Miyaoka, R.
Osaka J. Math.
30 (1993), 179–202

# THE LINEAR ISOTROPY GROUP OF $G_2/SO(4)$, THE HOPF FIBERING AND ISOPARAMETRIC HYPERSURFACES

Dedicated to Professor T. Nagano on his 60th birthday

REIKO MIYAOKA

(Received June 12, 1991)

## 1. Introduction

The classification problem of isoparametric hypersurfaces in a sphere with four or six principal curvatures is still open. A hypersurface in a sphere is called isoparametric if each principal curvature is constant. When the number $g$ of the principal curvatures is six, every principal curvature has the same multiplicity $m$ [9], which takes value 1 or 2 [1]. In either case, the known examples belong to the family of homogeneous hypersurfaces. Recently, Dorfmeister and Neher [4] proved that an isoparametric hypersurface with $(g, m)=(6, 1)$ are homogeneous. Their argument is, however, purely algebraic, because they classify isoparametric functions rather than the hypersurfaces themselves. Unfortunately, their proof does not work for $(g,m)=(6,2)$. So it seems significant to consider the problem from a different point of view, more geometrically.

Up to now, about homogeneous hypersurfaces with $g=6$, we know merely a general fact that they are orbits of isotropy actions of certain symmetric spaces. But as a special case of the recent result [7, Proposition 3], when $h: S^7 \rightarrow S^4$ is the Hopf fibering, the inverse image $\tilde{N}=h^{-1}(N)$ of an isoparametric hypersurface $N$ in $S^4$ is isoparametric with $g=2k$ where $k$ is the number of principal curvatures of $N$. When $k=3$, $N$ is known to be a tube of the Veronese surface [2] and certainly, $\tilde{N}$ is homogeneous. Since the family of homogeneous hypersurfaces in $S^7$ with $g=6$ is unique [13], this gives a new geometric characterization to it.

Now, it is interesting to know how the fibers $S^3$ of the Hopf fibering appear on $\tilde{N}$. Moreover, since $\tilde{N}$ is an orbit of the isotrpoy action of $G_2/SO(4)$ [13] and since $S^7$ is stratified by such orbits, it is interesting to know how this action is related with the Hopf fibration. In §2, we clarify this point in terms of a subgroup action of the linear isotropy group. In particular, concerning that $\tilde{N}$ is homeomorphic to $N \times S^3$, we show that $\tilde{N}$ is foliated by an isoparametric hypersurface with $(g, m)=(3, 1)$ which is diffeomorphic to $N$ (Proposition 2.4).

A similar correspondense exists between focal submanifolds $\tilde{N}_{\pm}$ (see the next paragraph) of $\tilde{N}$ and $N_{\pm}$ of $N$. For instance, $\tilde{N}_{\pm}$ is homeomorphic to a product of the Veronese surface $N_{\pm}$ and $S^3$. By the way, a fiber $S^3$ on $\tilde{N}_{\pm}$ is called an "equatorsphere" [1, 5.2], of which dimension turns out to be three. We show furthermore (Proposition 2.5) that $\tilde{N}_+$ is foliated by the Veronese surface, while $\tilde{N}_-$ is foliated by the minimal isoparametric hypersurface with $(g, m)=(3, 1)$. In contrast with that $N_+$ and $N_-$ are congruent to the Veronese surface in $S^4$, the following is remarkable:

**Proposition 3.3.** *$\tilde{N}_+$ and $\tilde{N}_-$ are not congruent in $S^7$. In particular, we have two minimal taut homogeneous embeddings of $P^2\boldsymbol{R}\times S^3$ into $S^7$ which are not congruent.*

The study of focal submanifolds of an isoparametric hypersurface $M$ is important since basic properties of the hypersurface condense into them. Here, by a focal submanifold, we mean the submanifold consisting of the first focal points of $M$ in a fixed normal direction. Thus we have two focal submanifolds $M_{\pm}$. §3 is devoted to the study of $M_{\pm}$, of which shape operator plays an important role when we investigate. $M$. In particular, on $\tilde{N}_{\pm}$, its null direction is constant for any normal vectors, and this is the second geometric characterization of the homogeneous hypersurface with $(g, m)=(6, 1)$ (Proposition 4.2). We note that this property is interpreted as an integrability condition of some unions of curvature distributions (Remark 4.2).

The whole argument in this article is independent of Dorfmeister-Neher's classification theorem.

The author wolud like to express her hearty thanks to Professors H. Ozeki and M. Takeuchi for their valuable suggestions and encouragement.

## 2. Isotropy representation of $G_2/SO(4)$ and the Hopf fibering

Using the quaternion field $\boldsymbol{H}$, let $S^7=\{\begin{pmatrix} u \\ v \end{pmatrix}\in\boldsymbol{H}^2 \mid \|u\|^2+\|v\|^2=1\}$ and $S^4=\{\begin{pmatrix} t \\ w \end{pmatrix}\in\boldsymbol{R}\times\boldsymbol{H} \mid t^2+\|w\|^2=1\}$. Consider the Hopf fibering $h\colon S^7\to S^4$, $h(\begin{pmatrix} u \\ v \end{pmatrix})=\begin{pmatrix} \|u\|^2-\|v\|^2 \\ 2u\bar{v} \end{pmatrix}$, which is associated with the action of $Sp(1)=\{s\in\boldsymbol{H} \mid \|s\|^2=1\}$ on $S^7$ given by

$$s\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} us^{-1} \\ vs^{-1} \end{pmatrix}, \quad s\in Sp(1), \quad \begin{pmatrix} u \\ v \end{pmatrix}\in\boldsymbol{H}^2. \tag{2.1}$$

**Proposition 2.1** [7]. *A homogeneous hypersurface in $S^7$ with six principal curvatures is the inverse image of an isoparametric hypersuface in $S^4$ with three principal curvatures under the Hopf fibering. This correspondence exists betweeen* *focal submanifolds of each hypersurface.*

In order to understand the relation of group actions on $S^4$ and $S^7$, we discuss for a while on this proposition and give a direct proof.

Recall that a homogeneous hypersurface in a sphere is a principal orbit of the linear isotropy action of some Riemmanian symmetric space of rank 2 [5]. A homogeneous hypersurface $M^h$ in $S^7$ with six principal curvatures is an orbit of the isotropy action of the symmetric space $G_2/SO(4)$, where $G_2$ is the automorphism group of the Cayley algebra $C$. Let $C$ be generated by $\{e_0, e_0, \cdots, e_7\}$ satisfying

$$\begin{cases} e_0 = 1 \\ e_i^2 = -1\,, \quad 1 \leq i \leq 7\,, \\ e_i e_j = -e_j e_i = e_k \end{cases}$$

where $(i, j, k)$ is a triple on some edge, middle segment or a circle of Fg. 1 put in order shown by its arrows. The automorphism group $G_2$ of $C$ is identified with a subgroup of $SO(7)$, where the metric on $C$ is given by

$$(x, y) = \Re x\bar{y} = \sum_{i=0}^{7} x^i y^i, \quad \text{for} \quad x = \sum_{i=0}^{7} x^i e_i, \quad y = \sum_{i=0}^{7} y^i e_i\,.$$

The Lie algebra $\mathfrak{g}$ of $G_2$ is given as follows [11]: Let $E_{ij}$ be the standard basis of $7\times 7$ matrices with $\boldsymbol{R}$-coefficients. Put $G_{ij}=E_{ij}-E_{ji}$, $i, j=1, \cdots, 7$ and put

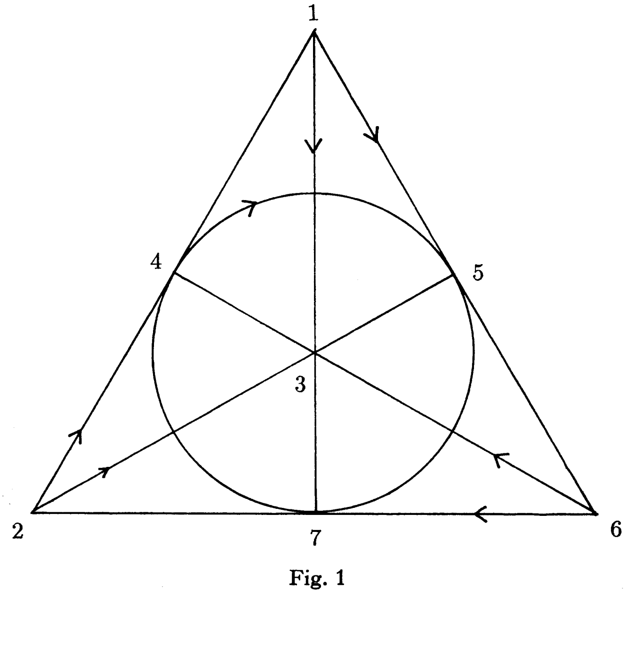


Fig. 1

$$\mathfrak{g}_i = \{\eta_1 G_{i+1\, i+3} + \eta_2 G_{i+2\, i+6} + \eta_3 G_{i+4\, i+5} \mid \eta_j \in \boldsymbol{R}, \sum_{j=1}^{3} \eta_j = 0\},$$

for $1 \leq i \leq 7$. Then $\mathfrak{g}$ is given by

$$\mathfrak{g} = \sum_{i=1}^{7} \mathfrak{g}_i \tag{2.2}$$

and satisfies

$$[\mathfrak{g}_i, \mathfrak{g}_i] = 0\,, \quad [\mathfrak{g}_i, \mathfrak{g}_j] = \mathfrak{g}_k \tag{2.3}$$

where $(i, j, k)$ is as before. Note that $[G_{ij}, G_{jk}] = G_{ik}$, for any $1 \leq i, j, k \leq 7$. Let $\tau$ be the involutive automorphism of $\mathfrak{g}$ given by $\tau(X) = -{}^tX$. Then $(\mathfrak{g}, \tau)$ is an effective orthogonal Lie algebra of compact type and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid \tau(X) = X\} = \mathfrak{g}_3 + \mathfrak{g}_4 + \mathfrak{g}_6 \simeq \mathfrak{so}(4)$$
$$\mathfrak{p} = \{X \in \mathfrak{g} \mid \tau(X) = -X\} = \mathfrak{g}_1 + \mathfrak{g}_2 + \mathfrak{g}_5 + \mathfrak{g}_7\,.$$

Note that (2.2) is an orthogonal decomposition with respect to the metric $\langle\,,\rangle$ on $\mathfrak{g}$ given by

$$\langle X, Y\rangle = -\frac{1}{2}\mathrm{Tr}XY\,.$$

Take a maximal abelian subspace $\mathfrak{a} = \mathfrak{g}_1 = \{\xi_1 G_{24} + \xi_2 G_{37} + \xi_3 G_{56} \mid \xi_i \in \boldsymbol{R}, \sum_{i=1}^{3} \xi_i = 0\}$ of $\mathfrak{p}$, whose dimension called the rank of $(\mathfrak{g}, \tau)$ is 2. Let $\kappa$ be a linear form on $\mathfrak{a}$ and put

$$\mathfrak{k}_\kappa = \{X \in \mathfrak{k} \mid (\mathrm{ad}H)^2(X) = -\kappa(H)^2 X\,, \quad \text{for all} \quad H \in \mathfrak{a}\}$$
$$\mathfrak{p}_\kappa = \{X \in \mathfrak{p} \mid (\mathrm{ad}H)^2(X) = -\kappa(H)^2 X\,, \quad \text{for all} \quad H \in \mathfrak{a}\}.$$

For $H \in \mathfrak{a}$, ad$H$ maps $\mathfrak{k}_\kappa$ (resp. $\mathfrak{p}_\kappa$) isomorphically onto $\mathfrak{p}_\kappa$ (resp. $\mathfrak{k}_\kappa$), if $\kappa(H) \neq 0$ [13, or (2.4) below]. We can select a suitable ordering in the dual space of $\mathfrak{a}$ such that the set $\Sigma_+$ of positive roots of $(\mathfrak{g}, \tau)$ with respect to $\mathfrak{a}$ is given by

$$\Sigma_+ = \{\kappa_1 = -\xi_2, \kappa_2 = \xi_1 - \xi_2, \kappa_3 = \xi_1, \kappa_4 = \xi_1 - \xi_3, \kappa_5 = -\xi_3, \kappa_6 = \xi_2 - \xi_3\}\,.$$

We define root vectors $X_i \in \mathfrak{k}_{\kappa_i}$ and $T_i \in \mathfrak{p}_{\kappa_i}$ as follows:

$$\begin{aligned}
X_1 &= G_{46} - G_{25} + 2G_{17}\,, & X_4 &= G_{46} + G_{25} & &\in \mathfrak{g}_3\\
X_2 &= -G_{27} - G_{34}\,, & X_5 &= -G_{27} + G_{34} - 2G_{15} & &\in \mathfrak{g}_6\\
X_3 &= G_{57} - G_{36} - 2G_{12}\,, & X_6 &= G_{57} + G_{36} & &\in \mathfrak{g}_4
\end{aligned}$$

$$\begin{aligned}
T_1 &= G_{26} + G_{45} - 2G_{13}\,, & T_4 &= G_{26} - G_{45} & &\in \mathfrak{g}_7\\
T_2 &= G_{23} + G_{47}\,, & T_5 &= -G_{23} + G_{47} - 2G_{16} & &\in \mathfrak{g}_5\\
T_3 &= G_{35} + G_{67} + 2G_{14}\,, & T_6 &= -G_{35} + G_{67} & &\in \mathfrak{g}_2\,.
\end{aligned}$$

It follows immediately

$$\mathrm{ad}\,HX_i = \kappa_i(H)T_i\,, \quad \mathrm{ad}\,HT_i = -\kappa_i(H)X_i\,. \tag{2.4}$$

Note that any two of above vectors are mutually orthogonal and $\|X_i\|=\|T_i\|$. Now, the connected subgroup of $G_2$ generated by $\mathfrak{k}$ is isomorphic to $SO(4)$. Let $\rho\colon SO(4)\to GL(\mathfrak{p})$ be the linear isotropy representation, i.e., $\rho(k)X=\mathrm{Ad}\,kX$ for $k\in SO(4)$ and $X\in\mathfrak{p}$. It is well known that $\rho(SO(4))$ is a subgroup of $SO(\mathfrak{p})$ $=SO(8)$ and that a principal orbit of this action in the unit sphere $S^7$ of $\mathfrak{p}$, is an isoparametric hypersurface with six principal curvatures [13, or Remark 2.2].

Now, put

$$\begin{aligned} Z_1 &= -\frac{1}{2}(X_1-X_4) = G_{25}-G_{17} &&\in\mathfrak{g}_3\\ Z_2 &= -\frac{1}{2}(X_2+X_5) = G_{27}+G_{15} &&\in\mathfrak{g}_6\\ Z_3 &= -\frac{1}{2}(X_3+X_6) = G_{12}-G_{57} &&\in\mathfrak{g}_4\,. \end{aligned}$$

Then $Z_1$, $Z_2$ and $Z_3$ span an ideal $\mathfrak{l}$ of $\mathfrak{so}(4)$. The orthogonal decomposition of $\mathfrak{so}(4)=\mathfrak{l}\oplus\mathfrak{l}^\perp$ defines another ideal $\mathfrak{l}^\perp$ spanned by

$$\begin{aligned} Z_4 &= \frac{1}{2}(X_1+3X_4) = 2G_{46}+G_{25}+G_{17} &&\in\mathfrak{g}_3\\ Z_5 &= \frac{1}{2}(3X_2-X_5) = -G_{27}-2G_{34}+G_{15} &&\in\mathfrak{g}_6\\ Z_6 &= \frac{1}{2}(X_3-3X_6) = -G_{57}-2G_{36}-G_{12} &&\in\mathfrak{g}_4\,. \end{aligned}$$

Certainly, $\mathfrak{l}\cong\mathfrak{l}^\perp\cong\mathfrak{sp}(1)\cong\mathfrak{so}(3)$. Now for any non-zero $H=\xi_1G_{24}+\xi_2G_{37}+\xi_3G_{56}\in\mathfrak{a}$, define $W_i=-[Z_i, H]\in\mathfrak{p}$, $i=1, 2, 3$, i.e.

$$\begin{aligned} W_1 &= -\xi_1G_{45}-\xi_2G_{13}-\xi_3G_{26} &&\in\mathfrak{g}_7\\ W_2 &= -\xi_1G_{47}+\xi_2G_{23}-\xi_3G_{16} &&\in\mathfrak{g}_5\\ W_3 &= -\xi_1G_{14}+\xi_2G_{35}+\xi_3G_{67} &&\in\mathfrak{g}_2\,, \end{aligned} \tag{2.5}$$

and put $V=\{H, W_1, W_2, W_3\}$. Similarly, for an orthogonal element $H^\perp=(\xi_3-\xi_2)G_{24}+(\xi_1-\xi_3)G_{37}+(\xi_2-\xi_1)G_{56}$ of $H$ in $\mathfrak{a}$, put $W_i^\perp=-[Z_i, H^\perp]$, $i=1, 2, 3$, which is orthogonal to $W_i$ in $\mathfrak{g}_{j_i}$, $j_1=7, j_2=5, j_3=2$. In fact, $W_i^\perp$ is given by changing $(\xi_1, \xi_2, \xi_3)$ into $(\xi_3-\xi_2, \xi_1-\xi_3, \xi_2-\xi_1)$ in (2.5). Let $V^\perp=\{H^\perp, W_1^\perp, W_2^\perp, W_3^\perp\}$. Then we have an orthogonal decomposition

$$\mathfrak{p} = V\oplus V^\perp.$$

Denoting the unit vector parallel to a vector $A$ by $\bar{A}$, define a linear map $\phi\colon\mathfrak{p}\to$

$\boldsymbol{H}^2$ by

$$\phi(\bar{H})=\begin{pmatrix}1\\0\end{pmatrix},\quad \phi(\bar{W}_1)=\begin{pmatrix}i\\0\end{pmatrix},\quad \phi(\bar{W}_2)=\begin{pmatrix}j\\0\end{pmatrix},\quad \phi(\bar{W}_3)=\begin{pmatrix}k\\0\end{pmatrix},$$
$$\phi(\bar{H}^\perp)=\begin{pmatrix}0\\1\end{pmatrix},\quad \phi(\bar{W}_1^\perp)=\begin{pmatrix}0\\i\end{pmatrix},\quad \phi(\bar{W}_2^\perp)=\begin{pmatrix}0\\j\end{pmatrix},\quad \phi(\bar{W}_3^\perp)=\begin{pmatrix}0\\k\end{pmatrix},$$

which is an isometry. We identify $\boldsymbol{H}^2$ with $\mathfrak{p}$ by this mapping.

Let $Sp(1)$ and $Sp(1)'$ be the connected, simply connected subgroups of $SO(4)$ corresponding to $\mathfrak{l}$ and $\mathfrak{l}^\perp$, respectively.

**Lemma 2.2.** *The action $\phi\cdot\rho(Sp(1))\cdot\phi^{-1}$ on $\boldsymbol{H}^2$ coincides with the action* (2.1) *of $Sp(1)$ on $\boldsymbol{H}^2$.*

Proof. We can check that the adjoint action of $\mathfrak{l}$ on $V$ is given by

(2.6)

| | $H$ | $W_1$ | $W_2$ | $W_3$ |
|---|---|---|---|---|
| $\mathrm{ad}\,Z_1$ | $-W_1$ | $H$ | $W_3$ | $-W_2$ |
| $\mathrm{ad}\,Z_2$ | $-W_2$ | $-W_3$ | $H$ | $W_1$ |
| $\mathrm{ad}\,Z_3$ | $-W_3$ | $W_2$ | $-W_1$ | $H$ |

A similar relation holds for the adjoint action of $\mathfrak{l}$ on $V^\perp$. On the other hand, let $\{1, i, j, k\}$ be the standard basis of $\boldsymbol{H}$. Consider the Lie algebra $\mathfrak{sp}(1)$ of $Sp(1)$ generated by $s_i$, $s_j$, $s_k$ where

$$s_i u=-ui\,,\quad s_j u=-uj\,,\quad s_k u=-uk\,,\quad u\in\boldsymbol{H}\,.$$

We see that the matrix representation of $s_i$, $s_j$, $s_k$ with respect to the basis $\{1, i, j, k\}$ coincides with the matrix representation of $\mathrm{ad}\,Z_1$, $\mathrm{ad}\,Z_2$, $\mathrm{ad}\,Z_3$ on $V$ ($V^\perp$, respectively) with respect to the basis $\{H, W_1, W_2, W_3\}$ ($\{H^\perp, W_1^\perp, W_2^\perp, W_3^\perp\}$, respectively). Since we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{p} \arrow[r, "\phi"] \arrow[d, "\mathrm{ad}\,Z"'] & \boldsymbol{H}^2 \arrow[d, "s_Z"] \\
\mathfrak{p} \arrow[r, "\phi"] & \boldsymbol{H}^2
\end{tikzcd}$$

where $Z\in\mathfrak{l}=\mathfrak{sp}(1)$, $s_Z u=-uZ$, $u\in\boldsymbol{H}$, and since $\exp\mathrm{ad}\,\mathfrak{l}=\mathrm{Ad}\exp\mathfrak{l}=\rho(Sp(1))$, while $\exp\mathfrak{sp}(1)=Sp(1)$, we obtain the Lemma. ∎

In order to investigate the action $\rho(Sp(1)')$ on $\mathfrak{p}$, recall that the Lie algebra $\mathfrak{sp}(2)$ of $Sp(2)$ standardly acting on $\boldsymbol{H}^2$ is generated by

$$E^i=\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad E^j=\begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \quad E^k=\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix},$$
$$E_i=\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, \quad E_j=\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, \quad E_k=\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix},$$
$$F_i=\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad F_j=\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad F_k=\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix},$$
$$F=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

**Lemma 2.3.** *The action of $\mathfrak{l}^\perp$ on $\boldsymbol{H}^2$ given by $\phi\cdot\mathrm{ad}\,\mathfrak{l}^\perp\cdot\phi^{-1}$ is expressed as*

$$(2.8)\quad \begin{aligned} \phi\cdot\mathrm{ad}\,Z_4\cdot\phi^{-1} &= \frac{1}{R}\{6\xi_1\xi_3E^i+2(\xi_1-\xi_2)(\xi_2-\xi_3)E_i-2\sqrt{3}\xi_2(\xi_3-\xi_1)F_i\}, \\ \phi\cdot\mathrm{ad}\,Z_5\cdot\phi^{-1} &= \frac{1}{R}\{-6\xi_1\xi_2E^j-2(\xi_2-\xi_3)(\xi_3-\xi_1)E_j+2\sqrt{3}\xi_3(\xi_1-\xi_2)F_j\}, \\ \phi\cdot\mathrm{ad}\,Z_6\cdot\phi^{-1} &= \frac{1}{R}\{6\xi_2\xi_3E^k+2(\xi_1-\xi_2)(\xi_3-\xi_1)E_k-2\sqrt{3}\xi_1(\xi_2-\xi_3)F_k\}, \end{aligned}$$

where $R=\xi_1^2+\xi_2^2+\xi_3^2$.

Proof. Using

$$\begin{cases} G_{24}-G_{56}=\dfrac{1}{R}\{-(\xi_3-\xi_1)H-\xi_2H^\perp\} \\ G_{37}-G_{56}=\dfrac{1}{R}\{(\xi_2-\xi_3)H+\xi_1H^\perp\}, \end{cases} \quad \begin{cases} G_{45}-G_{26}=\dfrac{1}{R}\{(\xi_3-\xi_1)W_1+\xi_2W_1^\perp\} \\ G_{13}-G_{26}=\dfrac{1}{R}\{-(\xi_2-\xi_3)W_1-\xi_1W_1^\perp\}, \end{cases}$$

$$\begin{cases} G_{47}-G_{16}=\dfrac{1}{R}\{(\xi_3-\xi_1)W_2+\xi_2W_2^\perp\} \\ G_{23}+G_{16}=\dfrac{1}{R}\{(\xi_2-\xi_3)W_2+\xi_1W_2^\perp\}, \end{cases} \quad \begin{cases} G_{14}+G_{67}=\dfrac{1}{R}\{(\xi_3-\xi_1)W_3+\xi_2W_3^\perp\} \\ G_{35}-G_{67}=\dfrac{1}{R}\{(\xi_2-\xi_3)W_3+\xi_1W_3^\perp\}, \end{cases}$$

we obtain, for instance,

$$\begin{aligned} \mathrm{ad}\,Z_4(H) &= \frac{1}{R}\{6\xi_1\xi_3W_1-2\xi_2(\xi_3-\xi_1)W_1^\perp\}, \\ \mathrm{ad}\,Z_4(W_1) &= \frac{1}{R}\{-6\xi_1\xi_3H+2\xi_2(\xi_3-\xi_1)H^\perp\}, \\ \mathrm{ad}\,Z_4(W_2) &= \frac{1}{R}\{6\xi_1\xi_3W_3-2\xi_2(\xi_3-\xi_1)W_3^\perp\}, \\ \mathrm{ad}\,Z_4(W_3) &= \frac{1}{R}\{-6\xi_1\xi_3W_2+2\xi_2(\xi_3-\xi_1)W_2^\perp\}, \end{aligned}$$

$$\operatorname{ad} Z_4(H^\perp) = \frac{1}{R}\{-6\xi_2(\xi_3-\xi_1)W_1+2(\xi_1-\xi_2)(\xi_2-\xi_3)W_1^\perp\},$$

$$\operatorname{ad} Z_4(W_1^\perp) = \frac{1}{R}\{6\xi_2(\xi_3-\xi_1)H-2(\xi_1-\xi_2)(\xi_2-\xi_3)H^\perp\},$$

$$\operatorname{ad} Z_4(W_2^\perp) = \frac{1}{R}\{-6\xi_2(\xi_3-\xi_1)W_3+2(\xi_1-\xi_2)(\xi_2-\xi_3)W_3^\perp\},$$

$$\operatorname{ad} Z_4(W_3^\perp) = \frac{1}{R}\{6\xi_2(\xi_3-\xi_1)W_2-2(\xi_1-\xi_2)(\xi_2-\xi_3)W_2^\perp\}.$$

By similar calculations for $\operatorname{ad} Z_5$ and $\operatorname{ad} Z_6$, we obtain (2.8), if we note that $\|H^\perp\| =\sqrt{3}\|H\|$ and $\|W_i^\perp\|=\sqrt{3}\|W_i\|$. ■

Proof of Proposition 2.1. From $\exp \operatorname{ad}\mathfrak{l}^\perp=\operatorname{Ad}\exp\mathfrak{l}^\perp=\rho(Sp(1)')$, while $\exp \mathfrak{sp}(2)=Sp(2)$, we obtain $\rho(Sp(1)')\subset Sp(2)$. The following argument connecting the standard $Sp(2)$-action on $S^7$ with an action $\sigma(Sp(2))$ on $S^4$ is due to M. Takeuchi. Identify $\boldsymbol{R}\times\boldsymbol{H}$ with $\boldsymbol{R}^5$, where

$$\begin{aligned}\boldsymbol{R}^5 &= \{X\in M_2(\boldsymbol{H}),\ {}^t\bar{X} = X,\ \operatorname{Tr}X = 0\}\\ &= \{X = \begin{pmatrix} t & w\\ \bar{w} & -t\end{pmatrix} \mid t\in\boldsymbol{R},\ w\in\boldsymbol{H}\}.\end{aligned}$$

by

$$\boldsymbol{R}\times\boldsymbol{H}\ni\begin{pmatrix} t\\ w\end{pmatrix}\mapsto\begin{pmatrix} t & w\\ \bar{w} & -t\end{pmatrix}\in\boldsymbol{R}^5.$$

Define an inner product on $\boldsymbol{R}^5$ by

$$\langle X, Y\rangle = \frac{1}{2}\Re\operatorname{Tr}(XY),$$

with which the above correspondence becomes an isometry. Consider the Veronese embedding

$$\iota\colon \boldsymbol{P}^1(\boldsymbol{H}) = S^7/Sp(1)\to S^4\subset\boldsymbol{R}^5,$$

by $\iota(x \bmod Sp(1))=2x{}^t\bar{x}-I_2$, that is, by

$$\boldsymbol{H}^2\ni x = \begin{pmatrix} u\\ v\end{pmatrix} \bmod Sp(1)\mapsto\begin{pmatrix}\|u\|^2-\|v\|^2 & 2u\bar{v}\\ 2v\bar{u} & -\|u\|^2+\|v\|^2\end{pmatrix},$$

where $u, v\in\boldsymbol{H}$ with $\|u\|^2+\|v\|^2=1$. Let $p\colon S^7\to\boldsymbol{P}^1(\boldsymbol{H})$ be the projection. Then the Hopf fibering

$$h\colon S^7\to S^4$$

is given by $h=\iota\cdot p$. Next, for $\alpha\in Sp(2)\subset SO(\boldsymbol{H}^2)=SO(8)$, define

$$\sigma\colon Sp(2) \to SO(5) = SO(\boldsymbol{R}^5)$$

by

$$\sigma(\alpha)X = \alpha X {}^t\bar{\alpha} \in \boldsymbol{R}^5\,, \quad X \in \boldsymbol{R}^5,\ \alpha \in Sp(2)\,.$$

To show that $h$ is $Sp(2)$-equivariant with respect to $\sigma\colon Sp(2) \to SO(5)$ is a standard exercise.

Now, denote by $1_G$ the identity element of a group $G$. From $SO(4) = Sp(1) \times_{Z_2} Sp(1)'$, we know that $-1_{Sp(1)'}$ is identified with $-1_{Sp(1)}$ in $SO(4)$. Thus we obtain $\rho(-1_{Sp(1)'}) = \rho(-1_{Sp(1)}) = -1_{SO(8)} = -1$ in $Sp(2)$, where we use Lemma 2.2. Since we have shown that $Sp(1)' \subset Sp(2)$ (here, we omit $\rho$), the quotient group $SO(4)/Sp(1) \cong Sp(1)'/\{\pm 1\} = SO(3)$ induces an action $\hat{\sigma}\colon SO(3) \to SO(5)$ by the restriction of $\sigma\colon Sp(2) \to SO(5)$ to $Sp(1)'$. Thus $h$ is $SO(4)$-equivariant with respect to $\pi\colon SO(4) \to SO(3)$ (natural projection). Here, we can check that $\hat{\sigma}$ is equivalent to the irreducible representation on traceless symmetric $3\times 3$ real matrices (see Remark 2.1). Thus a regular $SO(3)$-orbit in $S^4$ is a tube of Veronese surface $\boldsymbol{P}^2 \subset S^4$. The correspondence between singular orbits is obvious. ■

REMARK 2.1. Since $h$ is trivial on $S^4 - \{\text{a point}\}$, $M^h$ is homeomorphic to $N \times S^3$, where $N = h(M^h)$ is an $SO(3)$-orbit. The homogeneous hypersurfaces in $S^4$ are 3-spheres, products of two spheres, or tubes of the Veronese surface [5]. The topology of $M^h$, for instance, the sum of $\boldsymbol{Z}_2$-Betti numbers which is equal to 12 [9], implies $N$ must be the last one. Thus if we do not know $\hat{\sigma}$ exactly, we obtain the same result.

From now on, we write $M^h = \rho(SO(4))\bar{H} = \tilde{N} = h^{-1}(N)$, where $N$ is as above.

REMARK 2.2. Since a unit normal vector $\bar{H}^\perp$ at $\rho(k)\bar{H} \in \tilde{N}$, $k \in SO(4)$ can be given by $\mathrm{Ad}(k)\bar{H}^\perp$, the shape operator $A_{\bar{H}^\perp}$ of $\tilde{N}$ (cf. §3) with respect to $\bar{H}^\perp$ at $\bar{H}$ is given by

$$\begin{aligned} A_{\bar{H}^\perp} T_i &= -\tilde{\nabla}_{T_i} \bar{H}^\perp = \frac{1}{\kappa_i(\bar{H})} \frac{d}{dt}\Big|_{t=0} (\mathrm{Ad}\exp tX_i)\bar{H}^\perp \\ &= \frac{1}{\kappa_i(\bar{H})}[X_i, \bar{H}^\perp] = -\frac{\kappa_i(\bar{H}^\perp)}{\kappa_i(\bar{H})} T_i \\ &= -\frac{\kappa_i(H^\perp)}{\sqrt{3}\kappa_i(H)} T_i \end{aligned}$$

where we use (2.4) and $\|H^\perp\| = \sqrt{3}\|H\|$. Thus the principal curvatures of $\rho(SO(4))(\bar{H})$ are given by

$$\lambda_1 = -\frac{\xi_1-\xi_3}{\sqrt{3}\xi_2} = -\frac{1}{\lambda_4},$$
$$(2.9) \qquad \lambda_2 = -\frac{\sqrt{3}\xi_3}{\xi_1-\xi_2} = -\frac{1}{\lambda_5},$$
$$\lambda_3 = \frac{\xi_2-\xi_3}{\sqrt{3}\xi_1} = -\frac{1}{\lambda_6}.$$

The unit principal vector corresponding to $\lambda_i$ is $\bar{T}_i$. The tangent space of the fibre of $h$ at $\bar{H}$ is spanned by $W_1$, $W_2$, $W_3$, i.e., by

$$\{\bar{T}_1-\lambda_1\bar{T}_4,\ \bar{T}_2+\lambda_2\bar{T}_5,\ \bar{T}_3+\lambda_3\bar{T}_6\}.$$

In the following, as a regular element $H=H(\xi_1, \xi_2, \xi_3)$, we choose $(\xi_1, \xi_2, \xi_3)$ so that

$$(2.10) \qquad \lambda_1>\cdots>\lambda_6, \quad \text{i.e. } \xi_1>0>\xi_2>\xi_3.$$

The singular orbits correspond to

$$(2.11) \qquad H_+ = H(\xi_2=\xi_3) \quad \text{or} \quad H_- = H(\xi_2=0).$$

In each case, say, when $(\xi_1, \xi_2, \xi_3)=(2, -1, -1)$ ($(\xi_1, \xi_2, \xi_3)=(1, 0, -1)$, respectively), the tangent space of the fiber $S^3$ at $\bar{H}_+$ ($\bar{H}_-$, respectively) is spanned by

$$(2.12) \quad \{\bar{T}_1-\sqrt{3}\bar{T}_4,\ \sqrt{3}\bar{T}_2+\bar{T}_5,\ \bar{T}_3\} \ (\{\bar{T}_4,\ \bar{T}_2+\sqrt{3}\bar{T}_5,\ \sqrt{3}\bar{T}_3+\bar{T}_6\}, \text{ respectively}).$$

Now, we prove

**Proposition 2.4.** *A homogeneous hypersurface $\tilde{N}$ in $S^7$ with six principal curvatures is foliated by an isoparametric hypersurface with $(g, m)=(3, 1)$. A leaf has a unique intersection point with a fiber $S^3$ of $h$, if they intersect, by which we can define a section $\tau: N\to\tilde{N}$.*

Proof. Observing the signature of the Killing form, we see that the Lie algebra generated by $X_2$, $X_4$ and $X_6$ is isomorphic to $\mathfrak{so}(3)$. We denote by $SO(3)'$ the Lie subgroup of $SO(4)$ isomorphic to $SO(3)$, whose tangent space at $1_{SO(4)}$ is spanned by $X_2, X_4$ and $X_6$. It is easy to see that the subspace $\mathfrak{q}=$ span $\{H, H^{\perp}, T_2, T_4, T_6\}$ of $\mathfrak{p}$ is $\rho(SO(3)')$-invariant. Here, $\mathfrak{so}(3)+\mathfrak{q}$ gives a decomposition associated with the symmetric space $SU(3)/SO(3)'$, where the Cartan subalgebra is also given by $\mathfrak{a}$. A regular orbit of the isotropy action of $SO(3)'$ on $\mathfrak{q}$ is nothing but a tube of the Veronese surface. In fact, put $N_{\bar{H}}=\rho(SO(3)')\bar{H}$. The computation in Remark 2.2 provides the principal curvatures of $N_{\bar{H}}$:

$$(2.13) \qquad \lambda_2 = -\frac{\sqrt{3}\xi_3}{\xi_1-\xi_2}, \quad \lambda_4 = \frac{\sqrt{3}\xi_2}{\xi_1-\xi_3}, \quad \lambda_6 = -\frac{\sqrt{3}\xi_1}{\xi_2-\xi_3},$$

which shows, under (2.10), $H$ is a regular element for the isotropy action of $SU(3)/SO(3)'$, and $N_{\bar{H}}$ is an isoparametric hypersurface with $(g, m)=(3, 1)$. Then by the homogeneity of $\tilde{N}$, there passes an isoparametric hypersurface with $(g, m)=(3, 1)$ isometric to $N_{\bar{H}}$ through each point of $\tilde{N}$.

Now, we show that $\{gN_{\bar{H}}, g\in SO(4)\}$ (we omit $\rho$) gives a foliation on $\tilde{N}$. To see this, we must show that if $gN_{\bar{H}}\cap g'N_{\bar{H}}\neq\emptyset$, $g, g'\in SO(4)$, then $gN_{\bar{H}}=g'N_{\bar{H}}$, that is, the isotropy subgroup of $\bar{H}$ in $SO(4)$ belongs to $SO(3)'$. In fact, we have already seen that $\tilde{N}\simeq S^3\times N$ (Remark 2.1), and it is well known that $N\simeq SO(3)/\boldsymbol{Z}_2+\boldsymbol{Z}_2$. Let $L$ be the isotropy subgroup of $\bar{H}$ in $SO(4)$. Then $\tilde{N}\simeq SO(4)/L\simeq Sp(1)\times SO(3)/L$ by the argument before Remark 2.1. Thus we get $L\cong\boldsymbol{Z}_2+\boldsymbol{Z}_2$. Since the isotropy subgroup of $\bar{H}$ in $SO(3')'$ is $\boldsymbol{Z}_2+\boldsymbol{Z}_2$, and it is obvious that this is contained in $L$, $L\cong\boldsymbol{Z}_2+\boldsymbol{Z}_2$ implies $L\subset SO(3)'$.

Now, denote the fiber of $h$ at $\bar{H}$ by $S_{\bar{H}}$. We may show that $N_{\bar{H}}\cap S_{\bar{H}}=\{\bar{H}\}$, since $\tilde{N}$ is homogeneous and $Sp(1)$ is a normal subgroup in $SO(4)$. It is easy to see that $V\cap\mathfrak{q}=\boldsymbol{R}H$ and $N_{\bar{H}}\cap S_{\bar{H}}\subset\{\pm\bar{H}\}$. As is well-known [9] for an isoparametric hypersurface $M$ in $S^n$, a normal geodesic $\gamma$ at $x\in M$ intersects $M$ at $2g$ points $x_1=x$, $x_2$, $\cdots$, $x_{2g}$, where

$$\measuredangle(x_i, x_{i+1})=\begin{cases}\dfrac{\pi}{g}-\theta\,, & i=\text{odd}\\[2mm] \dfrac{\pi}{g}+\theta\,, & i=\text{even}\end{cases}$$

for some $\theta$, $0\leq\theta<\dfrac{\pi}{2g}$. Note that $\theta=0$ corresponds to the minimal isoparametric hypersurface. Thus $N_{\bar{H}}$ $(g=3)$ contains $-\bar{H}$ if and only if $N_{\bar{H}}$ is a regular minimal orbit. But (2.13) implies that $N_{\bar{H}}$ is minimal if and only if $\xi_1\xi_2\xi_3=0$, which dose not occur by (2.10). ■

REMARK 2.3. Certainly, another proof of Proposition 2.4 will be given by using $\mathfrak{l}^{\perp}\equiv\mathfrak{so}(3)$ mod $\mathfrak{l}$.

REMARK 2.4. Each of six curvature circles provides other foliation on $\tilde{N}$. Furthermore, $\tilde{N}$ is foliated by three kinds of Clifford tori as is shown in Remark 4.2.

**Proposition 2.5.** *Let $\tilde{N}_\pm$ be the focal submanifolds of $\tilde{N}$ coresponding to $H_\pm$. Then*

(1) *$\tilde{N}_+$ is foliated by the Veronese surface, and we have a section $\tau\colon N_+\to\tilde{N}_+$.*

(2) *$\tilde{N}_-$ is foliated by the minimal isoparametric hypersurface with $(g, m)=(3, 1)$. Moreover, $\tilde{N}_+$ and $\tilde{N}_-$ are not congruent in $S^7$.*

Proof. We know by (2.13) that $\rho(SO(3)')\bar{H}_+$ is a singular orbit, i.e. the

Veronese surface $\boldsymbol{P}_+^2$ and has a unique intersection with the fiber at $\bar{H}_+$ since $-\bar{H}_+\notin\boldsymbol{P}_+^2$. In particular, we can define a section $\tau: N_+\to\boldsymbol{P}_+^2\subset\tilde{N}_+$. On the other hand, in view of (2.13), $\rho(SO(3)')\bar{H}_-$ is the minimal isoparametric hypersurface with $(g, m)=(3, 1)$. The last assertion will be shown in the proof of Proposition 3.3. The fact that the isotropy subgroup of $\tilde{N}_+(\tilde{N}_-$, respectively) is generated by $X_6$ ($X_1$, respectively) suggests the non-congruence of $\tilde{N}_+$ and $\tilde{N}_-$. ■

## 3. Isoparametric hypersurfaces and focal submanifolds

### 3.1. Preliminaries

Let $M$ be an isometrically immersed orientable hypersurface in the unit sphere $S^{n+1}$ and let $\xi$ be a unit normal vector field on $M$. With respect to the riemannian connection $\tilde{\nabla}$ on $S^{n+1}$, the shape operator $A$ of $M$ is given by

$$AX=-\tilde{\nabla}_X\xi\,,\quad X\in T_pM\,,\quad p\in M\,,$$

of which eigenvalues $\lambda_1\geq\cdots\geq\lambda_n$ are called the principal curvatures. For $\lambda\in\{\lambda_1, \cdots, \lambda_n\}$, define

$$D_\lambda(p)=\{X\in T_pM\,|\,AX=\lambda X\}\,.$$

and let $m_\lambda(p)=\dim D_\lambda(p)$. The following is fundamental [12]:

**Fact 3.1.** *If $m_\lambda(p)$ is constant on $M$, say $m$, then*

(1) *$\lambda$ is a differentiable function on $M$.*

(2) *$D_\lambda$ is a completely integrable differentiable distribution on $M$.*

(3) *If $m\geq 2$, then $\lambda$ is constant along each leaf of $D_\lambda$.*

(4) *If $\lambda$ is constant along a leaf $L$ of $D_\lambda$, then $L$ is locally an $m$-sphere of $S^{n+1}$.*

From now on, we assume that $m_\lambda$ is constant on $M$ for all $\lambda$. Then by (2), we can choose a local orthonormal frame $(e_1, \cdots, e_n)$ so that each $e_\alpha$ is a unit principal vector with respect to $\lambda_\alpha$, $1\leq\alpha\leq n$. We call such frame *an adapted frame*. Now, we may express

$$\tilde{\nabla}_{e_\alpha}e_\beta=\Lambda_{\alpha\beta}^\sigma e_\sigma+\lambda_\alpha\delta_{\alpha\beta}\xi \tag{3.1}$$

where $1\leq\alpha, \beta\leq n$, and $\sigma$ always denotes the summation over $1\leq\sigma\leq n$. Obviously we have

$$\Lambda_{\alpha\beta}^\gamma=-\Lambda_{\alpha\gamma}^\beta\,.$$

The curvature tensor $R_{\alpha\beta\gamma\delta}$ of $M$ is given by

$$\begin{aligned}R_{\alpha\beta\gamma\delta}&=(1+\lambda_\alpha\lambda_\beta)(\delta_{\beta\gamma}\delta_{\alpha\delta}-\delta_{\alpha\gamma}\delta_{\beta\delta})\\&=e_\alpha(\Lambda_{\beta\gamma}^\delta)-e_\beta(\Lambda_{\alpha\gamma}^\delta)+\Lambda_{\beta\gamma}^\sigma\Lambda_{\alpha\sigma}^\delta-\Lambda_{\alpha\gamma}^\sigma\Lambda_{\beta\sigma}^\delta-\Lambda_{\alpha\beta}^\sigma\Lambda_{\sigma\gamma}^\delta+\Lambda_{\beta\alpha}^\sigma\Lambda_{\sigma\gamma}^\delta\,.\end{aligned} \tag{3.2}$$

The covarient derivatives of the coefficients of the second fundamental tensor $h_{\alpha\beta}=\langle Ae_\alpha, e_\beta\rangle=\lambda_\alpha\delta_{\alpha\beta}$ are given by

$$h_{\alpha\beta,\gamma}=e_\gamma(h_{\alpha\beta})-\Lambda^\sigma_{\gamma\alpha}h_{\sigma\beta}-\Lambda^\sigma_{\gamma\beta}h_{\alpha\sigma}$$

so that

$$h_{\alpha\beta,\gamma}=e_\gamma(\lambda_\alpha)\delta_{\alpha\beta}+\Lambda^\beta_{\gamma\alpha}(\lambda_\alpha-\lambda_\beta)\,. \tag{3.3}$$

The equation of Coddazi is written as

$$h_{\alpha\beta,\gamma}=h_{\beta\gamma,\alpha}=h_{\gamma\alpha,\beta} \tag{3.4}$$

from which we obtain

$$e_\beta(\lambda_\alpha)=\Lambda^\beta_{\alpha\alpha}(\lambda_\alpha-\lambda_\beta)\,,\quad \text{for}\quad \alpha\neq\beta\,. \tag{3.5}$$

For distinct principal curvatures $\lambda_\alpha$, $\lambda_\beta$, $\lambda_\gamma$, we get from (3.3) and (3.4),

$$\Lambda^\gamma_{\alpha\beta}(\lambda_\beta-\lambda_\gamma)=\Lambda^\beta_{\gamma\alpha}(\lambda_\alpha-\lambda_\beta)=\Lambda^\alpha_{\beta\gamma}(\lambda_\gamma-\lambda_\alpha)\,. \tag{3.6}$$

This implies also

$$\Lambda^\gamma_{\alpha\beta}\Lambda^\beta_{\gamma\alpha}+\Lambda^\beta_{\gamma\alpha}\Lambda^\alpha_{\beta\gamma}+\Lambda^\alpha_{\beta\gamma}\Lambda^\gamma_{\alpha\beta}=0\,. \tag{3.7}$$

Moreover from (3.3) and (3.4), we have

$$\Lambda^\gamma_{ab}=0\,,\quad \Lambda^\gamma_{aa}=\Lambda^\gamma_{bb},\quad \text{if}\quad \lambda_a=\lambda_b\neq\lambda_\gamma \text{ and } a\neq b. \tag{3.8}$$

Note that from (3.5) follows (3) of Lemma 3.1 immediately and that when $\lambda_\alpha$ is constant on $M$, we have

$$\Lambda^\gamma_{\alpha\alpha}=0\quad \text{if}\quad \lambda_\gamma\neq\lambda_\alpha\,. \tag{3.9}$$

DEFINITION. When each principal curvature is constant on $M$, $M$ is called *isoparametric*.

For fundamental facts on isoparametric hypersurfaces, see [8,9].

### 3.2. Focal submanifolds

Let $M$ be an embedded isoparametric hypersurface in $S^7$. By [8], we may assume that $M$ is closed. Moreover, we assume $(g, m)=(6, 1)$, and choose $e_1, \cdots, e_6$ as above. Note that we know from (3.8)

$$\Lambda^\gamma_{\alpha\beta}=0\quad \text{if}\quad \#\{\alpha,\beta,\gamma\}\leq 2\,,$$

and from (3.6), $\Lambda^\gamma_{\alpha\beta}$, $\Lambda^\alpha_{\beta\gamma}$, $\Lambda^\beta_{\gamma\alpha}$ vanish at the same time if $\#\{\alpha,\beta,\gamma\}=3$. For convenience, we put $\lambda=\lambda_1$, $\mu=\lambda_2$, $\nu=\lambda_3$, $\rho=\lambda_4$, $\sigma=\lambda_5$ and $\tau=\lambda_6$, where, as is well known, $\lambda_i=\cot\theta_i$, $1\leq i\leq 6$, $|\theta_i|<\frac{\pi}{2}$, $\theta_{i+1}-\theta_i\equiv\frac{\pi}{6}\bmod\pi$. In particular,

$\cot(\alpha+\frac{\pi}{2})=-\tan\alpha$ implies

$$\rho=-\frac{1}{\lambda}\,,\quad \sigma=-\frac{1}{\mu}\,,\quad \tau=-\frac{1}{\nu}\,. \tag{3.10}$$

Note that each leaf $L^i$ of $D_{\lambda_i}$ is a circle of $S^7$. Let $M_\tau$ be the focal submanifold of $M$ corresponding to $\tau=\cot\theta(\theta=\theta_6)$, that is

$$M_\tau=\{\cos\theta p+\sin\theta\xi_p\,|\,p\in M\}\,.$$

We define the projection map $f\colon M\to M_\tau$ by

$$f(p)=\cos\theta p+\sin\theta\xi_p\,.$$

In the following, we use the indices $1\leq i,j,k\leq 5$, and the Einstein convention in this region. Denote $\bar{p}=f(p)$. Then we have $T_{\bar{p}}M_\tau=\mathrm{span}\{e_i\in T_pM,\ 1\leq i\leq 5\}$, since

$$f_*e_i=\sin\theta(\tau-\lambda_i)e_i\,,\quad 1\leq i\leq 6\,,$$

where the right hand side is considered as a vector in $T_{\bar{p}}S^7$ by a parallel translation in $\boldsymbol{R}^8$. An orthonormal basis of the normal space of $M_\tau$ at $\bar{p}$ is given by $\{\eta_p, \zeta_p\}$, where

$$\eta_p=-\sin\theta p+\cos\theta\xi_p\,,\quad \zeta_p=e_6(p)\,,$$

under the identification by a parallel translation in $\boldsymbol{R}^8$. Let $\bar{\nabla}$ denote the connection on $M_\tau$ acting on tangent fields of $\boldsymbol{R}^8$ along $M_\tau$ induced from the euclidean connection $D$. Then

$$\bar{\nabla}_{e_i}\tilde{X}=\frac{1}{\sin\theta(\tau-\lambda_i)}D_{e_i}X\,,\quad 1\leq i\leq 5\,,$$

where $X$ is a tangent field of $\boldsymbol{R}^8$ along $L_p^{\lambda_i}$ in a neighborhood of $p$, and $\tilde{X}$ is a tangent field along $f(L_p^{\lambda_i})$ near $\bar{p}$ defined by $\tilde{X}(\bar{q})=X(q)$ for $q\in L_p^{\lambda_i}$, $\bar{q}=f(q)$. In particular, we have

$$\bar{\nabla}_{e_i}\tilde{e}_j=\frac{1}{\sin\theta(\tau-\lambda_i)}\{\sum_{\alpha=1}^{6}\Lambda_{ij}^{\alpha}e_\alpha+\delta_{ij}(\lambda_i\xi_p-p)\}\,.$$

We denote by $\bar{\nabla}_{e_i}^{\top}\tilde{e}_j$ ($\bar{\nabla}_{e_i}^{\perp}\tilde{e}_j$, resp.) the tangential (normal, resp.) component of $\bar{\nabla}_{e_i}\tilde{e}_j$ for $1\leq i,j\leq 5$ in $S^7$. Then we have

$$\begin{aligned}\bar{\nabla}_{e_i}^{\top}\tilde{e}_j&=\frac{1}{\sin\theta(\tau-\lambda_i)}\sum_{k=1}^{5}\Lambda_{ij}^{k}e_k\,,\\ \bar{\nabla}_{e_i}^{\perp}\tilde{e}_j&=\frac{1}{\sin\theta(\tau-\lambda_i)}\{\Lambda_{ij}^{6}e_6+\sin\theta(1+\lambda_i\tau)\delta_{ij}\eta_p\}\,,\end{aligned} \tag{3.11}$$

since $\langle \lambda_i \xi_p - p, \eta_p \rangle = \sin\theta(1+\lambda_i \tau)$.

For later use, we give the matrix representation of the shape operators $B_{\eta_p}$ and $B_{\zeta_p}$ of $M_\tau$:

**Lemma 3.1.** *The shape operators $B_{\eta_p}$ and $B_{\zeta_p}$ of $M_\tau$ with respect to the normal vectors $\eta_p$ and $\zeta_p$ at $\bar{p}$ are given respectively by symmetric matrices:*

$$B_{\eta_p} = \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} \end{pmatrix}$$

$$B_{\zeta_p} = \frac{1}{\sin\theta}$$

$$\begin{pmatrix} 0 & (\lambda-\tau)^{-1}\Lambda_{16}^2 & (\lambda-\tau)^{-1}\Lambda_{16}^3 & (\lambda-\tau)^{-1}\Lambda_{16}^4 & (\lambda-\tau)^{-1}\Lambda_{16}^5 \\ (\mu-\tau)^{-1}\Lambda_{26}^1 & 0 & (\mu-\tau)^{-1}\Delta_{26}^3 & (\mu-\tau)^{-1}\Lambda_{26}^4 & (\mu-\tau)^{-1}\Lambda_{26}^5 \\ (\nu-\tau)^{-1}\Lambda_{36}^1 & (\nu-\tau)^{-1}\Lambda_{36}^2 & 0 & (\nu-\tau)^{-1}\Lambda_{36}^4 & (\nu-\tau)^{-1}\Lambda_{36}^5 \\ (\rho-\tau)^{-1}\Lambda_{46}^1 & (\rho-\tau)^{-1}\Lambda_{46}^2 & (\rho-\tau)^{-1}\Lambda_{46}^3 & 0 & (\rho-\tau)^{-1}\Lambda_{46}^5 \\ (\sigma-\tau)^{-1}\Lambda_{56}^1 & (\sigma-\tau)^{-1}\Lambda_{56}^2) & (\sigma-\tau)^{-1}\Lambda_{56}^3 & (\sigma-\tau)^{-1}\Lambda_{56}^4 & 0 \end{pmatrix},$$

*where we use the basis $e_i \in T_pM$, $1 \leq i \leqq 5$ of $T_{\bar{p}}M_\tau$.*

Proof. Let $\tilde{\eta}$ be the vector field along $f(L_p^{\lambda_i})$ given by $\tilde{\eta}(\bar{q}) = \eta(q)$, where $q \in L_p^{\lambda_i}$, $\bar{q} = f(q)$. Since

$$B_{\eta_p}(e_i) = -\nabla_{e_i}^{\top}\tilde{\eta} = -\frac{1}{\sin\theta(\tau-\lambda_i)} D_{e_i}^{\top}\eta = \frac{1+\lambda_i\tau}{\tau-\lambda_i} e_i\,,$$

where, $D^{\top}$ denotes the component of $D_{e_i}\eta$ parallel with $T_{\bar{p}}M_\tau$, noting that $\dfrac{1+\lambda_i\tau}{\tau-\lambda_i}$ $=\cot(\theta_i-\theta)=\cot(-\dfrac{(6-i)\pi}{6})$, $i=1, \cdots, 5$, we get $B_{\eta_p}$. Similarly follows

$$B_{\zeta_p}(e_i) = -\nabla_{e_i}^{\top}\tilde{e}_6 = -\frac{1}{\sin\theta(\tau-\lambda_i)}\tilde{\nabla}_{e_i}e_6 = \frac{1}{\sin\theta(\lambda_i-\tau)}\sum_{j=1}^{5}\Lambda_{i6}^j e_j\,.$$

In particular, $B_{\zeta_p}$ is symmetric by virtue of (3.6). ■

Next, we apply above argument to the focal submanifold $M_\lambda$ corresponding to $\lambda = \cot\theta_1$. With respect to the tangent basis $e_2, e_3, \cdots, e_6$ and the normal basis $\eta_p' = -\sin\theta_1 p + \cos\theta_1 \xi$, $\zeta' = e_1(p)$ at $\bar{p} = \cos\theta_1 p + \sin\theta_1 \xi_p$, we have

(3.12)

$$B_{\eta'_p} = \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{3} \end{pmatrix}$$

$$B_{\zeta'_p} = \frac{1}{\sin\theta_1} \begin{pmatrix} 0 & (\mu-\lambda)^{-1}\Lambda^3_{21} & (\mu-\lambda)^{-1}\Lambda^4_{21} & (\mu-\lambda)^{-1}\Lambda^5_{21} & (\mu-\lambda)^{-1}\Lambda^6_{21} \\ (\nu-\lambda)^{-1}\Lambda^2_{31} & 0 & (\nu-\lambda)^{-1}\Lambda^4_{31} & (\nu-\lambda)^{-1}\Lambda^5_{31} & (\nu-\lambda)^{-1}\Delta^6_{31} \\ (\rho-\lambda)^{-1}\Lambda^2_{41} & (\rho-\lambda)^{-1}\Lambda^3_{41} & 0 & (\rho-\lambda)^{-1}\Lambda^5_{41} & (\rho-\lambda)^{-1}\Lambda^6_{41} \\ (\sigma-\lambda)^{-1}\Lambda^2_{51} & (\sigma-\lambda)^{-1}\Lambda^3_{51} & (\sigma-\lambda)^{-1}\Lambda^4_{51} & 0 & (\sigma-\lambda)^{-1}\Lambda^6_{51} \\ (\tau-\lambda)^{-1}\Lambda^2_{61} & (\tau-\lambda)^{-1}\Lambda^3_{61} & (\tau-\lambda)^{-1}\Lambda^4_{61} & (\tau-\lambda)^{-1}\Lambda^5_{61} & 0 \end{pmatrix}.$$

By these data, we know immediately that $M_\tau$ and $M_\lambda$ are minimal [10].

**Corollary 3.2.** [8]. *For any unit normal vector $n$ of $M_\tau$ or of $M_\lambda$ at $\bar{p}$, the eigenvalues of $B_n$ are given by* $\pm\sqrt{3}$, $\pm\dfrac{1}{\sqrt{3}}$, 0.

This is shown immediately by choosing $q \in L^\tau_p$ so that $n = \eta_q$ (or $q \in L^\lambda_p$ so that $n = \eta'_q$).

### 3.3. Geometric data of $\tilde{N}$

For later use, we calculate $\Lambda^\gamma_{\alpha\beta}$ and components of $B_n$ for the homegeneous hypersurface $\tilde{N}$. Recall that we choose $\xi_1, \xi_2, \xi_3$ satisfying (2.10). Note that $||H||^2 = -\frac{1}{2}\mathrm{Tr}\,H^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 = 2(\xi_1^2 + \xi_2^2 + \xi_1\xi_2)$ and so $||H|| = \sqrt{2}\sqrt{\xi_1^2 + \xi_2^2 + \xi_1\xi_2}$. For tangent field $\tilde{T}_\beta$ in a neighborhood of $\bar{H}$ given by $\tilde{T}_\beta(\rho(k)\bar{H}) = \rho(k)T_\beta$, $k \in SO(4)$, we have at $\bar{H}$, using (2.4),

$$\begin{aligned} D_{T_\alpha}\tilde{T}_\beta &= -\frac{1}{\kappa_\alpha(\bar{H})}\frac{d}{dt}\Big|_{t=0}(\mathrm{Ad}\exp tX_\alpha)T_\beta \\ &= -\frac{1}{\kappa_\alpha(\bar{H})}[X_\alpha, T_\beta] \\ &= -\frac{||H||}{\kappa_\alpha(H)}[X_\alpha, T_\beta], \end{aligned}$$

and so

$$\Lambda^\gamma_{\alpha\beta} = -\frac{||H||}{\kappa_\alpha(H)}\langle[\bar{X}_\alpha, \bar{T}_\beta], \bar{T}_\gamma\rangle,$$

which is constant in a neighborhood of $\bar{H}$. By virtue of (2.3) and $\mathfrak{g}_i \perp \mathfrak{g}_j$ for $i \neq j$, it is easy to see that $\Lambda^{\gamma}_{\alpha\beta}=0$, $1 \leq \alpha < \beta < \gamma \leq 6$, except for $\Lambda^3_{12}$, $\Lambda^6_{12}$, $\Lambda^5_{13}$, $\Lambda^6_{15}$, $\Lambda^4_{23}$, $\Lambda^6_{24}$, $\Lambda^5_{34}$ and $\Lambda^6_{45}$. Now, from $[X_1, T_2]=-T_3$ and $[X_4, T_5]=-T_3$ follow $\Lambda^6_{12}=\Lambda^6_{45}=0$, $\Lambda^3_{12}=\frac{||H||}{\sqrt{2}\kappa_1(H)}=-\frac{||H||}{\sqrt{2}\xi_2}$ and $\Lambda^5_{34}=\Lambda^3_{45}\frac{\sigma-\nu}{\rho-\sigma}=-\frac{\sqrt{3}||H||}{\sqrt{2}\kappa_4(H)}=\frac{\sqrt{3}||H||}{\sqrt{2}(\xi_1-\xi_3)}$ (see (3.6)). Similarly, we get

$$
\begin{aligned}
&[X_1, T_3] = 2T_5+3T_2 && \Rightarrow\quad \Lambda^5_{13} = -\frac{2||H||}{\sqrt{6}\kappa_1(H)} = \frac{2||H||}{\sqrt{6}\xi_2}\\
&[X_1, T_5] = -2T_3-3T_6 && \Rightarrow\quad \Lambda^6_{15} = \frac{||H||}{\sqrt{2}\kappa_1(H)} = -\frac{||H||}{\sqrt{2}\xi_2}\\
&[X_2, T_3] = T_1 && \Rightarrow\quad \Lambda^4_{23} = 0\\
&[X_2, T_4] = T_6 && \Rightarrow\quad \Lambda^6_{24} = -\frac{||H||}{\sqrt{2}\kappa_2(H)} = -\frac{||H||}{\sqrt{2}(\xi_1-\xi_2)}\,.
\end{aligned}
$$

Next, since $\cot\theta=\tau=-\frac{\sqrt{3}\xi_1}{\xi_2-\xi_3}$ by (2.9), we obtain

$$\frac{1}{\sin\theta}=-\sqrt{\left(\frac{\sqrt{3}\xi_1}{\xi_2-\xi_3}\right)^2+1}=-\frac{\sqrt{2}||H||}{\xi_2-\xi_3}\,.$$

Moreover, from (2.9), we have

$$
\begin{aligned}
\lambda-\tau &= -\frac{\xi_1-\xi_3}{\sqrt{3}\xi_2}+\frac{\sqrt{3}\xi_1}{\xi_2-\xi_3} = \frac{-||H||^2}{\sqrt{3}\xi_2(\xi_2-\xi_3)}\\
\mu-\tau &= -\frac{\sqrt{3}\xi_3}{\xi_1-\xi_2}+\frac{\sqrt{3}\xi_1}{\xi_2-\xi_3} = \frac{\sqrt{3}||H||^2}{(\xi_1-\xi_2)(\xi_2-\xi_3)}\,.
\end{aligned}
$$

Thus we get

$$\frac{1}{\sin\theta}\,\frac{\Lambda^5_{16}}{\lambda-\tau}=\sqrt{3}$$

and

$$\frac{1}{\sin\theta}\,\frac{\Lambda^4_{26}}{\mu-\tau}=-\frac{1}{\sqrt{3}}\,.$$

Finally we obtain

$$
\text{(3.13)}\qquad B_{\zeta_p}=\begin{pmatrix}
0 & 0 & 0 & 0 & \sqrt{3}\\
0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0\\
0 & 0 & 0 & 0 & 0\\
0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0\\
\sqrt{3} & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

REMARK 3.1. In computation, it is convenient to consider the minimal isoparametric hypersurface case, that is, the case when $(\xi_1, \xi_2, \xi_3)=(2+\sqrt{3}, -1, -1-\sqrt{3})$ (see (2.9)). In this case, $\|H\|=\sqrt{3}(\sqrt{3}+1)$ and above constants become

$$
(3.14)\qquad \begin{aligned}
&\Lambda_{12}^{3}=\frac{\sqrt{3}(\sqrt{3}+1)}{\sqrt{2}}, \quad \Lambda_{34}^{5}=\frac{\sqrt{6}}{\sqrt{3}+1}, \quad \Lambda_{13}^{5}=-\sqrt{2}(\sqrt{3}+1), \\
&\Lambda_{15}^{6}=\frac{\sqrt{3}(\sqrt{3}+1)}{\sqrt{2}}, \quad \Lambda_{24}^{6}=-\frac{1}{\sqrt{2}}.
\end{aligned}
$$

Thus for $M_\lambda=\tilde{N}_-$, we obtain

$$
(3.15)\qquad B_{\zeta_p'}=\begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & \frac{2}{\sqrt{3}} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{\sqrt{3}} & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}.
$$

Now, the last assertion of Proposition 2.5 is proved as follows. For any normal vector $n$ of $\tilde{N}_\pm$, let $E_\pm^n(\alpha)$ be the eigenspace of $B_n$ with respect to the eigenvalue $\alpha$. Then from Lemma 3.1 and (3.13), we see that the spaces $E_+^n(0)$, $E_+^n(\sqrt{3})\oplus E_+^n(-\sqrt{3})$, $E_+^n(\frac{1}{\sqrt{3}})\oplus E_+^n(-\frac{1}{\sqrt{3}})$ are independent of $n$. On the other hand, in view of (3.12) and (3.15), the spaces $E_-^n(\sqrt{3})\oplus E_-^n(-\sqrt{3})$ and $E_-^n(\frac{1}{\sqrt{3}})\oplus E_-^n(-\frac{1}{\sqrt{3}})$ depend on $n$, though the space $E_-^n(0)$ dose not. Finally, we conclude

**Proposition 3.3.** *$\tilde{N}_+$ and $\tilde{N}_-$ are not congruent in $S^7$. In particular, we have two minimal taut homogeneous embeddings of $P^2\mathbf{R}\times S^3$ into $S^7$ which are not congruent.*

REMARK 3.1. As is well-known, the Veronese surface is rigid in $S^4$. Thus $N_+$ is congruent to $N_-$ in $S^4$. But this congruence is given by $\sigma\in O(5)$ with $\det\sigma=-1$. This $\sigma$ is not lifted to an isometry of $S^7$.

## 4. A characterization of the homogeneous hypersurfaces

As we have seen above, $\Lambda_{14}^{\alpha}=\Lambda_{25}^{\alpha}=\Lambda_{36}^{\alpha}=0$, $1\leq\alpha\leq 6$ on $\tilde{N}$. The following argument is independent of the homogeneity.

**Lemma 4.1.** *On an embedded closed isoparametric hypersurface $M$ with $(g, m)=(6, 1)$, the following* (1), (2), (3) *are mutually equivalent:*

(1) $\Lambda_{14}^{\alpha}\equiv 0,\quad 1\leq\alpha\leq 6,\qquad$ *on* $M$.
(2) $\Lambda_{25}^{\alpha}\equiv 0,\quad 1\leq\alpha\leq 6,\qquad$ *on* $M$.
(3) $\Lambda_{36}^{\alpha}\equiv 0,\quad 1\leq\alpha\leq 6,\qquad$ *on* $M$.

REMARK 4.1. Any local adapted frame $\{e_1, e_2, \cdots, e_6\}$ differs from another at most in the directions of some $e_i'$s. Thus $\Lambda_{\alpha\beta}^{\gamma}$ differs at most in its signature by a change of local frames. We mean by "$\Lambda_{\alpha\beta}^{\gamma}\equiv 0$ on $M$" a global condition, that is, for any local adapted frame in any neighborhood of $M$, $\Lambda_{\alpha\beta}^{\gamma}$ vanishes.

Proof. Let $p\in M$ and let $\gamma$ be the normal geodesic at $p$. As we have seen in the proof of Proposition 2.4, $\gamma\cap M$ consists of twelve points $p_1, \cdots, p_{12}$ which are vertices of certain dodecagon. Using the fact that $M$ is taut, we have shown in [6] that the tangent spaces of the leaves at each point of $\gamma\cap M$ are decomposed into parallel families as follows ("parallel" means parallel with respect to the connection of $S^7$):

$$\{T_i^6, T_{i+1}^2, T_{i+2}^4, T_{i+3}^4, T_{i+4}^2, T_{i+5}^6\}$$
$$\{T_{i+1}^1, T_{i+2}^5, T_{i+3}^3, T_{i+4}^3, T_{i+5}^5, T_{i+6}^1\},$$

where $i=2, 4, 6$, and $T_j^k$ denotes the tangent space of the $\lambda_k$-leaf at $p_j$ (see Fig. 2). Now, parametrize $L_{p_1}^{\lambda}$ by the angle $\varphi$ between $p_1$ and $p\in L_{p_1}^{\lambda}$ with respect to the center $\bar{p}_1$ of $L_{p_1}^{\lambda}$ in $\boldsymbol{R}^8$, so that we have

$$\frac{d}{d\varphi}p(\varphi) = \frac{d}{d\varphi}(p(\varphi)-\bar{p}_1) = \sin\theta_1 e_1(p(\varphi)) .$$

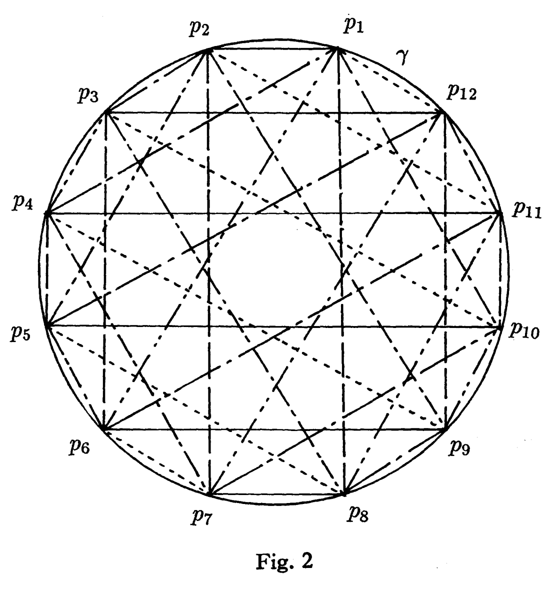


Fig. 2

Similar parametrization of $L^{\sigma}_{p_2} \ni q = q(\varphi)$, where $\varphi$ is the angle from $p_2$ in the same direction as $L^{\lambda}_{p_1}$ gives

$$\frac{d}{d\varphi} q(\varphi) = \varepsilon \sin\theta_5 e_5(q(\varphi)),$$

where $\varepsilon$ is 1 or $-1$. Using the normal geodesic at $p(\varphi)$, the above investigation implies that $e_4(p(\varphi))$ is parallel with $e_2(q(\varphi))$, hence we have

$$\begin{aligned}\tilde{\nabla}_{e_1} e_4(p(\varphi)) &= \frac{1}{\sin\theta_1} \frac{d}{d\varphi} e_4(p(\varphi)) \\ &= \frac{\sin\theta_5}{\sin\theta_1} \frac{1}{\sin\theta_5} \frac{d}{d\varphi} e_2(q(\varphi)) \\ &= \varepsilon \frac{\sin\theta_5}{\sin\theta_1} \tilde{\nabla}_{e_5} e_2(q(\varphi)).\end{aligned}$$

Since $e_2(p(\varphi)) = \varepsilon e_4(q(\varphi))$, $e_3(p(\varphi)) = \varepsilon e_3(q(\varphi))$, $e_5(p(\varphi)) = \varepsilon e_1(q(\varphi))$ and $e_6(p(\varphi)) = \varepsilon e_6(q(\varphi))$ where $\varepsilon = \pm 1$, we obtain, (continuing similar arguments also along $L^{\nu}_{p_3}, \cdots$),

$$\begin{aligned}&\Lambda^2_{14}(p_1) \sim \Lambda^4_{52}(p_2) \sim \Lambda^4_{36}(p_3) \sim \cdots \\ &\Lambda^3_{14}(p_1) \sim \Lambda^3_{52}(p_2) \sim \Lambda^5_{36}(p_3) \sim \cdots \\ &\Lambda^5_{14}(p_1) \sim \Lambda^1_{52}(p_2) \sim \Lambda^1_{36}(p_3) \sim \cdots \\ &\Lambda^6_{14}(p_1) \sim \Lambda^6_{52}(p_2) \sim \Lambda^2_{36}(p_3) \sim \cdots,\end{aligned}$$

where $\sim$ means "be equal to a non-zero constant multiple of". Since we have these relations among any corresponding points, we obtain Lemma 4.1. ■

REMARK 4.2. By virtue of (3.6), $D_\lambda + D_\rho$ is integrable if and only if $\Lambda^{\alpha}_{14} = 0$, $1 \leq \alpha \leq 6$. Lemma 4.1 implies that $D_\lambda + D_\rho$, $D_\mu + D_\sigma$, $D_\nu + D_\tau$ are integrable at the same time if one of them is integrable, which is the case for the homogeneous $\tilde{N}$. It is easy to see that each leaf is a Clifford torus.

Note that (3), or equivalently, $\Lambda^{\sigma}_{63} \equiv 0$ holds if and only if the null direction $e_3$ of $B_\eta$ is constant along $L^{\tau}_{p}$. Concerning the fact stated before Proposition 3.3, we show:

**Proposition 4.2.** *Let $M$ be an embedded closed isoparametric hypersurface in $S^7$ with six principal curvatures. Then $M$ is homogensous if and only if $e_3$ is constant on $L^{\tau}_p$ for any $p \in M$.*

Proof. Since we may show the sufficiency, assume

$$\Lambda^{\alpha}_{63} \equiv 0, \quad 1 \leq \alpha \leq 6 \qquad \text{on } M. \tag{4.1}$$

Then by Lemma 4.1, we have

$$\Lambda_{14}^{\alpha}=\Lambda_{25}^{\alpha}=\Lambda_{36}^{\alpha}\equiv 0, \quad 1\leq\alpha\leq 6 \qquad \text{on } M.$$

Using Lemma 3.1, we may write

$$B_{\zeta_p}=\begin{pmatrix} 0 & a & 0 & 0 & b \\ a & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & d \\ b & 0 & 0 & d & 0 \end{pmatrix}. \tag{4.2}$$

Now, parametrizing $L_p^{\tau}$ by the angle $\varphi$ between $p$ and $p(\varphi)\in L_p^{\tau}$ with respect to the center of $L_p^{\tau}$ in $\boldsymbol{R}^8$, we see that $\eta_\varphi=\eta_{p(\varphi)}$ is given by $\cos\varphi\eta_p-\sin\varphi\zeta_p$. Since the eigenvalues of $B_{\eta_\varphi}=\cos\varphi B_{\eta_p}-\sin\varphi B_{\zeta_p}$ are $\pm\sqrt{3}$, $\pm\frac{1}{\sqrt{3}}$, we get $\det(xI-B_{\eta_\varphi})=x(x^2-3)(x^2-\frac{1}{3})$, and so

$$\sin^2\varphi(a^2+b^2+c^2+d^2)+\frac{10}{3}\cos^2\varphi=\frac{10}{3} \tag{1}$$

$$\cos\varphi\sin^2\varphi(a^2-d^2)=0 \tag{2}$$

$$\sin^2\varphi\{-2+\sin^2\varphi+\cos^2\varphi(-a^2+\frac{b^2}{3}+3c^2-d^2)+\sin^2\varphi(aa-bc)^2\}=0 \tag{3}$$

for any $\varphi$, $0\leq\varphi<2\pi$. Solving this system of equations where we choose the direction of $e_1$ and $e_4$ so that $a, c\leq 0$, we have the following two cases:

$$(a, b, c, d)=(0, \pm\sqrt{3}, -\frac{1}{\sqrt{3}}, 0) \tag{i}$$

$$(a, b, c, d)=(-1, 0, -\frac{2}{\sqrt{3}}, \pm 1). \tag{ii}$$

In fact, we have from (1) and (2)

$$a^2+b^2+c^2+d^2=\frac{10}{3} \tag{4}$$

$$a^2=d^2. \tag{5}$$

Putting $\varphi=\frac{\pi}{2}$ and $\frac{\pi}{4}$ in (3), we obtain

$$(ad-bc)^2=1 \tag{6}$$

$$2a^2-\frac{b^2}{3}-3c^2=-2. \tag{7}$$

Then from (4)$\sim$(7), follow (i) and (ii). In case (i), we may assume $(a, b, c, d) =(0, \sqrt{3}, -\frac{1}{\sqrt{3}}, 0)$, changing $e_5$ to $-e_5$ if necessary. Note that then, $B_{\zeta_p}$ is equal to the one in the homogeneous case. When (ii) is the case, take the focal submanifold $M_\lambda$ corresponding to $\lambda$. From the second matrix in (3.12), using $\Lambda^\alpha_{36}=\Lambda^\alpha_{25}=\Lambda^\alpha_{14}\equiv 0$, $B_{\zeta'_p}$ must be of the form (4.2), but from $\Lambda^5_{16}=0$, using (5), it must be

$$B_{\zeta'_p}=\begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & \pm\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \pm\frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then (ii) is reduced to case (i) if we change the directions of the normal vector, and of some principal vectors, if necessary. Now, we investigate the case (i), i.e. the case when

$$\Lambda^\alpha_{14} = \Lambda^\alpha_{25} = \Lambda^\alpha_{36} \equiv 0\,, \quad \Lambda^2_{16} = \Lambda^5_{46} \equiv 0\,,$$
$$\frac{1}{\sin\theta}\,\frac{\Lambda^5_{16}}{\lambda-\tau} = \sqrt{3}\,, \quad \frac{1}{\sin\theta}\,\frac{\Lambda^4_{26}}{\mu-\tau} = -\frac{1}{\sqrt{3}}\,.$$

Recall the Gauss equation (3.2):

$$\text{(8)} \qquad \begin{aligned} 1+\lambda\mu = R_{1221} &= -\Lambda^\sigma_{12}\Lambda^1_{2\sigma}-\Lambda^\sigma_{12}\Lambda^1_{\sigma 2}+\Lambda^\sigma_{21}\Lambda^1_{\sigma 2} \\ &= -2\Lambda^3_{12}\Lambda^1_{23} \end{aligned}$$

where we use (3.7), and

$$\text{(9)} \qquad 1+\lambda\nu = R_{1331} = -2(\Lambda^2_{13}\Lambda^1_{32}+\Lambda^5_{13}\Lambda^1_{35})\,,$$
$$\text{(10)} \qquad 1+\rho\sigma = R_{4554} = -2\Lambda^3_{45}\Lambda^4_{53}\,,$$
$$\text{(11)} \qquad 1+\mu\rho = R_{2442} = -2(\Lambda^3_{24}\Lambda^2_{43}+\Lambda^6_{24}\Lambda^2_{46})\,.$$

Without loss of generality, we may assume that $M$ is minimal, so $\Lambda^5_{16}= -\frac{\sqrt{3}(\sqrt{3}+1)}{\sqrt{2}}$, $\Lambda^4_{26} = \frac{1}{\sqrt{2}}$. Then (11) implies

$$\Lambda^4_{23} = 0\,,$$

while (8) and (9) imply

$$(\Lambda^3_{12})^2 = \frac{3(\sqrt{3}+1)^2}{2}, \quad (\Lambda^5_{13})^2 = 2(\sqrt{3}+1)^2.$$

Since we have a freedom to choose the direction of $e_3$, we choose it so that $\Lambda_{13}^5 = -\sqrt{2}(\sqrt{3}+1)$. When it becomes $\Lambda_{12}^3 = -\frac{\sqrt{2}(\sqrt{3}+1)}{2}$, change the directions of $e_2$ and $e_4$ at the same time in order to preserve $\Lambda_{24}^6$ and $\Lambda_{13}^5$. Finally, $\Lambda_{34}^5$ is obtained uniquely from (10) and

$$0 = R_{1245} = \Lambda_{24}^{\sigma}\Lambda_{1\sigma}^5 - \Lambda_{12}^{\sigma}\Lambda_{\sigma 4}^5 + \Lambda_{21}^{\sigma}\Lambda_{\sigma 4}^5 = \Lambda_{24}^6\Lambda_{16}^5 - \Lambda_{12}^3\Lambda_{34}^5 + \Lambda_{21}^3\Lambda_{34}^5 .$$

Thus all structure constants $\Lambda_{\alpha\beta}^{\gamma}$ and the coefficients of the second fundamental tensor $h_{\alpha\beta}$ coincide with those in the homogeneous case (see (3.14)), and we conclude that $M$ is locally homogeneous. Since any embedded isoparametric hypersurface extends to a unique complete isoparametric hypersurface [8], we obtain the proposition. ■

Finally, in order to give a new proof of Dorfmeister-Neher's theorem, we may show that on any isoparametric hypersurface with $(g, m)=(6, 1)$, the shape operator of its focal submanifold has a constant null direction. This is by no means easy.

## References


[1] U. Abresch: *Isoparametric hypersurfaces with four and six principal curvatures,* Math. Ann. **264** (1983). 283–302.

[2] E. Cartan: *Familles de surfaces isoparamétriques dans les espaces à courbure constante,* Ann. di Mat. **17** (1938), 177–191.

[3] T. Cecil and P. Ryan: *Tight spherical embeddings,* Lecture Notes in Math. **838** (1981), 94–104.

[4] J. Dorfmeister and E. Neher: *Isoparametric hypersufaces, case $g=6$, $m=1$,* Comm. in Alg. **13** (1985), 2299–2368.

[5] W.Y. Hsiang and H.B. Lawson: *Minimal submanifolds of low cohomogeneity,* J. Diff. Geom. **5** (1971), 1–38.

[6] R. Miyaoka: *Dupin hypersurfaces with six principal curvatures,* Kodai Math. J. **12** (1989), 308–315.

[7] R. Miyaoka and T. Ozawa: *Construction of taut embeddings and Cecil-Ryan conjecture,* in "Geometry of manifolds (edited by K. Shiohama)," Acad. Press, London, 1989, 181–189.

[8] M.F. Münzner: *Isoparametrische Hyperflächen in Sphären I,* Math. Ann. **251** (1980), 57–71.

[9] ————: *Isoparametrische Hyperflächen in Sphären II,* Math. Ann. **256** (1981), 215–232.

[10] K. Nomizu: *Elie Cartan's work on isoparametric families of hypersurfaces,* Proc. of Symp. in Pure Math. **27** (1975), 191–200.

[11] H. Ozeki and M. Takeuchi: *On some types of isoparametric hypersurfaces in spheres II,* Tohoku Math. J. **28** (1976), 7–55.

[12] H. Reckziegel: *Krümmungsfläschen von isometrischen Immersionen in Räumen Konstanter Krümmung,* Math. Ann. **223** (1976), 169–181.

[13] R. Takagi and T. Takahashi: *On the principal curvatures of homogeneous hypersurfaces in a sphere,* in "Differentirl Geometry, in honor of K. Yano," Kinokuniya, Tokyo, 1972, 469–481.

Department of Mathematics,
Tokyo Institute of Technology,
Oh-okayama, Meguroku,
Tokyo 152, Japan