nu_2(n+1)-2$ *and*

$$W=\begin{cases} 4T_4+2T_2+T_1 & (j\equiv n+1\equiv 0 \pmod 8)\\ 8T_4+4T_2-T_1 & (j\equiv n-3\equiv 4 \pmod 8)\\ 6T_3+4T_2+4T_1 & (j\equiv n-3\equiv 0 \pmod 8)\\ 3T_3+2T_2+3T_1 & (j\equiv n+1\equiv 4 \pmod 8)\,. \end{cases}$$

Suppose that $m\geqq n+5+2h_4(j+4,j)\,h_4(n+1,n)$. It follows from the proof for the case $n\not\equiv 3 \pmod 4$ that we have

$$W\equiv \Sigma_{i=1}^3 m_i\,z_i \pmod{UO^j_{m,n+1}},$$

where $z_i=\mu_j(v_i)$ $(1\leqq i\leqq 3)$, $\nu_2(m_i)=2+h_4(n+j,n+j-4)-i$ $(1\leqq i\leqq 2)$ and $\nu_2(m_3)=2h_4(j,j-4)$. Therefore

$$J''(Y)\cong F(v)/\langle \{\Sigma_{i=0}^3 M_i\,v_i\} \cup \{B_i \mid 1\leqq i\leqq 4\}\rangle,$$

where $F(v)$ is the free abelian group generated by $\{v_i \mid 0\leqq i\leqq 4\}$, $M_0=\mathfrak{m}((n+j+1)/2)$,

$$B_i = 2^{b_i(j,m,n)}\,v_i \quad (1\leqq i\leqq 3),$$
$$B_4 = (h_4(j+4,j)\,h_4(n+5,n+1)+1)\,v_4$$

and $M_i=-m_i\,M$ $(1\leqq i\leqq 3)$. Set

$$\text{(4.14)}\qquad i_k=\min\{b_k(j,m,n),\ \nu_2(n+1)+\nu_2(m_k)-2\}\quad (1\leqq k\leqq 3)\,.$$

For the sake of simplicity, we put $b_k=b_k(j,m,n)$ $(1\leqq k\leqq 3)$ in the following calculation. Since $\nu_2(M)=\nu_2(n+1)-2$, the greatest common divisor of M_k and 2^{b_k} is equal to 2^{i_k} $(1\leqq k\leqq 3)$. Choose integers e_{11}, e_{12}, e_{21}, e_{22}, e_{31} and e_{32} with $e_{k1}\,2^{b_k}+e_{k2}\,M_k=2^{i_k}$ $(1\leqq k\leqq 3)$. If $b_1-i_1>b_2-i_2$ and $i_2\leqq \nu_2(M_3)$, then we have

$$A_1\begin{pmatrix} \Sigma_{i=0}^3 M_i\,v_i\\ B_1\\ B_2\\ B_3\end{pmatrix}=\begin{pmatrix} 2^{b_1-i_1}M_0\,v_0\\ 2^{b_2-i_2}(e_{12}\,M_0\,v_0+2^{i_1}v_1)\\ e_{22}(M_0\,v_0+M_1\,v_1+M_3\,v_3)+2^{i_2}v_2\\ B_3\end{pmatrix},$$

where

$$A_1=\begin{pmatrix} 2^{b_1-i_1} & -M_1/2^{i_1} & -2^{b_1-b_2-i_1}M_2 & -2^{b_1-b_3-i_1}M_3\\ e_{12}\,2^{b_2-i_2} & e_{11}\,2^{b_2-i_2} & -e_{12}\,M_2/2^{i_2} & -2^{b_2-b_3-i_2}e_{12}\,M_3\\ e_{22} & 0 & e_{21} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and det $A_1=1$. This implies that

$$J''(Y)\cong\begin{cases}\boldsymbol{Z}/2^{b_1-1}M_0\oplus\boldsymbol{Z}/2^{b_2+1}\oplus\boldsymbol{Z}/2^{b_3}\oplus\boldsymbol{Z}/2 & (j\equiv n+1\equiv 4\ (\mathrm{mod}\ 8))\\ \boldsymbol{Z}/2^{b_1-i_1}M_0\oplus\boldsymbol{Z}/2^{b_2-i_2+i_1}\oplus\boldsymbol{Z}/2^{i_2}\oplus\boldsymbol{Z}/2^{b_3} & (\text{otherwise}).\end{cases}$$

If $b_2-i_2\geqq b_1-i_1$ and $\nu_2(M_3)\geqq i_2$, then we have

$$A_2\begin{pmatrix}\sum_{i=0}^3 M_i v_i\\ B_1\\ B_2\\ B_3\end{pmatrix}=\begin{pmatrix}2^{b_2-i_2}M_0 v_0\\ B_1\\ e_{22}(M_0 v_0+M_1 v_1+M_3 v_3)+2^{i_2}v_2\\ B_3\end{pmatrix},$$

where

$$A_2=\begin{pmatrix}2^{b_2-i_2} & -M_1 2^{b_2-b_1-i_2} & -M_2/2^{i_2} & -2^{b_2-b_3-i_2}M_3\\ 0 & 1 & 0 & 0\\ e_{22} & 0 & e_{21} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and det $A_2=1$. This implies that

$$J''(Y)\cong\begin{cases}\boldsymbol{Z}/2^{b_2}M_0\oplus\boldsymbol{Z}/2^{b_1}\oplus\boldsymbol{Z}/2^{b_3}\oplus\boldsymbol{Z}/2 & (j\equiv n+1\equiv 4\ (\mathrm{mod}\ 8))\\ \boldsymbol{Z}/2^{b_2-i_2}M_0\oplus\boldsymbol{Z}/2^{b_1}\oplus\boldsymbol{Z}/2^{i_2}\oplus\boldsymbol{Z}/2^{b_3} & (\text{otherwise}).\end{cases}$$

If $i_2>\nu_2(M_3)$, then we necessarily have $j\equiv n-3\equiv 4\ (\mathrm{mod}\ 8)$, $b_1=4$, $i_1\geqq 3$, $b_2=3$ and $i_2=i_1-1=\nu_2(M_3)+1$. If $i_2>\nu_2(M_3)$ and $b_2-i_2>b_3-i_3$, then we have $i_2=2$, $b_3=i_3$ and

$$A_3\begin{pmatrix}\sum_{i=0}^3 M_i v_i\\ B_1\\ B_2\\ B_3\end{pmatrix}=\begin{pmatrix}2M_0 v_0\\ B_1\\ e_{22}(M_0 v_0+M_1 v_1)+4v_2\\ B_3\end{pmatrix},$$

where

$$A_3=\begin{pmatrix}2 & -M_1/8 & -M_2/4 & -2M_3/2^{b_3}\\ 0 & 1 & 0 & 0\\ e_{22} & 0 & e_{21} & -e_{22}M_3/2^{b_3}\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and det $A_3=1$. This implies that

$$J''(Y)\cong\boldsymbol{Z}/2M_0\oplus\boldsymbol{Z}/16\oplus\boldsymbol{Z}/4\oplus\boldsymbol{Z}/2^{b_3}.$$

If $i_2>\nu_2(M_3)$ and $b_3-i_3\geqq b_2-i_2$, then we have

$$A_4\begin{pmatrix}\sum_{i=0}^{3} M_i v_i \\ B_1 \\ B_2 \\ B_3\end{pmatrix}=\begin{pmatrix}2^{b_3-i_3} M_0 v_0 \\ B_1 \\ B_2 \\ e_{32}(M_0 v_0+M_1 v_1+M_2 v_2)+2^{i_3} v_3\end{pmatrix},$$

where

$$A_4=\begin{pmatrix}2^{b_3-i_3} & -M_1 2^{b_3-i_3-4} & -M_2 2^{b_3-i_3-3} & -M_3/2^{i_3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e_{32} & 0 & 0 & e_{31}\end{pmatrix}$$

and $\det A_4=1$. This implies that

$$J''(Y) \cong \boldsymbol{Z}/2^{b_3-i_3}M_0 \oplus \boldsymbol{Z}/16 \oplus \boldsymbol{Z}/8 \oplus \boldsymbol{Z}/2^{i_3}.$$

If $n+7>m\geqq n+5$ and $j\equiv n-3\equiv 4 \pmod 8$, then we have $W\equiv m_1T_3$ (mod $UO^j_{m,n+1}$), where $\nu_2(m_1)=2$. Therefore

$$J''(Y) \cong F(v)/\langle\{M_0 v_0+M_1 v_1, B_1\}\rangle,$$

where $F(v)$ is the free abelian group generated by $\{v_i \mid 0\leqq i\leqq 1\}$, $M_0=\mathfrak{m}((n+j+1)/2)$, $B_1=2^b v_1$, $b=b_1(j,m,n)$ and $M_1=-m_1M$. Set $i=\min\{b, \nu_2(n+1)\}$. Choose integers e_1 and e_2 with $e_1 2^b+e_2 M_1=2^i$. Then we have

$$\begin{pmatrix}2^{b-i} & -M_1/2^i \\ e_2 & e_1\end{pmatrix}\begin{pmatrix}M_0 v_0+M_1 v_1 \\ B_1\end{pmatrix}=\begin{pmatrix}2^{b-i} M_0 v_0 \\ e_2 M_0 v_0+2^i v_1\end{pmatrix}.$$

This implies that $J''(Y)\cong\boldsymbol{Z}/2^{b-i}M_0\oplus\boldsymbol{Z}/2^i$.

If $n+5>m>n$, then we have $Mf_1(W)\equiv 0$ (mod $f_1(UO^j_{m,n})$) and

$$J''(Y) \cong \boldsymbol{Z}/\mathfrak{m}((n+j+1)/2)\oplus(VO^j_{m,n+1}/UO^j_{m,n+1}).$$

According to [2], [3] and [12], we have $\tilde{J}(S^j(L_8^m/L_8^n))\cong J''(Y)$. Thus, the proof for the parts of the case $n+1\equiv j\equiv 0 \pmod 8$ of Theorem 2 is completed.

5. Proof for the case $j\equiv 2 \pmod 4$

In this section we prove the parts of the case $j\equiv 2 \pmod 4$ of Theorems 1 and 2. Throughout this section j denotes a positive integer with $j\equiv 2 \pmod 4$. Consider the elements S_i $(1\leqq i\leqq 7)$ of $\tilde{K}(S^j L_8^m)$ defined by

$$S_i = I^{j/2}(\varphi(X_i(n+1))) \quad (1\leqq i\leqq 7), \tag{5.1}$$

where $\varphi\colon F(x)\to\tilde{K}(L_8^m)$ is the homomorphism defined in (3.7). For each integer n with $0\leqq n\leqq m$, we denote the kernel of the homomorphism

$$(i_n^m)^!\colon \tilde{K}(S^j L_8^m) \to \tilde{K}(S^j L_8^n)$$

by $V_{m,n}^j$. By Proposition 3.8, we have

(5.2) $$V_{m,2[(n+1)/2]}^j = \langle\{S_i \mid 1 \leqq i \leqq 7\}\rangle .$$

Consider the Bott exact sequence (cf. [5] and [6, (12.2)])

(5.3) $$\to \widetilde{KO}(S^{j+2}X) \xrightarrow{c} \tilde{K}(S^{j+2}X) \xrightarrow{r \circ I^{-1}} \widetilde{KO}(S^j X) \xrightarrow{\partial} \widetilde{KO}(S^{j-1}X) \to$$

for $X = L_8^m / L_8^n$, where ∂ is the homomorphism defined by the exterior product with the generator of $\widetilde{KO}(S^1)$. Using the isomorphisms

$$VO_{m,2[(n+1)/2]}^{j+2} \cong \widetilde{KO}(S^{j+2}(L_8^m / L_8^{2[(n+1)/2]}))$$

and

$$V_{m,2[(n+1)/2]}^j \cong \tilde{K}(S^j(L_8^m / L_8^{2[(n+1)/2]})) ,$$

we obtain the exact sequence

(5.4) $$\to VO_{m,2u}^{j+2} \xrightarrow{I^{-1}\circ c} V_{m,2u}^j \xrightarrow{r_1} \widetilde{KO}(S^j(L_8^m / L_8^{2u})) \xrightarrow{\partial} G \to 0 ,$$

where $u = [(n+1)/2]$ and

$$G = \begin{cases} \widetilde{KO}(S^{j+1}(L_8^m / L_8^{2u})) & (m+j \equiv 0,\ 1 \text{ or } 2 \pmod 8) \\ 0 & (\text{otherwise}) . \end{cases}$$

Using Lemma 3.15, we obtain the following lemma.

Lemma 5.5. *For the homomorphism r_1 in the exact sequence* (5.4), *we have*

$$\operatorname{Ker} r_1 = \begin{cases} \langle\{S_1, S_2, S_4, S_6\}\rangle & (2u+j \equiv 4 \text{ or } 6 \pmod 8) \\ \langle\{2S_1, S_2, S_4, S_6\}\rangle & (2u+j \equiv 2 \pmod 8) \\ \langle\{S_1, S_2, S_4, 2S_6\}\rangle & (2u \equiv j-4 \equiv 2 \pmod 8) \\ \langle\{S_1, S_2, 2S_4, S_6\}\rangle & (2u \equiv j-4 \equiv 6 \pmod 8) , \end{cases}$$

where $u = [(n+1)/2]$.

It follows from Lemmas 3.9 and 5.5 that we have

Lemma 5.6. *The Adams operations are given by the following formulae, where* $u = [(n+1)/2]$, $T_i = r_1(S_i)$ $(1 \leqq i \leqq 7)$ *and* $k \equiv 1 \pmod 2$.

(1) $$\psi^k(T_3) = (-1)^{h_3(k+2,k)} k^{j/2}\, T_3 .$$

(2) $$\psi^k(T_5) = (-1)^{h_4(k+4,k)} k^{j/2}\, \tilde{T}_5 ,$$

where $\tilde{T}_5 = T_5 - h_4(k+5, k+1)\,(h_3(u+1, u-1)\, T_3 + h_4(j+4, j)\, h_3(u+2, u+1)\, T_1)$.

(3) $$\psi^k(T_7) = (-1)^{h_3(k+2,k)} k^{j/2} \tilde{T}_7,$$

where $\tilde{T}_7 = T_7 - h_4(k+5, k+1)\, h_4(j+4, j)\, h_3(u+2, u+1)\, T_1$.

(4) $$\psi^k(T_i) = T_i \quad (i = 1, 2, 4 \text{ or } 6).$$

We set

(5.7) $$U^j_{m,2u} = \sum_{k\,:\,\mathrm{odd}} (\psi^k - 1)\, r_1(V^j_{m,2u}).$$

It follows from Lemma 5.6 that we have

(5.8) $$U^j_{m,2u} = \langle \{4T_3, R_1, R_2\} \rangle,$$

where $u = [(n+1)/2]$,

$$R_1 = 2T_5 + h_3(u+1, u-1)\, T_3 - h_4(j+4, j)\, h_3(u+2, u+1)\, T_1,$$
$$R_2 = 4T_7 - h_4(j+4, j)\, h_3(u+2, u+1)\, T_1$$

and $T_i = r_1(S_i)$ $(1 \leqq i \leqq 7)$.

5.1. Proof for the case $n \equiv 0$ (mod 2). Suppose $n \equiv 0$ (mod 2). If $m \geqq n+2$, then by Proposition 3.8 and Lemma 5.5, we have

$$r_1(V^j_{m,n}) \cong \langle \{w_i \mid 0 \leqq i \leqq 3\} \rangle / \langle \{R_i \mid 4 \leqq i \leqq 7\} \rangle,$$

where $w_0 = h_3(n, n-2)\, X_1(n) + h_4(n+6, n+4)\, X_6(n) + h_4(n+2, n)\, X_4(n)$,

$$w_1 = (1 - 2h_4(n+6, n+2))\, X_3(n),$$
$$w_2 = h_4(n+2, n-2)\, X_5(n) + h_4(n+6, n+2)\, X_7(n),$$
$$w_3 = h_4(n+6, n+2)\, X_5(n) + h_4(n+2, n-2)\, X_7(n),$$
$$R_4 = h_4(n+j, n+j-12)\, w_0,$$
$$R_5 = 2^{h_3(m+j, n+j)-1}\, \tilde{w}_1,$$
$$R_6 = 2^{a_8(m+j, n+j)}(\tilde{w}_2 + 2^{a_8(m+j, n+j)}\, \tilde{w}_1),$$
$$R_7 = 2^{a_9(m+j, n+j)-1}(\tilde{w}_3 - 2^{a_9(m+j, n+j)}\, \tilde{w}_1),$$
$$\tilde{w}_1 = (1 - 2h_4(n+6, n+2))\, (2X_3(n) - h_4(n+j+6, n+j+4)\, w_0),$$
$$\tilde{w}_2 = 2w_2 - w_1 + h_4(j, j-4)\, h_4(n, n-2)\, w_0,$$
$$\tilde{w}_3 = 2w_3 - h_4(n+j+6, n+j+4)\, w_0,$$
$$a_8(m+j, n+j) = \Sigma^1_{i=0}\, h_4(n+j+4i, n+j+4i-4)\, h_4(m+j+4i-4, n+j+4i)$$

and $a_9(m+j, n+j) = a_8(m+j+8, n+j+4)$. If $m \geqq n+2+12h_4(n+j+6, n+j+4) + 2h_3(n+2, n)$, then we have

$$r_1(V^j_{m,n}) \cong \langle \{u_i \mid 0 \leqq i \leqq 3\} \rangle / \langle \{A_i \mid 0 \leqq i \leqq 3\} \rangle,$$

where $u_0 = w_0$, $u_1 = w_2$, $A_0 = R_4$, $A_1 = 2^{h_3(m+j, n+j)+1}\, u_1$,

$$A_i = 2^{a_{i+6}(m+j,n+j)} u_i \quad (2 \leqq i \leqq 3),$$
$$u_2 = 2w_2 - w_1 + 2^{a_8(m+j,n+j)+2} w_2$$

and $u_3 = w_3 - 2^{a_9(m+j,n+j)+1} w_2$. If $n+2+12h_4(n+j+6, n+j+4)+2h_3(n+2, n) > m \geqq n+2+2h_4(n+j+4, n+j+2)$, then we have

$$r_1(V^j_{m,n}) \cong \langle \{u_i \mid 0 \leqq i \leqq 3\} \rangle / \langle \{A_i \mid 0 \leqq i \leqq 3\} \rangle,$$

where $u_1 = w_2$,

$$u_0 = \begin{cases} 2w_3 - 4w_1 - w_0 & (n+j \equiv 2 \ (\mathrm{mod}\ 8) \text{ and } n+14 > m \geqq n+6) \\ 2w_1 - w_0 & (n+j \equiv 2 \ (\mathrm{mod}\ 8) \text{ and } n+6 > m \geqq n+2) \\ w_0 & (n+j \equiv 0 \ (\mathrm{mod}\ 8) \text{ and } n+4 > m \geqq n+2), \end{cases}$$

$$u_2 = \begin{cases} 2w_2 - w_1 & (n+j \equiv 2 \ (\mathrm{mod}\ 8) \text{ and } n+14 > m \geqq n+10) \\ 2w_2 + w_1 - h_4(n+4, n)\, w_0 & (n+j \equiv 2 \ (\mathrm{mod}\ 8) \text{ and } n+10 > m \geqq n+2) \\ w_1 & (n+j \equiv 0 \ (\mathrm{mod}\ 8) \text{ and } n+4 > m \geqq n+2), \end{cases}$$

$$u_3 = \begin{cases} w_3 - w_1 - h_4(n+4, n)\, w_0 & (n+j \equiv 2 \ (\mathrm{mod}\ 8) \text{ and } n+6 > m \geqq n+2) \\ w_3 - 2w_1 & (\text{otherwise}), \end{cases}$$

$$A_0 = \begin{cases} 2u_0 & (n+j \equiv 0 \ (\mathrm{mod}\ 8) \text{ and } n+4 > m \geqq n+2) \\ u_0 & (\text{otherwise}), \end{cases}$$

$$A_1 = \begin{cases} 2^{h_4(m+j+20, n+j)} u_1 & (n+j \equiv 2 \ (\mathrm{mod}\ 8)) \\ u_1 & (n+j \equiv 0 \ (\mathrm{mod}\ 8)), \end{cases}$$

$$A_2 = \begin{cases} 2u_2 & (n+j \equiv 2 \ (\mathrm{mod}\ 8) \text{ and } n+14 > m \geqq n+10) \\ u_2 & (\text{otherwise}) \end{cases}$$

and

$$A_3 = \begin{cases} 4u_3 & (n+j \equiv 2 \ (\mathrm{mod}\ 8) \text{ and } n+14 > m \geqq n+6) \\ u_3 & (\text{otherwise}). \end{cases}$$

If $n+2+2h_4(n+j+4, n+j+2) > m > n$, then we have $r_1(V^j_{m,n}) \cong 0$.

Suppose $j \equiv 2 \ (\mathrm{mod}\ 8)$. If $m \geqq 8\,[(n+2)/8]+10+4h_4(n, n-2)$, then we have

$$J''(r_1(V^j_{m,n})) \cong \langle \{v_i \mid 0 \leqq i \leqq 3\} \rangle / \langle \{B_i \mid 0 \leqq i \leqq 3\} \rangle,$$

where $v_0 = w_0$, $v_1 = w_2$, $v_2 = 2w_2 + w_1$, $v_3 = w_3$, $B_0 = R_4$,

$$B_1 = 4h_4(n+10, n-2)\, v_1,$$
$$B_2 = 2^{h_4(n+6, n+2) b_3(j, m+2, n)}\, v_2$$

and $B_3 = 2^{h_4(n+6, n+2)+h_4(n+2, n-2) b_3(j, m+2, n)}\, v_3$. If $8\,[(n+2)/8]+10+4h_4(n,n-2) > m \geqq 8\,[(n+2)/8]+10$, then we have

$$J''(r_1(V^j_{m,n})) \cong \langle \{v_i \mid 0 \leqq i \leqq 3\} \rangle / \langle \{B_i \mid 0 \leqq i \leqq 3\} \rangle,$$

where $v_0=2w_3-w_0$, $v_1=w_2$, $v_2=2w_2+w_1$, $v_3=w_3$, $B_0=v_0$, $B_1=8v_1$, $B_2=v_2$ and $B_3=4v_3$. If $8[(n+2)/8]+10>m<n$, then we have $U^j_{m,n}\cong 0$.

Suppose $j\equiv 6 \pmod 8$. If $m\geqq 8[(n+2)/8]+10$, then we have

$$J''(r_1(V^j_{m,n}))\cong \langle\{v_i\,|\,0\leqq i\leqq 3\}\rangle/\langle\{B_i\,|\,0\leqq i\leqq 3\}\rangle,$$

where $v_0=w_0-4h_4(n+4,n+2)\,w_2$, $v_1=w_2$, $v_2=2w_2+(1-2h_4(n+4,n+2))\,w_1$,

$$\begin{aligned} v_3 &= w_3-2h_4(n+4,n+2)\,w_2\,,\\ B_0 &= h_4(n+14,n+4)\,v_0\,,\\ B_1 &= 4h_4(n+12,n-2)\,v_1\,,\\ B_2 &= 2^{h_4(n+6,n+2)b_3(j,m+2,n)}\,v_2 \end{aligned}$$

and $B_3=2^{h_4(n+6,n+2)+h_4(n+2,n-2)b_3(j,m+2,n)}\,v_3$. If $8[(n+2)/8]+10>m<n$, then we have $U^j_{m,n}\cong 0$. Thus we obtain

(5.9) *Suppose that* $n\equiv 0 \pmod 2$.

(1) *If* $m\geqq n+2+12h_4(n+j+6,n+j+4)+2h_3(n+2,n)$, *then we have*

$$r_1(V^j_{m,n})\cong \boldsymbol{Z}/2^{h_3(m+j,n+j)+1}\oplus(\oplus_{i=0}^{1}\boldsymbol{Z}/2^{h_4(m+j-4i,n+j-4i+4)})\oplus G(n+j)\,,$$

where $G(n+j)$ *is the group defined by* (2.3).

(2) *If* $n+2+12h_4(n+j+6,n+j+4)+2h_3(n+2,n)>m>n$, *then we have*

$$r_1(V^j_{m,n})\cong\begin{cases}\boldsymbol{Z}/2^{h_3(m+j,n+j)+1}\oplus\boldsymbol{Z}/2^{h_4(m+j-4,n+j)}\oplus\boldsymbol{Z}/4 & (m\geqq n+6)\\ \boldsymbol{Z}/8 & (n+j\equiv 2 \pmod 8 \textit{ and } n+6>m\geqq n+2)\\ \boldsymbol{Z}/2 & (n+j\equiv 0 \pmod 8 \textit{ and } n+4>m\geqq n+2)\\ 0 & (\textit{otherwise})\,.\end{cases}$$

(3) *If* $m\geqq 8[(n+2)/8]+10+4h_4(n,n-2)\,h_4(j,j-4)$, *then we have*

$$J''(r_1(V^j_{m,n}))\cong\begin{cases}\boldsymbol{Z}/8\oplus\boldsymbol{Z}/2^b\oplus G(n+j) & (h_4(n+6,n+2)\,h_4(n+j+4,n+j-2)=0)\\ \boldsymbol{Z}/4\oplus\boldsymbol{Z}/2^b\oplus\boldsymbol{Z}/2\oplus G(n+j) & (\textit{otherwise})\,,\end{cases}$$

where $b=b_3(j,m+2,n)$ *and* $G(n+j)$ *is the group defined by* (2.3).

(4) *If* $j\equiv n+2\equiv 2 \pmod 8$ *and* $n+14>m\geqq n+10$, *then we have*

$$J''(r_1(V^j_{m,n}))\cong\boldsymbol{Z}/8\oplus\boldsymbol{Z}/4\,.$$

(5) *If* $8[(n+2)/8]+10>m>n$, *then we have* $J''(r_1(V^j_{m,n}))\cong r_1(V^j_{m,n})$.

By (5.4) and (5.9), we obtain the results for the cases $j\equiv 2 \pmod 4$, $n\equiv 0 \pmod 2$ and $m+j\equiv 3, 4, 5, 6$ or $7 \pmod 8$.

We now turn to the case $m+j\equiv 0 \pmod 8$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \uparrow & & \uparrow & & & \\
 & V^j_{m-1,n} & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^{m-1}/L_8^n)) & \longrightarrow & 0 & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & V^j_{m,n} & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) & \longrightarrow 0 \\
 & f\uparrow & & g\uparrow & & \uparrow & \\
0 \to & \tilde{K}(S^{m+j}) & \xrightarrow{r} & \widetilde{KO}(S^{m+j}) & \xrightarrow{\partial_1} & \widetilde{KO}(S^{m+j+1}) & \longrightarrow 0 \\
 & & & & & \uparrow & \\
 & & & & & 0 &
\end{array}$$

of exact sequences. Since $\widetilde{KO}(S^{m+j+1}) \cong \boldsymbol{Z}/2$, Lemma 3.10 implies that $\tilde{K}(S^{m+j}) \cong \boldsymbol{Z}$ has a generator γ with

$$f(\gamma) = \begin{cases} I^{j/2}(\varphi(X_5(m-2))) & (j \equiv 6 \pmod 8) \\ I^{j/2}(\varphi(X_7(m-2))) & (j \equiv 2 \pmod 8) \end{cases}$$

and $r(\gamma) = 2\beta$, where β is a generator of the group $\widetilde{KO}(S^{m+j}) \cong \boldsymbol{Z}$. It follows from Lemma 5.5 that we have

$$2g(\beta) = r_1(f(\gamma)) = 2^{h_4(m+j-16,n+j)}\, \tilde{W}_2 + 2^{h_3(m+j-12,n+j)}\, \tilde{W}_1\,,$$

where $\tilde{W}_1 = (1-2h_4(n+j, n+j-4))\, r_1(I^{j/2}(\varphi(\tilde{w}_1)))$ and

$$\tilde{W}_2 = \begin{cases} r_1(I^{j/2}(\varphi(\tilde{w}_2))) & (h_4(n+j, n+j-4) = 0) \\ r_1(I^{j/2}(\varphi(\tilde{w}_3))) & (\text{otherwise})\,. \end{cases}$$

If $m > n+12$, we set

$$\alpha = g(\beta) - \begin{cases} 2^{[(m-n-14)/8]}\, W_3 - 2^{[(m-n-10)/4]}\, W_1 & (h_4(n+j+4, n+j) = 0) \\ 2^{[(m-n-14)/8]}\, (2W_2 - W_1) + 2^{[(m-n-10)/4]}\, W_1 & (\text{otherwise})\,, \end{cases}$$

where $W_i = r_1(I^{j/2}(\varphi(w_i)))$ $(1 \leqq i \leqq 3)$. Then we have $\partial(\alpha) \neq 0$ and $2\alpha = 0^{m-n-14} W_0$, where $W_0 = r_1(I^{j/2}(\varphi(w_0)))$. By Lemma 3.9, Lemma 5.5, (5.8) and the fact $8g(\beta) = 0$, we have

$$\psi^k(\alpha) \equiv (k-2\,[k/2])\,\alpha \pmod{U^j_{m,n}}\,. \tag{5.10}$$

According to [3, II], the Adams operations on $\widetilde{KO}(S^{m+j+1})$ are given by $\psi^k = k-2\,[k/2]$. If $m > n+14$, then the short exact sequence

$$0 \to r_1(V^j_{m,n}) \to \widetilde{KO}(S^j(L_8^m/L_8^n)) \xrightarrow{\partial} \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) \to 0$$

splits. Hence we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong r_1(V^j_{m,n}) \oplus \boldsymbol{Z}/2\,.$$

It follows from (5.10) that we have

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong J''(r_1(V_{m,n}^j))\oplus Z/2\,.$$

If $m=n+14$, then we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) = \langle r_1(V_{m,n}^j)\cup\{\alpha\}\rangle = \langle\{W_1, W_2, W_3, \alpha\}\rangle\,.$$

Since ord $\widetilde{KO}(S^j(L_8^m/L_8^n))=1024$ by [13], ord $\langle\{W_1, W_2, W_3\}\rangle=256$ and ord $\langle\alpha\rangle=4$, we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong Z/32\oplus Z/4\oplus Z/4\oplus Z/2$$

and

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} Z/8\oplus Z/4\oplus Z/4 & (j\equiv 2 \pmod 8) \\ Z/8\oplus Z/2\oplus Z/2\oplus Z/2 & (j\equiv 6 \pmod 8)\,. \end{cases}$$

If $n+12\geqq m\geqq n+2$, we set

$$\alpha = g(\beta)-([(m-n-2)/8]+[(m-n+2)/8])\,W_2\,.$$

Then we have

$$2\alpha = W_2+[(m-n+2)/8]\,(W_3-W_2)+[(m-n-2)/8]\,(W_1-2W_2-W_3)\,.$$

Hence $\widetilde{KO}(S^j(L_8^m/L_8^n)) = \langle r_1(V_{m,n}^j)\cup\{\alpha\}\rangle$

$$= \begin{cases} \langle\{W_2, W_3, \alpha\}\rangle & (m\geqq n+10) \\ \langle\{W_0, W_2, \alpha\}\rangle & (m=n+8) \\ \langle\{W_2, \alpha\}\rangle & (m=n+6) \\ \langle\alpha\rangle & (n+6>m\geqq n+2)\,, \end{cases}$$

$$\text{ord}\,\langle\{W_2, W_3\}\rangle = 32 \quad (m\geqq n+10)\,,$$

$$\text{ord}\,\langle W_2\rangle = 8 \quad (n+10>m\geqq n+6)$$

and

$$\text{ord}\,\langle\alpha\rangle = 2^{[(n-m+31)/8]}\,.$$

Since ord $\widetilde{KO}(S^j(L_8^m/L_8^n))=2^{2h_3(m+j,n+j)+h_4(n+j,n+j-4)+1}$ by [13], we have

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} Z/16\oplus Z/4\oplus Z/2 & (m\geqq n+10) \\ Z/8\oplus Z/4\oplus Z/2 & (m=n+8) \\ Z/8\oplus Z/8 & (m=n+6) \\ Z/8 & (n+6>m\geqq n+2) \end{cases}$$

and

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} Z/8\oplus Z/2\oplus Z/2 & (m\geqq n+10 \text{ and } j\equiv 6 \pmod 8) \\ Z/4\oplus Z/4\oplus Z/2 & (m\geqq n+8 \text{ and } h_4(n+2, n-2)=0) \\ Z/8\oplus Z/4 & (m=n+6 \text{ and } j\equiv 6 \pmod 8) \\ \widetilde{KO}(S^j(L_8^m/L_8^n)) & (\text{otherwise})\,. \end{cases}$$

Thus we obtain the results for the case $m+j\equiv 0 \pmod 8$.

Modifying the proof above, we obtain the results for the case $m+j\equiv 1 \pmod 8$ (cf. [10]).

Finally we consider the case $m+j\equiv 2 \pmod 8$. Since

$$\widetilde{KO}(S^j(L_8^m/L_8^{m-2})) \cong \widetilde{KO}(S^{j+m-2}L_8^2) \cong \boldsymbol{Z}/2\oplus\boldsymbol{Z}/2$$

by Proposition 3.13, and the Adams operations on $\widetilde{KO}(S^j(L_8^m/L_8^{m-2}))$ are given by $\psi^k=k-2[k/2]$, the proof for this case is similar to the corresponding proof of [10].

Thus the proof for the case $j\equiv 2 \pmod 4$ and $n\equiv 0 \pmod 2$ is completed.

5.2. Proof for the case $n\equiv 1 \pmod 2$. Consider the following commutative diagram, in which the row is exact.

$$\begin{array}{ccccccc} 0 \to & V_{m,n+1}^j & \xrightarrow{f_1} & \tilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{f_2} & \tilde{K}(S^{n+j+1}) \to 0 \\ & \| & & \downarrow f_3 & & \\ & V_{m,n+1}^j & \hookrightarrow & \tilde{K}(S^jL_8^m)\,. & & \end{array}$$

By Lemma 3.10, we can choose an element $x\in\tilde{K}(S^j(L_8^m/L_8^n))$ such that $f_2(x)$ generates the group $\tilde{K}(S^{n+j+1})\cong\boldsymbol{Z}$ and

$$f_3(x)=\begin{cases} I^{j/2}(\varphi(X_5(n))) & (n\equiv 1 \pmod 8)\\ I^{j/2}(\varphi(X_6(n)+X_1(n))) & (n\equiv 3 \pmod 8)\\ I^{j/2}(\varphi(X_7(n))) & (n\equiv 5 \pmod 8)\\ I^{j/2}(\varphi(X_4(n))) & (n\equiv 7 \pmod 8)\,. \end{cases}$$

Inspect the following commutative diagram

$$\begin{array}{ccccccc} 0 & & 0 & & \widetilde{KO}(S^{n+j+2}) & & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ VO_{m,n+1}^{j+2} & \xrightarrow{I^{-1}\circ c} & V_{m,n+1}^j & \xrightarrow{r_1} & \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^{n+1})) \\ \downarrow & & f_1\downarrow & & g_1\downarrow & & h_1\downarrow \\ \widetilde{KO}(S^{j+2}(L_8^m/L_8^n)) & \to & \tilde{K}(S^j(L_8^m/L_8^n)) & \xrightarrow{r_2} & \widetilde{KO}(S^j(L_8^m/L_8^n)) & \xrightarrow{\partial} & \widetilde{KO}(S^{j+1}(L_8^m/L_8^n)) \\ \downarrow & & f_2\downarrow & & g_2\downarrow & & h_2\downarrow \\ \widetilde{KO}(S^{n+j+3}) & \xrightarrow{I^{-1}\circ c} & \tilde{K}(S^{n+j+1}) & \xrightarrow{r} & \widetilde{KO}(S^{n+j+1}) & \xrightarrow{\partial} & \widetilde{KO}(S^{n+j+2}) \to 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & \widetilde{KO}(S^{j-1}(L_8^m/L_8^{n+1})) & & \end{array}$$

of exact sequences. By Lemma 3.9, Proposition 3.13 and Lemma 3.15, we obtain

$$\text{(5.11)}\qquad \operatorname{Ker} r_2=\begin{cases} \langle\{f_1(S_i)\,|\,i=1,2,4 \text{ or } 6\}\rangle & (n\equiv 1 \pmod 4)\\ \langle f_1(\operatorname{Ker} r_1)\cup\{2x\}\rangle & (n+j\equiv 1 \pmod 8)\\ \langle f_1(\operatorname{Ker} r_1)\cup\{x\}\rangle & (n+j=5 \pmod 8)\,. \end{cases}$$

Suppose $m \geqq n+3$. Then we have

$$\operatorname{Coker} g_2 \cong \widetilde{KO}(S^{n+j+2}) \cong \begin{cases} \boldsymbol{Z}/2 & (n+j \equiv 7 \pmod 8) \\ 0 & (\text{otherwise}), \end{cases}$$

and hence

$$r(\tilde{K}(S^{n+j+1})) = g_2(\widetilde{KO}(S^j(L_8^m/L_8^n))) = \begin{cases} 2\widetilde{KO}(S^{n+j+1}) & (n+j \equiv 7 \pmod 8) \\ \widetilde{KO}(S^{n+j+1}) & (\text{otherwise}). \end{cases}$$

Since h_1 is a monomorphism, we have $\operatorname{Ker} g_1 \subset r_1(V^j_{m,n+1})$. Thus we obtain a split short exact sequence

$$0 \to \widetilde{KO}(S^j(L_8^m/L_8^{n+1}))/\operatorname{Ker} g_1 \xrightarrow{\bar{g}_1} \widetilde{KO}(S^j(L_8^m/L_8^n)) \xrightarrow{\bar{g}_2} H \to 0,$$

where $\operatorname{Ker} g_1 = \langle \{r_1(S_4), r_1(S_6)\} \rangle$ and

$$H = \begin{cases} 2\widetilde{KO}(S^{n+j+1}) & (n+j \equiv 7 \pmod 8) \\ \widetilde{KO}(S^{n+j+1}) & (\text{otherwise}). \end{cases}$$

Applying the method used in the proof of Lemma 4.11 to x, we obtain the following result by Lemma 3.9 and (5.11).

(5.12) (1) *If* $n \equiv 3 \pmod 4$, *then the Adams operations are given by*

$$\psi^k(r_2(x)) = (k-2[k/2])\, r_2(x).$$

(2) *Suppose* $n \equiv 1 \pmod 4$ *and* $k \equiv \varepsilon \pmod 8$, *where* ε *is an odd integer with* $-3 \leqq \varepsilon \leqq 3$. *Then the Adams operation* ψ^k *is given by*

$$\psi^k(r_2(x)) = k^u\, r_2(x) + ((\varepsilon k^{j/2} - k^u)/8)\, r_2(f_1(8f_3(x))) + k^{j/2}\, w,$$

where $u = (n+j+1)/2$, $w = -(\varepsilon/3)\, h_4(\varepsilon+5, \varepsilon+1)\, W$ *and*

$$W = \begin{cases} r_2(f_1(S_3+S_5)) & (n \equiv 1 \pmod 8) \\ r_2(f_1(2S_7-S_3)) & (n \equiv 5 \pmod 8). \end{cases}$$

Suppose $n \equiv 3 \pmod 4$. Then using (5.11) and (5.12) (1), we see that the short exact sequence

$$0 \to \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) \to \widetilde{KO}(S^j(L_8^m/L_8^n)) \to \widetilde{KO}(S^{n+j+1}) \to 0$$

of ψ-maps splits. This implies that

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) \oplus \widetilde{KO}(S^{n+j+1})$$

and

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \tilde{J}(S^j(L_8^m/L_8^{n+1})) \oplus \tilde{J}(S^{n+j+1}).$$

Thus, results of the case $j\equiv 2 \pmod 4$, $n\equiv 3 \pmod 4$ and $m\geqq n+3$ follow from those of the case $j\equiv 2 \pmod 4$ and $n\equiv 0 \pmod 4$.

Now, we turn to the case $n\equiv 1 \pmod 4$. If $m\geqq n+5$, then we have

$$\mathrm{Im}\, r_2 \cong \boldsymbol{Z}\oplus \boldsymbol{Z}/2^{h_3(m+j,n+j)}\oplus(\oplus_{i=0}^{1} \boldsymbol{Z}/2^{h_4(m+j-4i,n+j-4i+5)})\,.$$

If $n+5>m\geqq n+3$, then we have $\mathrm{Im}\, r_2\cong \boldsymbol{Z}$. Under the assumption of (5.12) (2), we have

$$\begin{aligned}&((\varepsilon k^{j/2}-k^u)/8)\, r_2(f_1(8f_3(x)))+k^{j/2}\, w\\ &\quad \equiv ((n+1)/2)\, N(u/2^{\nu_2(u)})\, ((k^u-1)/2^{\nu_2(u)+2})\, W \pmod{g_1(U^j_{m,n+1})}\,,\end{aligned}$$

where $u=(n+j+1)/2$. Thus we have $\tilde{J}(S^j(L_8^m/L_8^n))\cong \widetilde{KO}(S^j(L_8^m/L_8^n))/U_1$, where U_1 is the subgroup of $\widetilde{KO}(S^j(L_8^m/L_8^n))$ generated by

$$g_1(U^j_{m,n+1})\cup \{\mathfrak{m}((n+j+1)/2)\, r_2(x)+MW\}$$

with $M\equiv 1 \pmod 2$. By (5.8), we have

$$(5.13)\quad U_1=\begin{cases}\langle\{4v_1,\, 2v_2,\, 4v_3,\, M_0\, v_0+M(v_1+v_2)\}\rangle & (n\equiv 1 \pmod 8)\\ \langle\{8v_2,\, 2v_2+v_1,\, 4v_3,\, M_0\, v_0+2M(v_2+v_3)\}\rangle & (n\equiv 5 \pmod 8)\,,\end{cases}$$

where $M_0=\mathfrak{m}((n+j+1)/2)$, $\nu_2(M)=0$, $v_i=g_1(r_1(S_{2i+1}))$ $(1\leqq i\leqq 3)$ *and* $v_0=r_2(x)$.

This implies that we have

$$J''(\mathrm{Im}\, r_2)\cong F(v)/\langle\{B_i\,|\,0\leqq i\leqq 3\}\rangle\,,$$

where $F(v)$ is the free abelian group generated by $\{v_i\,|\,0\leqq i\leqq 3\}$,

$$B_0=\begin{cases}4M_0\, v_0 & (m\geqq n+9+12h_4(n,\, n-4))\\ 2M_0\, v_0 & (n+9+12h_4(n,\, n-4)>m\geqq n+5+4h_4(n,\, n-4))\\ M_0\, v_0 & (n+5+4h_4(n,\, n-4)>m)\,,\end{cases}$$

$$B_1=\begin{cases}v_1+2v_2 & (n\equiv 5 \pmod 8)\\ 2M_0\, v_0+2v_1 & (n\equiv 1 \pmod 8 \text{ and } m\geqq n+13)\\ v_1-2v_3 & (n\equiv 1 \pmod 8 \text{ and } n+13>m)\,,\end{cases}$$

$$B_2=\begin{cases}2M_0\, v_0+4v_2 & (n\equiv 5 \pmod 8 \text{ and } m\geqq n+17)\\ MM_0\, v_0+2v_2 & (n\equiv 5 \pmod 8 \text{ and } n+17>m\geqq n+5)\\ M_0\, v_0+Mv_1+v_2 & (n\equiv 1 \pmod 8 \text{ and } m\geqq n+9)\\ v_2+2h_4(n,\, n-4)\, v_3 & (n+5+4h_4(n,\, n-4)>m)\end{cases}$$

and

$$B_3=\begin{cases}M_0\, v_0+2Mv_0+2v_3 & (n\equiv 5 \pmod 8 \text{ and } m\geqq n+9)\\ 4v_3 & (n\equiv 1 \pmod 8 \text{ and } m\geqq n+5)\\ v_3+2h_4(n+4,\, n)\, v_2 & (n+5+4h_4(n+4,\, n)>m)\,.\end{cases}$$

By the proof for the case $n\equiv 2 \pmod 4$, we obtain

(5.14) *Suppose* $j-1\equiv n\equiv 1 \pmod 4$ *and* $m\geqq n+3$.
(1) *If* $m+j\equiv 0 \pmod 8$, *then we have*

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \begin{cases} (\mathrm{Im}\, r_2)\oplus \boldsymbol{Z}/2 & (m\geqq n+17) \\ \boldsymbol{Z}\oplus\boldsymbol{Z}/16\oplus\boldsymbol{Z}/4\oplus\boldsymbol{Z}/2 & (m=n+13) \\ \boldsymbol{Z}\oplus\boldsymbol{Z}/8\oplus\boldsymbol{Z}/4 & (m=n+9) \\ \boldsymbol{Z}\oplus\boldsymbol{Z}/8 & (m=n+5) \end{cases}$$

and

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} J''(\mathrm{Im}\, r_2)\oplus\boldsymbol{Z}/2 & (m\geqq n+17) \\ \boldsymbol{Z}/2M_0\oplus\boldsymbol{Z}/4\oplus\boldsymbol{Z}/4 & (h_4(n+4,n)=m-n-13=0) \\ \boldsymbol{Z}/4M_0\oplus\boldsymbol{Z}/2\oplus\boldsymbol{Z}/2\oplus\boldsymbol{Z}/2 & (h_4(n,n-4)=m-n-13=0) \\ \boldsymbol{Z}/2M_0\oplus\boldsymbol{Z}/4\oplus\boldsymbol{Z}/2 & (h_4(n+4,n)=m-n-9=0) \\ \boldsymbol{Z}/4M_0\oplus\boldsymbol{Z}/4\oplus\boldsymbol{Z}/2 & (h_4(n,n-4)=m-n-9=0) \\ \boldsymbol{Z}/M_0\oplus\boldsymbol{Z}/8 & (h_4(n+4,n)=m-n-5=0) \\ \boldsymbol{Z}/2M_0\oplus\boldsymbol{Z}/4 & (h_4(n,n-4)=m-n-5=0)\,, \end{cases}$$

where $M_0=\mathfrak{m}((n+j+1)/2)$.
(2) *If* $m+j\equiv 1 \pmod 8$, *then we have*

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong \widetilde{KO}(S^j(L_8^{m-1}/L_8^n))\oplus\boldsymbol{Z}/2$$

and

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \tilde{J}(S^j(L_8^{m-1}/L_8^n))\oplus\boldsymbol{Z}/2\,.$$

(3) *If* $m+j\equiv 2 \pmod 8$, *then we have*

$$\widetilde{KO}(S^j(L_8^m/L_8^n)) \cong (\mathrm{Im}\, r_2)\oplus\boldsymbol{Z}/2$$

and

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong J''(\mathrm{Im}\, r_2)\oplus\boldsymbol{Z}/2\,.$$

Noting the fact that we have $S^j(L_8^{n+1}/L_8^n)\approx S^{n+j+1}$ and

$$S^j(L_8^{n+2}/L_8^n)\simeq\begin{cases} S^{n+j+2}\vee S^{n+j+1} & (n\equiv 1 \pmod 2)) \\ S^{n+j}L_8^2 & (n\equiv 0 \pmod 2))\,, \end{cases}$$

we obtain the result for the case $n\equiv 1 \pmod 2$.

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