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ON SOME NEW CLASSES OF SEMIFIELD PLANES

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1. Introduction

In [9] Hiramine, Matsumoto and Oyama introduced a construction method that associates with any translation plane of order q^2 (q odd) and kernel $K \cong GF(q)$, translation planes of order q^4 and kernel $K' \cong GF(q^2)$. In this article we study the class of semifield planes of order q^4 obtained from this method and show that with a few exceptions, the members of this class are new semifield planes. This class includes some recently constructed classes of planes; namely the class presented by Boerner-Lantz in [4] and the one by Cordero in [6].

2. Notation and preliminary results

Let $\mathcal{S}=(\mathcal{S}, +, \cdot)$ be a finite semifield which is not a field. We denote by $\pi(\mathcal{S})$ the semifield plane coordinatized by $\mathcal{S}$ with respect to the points (0), (∞), $(0, 0)$ and $(1, 1)$. The dual translation plane of $\pi(\mathcal{S})$ is also a semifield plane and it is coordinatized by the semifield $\mathcal{S}^*=(\mathcal{S}, +, *)$, where $a*b=b\cdot a$. Let q be an odd prime power, $\mathcal{F}=GF(q^2)$ and $x^\tau=\bar{x}=x^q$ for $x\in\mathcal{F}$. Let π be a semifield plane obtained by the construction method of Hiramine, Matsumoto and Oyama. Then π admits a matrix spread set of the form

$$\mathcal{M} = \left\{ M(u,v) = \begin{bmatrix} u & v \\ f(v) & \bar{u} \end{bmatrix} : u, v \in \mathcal{F} \right\}$$

where $f\colon \mathcal{F}\to\mathcal{F}$ is an additive function. π is coordinatized by the semifield $\mathcal{P}=\mathcal{P}_f=(\mathcal{P}, +, \cdot)$, where $\mathcal{P}=\mathcal{F}\times\mathcal{F}$ and

$$(x, y)\cdot(u, v) = (x, y) \begin{bmatrix} u & v \\ f(v) & \bar{u} \end{bmatrix}.$$

We shall denote this plane by π_f. We define the following classes:

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$\Omega(\mathcal{F})=\{f: \mathcal{F}\to\mathcal{F}: f$ is an additive function and $\mathcal{P}_f$ is a proper semifield$\}$.

$\Lambda(\mathcal{F})=\{f\in\Omega(\mathcal{F})$: either $f(v)=a\bar{v}$ for some $a\in\mathcal{F}-GF(q)$, or $f(v)=av^{\theta}$ for some nonsquare $a\in\mathcal{F}$ and $\theta\in\mathrm{Aut}(\mathcal{F})$, $\theta\neq\tau\}$.

$\Pi(\mathcal{F})=\{\pi_f: f\in\Omega(\mathcal{F})\}$.

$\Sigma(\mathcal{F})=\{\mathcal{P}_f: \pi(\mathcal{P}_f)\in\Pi(\mathcal{F})\}$.

Notice that $\Pi(\mathcal{F})$ is the class of semifield planes of order q^4 which are obtained from the construction method of Hiramine, Matsumoto and Oyama applied to translation planes of order q^2.

Among the known classes of proper finite semifields we have the following:

(i) Cohen and Ganley commutative semifields [5]
(ii) Kantor semifields [13]
(iii) Knuth semifields of characteristic 2 [14]
(iv) Twisted fields [1] and Generalized twisted fields [2]
(v) Sandler semifields [15]
(vi) Knuth four-type semifields [14], these include the Hughes-Kleinfeld semifields [10]
(vii) Generalized Dickson semifields [8]
(viii) Boerner-Lantz semifields [4]
(ix) p-primitive type IV and type V semifields [6]

The semifield planes coordinatized by the semifields on class (viii) belong to the class $\Pi(\mathcal{F})$, see [12], Theorem 4.3, and those coordinatized by semifields on class (ix) belong to $\Pi(F)$ where $F=GF(p^2)$ and p is a prime number, see [6]. The two main results on this paper state that the only known semifields (from classes (i) to (vii)) which belong to $\Sigma(\mathcal{F})$ are the Knuth semifields which are of all four types and the Generalized Dickson semifields.

We now state some properties of $\mathcal{P}_f$ and π_f.

Lemma 1. *Let $f\in\Omega(\mathcal{F})$ and $\mathcal{P}=\mathcal{P}_f$. The nuclei of $\mathcal{P}$ are:*
(i) $\mathcal{N}_l(\mathcal{P})=\{(x, 0) : x\in\mathcal{F}\}$,
(ii) $\mathcal{N}_m(\mathcal{P})=\mathcal{N}_r(\mathcal{P})=\{(x, 0) : f(xy)=\bar{x}f(y)$, *for any* $y\in\mathcal{F}\}$

Proof. For $a=(x, y)$, $b=(u, v)$ and $c=(r, s)$ in $\mathcal{P}$ the condition $(a\cdot b)\cdot c=a\cdot(b\cdot c)$ is equivalent to the two equations

$$y(rf(v)+\bar{u}f(s)) = yf(us+v\bar{r}) \tag{2.1}$$

and

$$ys\,f(v) = y\bar{v}\,\overline{f(s)}\,. \tag{2.2}$$

Clearly, from 2.1 and 2.2, $(x, 0)\in\mathcal{N}_l(\mathcal{P})$ for $x\in\mathcal{F}$ and since $\mathcal{P}$ is not a field, (i) follows.

Assume now that $(u, v)\in\mathcal{N}_m(\mathcal{P})$. If $v\neq 0$, then from 2.2 with $y=1$ we have that $\overline{f(s)}=\frac{sf(v)}{\bar{v}}$, for any $s\in\mathcal{F}$, which implies that $c=\frac{f(v)}{\bar{v}}\in GF(q)$ and $f(s)=c\bar{s}$. This implies that $\mathcal{P}$ is a field, which is not the case. Thus, $v=0$ and from 2.1 we get that $\mathcal{N}_m(\mathcal{P})=\{(u,0): f(us)=\bar{u}f(s)$, for any $s\in\mathcal{F}\}$.

Let $(r,s)\in\mathcal{N}_r(\mathcal{P})$. Then, as above, $s=0$ and from 2.1 we get that $rf(v)=f(v\bar{r})$, for any $v\in\mathcal{F}$. By taking $x=\bar{r}$ (so $\bar{x}=r$), we have $\bar{x}f(v)=f(vx)$, for any $v\in\mathcal{F}$. This completes the proof of (ii).

The following lemma is a consequence of the previous one.

Lemma 2. *Let $f\in\Omega(\mathcal{F})$. Then $f(v)=a\bar{v}$ for some $a\in\mathcal{F}-GF(q)$ if and only if $\mathcal{N}_l(\mathcal{P})=\mathcal{N}_m(\mathcal{P})=\mathcal{N}_r(\mathcal{P})\cong\mathcal{F}$.*

3. On the class $\Pi(\mathcal{F})$

Let $f\in\Omega(\mathcal{F})$ and let π_f^* denote the dual translation plane of π_f with respect to (∞). We begin this section by showing that the semifields on classes (i)-(v) above do not coordinatize planes in $\Pi(\mathcal{F})$.

Lemma 3. *Let $f\in\Omega(\mathcal{F})$ and let S be a semifield belonging to any one of the classes* (i)-(v) *above. Then neither π_f nor π_f^* is isomorphic to $\pi(S)$.*

Proof. If $\mathcal{P}$ ($\mathcal{P}^*$) is a semifield which coordinatizes π_f (π_f^*), then $\mathcal{P}$ ($\mathcal{P}^*$) has characteristic $\neq 2$. On the other hand, if S belongs to classes (ii) or (iii), then the characteristic of S is 2 and therefore S is not isotopic to $\mathcal{P}$(or $\mathcal{P}^*$). If S belongs to class (i), then S is commutative and by using Exercise 8.10 in [11] we conclude that $\mathcal{P}$(or $\mathcal{P}^*$) is not isotopic to S. Thus in these cases $\pi_f\not\cong\pi(S)\not\cong\pi_f^*$. In [3] it is shown that a generalized twisted field plane of order p^n, p an odd prime, $n\geq 3$, admits an autotopism g whose order is a p-primitive divisor of p^n-1, i.e. $|g|\mid p^n-1$ but $|g|\nmid p^i-1$ for $1\leq i\leq n-1$. From Propositions 6.3 and 6.4 in [9] it follows that if g is an autotopism of π_f then $|g|\mid 4(q^2-1)$. Therefore if S is a generalized twisted field plane then $\pi_f\not\cong\pi(S)\not\cong\pi_f^*$. (Recall that every twisted field palne is a generalized twisted field plane, [2].)

Assume now that S belongs to class (v) above. Then the dimension of S over $\mathcal{N}_l(S)$ is ≥ 4 and $\mathcal{N}_m(S)=\mathcal{N}_r(S)$ ([15], Theorem 1). Since $\mathcal{P}$ is a 2-dimensional vector space over $\mathcal{N}_l(\mathcal{P})$, we have that $\pi_f\not\cong\pi(S)$. If $\pi_f^*\cong\pi(S)$, then by Theorem 8.2 in [11] we would have $\mathcal{F}\cong\mathcal{N}_l(\mathcal{P})\cong\mathcal{N}_r(S)=\mathcal{N}_m(S)\cong\mathcal{N}_m(\mathcal{P})=\mathcal{N}_r(\mathcal{P})\cong\mathcal{N}_l(S)$. From here we conclude that S is a 2-dimensional vector space over $\mathcal{N}_l(S)$ which is a contradiction. Thus $\pi_f^*\not\cong\pi(S)$.

Next we deal with the Knuth four-type semifields. These semifields were defined in [14]. The semifields of type II, III and IV are characterized by their nuclei; type II: $\mathcal{N}_r=\mathcal{N}_m\cong\mathcal{F}$; type III: $\mathcal{N}_l=\mathcal{N}_m\cong\mathcal{F}$ and type IV: $\mathcal{N}_l=\mathcal{N}_r\cong\mathcal{F}$. A semifield of type I has multiplication given by:

$$(x, y)\cdot(u, v) = (xu+y^{\sigma^{-2}}v^{\sigma}h,\ xv+yu^{\sigma}+y^{\sigma^{-1}}v^{\sigma}g) \tag{3.3}$$

where $(x, y), (u, v)\in\mathcal{F}\times\mathcal{F}$, $1\neq\sigma\in\mathrm{Aut}(\mathcal{F})$ and h and g are elements in $\mathcal{F}$ such that the polynomial $x^{\sigma+1}+gx-h$ is irreducible in $\mathcal{F}$. The next lemma gives the condition under which a semifield plane coordinatized by a Knuth semifield plane belongs to the class $\Pi(\mathcal{F})$.

Lemma 4. *Let $f\in\Omega(\mathcal{F})$ and let $\boldsymbol{K}$ be a Knuth four-type semifield. Then π_f or π_f^* is isomorphic to $\pi(\boldsymbol{K})$ if and only if $f(v)=a\bar{v}$ for some $a\in\mathcal{F}-GF(q)$.*

Proof. Assume that $f(v)=a\bar{v}$. Then by Lemma 2 and Corollary 7.4.2 in [14] we have that $\mathcal{P}_f$ is of all four types I, II, III, IV where $\sigma^2=1$ and $g=0$.

Let $\boldsymbol{K}$ be a Knuth four-type semifield. If $\boldsymbol{K}$ is of type II, III, or IV and if $\pi_f\cong\pi(\boldsymbol{K})$ or $\pi_f^*\cong\pi(\boldsymbol{K})$ then by ([11], Theorem 8.2) and Lemmas 1 and 2 it follows that $f(v)=a\bar{v}$. Suppose that $\boldsymbol{K}$ is of type I. If $g=0$ and $\sigma^2=1$ then from 3.3 we get that $\boldsymbol{K}=\mathcal{P}_{f_1}$ where $f_1(v)=hv^{\sigma}=h\bar{v}$. Hence, by Lemma 2, $\mathcal{F}\cong\mathcal{N}_l(\boldsymbol{K})=\mathcal{N}_m(\boldsymbol{K})=\mathcal{N}_r(\boldsymbol{K})$. Now if $\pi_f\cong\pi(\boldsymbol{K})$ or $\pi_f^*\cong\pi(\boldsymbol{K})$, then by ([11], Theorem 8.2) and Lemma 2 we have that $f(v)=a\bar{v}$. We now show that the case when $g=0$ and $\sigma^2\neq 1$ and the case when $g\neq 0$ are not possible.

Let $\mathcal{P}=\mathcal{P}_f$ and suppose that $\pi_f\cong\pi(\boldsymbol{K})$. Then $\mathcal{F}\cong\mathcal{N}_l(\mathcal{P})\cong\mathcal{N}_l(\boldsymbol{K})$. Let $(x, y)\in\mathcal{N}_l(\boldsymbol{K})$. The condition $((x, y)\cdot(0, 1))\cdot(0, s)=(x, y)\cdot((0, 1)\cdot(0, s))$, for all s is $\mathcal{F}$ is equivalent to

$$(x+y^{\sigma^{-1}}g)^{\sigma^{-2}}s^{\sigma}h = xs^{\sigma}h+y^{\sigma^{-2}}s^{\sigma 2}g^{\sigma}h\,, \tag{3.4}$$

and

$$y^{\sigma^{-2}}hs+(x+y^{\sigma^{-1}}g)^{\sigma^{-1}}s^{\sigma}g = xs^{\sigma}g+ys^{\sigma 2}h^{\sigma}+y^{\sigma^{-1}}s^{\sigma 2}g^{\sigma}g\,, \tag{3.5}$$

for any s in $\mathcal{F}$. If $g=0$ and $\sigma^2\neq 1\neq\sigma$, then 3.5 implies that $y=0$ and from 3.4 we get that $x^{\sigma^{-2}}=x$. Therefore $\mathcal{F}\cong\mathcal{N}_l(\boldsymbol{K})\subset\{(x, 0)\ x\in\mathcal{F}$ and $x^{\sigma 2}=x\}$ which implies that $\sigma^2=1$, but $\sigma^2\neq 1$. If $g\neq 0$ then from 3.4 we get that $y=0$, and from 3.5 we have that $x^{\sigma^{-1}}s^{\sigma}g=xs^{\sigma}g$. Hence $\mathcal{F}\cong\mathcal{N}_l(\boldsymbol{K})\subset\{(x, 0)\colon x\in\mathcal{F}$ and $x^{\sigma}=x\}$ and therefore $\sigma=1$, which is a contradiction. A similar argument shows that $\pi_f^*\cong\pi(\boldsymbol{K})$ is not possible.

The last class to consider is the class of generalized Dickson semifields. Let $\pi(\mathcal{D})$ be a generalized Dickson semifield plane of order q^4 which is coordinatized by the semifield $\mathcal{D}=(\mathcal{D}, +, \cdot)$ where $\mathcal{D}=\mathcal{F}\times\mathcal{F}$ and the product is given by (cf [8])

$$(x, y)\cdot(r, s) = (xr+y^{\alpha}s^{\beta}\omega,\ xs+yr^{\sigma}) \tag{3.6}$$

where α, β, σ are arbitrary automorphisms of $\mathcal{F}$ but not all the identity, and ω is a nonsquare in $\mathcal{F}$. If $(u, v)\cdot((x, y)\cdot(r, s))=((u, v)\cdot(x, y))\cdot(r, s)$ then the following two conditions must be satisfied:

$$uy^{\alpha}s^{\beta}\omega+v^{\alpha}(xs+yr^{\sigma})^{\beta}\omega = v^{\alpha}y^{\beta}r\omega+(uy+vx^{\sigma})^{\alpha}s^{\beta}\omega\ , \tag{3.7}$$

and

$$vy^{\alpha\sigma}s^{\beta\sigma}\omega^{\sigma} = v^{\alpha}y^{\beta}s\omega \tag{3.8}$$

From now on $\mathcal{D}$ will denote a generalized Dickson semifield plane of order q^4 with multiplication given by (3.6).

Under certain conditions a generalized Dickson semifield is a Knuth four-type semifield. In the next lemma we give the necessary conditions on the automorphisms α, β, σ under which $\mathcal{D}$ is a Knuth four-type semifield.

Lemma 5. *If any of the following conditions are satisfied:*
(i) $\beta=\alpha\sigma$ *and* $\beta\sigma=1$*, or*
(ii) $\alpha=1$ *and* $\sigma=\beta$*, or*
(iii) $\alpha=1$ *and* $\sigma\beta=1$
then $\mathcal{D}$ *is a Knuth four-type semifield.*

Proof. Assume that (i) is true. Then 3.7 and 3.8 become, respectively,

$$uy^{\alpha}s^{\beta}\omega = u^{\alpha}y^{\alpha}s^{\beta}\omega\ , \tag{3.9}$$

and,

$$vy^{\beta}s\omega^{\sigma} = v^{\alpha}y^{\beta}s\omega \tag{3.10}$$

From these equations we get that $(x, 0)\in\mathcal{N}_m(\mathcal{D})$ for any $x\in\mathcal{F}$ and $(r, 0)\in\mathcal{N}_r(\mathcal{D})$ for any $r\in\mathcal{F}$. Since $\mathcal{D}$ is not a field we have that $\mathcal{N}_m(\mathcal{D})=\mathcal{N}_r(\mathcal{D})\cong\mathcal{F}$ and $\mathcal{D}$ is a Knuth semifield of type II. In a similar way if (ii) or (iii) occur then $\mathcal{D}$ is a Knuth semifield of type III or IV, respectively.

In the following lemma the nuclei of $\mathcal{D}$ are given.

Lemma 6. *Assume that* $\mathcal{D}$ *is not a Knuth four-type semifield. Then the nuclei of* $\mathcal{D}$ *are:*
(i) $\mathcal{N}_l(\mathcal{D})=\{(u, 0)\in\mathcal{D}: u^{\alpha}=u\}$,
(ii) $\mathcal{N}_m(\mathcal{D})=\{(x, 0)\in\mathcal{D}: x^{\beta}=x^{\sigma\alpha}\}$, *and*
(iii) $\mathcal{N}_r(\mathcal{D})=\{(r, 0)\in\mathcal{D}: r^{\sigma\beta}=r\}$.

Proof. Let $(u, v)\in\mathcal{N}_l(\mathcal{D})$ and suppose that $v\neq0$. Then from 3.8 we get that $v\omega^{\sigma}=v^{\alpha}\omega$, $y^{\alpha\sigma}=y^{\beta}$ and $s^{\beta\sigma}=s$, for all $y, s\in\mathcal{F}$. Hence, $\alpha\sigma=\beta$ and $\beta\sigma=1$, which is a contradiction by Lemma 5 (i). Thus, $v=0$ and from 3.8 we have that $uy^{\alpha}s^{\beta}\omega=u^{\alpha}y^{\alpha}s^{\beta}\omega$ for all $y, s\in\mathcal{F}$; from this (i) follows. (ii) and (iii) are proved similarly.

In the next two lemmas the question of when a generalized Dickson semifield plane belongs to the class $\Pi(\mathcal{F})$ is answered.

Lemma 7. *Let* $f\in\Omega(\mathcal{F})$ *and* $\mathcal{P}=\mathcal{P}_f$. *Assume that* $\mathcal{U}$ *is a non-*

desarguesian semifield plane that admits a matrix spread set of the form

$$\mathcal{M}_1 = \left\{ Q(x,y) = \begin{pmatrix} x & y \\ ky^\theta & x^\varphi \end{pmatrix} : x, y \in \mathcal{F} \right\}$$

where θ, φ *are automorphisms of* $\mathcal{F}$ *and* k *is a nonsquare in* $\mathcal{F}$. *Then, if* $\pi_f \cong \mathcal{U}$, *one of the following must be true:*

(i) $\theta = \varphi = \tau$, *where* $x^\tau = \bar{x}$, *and* $f(v) = c\bar{v}$ *for some* $c \in \mathcal{F} - GF(q)$.

(ii) $f(v) = cv^\psi$, *for some* $\psi \in \mathrm{Aut}(\mathcal{F})$ *and some nonsquare* c *in* $\mathcal{F}$.

Proof. Let $\mathcal{X} = \mathcal{F} \times \mathcal{F}$. Then $\mathcal{M}^* = \{(X, XM(u, v)) : M(u, v) \in \mathcal{M}\} \cup \{(0, X)\}$ is a spread for π_f in $\mathcal{X} \oplus \mathcal{X}$. Let $\mathcal{M}_1^*$ be the spread for $\mathcal{U}$ in $\mathcal{X} \oplus \mathcal{X}$ associated with $\mathcal{M}_1$. Since $\pi_f \cong \mathcal{U}$, there is a semilinear transformation T from the $\mathcal{F}$-vector space $\mathcal{X} \oplus \mathcal{X}$ into itself that maps $\mathcal{M}^*$ onto $\mathcal{M}_1^*$. We may assume that $(X, 0)^T = (X, 0)$ and $(0, X)^T = (0, X)$, so the linear part of T has the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, for some $A, B \in GL(2, q^2)$. Let δ be the automorphism of $\mathcal{F}$ associated with T. Since T maps $(X, XM(u, v)) \in \mathcal{M}^*$ onto $(X, XA^{-1}M(u, v)^\delta B) \in \mathcal{M}_1^*$, where $(a_{ij})^\delta = (a_{ij}^\delta)$, we have that for each $M(u, v) \in \mathcal{M}$ there is a unique $Q(x, y) \in \mathcal{M}_1$ such that

$$A^{-1}M(u, v)^\delta B = Q(x, y). \tag{3.11}$$

Let $Q(a, b) = A^{-1}M(1, 0)^\delta B = A^{-1}B$, $u \in GF(q) - \{0\}$ and $u' = u^\delta$. Then $A^{-1}M(u, 0)^\delta B = u'A^{-1}B = u'Q(a, b) \in \mathcal{M}_1$, for all $u' \in GF(q) - \{0\}$. Thus, if $a \neq 0$, then $u' = (u')^\varphi$, which implies that $\varphi \in \{1, \tau\}$. Similarly, if $b \neq 1$, then $\theta \in \{1, \tau\}$. Since $A^{-1} = Q(a, b)B^{-1}$, 3.11 becomes

$$B^{-1}M(u, v)^\delta B = Q(a, b)^{-1}Q(x, y). \tag{3.12}$$

Let $\Delta = \det Q(a, b)^{-1}$ and $\mathrm{tr}(N) =$ trace of a matrix N. Since $\mathrm{tr}(B^{-1}M(u, v)^\delta B) = (u + \bar{u})^\delta \in GF(q)$, from 3.12 we have that $\mathrm{tr}(Q(a, b)^{-1}Q(x, 0)) = \Delta(a^\varphi x + ax^\varphi) \in GF(q)$, for all $x \in \mathcal{F}$. If $\varphi = 1$, then we have that $2ax\Delta \in GF(q)$, for any $x \in \mathcal{F}$, which implies that $a = 0$. Therefore if $a \neq 0$ then $\varphi = \tau$. Likewise, considering $Q(0, y)$ we get that if $b \neq 0$ then $\theta = \tau$.

First we assume that $a \neq 0$ and $b \neq 0$. Then $\theta = \varphi = \tau$ and $\mathcal{U} = \pi(\mathcal{P}_g)$, where $g(y) = k\bar{y}$. By Lemma 2 the three nuclei of $\mathcal{P}_g$ are equal and isomorphic to $\mathcal{F}$. Now (i) follows from Lemma 2. Assume that $a = 0$ and $b \neq 0$. Then $\theta = \tau$ and we may assume that $\varphi \neq \tau$. Letting $r = (yb^{-1})^\tau$, $s = x^\varphi(kb^\tau)$, $g(s) = ds^{\varphi^{-1}}$ where $d = (k\bar{b})^{\varphi^{-1}}b^{-1}$ is a nonsquare in $\mathcal{F}$ and $Q_1(r, s) = \begin{pmatrix} r & s \\ g(s) & \bar{r} \end{pmatrix}$ we have that $Q(0, b)^{-1}Q(x, y) = Q_1(r, s)$. Now 3.12 becomes

$$M(u, v)^\delta = BQ_1(r, s)B^{-1}. \tag{3.13}$$

Let $B=\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ and $e=\det B$. Then $u^\delta=e^{-1}(b_1b_4r+b_4b_2g(s)-b_1b_3s-b_2b_3\bar{r})$ and $\bar{u}^\delta=e^{-1}(-b_2b_3r-b_2b_4g(s)+b_1b_3s+b_1b_4\bar{r})$. Since $\overline{u^\delta}=\bar{u}^\delta$, with $s=0$ we get $\overline{b_1b_4e^{-1}}=b_1b_4e^{-1}$ and $\overline{b_2b_3e^{-1}}=b_2b_3e^{-1}$. Thus $b_1b_4e^{-1}$ and $b_2b_3e^{-1}$ are in $GF(q)$. Taking $r=0$ we get $\overline{b_4b_2e^{-1}d}\bar{s}^{\varphi^{-1}}-(\overline{b_1b_3e^{-1}})\bar{s}=b_1b_3e^{-1}s-b_2b_4e^{-1}ds^{\varphi^{-1}}$. If $\varphi^{-1}\neq 1$ (also $\varphi^{-1}\neq\tau$), then $b_1b_3=0$ and $b_2b_4=0$. If $\varphi^{-1}=1$ then $b_1b_3=b_2b_4d$ and $(b_1b_3e^{-1})^2=zd$ where $z=b_1b_2b_3b_4e^{-2}$. Since $z\in GF(q)$, z is a square in $GF(q^2)$, then since d is a nonsquare in $GF(q^2)$ we must have $b_1b_3=0$ and $b_3b_4=0$. So for any $\varphi\neq\tau$ we conclude that $b_1b_3=0$ and $b_2b_4=0$. Since $e\neq 0$, then $b_2=b_3=0$ or $b_1=b_4=0$. If $b_2=b_3=0$, then from 3.13 we have $v^\delta=b_1b_4^{-1}s$ abd $f(v)^\delta=e^{-1}b_4^2g(s)$. From these equations it follows that $f(v)=cv^{\varphi^{-1}}$ where c is a nonsquare in $\mathcal{F}$. If $b_1=b_4=0$, then a similar argument shows that $f(v)=cv^\varphi$ where again c is a nonsquare in $\mathcal{F}$. Thus in either case (ii) follows. The case when $a\neq 0$ and $b=0$ is handled similarly.

Lemma 8. *Let $f\in\Omega(\mathcal{F})$ and assume that $\mathcal{D}$ is not a Knuth four-type semifield. If either π_f or π_f^* is isomorphic to $\pi(\mathcal{D})$, then $f(v)=cv^\psi$ for some nonsquare c in $\mathcal{F}$ and $\psi\in\mathrm{Aut}(\mathcal{F})$, $\psi\neq\tau$.*

Proof. Assume that $\pi_f\cong\pi(\mathcal{D})$. Then from Lemmas 1 (i) and 6 (i) we have that $\mathcal{F}\cong\mathcal{N}_l(\mathcal{P}_f)\cong\mathcal{N}_l(\mathcal{D})$; this implies that $u^\alpha=u$ for all $u\in\mathcal{F}$. Hence $\alpha=1$ and 3.6 becomes $(x,y)\cdot(r,s)=(x,y)\begin{pmatrix} r & s \\ s^\beta\omega & r^\sigma \end{pmatrix}$. Let $Q(r,s)=\begin{pmatrix} r & s \\ s^\beta\omega & r^\sigma \end{pmatrix}$. Then $\{Q(r,s)\colon r,s\in\mathcal{F}\}$ is a matrix spread set for $\pi(\mathcal{D})$. Suppose now that $\pi_f^*\cong\pi(\mathcal{D})$. Then $\pi_f\cong\pi(\mathcal{D}^*)$ and $\mathcal{F}\cong\mathcal{N}_l(\mathcal{P}_f)\cong\mathcal{N}_l(\mathcal{D}^*)$, so $\mathcal{D}^*$ is a 2-dimensional vector space over $\mathcal{N}_l(\mathcal{D}^*)$. Since $\mathcal{N}_l(\mathcal{D}^*)=\mathcal{N}_r(\mathcal{D})$, from Lemma 6 (iii) we get that $\sigma\beta=1$. Let $z\in\mathcal{D}^*$ and let $(u,v)'$ be the coordinates of z with respect to the basis $(0,1)$, $(1,0)$ of $\mathcal{D}^*$ over $\mathcal{N}_l(\mathcal{D}^*)$, i.e. $(u,v)'=(u,0)*(1,0)+(v,0)*(0,1)$ where $*$ is the product in $\mathcal{D}^*$. Then $(u,v)'=(u,v^\sigma)$. Now $(r,s)'*(x,y)'=(r,s^\sigma)*(x,y^\sigma)=(x,y^\sigma)\cdot(r,s^\sigma)=(xr+y^{\alpha\sigma}s^{\sigma\beta}\omega,\ xs^\sigma+y^\sigma r^\sigma)=(xr+y^{\alpha\sigma}s\omega,\ x^{\sigma^{-1}}s+yr)'$. Letting $Q'(x,y)=\begin{pmatrix} x & y \\ y^{\alpha\sigma}\omega & x^{\sigma^{-1}} \end{pmatrix}$ we have that $(r,s)'*(x,y)'=(r,s)Q'(x,y)$. Hence, $\{Q'(x,y)\colon x,y\in\mathcal{F}\}$ is a matrix spread set for $\pi(\mathcal{D}^*)$. Therefore in either case $((\pi_f\cong\pi(\mathcal{D})$ or $\pi_f^*\cong\pi(\mathcal{D}^*)$ we may apply Lemma 7. Since $\mathcal{D}$ (and therefore $\mathcal{D}^*$) is not a Knuth four-type semifield, by Lemmas 2 and 4, case (i) of Lemma 7 does not occur; therefore the proof is complete.

We can now state our main results; their proofs follow from the lemmas.

Theorem 3.1. *Let $f\in\Omega(\mathcal{F})-\Lambda(\mathcal{F})$. Then neither π_f nor π_f^* is isomorphic to a semifield plane coordinatized by a semifield belonging to any one of the classes* (i)-(vii).

Theorem 3.2. *Let $f \in \Lambda(\mathcal{F})$. Then*

(i) *$f(v)=a\bar{v}$ for some $a \in \mathcal{F}-GF(q)$ if and only if π_f or π_f^* is isomorphic to a semifield plane coordinatized by a Knuth four-type semifield.*

(ii) *$f(v)=av^\theta$ for some nonsquare $a \in \mathcal{F}$ and $\theta \in \mathrm{Aut}(\mathcal{F})$, $\theta \neq \tau$ if and only if π_f or π_f^* is isomorphic to a semifield plane coordinatized by a generalized Dickson semifield.*

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