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ON AUSLANDER-REITEN COMPONENTS FOR CERTAIN GROUP MODULES

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Let G be a finite group and k a field of characteristic $p>0$. Let $\Gamma_s(kG)$ be the stable Auslander-Reiten quiver of the group algebra kG. By Webb's theorem , the tree class of a connected component Δ of $\Gamma_s(kG)$ is a Euclidean diagram, a Dynkin diagram or one of the infinite trees A_∞, B_∞, C_∞, D_∞, or A_∞^∞. Moreover if Δ contains the trivial kG-module k, then the graph structure of Δ has been investigated (see [21], [16] and [17]). In this paper we study a connected component of $\Gamma_s(kG)$ containing an indecomposable kG-module whose k-dimension is not divisible by p. Suppose that M is an indecomposable kG-module and $p \nmid \dim_k M$. In Section 2, we will show that M lies in a connected component isomorphic to $\boldsymbol{Z}A_\infty$ if k is algebraically closed and a Sylow p-subgroup of G is not cyclic, dihedral, semidihedral or generalized quaternion. In Section 3 we make some remarks on tensoring the component containing the trivial kG-module k with M. In Sections 4 and 5 we consider the situation where $p=2$ and a Sylow 2-gubgroup of G is dihedral of order at least 8 or semidihedral.

The notation is almost standard. All modules considered here are finite dimensional over k. We write $W \cong W'$ (*mod projectives*) for kG-modules W and W' if the projective-free part of W is isomorphic to that of W'. For an indecomposable non-projective kG-module W, we write $\mathcal{A}(W)$ to denote the Auslander-Reiten sequence (AR-sequence) $0 \to \Omega^2 W \to m(W) \to W \to 0$ terminating at W, where Ω is the Heller operator, and we write $m(W)$ to denote the middle term of $\mathcal{A}(W)$. If an exact sequence of kG-modules $\mathcal{S}$ is of the form $0 \to \Omega^2 W \oplus U' \to m(W) \oplus U \oplus U' \to W \oplus U \to 0$, where W is an indecomposable non-projective kG-module, and U, U' are proejctive or 0, we say that $\mathcal{S}$ is the AR-sequecne $\mathcal{A}(W)$ *modulo projectives*. The symbol $\otimes$ denotes the tensor product over the coefficient field k. For an exact sequence of kG-modules $\mathcal{S}$: $0 \to A \to B \to C \to 0$ and a kG-module W, we write $\mathcal{S} \otimes W$ to denote the tensor sequence $0 \to A \otimes W \to B \otimes W \to C \otimes W \to 0$. Concerning some basic facts and terminologies used here, we refer to [2], [10] and [11].

1. Preliminaries

We start by summarizing results on the graph structure of connected components of $\Gamma_s(kG)$.

Theorem 1.1 ([21], [17], [5], [9]). *Let Δ be a connected component of $\Gamma_s(kG)$. Then the tree class of Δ is A_n, $\tilde{A}_{1,2}$, $\tilde{B}_3$, A_∞, B_∞, C_∞, D_∞ or A_∞^∞. If k is algebraically closed, then the tree class is not $\tilde{B}_3$, B_∞ or C_∞. Moreover if the tree class or the reduced graph of Δ is Euclidean, then the modules in Δ lie in a block whose defect group is a Klein four group.*

Theorem 1.2 ([21], [16], [17], [7]). *Let Δ_0 be the connected component containing the trivial kG-module k, and let P be a Sylow p-subgroup of G. Then;*

(1) *If P is not cyclic, dihedral, semidihedral or generalized quaternion, then $\Delta_0 \cong \boldsymbol{Z}A_\infty$ and k lies at the end of Δ_0.*

(2) *If P is a dihedral 2-group of order at least 8, then $\Delta_0 \cong \boldsymbol{Z}A_\infty^\infty$.*

(3) *If P is a semidihedral 2-group, then $\Delta_0 \cong \boldsymbol{Z}D_\infty$ and k lies at the end of Δ_0.*

(4) *If P is a generalized quaternion 2-group, then Δ_0 is a 2-tube.*

We will need the following result on tensoring the AR-sequence by Auslander and Carlson [1].

Theorem 1.3 ([1], see also [3]). *Assume that k is algebraically closed. Let $\mathcal{A}(k)\colon 0 \to \Omega^2 k \to m(k) \to k \to 0$ be the AR-sequence terminating at the trivial kG-module k. Let M be an indecomposable kG-module. Then the tensor sequence $\mathcal{A}(k) \otimes M$: $0 \to \Omega^2 k \otimes M \to m(k) \otimes M \to M \to 0$ has the following properties.*

(i) *If $p \nmid \dim_k M$, the tensor sequence $\mathcal{A}(k) \otimes M$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives.*

(ii) *If $p \mid \dim_k M$, then the tensor sequence $\mathcal{A}(k) \otimes M$ is split.*

Concerning tensor products, we will also need the following result by Benson and Carlson [3].

Theorem 1.4([3], see also [1]). *Assume that k is algebraically closed. Let M and N be indecomposable kG-modules. Then;*

(1) *The following are equivalent.*

(a) $k \mid M \otimes N$.

(b) *$p \nmid \dim_k M$ and $N \cong M^*$. Here $M^* = \mathrm{Hom}_k(M, k)$ is the dual of M.*
Moreover if $p \nmid \dim_k M$, then the multiplicity of k in $M \otimes M^*$ is one.

(2) *Suppose that $p \mid \dim_k M$. Then for any indecomposable direct summand U of $M \otimes N$, we have $p \mid \dim_k U$.*

As an immediate consequence of Theorem 1.3, we have;

Lemma 1.5. *Assume that k is algebraically closed. Let M be an indecom-*

posable kG-module with $p \nmid \dim_k M$ *and* $\mathcal{A}(M)\colon 0\to\Omega^2M\to m(M)\to M\to 0$ *be the AR-sequence terminating at* M. *Let* W *be a kG-module, and let* $M\otimes W=(\oplus_i M_i)\oplus(\oplus_j N_j)\oplus U$, *where* M_i *and* N_j *are non-projective indecomposable kG-modules (possibly* 0) *such that* $p\nmid\dim_k M_i$ *and* $p\mid\dim_k N_j$, *and* U *is projective or* 0. *Then the tensor sequence* $\mathcal{A}(M)\otimes W\colon 0\to\Omega^2M\otimes W\to m(M)\otimes W\to M\otimes W\to 0$ *is a direct sum* $\oplus_i\mathcal{A}(M_i)$ *of the AR-sequences* $\mathcal{A}(M_i)$ *plus a split sequence* $0\to(\oplus_j\Omega^2N_j)\oplus U'\to(\oplus_j\Omega^2N_j)\oplus(\oplus_j N_j)\oplus U\oplus U'\to(\oplus_j N_j)\oplus U\to 0$, *where* U *and* U' *are projective or* 0.

Let (,) denote the inner product of the Green ring $a(kG)$ induced from $\dim_k\mathrm{Hom}(\ ,\ )$ (see [4]). For an exact sequence of kG-modlules $\mathcal{S}\colon 0\to A\to B\to C\to 0$, let $[\mathcal{S}]\in a(kG)$ be the element $[\mathcal{S}]=B-A-C$. Using the results of Benson and Parker [4, Section 3], we have the following two lemmas.

Lemma 1.6. *Assume that* k *is an algebraically closed field. Let* M *be a non-projective indecomposable kG-module and* H *a subgroup of* G. *Suppose that exactly* n *non-isomorphic indecomposable kH-modules* L_i $(i=1,2,\cdots,n)$ *satisfy* $M\mid L_i\uparrow^G$. *Let* t_i *be the multiplicity of* M *in* $L_i\uparrow^G$. *Then* $[\mathcal{A}(M)\downarrow_H]=\Sigma_{i=1}^n t_i[\mathcal{A}(L_i)]$ *as elements of the Green ring* $a(kH)$. *(*n *may be zero, and in this case, the right hand side of the above is understood to be zero.) In particular we have;*

(1) *Let* Q *be a vertex of* M *and* S *a* Q*-source of* M. *Let* $N=N_G(Q)$ *and* $T=\{g\in N\mid S^g\simeq S\}$. *Let* t *be the multiplicity of* M *in* $S\uparrow^G$. *Then* $[\mathcal{A}(M)\downarrow_Q]=t(\Sigma_{g\in N/T}[\mathcal{A}(S^g)])$.

(2) ([14, Lemma 2.3]) *Suppose that* H *is a normal subgroup of* G *and* M *is* H*-projective. Let* S *be an* H*-source of* M. *Let* $T=\{g\in G\mid S^g\simeq S\}$ *and* t *the multiplicity of* M *in* $S\uparrow^G$. *Then* $[\mathcal{A}(M)\downarrow_H]=t(\Sigma_{g\in G/T}[\mathcal{A}(S^g)])$.

(3) ([2, Proposition 2.17.10]) *The AR-sequence* $\mathcal{A}(M)$ *splits on restriction to* H *if and only if* M *is not* H*-projective.*

Proof. By [4, Theorem 3.4], it suffices to show that $(V, [\mathcal{A}(M)\downarrow_H]-\Sigma_{i=1}^n t_i[\mathcal{A}(L_i)])=0$ for any indecomposable kH-module V. Using the Frobenius reciprocity, we have $(V, [\mathcal{A}(M)\downarrow_H]-\Sigma_{i=1}^n t_i[\mathcal{A}(L_i)])=(V, [\mathcal{A}(M)\downarrow_H])-(V, \Sigma_{i=1}^n t_i[\mathcal{A}(L_i)])=(V\uparrow^G, [\mathcal{A}(M)])-\Sigma_{i=1}^n t_i(V, [\mathcal{A}(L_i)])$. Now $M\mid V\uparrow^G$ if and only if V is isomorphic to some L_i. Since k is algeblaically closed, we have $(V\uparrow^G, [\mathcal{A}(M)])=t_i$ in this case, and hence $(V, [\mathcal{A}(M)\downarrow_H]-\Sigma_{i=1}^n t_i[\mathcal{A}(L_i)])=0$ as desired.

Lemma 1.7. *Let* M *be a non-projective indecomposable kG-module. Let* $\mathcal{E}\colon 0\to\Omega^2M\to X\to M\to 0$ *be an exact sqeuence. Then;*

(1) $\mathcal{E}$ *is the AR-sequence* $\mathcal{A}(M)$ *if and only if* $(M,[\mathcal{E}])=d_M$. *Here* $d_M=\dim_k(\mathrm{End}_{kG}(M)/\mathrm{Rad}(\mathrm{End}_{kG}(M)))$.

(2) $\mathcal{E}$ *is the AR-sequence* $\mathcal{A}(M)$ *if and only if* $\mathcal{E}$ *does not split and* $(m(M),$

$[\mathcal{E}])=0$.

Proof. (1) Suppose that $\mathcal{E}$ is the AR-sequence. Then by [2, 2.18.4 Theorem] we have $(M, [\mathcal{E}])=d_M$. To show the converse assume by way of contradiction that $(M, [\mathcal{E}])=d_M$ but $\mathcal{E}$ is not the AR-sequence $\mathcal{A}(M)$. Now the exact sequence $\mathcal{E}$ does not split since $(M, [\mathcal{E}])>0$. Letting $\mathcal{A}(M)$: $0\to\Omega^2M\to m(M)\to M\to 0$ be the AR-sequence terminating at M, we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & \Omega^2M & \to & X & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & \Omega^2M & \to & m(M) & \to & M & \to & 0 \end{array}$$

Since the left-hand square is a pushout diagram, we get an exact sequence $\mathcal{E}'$: $0\to\Omega^2M\to X\oplus\Omega^2M\to m(M)\to 0$. Since $\mathcal{E}$ is not the AR-sequence $\mathcal{A}(M)$, $\mathcal{E}'$ does not split: if $\mathcal{E}'$ is a split sequence, then X is isomorphic to $m(M)$ but this implies that $\mathcal{E}$ is the AR-sequence $\mathcal{A}(M)$, a contradiction. Thus we also have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & \Omega^2M & \to & m(M) & \to & M & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \Omega^2M & \to & X\oplus\Omega^2M & \to & m(M) & \to & 0 \end{array}$$

Since the right-hand square is a pullback diagram, we get an exact sequence $\mathcal{E}''$: $0\to m(M)\to X\oplus\Omega^2M\oplus M\to m(M)\to 0$. Thus we get $[\mathcal{E}]=[\mathcal{A}(M)]+[\mathcal{E}']=[\mathcal{A}(M)]+[\mathcal{A}(M)]+[\mathcal{E}'']$. Hence we have $(M, [\mathcal{E}])=(M, [\mathcal{A}(M)]+[\mathcal{A}(M)]+[\mathcal{E}''])=2d_M+(M, [\mathcal{E}''])>d_M$, a contradiction.

(2) Suppose that $\mathcal{E}$ is the AR-sequence. Then by [2, 2.18.4 Theorem] we have $(m(M), [\mathcal{E}])=0$ since $M \not| m(M)$. Conversely suppose that $\mathcal{E}$ does not split and $(m(M), [\mathcal{E}])=0$. Let $[\mathcal{E}']$ be as in the proof of (1). Since $[\mathcal{E}]=[\mathcal{A}(M)]+[\mathcal{E}']$ and $(m(M), [\mathcal{E}])=0$, it follows that $(m(M), [\mathcal{E}'])=0$, which implies that $\mathcal{E}'$ splits. Thus X is isomorphic to $m(M)$, and hence $\mathcal{E}$ is the AR-sequence $\mathcal{A}(M)$.

REMARK. If k is algebraically closed, then $d_M=1$ for any indecomposable kG-module M.

The following two lemmas are useful for our investigation.

Lemma 1.8. *Let Δ be a connected component of $\Gamma_s(kG)$. Suppose that the tree class of Δ is A_∞. Let T: $M_1\leftarrow M_2\leftarrow\cdots\leftarrow M_n\leftarrow\cdots$ be a tree in Δ such that $\Delta\cong \boldsymbol{Z}T/\Pi$ for some admissible group of automorphisms $\Pi\subseteq \mathrm{Aut}\,\boldsymbol{Z}T$. Then $\dim_k M_n \equiv n(\dim_k M_1) \pmod p$ for all $n\geq 1$.*

Proof. We proceed by induction on n. Clearly $\dim_k M_1=1\times\dim_k M_1$ and

$\dim_k\Omega^2M_1\equiv\dim_kM_1 \pmod p$. Since the AR-sequence $\mathcal{A}(M_1)$ is of the form $0\to\Omega^2M_1\to M_2\oplus U\to M_1\to 0$, where U is projective or 0, we have $\dim_k M_2\equiv 2(\dim_k M_1) \pmod p$.

Suppose then that $\dim_k M_i\equiv\dim_k\Omega^2M_i\equiv i(\dim_k M_1) \pmod p$ for all i with $1\leq i\leq n-1$. Now we have the AR-sequence $\mathcal{A}(M_{n-1})$: $0\to\Omega^2M_{n-1}\to\Omega^2M_{n-2}\oplus M_n\oplus U\to M_{n-1}\to 0$, where U is projective or 0. Therefore $\dim_k M_n\equiv\dim_k M_{n-1}+\dim_k\Omega^2M_{n-1}-\dim_k\Omega^2M_{n-2}\equiv n(\dim_k M_1) \pmod p$.

Lemma 1.9. *Let Θ be a connected component of $\Gamma_s(kG)$.*

(1) *If the tree class of Θ is A_∞^∞, then* $\dim_k M\equiv\dim_k M' \pmod p$ *for all indecomposable kG-modules M and M' in Θ.*

(2) *Suppose that the tree class of Θ is D_∞. Let*

$$T:\ \begin{array}{l} M\leftarrow M_2\leftarrow M_3\leftarrow\cdots\leftarrow M_n\leftarrow\cdots \\ \qquad\ \downarrow \\ \qquad M' \end{array}$$

be a tree in Θ with $\Theta\simeq \boldsymbol{Z}T$. Then $\dim_k M\equiv\dim_k M' \pmod p$ *and* $\dim_k M_n\equiv 2(\dim_k M) \pmod p$ *for all $n\geq 2$.*

Proof. Let x be an element of G of order p and let $H=\langle x\rangle$. Then the group algebra kH has only p non-isomorphic indecomposable modules, say $V_1, V_2, \cdots, V_{p-1}$ and V_p, where $\dim_k V_t=t\ (1\leq t\leq p)$ and V_p is projective. For a kG-module M, let $a(t, M)$ be the multiplicity of V_t in $M\downarrow_H$.

(1) We show that $a(t, M)=a(t, M')$ for any indecomposable kG-modules M and M' in Θ and $1\leq t\leq p-1$. Let a_t be the smallest integer in $\{a(t, M)\,|\, M\in\Theta\}$ and let M_1 be a kG-module in Θ such that $a(t, M_1)=a_t$. Let T: $\cdots\to W_n\to\cdots\to W_2\to M_1\leftarrow M_2\leftarrow M_3\leftarrow\cdots\leftarrow M_n\leftarrow\cdots$ be a tree in Θ such that $\Theta\simeq\boldsymbol{Z}T/\Pi$ for some admissible group of automorphisms $\Pi\subseteq\mathrm{Aut}\,\boldsymbol{Z}T$. Then we have the AR-sequence $\mathcal{A}(M_1)$: $0\to\Omega^2M_1\to W_2\oplus M_2\oplus U\to M_1\to 0$, where U is projective or 0. Since the connected component containing M_1 is not a tube, M_1 is not periodic and in particular M_1 is not H-projective. Thus $\mathcal{A}(M_1)$ splits on restriction to H by Lemma 1.6(3) and it follows that $W_2\downarrow_H\oplus M_2\downarrow_H\oplus U\downarrow_H\simeq M_1\downarrow_H\oplus\Omega^2M_1\downarrow_H$. This implies that $a(t, W_2)+a(t, M_2)=a_t+a(t, \Omega^2M_1)$. Since $a(t, W_2)\geq a_t$, $a(t, M_2)\geq a_t$ and $a(t, \Omega^2M_1)=a_t$, we have $a(t, W_2)=a(t, M_2)=a_t$. Proceeding inductively, we obtain $a(t, M_n)=a(t, W_n)=a_t$ for all $n\geq 2$ and all t with $1\leq t\leq p-1$. Thus the result follows.

(2) Since the tree class of Θ is D_∞, all indecomposable modules in Θ are not H-projective. Hence for any indecomposable kG-module M in Θ, the AR-sequence $\mathcal{A}(M)$ splits on restriction to H by Lemma 1.6(3). We have the AR-sequences $\mathcal{A}(M)$: $0\to\Omega^2M\to M_2\oplus U\to M\to 0$ and $\mathcal{A}(M')$: $0\to\Omega^2M'\to M_2\oplus U'\to M'\to 0$, where U and U' are projective or 0. Since both $\mathcal{A}(M)$ and $\mathcal{A}(M')$ split on restriction to H, we have $\Omega^2M\downarrow_H\oplus M\downarrow_H\simeq M_2\downarrow_H\oplus U\downarrow_H$ and $\Omega^2M'\downarrow_H\oplus M'\downarrow_H\simeq M_2\downarrow_H\oplus U'\downarrow_H$. Thus we get $a(t, M_2)=2a(t, M)=2a(t, M')$ for $1\leq t\leq p-1$.

Next we show that $a(t, M_n)=a(t, M_2)=2a(t, M)$ for $1\leq t\leq p-1$ and all $n\geq 2$ by indiction on n. We have the AR-sequence $\mathcal{A}(M_2)$: $0\to\Omega^2M_2\to M_3\oplus\Omega^2M\oplus\Omega^2M'\oplus U_2\to M_2\to 0$, where U_2 is projective or 0. Since $\mathcal{A}(M_2)$ splits on restriction to H, we get $a(t, M_3)=a(t, M_2)+a(t, \Omega^2M_2)-a(t, \Omega^2M)-a(t, \Omega^2M')=a(t, M_2)$ for $1\leq t\leq p-1$. Suppose then that $a(t, M_i)=a(t, M_2)$ for all i with $2\leq i\leq n-1$. We have the AR-sequence $\mathcal{A}(M_{n-1})$: $0\to\Omega^2M_{n-1}\to\Omega^2M_{n-2}\oplus M_n\oplus U''\to M_{n-1}\to 0$, where U'' is projective or 0. As $\mathcal{A}(M_{n-1})$ splits on restriction to H, we get $a(t, M_n)=a(t, M_{n-1})+a(t, \Omega^2M_{n-1})-a(t, M_{n-2})=a(t, M_2)$ for $1\leq t\leq p-1$. Hence the result follows.

In the rest of this section, we consider the following situation.

(*) Assume that k is an algebraically closed field of characteristic $p>0$ and a Sylow p-subgroup P of G is normal. Let Ξ be a connected component of $\Gamma_s(kP)$. Assume that every module in Ξ is G-invariant. Assume furthermore that Ξ is not a tube and every arrow in Ξ is multiplicity free. Let S be an indecomposable kP-module in Ξ and M an indecomposable kG-module having S as a P-source. Let Θ be the connected component of $\Gamma_s(kG)$ containing M.

REMARK. The assumption (*) implies that P is not a Klein four group and Ξ is isomorphic to $\boldsymbol{Z}A_\infty$, $\boldsymbol{Z}D_\infty$ or $\boldsymbol{Z}A_\infty^\infty$.

Lemma 1.10. *Assume* (*). *Then all the P-sources of the indecomposable modules in Θ lie in Ξ.*

Proof. Let W be an indecomposable kG-module in Θ. Then there is a sequence of indecomposable kG-modules $M=M_1, M_2, \cdots, M_n=W$ such that M_i and M_{i+1} are connected by an irreducible map $(1\leq i\leq n-1)$. We proceed by induction on n.

By the assumption, a P-source S of $M=M_1$ lies in Ξ. Suppose then that a P-source S_{n-1} of M_{n-1} lies in Ξ. Now $M_n\mid m(M_{n-1})$ or $M_n\mid m(\Omega^{-2}M_{n-1})$, where $m(M_{n-1})$ (resp. $m(\Omega^{-2}M_{n-1})$) is the middle term of the AR-sequence $\mathcal{A}(M_{n-1})$ (resp. $\mathcal{A}(\Omega^{-2}M_{n-1})$). By Lemma 1.6 (2), we have $[\mathcal{A}(M_{n-1})\downarrow_P]=t[\mathcal{A}(S_{n-1})]$ and $[\mathcal{A}(\Omega^{-2}M_{n-1})\downarrow_P]=t[\mathcal{A}(\Omega^{-2}S_{n-1})]$, where t is the multiplicity of M_{n-1} in $S_{n-1}\uparrow^G$. This implies that a P-source of $M_n=W$ lies in Ξ.

For an indecomposable kG-module W in Θ, let φW be a (unique) P-source of W. The following fact is an immediate consequence of the result of Uno[20, Section 3].

Lemma 1.11. *Assume* (*). *Then φ induces a graph isomorphism from Θ onto Ξ.*

Proof. By [20, Theorem 3.5], the multiplicity of S in $M\downarrow_P$ is equal to that of M in $S\uparrow^G$. From Lemma 1.10 and [20, Theorem 3,7], we get the result.

2. ZA_∞-Components

In this section we consider a connected component of $\Gamma_s(kG)$ containing an indecomposable kG-module whose k-dimension is not divisible by p under the following hypothesis:

($\sharp$) k is an algebraically closed field of characteristic $p>0$ and a Sylow p-subgroup P of G is not cyclic, dihedral, semidihedral or generalized quaternion.

Theorem 2.1. *Assume ($\sharp$). Suppose that Θ is a connected component of $\Gamma_s(kG)$ and Θ contains an indecomposable kG-module whose k-dimension is not divisible by p. Then Θ is isomorphic to ZA_∞.*

Proof. The tree class of Θ is A_∞, D_∞ or A_∞^∞ by Theorem 1.1.

Step 1. The tree class of Θ is not A_∞^∞.

Proof. We shall derive a contradiction assuming that the tree class of Θ is A_∞^∞. Let T: $\cdots\to W_n\to\cdots\to W_2\to M_1\leftarrow M_2\leftarrow M_3\leftarrow\cdots\leftarrow M_n\leftarrow\cdots$ be a tree in Θ with $\Theta\cong ZT$. Note that $p\nmid \dim_k M$, $p\nmid \dim_k M_n$ and $p\nmid \dim_k W_n$ for all $n\geq 2$ from Lemma 1.9(1). On the other hand the connected component Δ_0 containing k is isomorphic to ZA_∞ by Theorem 1.2. Let T_0: $k=L_1\leftarrow L_2\leftarrow\cdots\leftarrow L_n\leftarrow\cdots$ be a tree in Δ_0 with $\Delta_0\cong ZT_0$. Let $\mathcal{A}(k)$: $0\to\Omega^2k\to L_2\oplus U\to k\to 0$ be the AR-sequence terminating at k, where U is projective or 0. Then the tensor sequence $\mathcal{A}(k)\otimes M$: $0\to\Omega^2k\otimes M\to(L_2\oplus U)\otimes M\to M\to 0$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives by Theorem 1.3. Hence it follows that $L_2\otimes M\cong M_2\oplus W_2$ (mod projectives).

In case $p=2$, this is a contradiction, since $2\mid\dim_k L_2$ by Lemma 1.8 and thus $L_2\otimes M$ does not have any odd dimensional indecomposable direct summand from Theorem 1.4(2).

In case $p>2$, applying Lemma 1.5, we have the tensor sequence $\mathcal{A}(L_2)\otimes M$: $0\to\Omega^2L_2\otimes M\to(\Omega^2k\oplus L_3)\otimes M\to L_2\otimes M\to 0$, which is a direct sum $\mathcal{A}(M_2)\oplus\mathcal{A}(W_2)$ modulo projectives, as $p\nmid\dim_k L_2$, $p\nmid\dim_k M_2$ and $p\nmid\dim_k W_2$. Hence we have $L_3\otimes M\cong M_3\oplus W_3\oplus\Omega^2M$ (mod projectives). Repeating this argument until $n=p$, we have $\mathcal{A}(L_{n-1})\otimes M$ is a direct sum of the AR-sequences modulo projectives and $M_n\oplus W_n\mid L_n\otimes M$ for $n\leq p$. In particular we obtain $M_p\oplus W_p\mid L_p\otimes M$. But this is also a contradiction, since $p\mid\dim_k L_p$ from Lemma 1.8 and thus $L_p\otimes M$ has no indecomposable direct summand whose k-dimension is not divisible by p from Theorem 1.4(2).

Step 2. The tree class of Θ is not D_∞.

Proof. Assume contrary that the tree class of Θ is D_∞. Let

$$\begin{array}{l} T\colon M\leftarrow M_2\leftarrow M_3\leftarrow\cdots\leftarrow M_n\leftarrow\cdots \text{ be a tree in } \Theta \text{ with } \Theta\in \boldsymbol{Z}T.\\ \qquad\ \ \downarrow\\ \qquad\ \ W \end{array}$$

Note that $p\not| \dim_k M$ and $p\not| \dim_k W$ from Lemma 1.9(2). Let $\mathcal{A}(k)\colon 0\to\Omega^2 k \to m(k)\to k\to 0$ be the AR-sequence terminating at k. By Theorem 1.3 the tensor sequences $\mathcal{A}(k)\otimes M$ and $\mathcal{A}(k)\otimes W$ are the AR-sequences $\mathcal{A}(M)$ modulo projectives and $\mathcal{A}(W)$ modulo projectives respectively. Hence we have $M_2\cong m(k)\otimes M\cong m(k)\otimes W$ (mod projectives). Thus $m(k)\otimes M\otimes M^*\cong m(k)\otimes W\otimes M^*$ (mod projectives). Note that $m(k)\otimes M\otimes M^*$ and $m(k)\otimes W\otimes M^*$ are the middle terms of the tensor sequences $\mathcal{A}(k)\otimes M\otimes M^*$ and $\mathcal{A}(k)\otimes W\otimes M^*$ respectively.

Let $M\otimes M^*=k\oplus(\oplus_i L_i)\oplus(\oplus_j L'_j)\oplus N$, where L_i is an indecomposable kG-module lying in Δ_0 such that $p\not| \dim_k L_i$ and L'_j is an indecomposable kG-module lying in Δ_0 such that $p|\dim_k L'_j$ and N has no indecomposable direct summand lying in Δ_0. Since the multiplicity of k in $M\otimes M^*$ is one, L_i is not isomorphic to k. By Lemma 1.5, we have $m(k)\otimes M\otimes M^*\cong m(k)\oplus(\oplus_i m(L_i))\oplus(\oplus_j(\Omega^2 L'_j\oplus L'_j))\oplus N'$ for some kG-module N'. Note that N' does not have any indecomposable direct summand lying in Δ_0. Therefore the number of indecomposable direct summands of $m(k)\otimes M\otimes M^*$ lying in Δ_0 is odd. On the other hand k is not a direct summand of $W\otimes M^*$. Therefore the number of indecomposable direct summands of $m(k)\otimes W\otimes M^*$ lying in Δ_0 is even, a contradiction.

By Steps 1 and 2, the tree class of Θ is A_∞. Since a Sylow p-subgroup P of G is not generalized quaternion, indecomposable kG-modules whose k-dimension is not divisible by p are not periodic. Hence Θ is isomorphic to $\boldsymbol{Z}A_\infty$.

Lemma 2.2. *Assume (♯). Suppose that Θ is a connected component of $\Gamma_s(kG)$ and Θ contains an indecomposable kG-module whose k-dimension is not divisible by p. Then all modules in Θ have the same vertex P.*

Proof. By Theorem 2.1, Θ is isomorphic to $\boldsymbol{Z}A_\infty$. Let M_1 be an indecomposable kG-module lying at the end of Θ. Then Lemma 1.8 implies that $p\not| \dim_k M_1$. Hence a Sylow p-subgroup P of G is a vertex of M_1 and the result follows from [20, Theorem 4.3].

Let M be an indecomposable kG-module having a Sylow p-subgroup P of G as vertex, and let S be a P-source of M. Then $p\not| \dim_k M$ if and only if $p\not| \dim_k S$ from [3, Proposition 2.4].

Proposition 2.3. *Assume (♯). Suppose that Θ is a connected component of $\Gamma_s(kG)$ containing an indecomposable kG-module whose k-dimension is not divisible by p, and let $T\colon M_1\leftarrow M_2\leftarrow\cdots\leftarrow M_n\leftarrow\cdots$ be a tree in Θ with $\Theta\cong\boldsymbol{Z}T$. Let S_1 be a P-source of M_1 and Ξ the connected component of $\Gamma_s(kP)$ containing S_1. Then we* *have P-source S_n of M_n $(n \geq 1)$ and a tree T': $S_1 \leftarrow S_2 \leftarrow \cdots \leftarrow S_n \leftarrow \cdots$ with $\Xi \simeq \boldsymbol{Z}T'$.*

Proof. Lemma 1.8 implies that $p \nmid \dim_k M_1$, and thus by the remark preceding Proposition 2.3 we have $p \nmid \dim_k S_1$. Hence both Θ and Ξ are isomorphic to $\boldsymbol{Z}A_\infty$ by Theorem 2.1.

Step 1. We may assume that P is a normal subgroup of G.

Proof. Let $N=N_G(P)$ and f the Green correspondence with respect to (G, P, N). Let Θ' be the connected component of $\Gamma_s(kN)$ containing fM. Since $p \nmid \dim_k fM_1$, Θ' is isomorphic to $\boldsymbol{Z}A_\infty$ and all modules in Θ' have the same vertex P by Theorem 2.1 and Lemma 2.2. Therefore f induces a graph isomorphism between Θ and Θ' by [13, Theorem].

Step 2. We may assume that every module in Ξ is G-invariant.

Proof. Let $H=\{g \in G \mid W^g \in \Xi$ for all $W \in \Xi\}$ be the inertia group of Ξ. Since $\Xi \simeq \boldsymbol{Z}A_\infty$, H acts on Ξ trivially. Hence H is the inertia group of S_1 and all modules in Ξ are H-invariant.

Suppose that $S_1 \uparrow^H = R_1 \oplus R_2 \oplus \cdots \oplus R_n$ is an indecomposable direct sum decomposition such that $R_1 \uparrow^G = M_1$(Note that each $R_i \uparrow^G$ is indecomposable by [12, VII. 9.6 Theorem]). Let Θ'' be the connected component of $\Gamma_s(kH)$ containing R_1. Then the inducing from H to G gives a graph isomorphism from Θ'' onto Θ by [14, Theorem].

Now we may assume that P is normal and every module in Ξ is G-invariant. Hence we can apply Lemma 1.11 and the conclusion holds.

As an immediate consequence of Proposition 2.3, we have;

Corollary 2.4. *Assume (♯). Let M be an indecomposable kG-module whose k-dimension is not divisible by p, and let S be a P-source of M. Then M lies at the end of a $\boldsymbol{Z}A_\infty$-component if and only if S lies at the end of a $\boldsymbol{Z}A_\infty$-component.*

In the rest of this section, we give examples of indecomposable kG-modules lying at the end of a $\boldsymbol{Z}A_\infty$-component.

Lemma 2.5. *Suppose that Θ is a connected component isomorphic to $\boldsymbol{Z}A_\infty$. Let T: $M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_n \leftarrow \cdots$ be a tree in Θ with $\Theta \simeq \boldsymbol{Z}T$. Suppose that all modules in Θ have the same vertex P. Let Q be a proper subgroup of P, and let N be the projective-free part of $M_1 \downarrow_Q$. Then $M_n \downarrow_Q = \oplus_{i=0}^{n-1} \Omega^{2i} N$ (mod projectives) for all $n \geq 1$.*

Proof. We proceed by induction on n. Clearly $M_1 \downarrow_Q = N$ (mod projectives) and $\Omega^2 M_1 \downarrow_Q = \Omega^2 N$ (mod projectives). Now the AR-sequence $\mathcal{A}(M_1)$ is of the form $0 \to \Omega^2 M_1 \to M_2 \oplus U \to M_1 \to 0$, where U is projective or 0. Since $\mathcal{A}(M_1)$ splits on restriction to Q by Lemma 1.6(3), we have $M_2 \downarrow_Q = \oplus_{i=0}^{1} \Omega^{2i} N$ (mod projectives).

Suppose then that $M_i\downarrow_Q=\oplus_{t=0}^{i-1}\Omega^{2t}N$ (mod projectives) for all i with $1\leq i\leq n-1$. We have the AR-seqeunce $\mathcal{A}(M_{n-1})$: $0\to\Omega^2M_{n-1}\to M_n\oplus\Omega^2M_{n-2}\oplus U\to M_{n-1}\to 0$, where U is projective or 0. Since $\mathcal{A}(M_{n-1})$ splits on restriction to Q by Lemma 1.6(3), we have $(M_n\oplus\Omega^2M_{n-2}\oplus U)\downarrow_Q\cong M_{n-1}\downarrow_Q\oplus\Omega^2M_{n-1}\downarrow_Q$. This implies that $M_n\downarrow_Q=\oplus_{t=0}^{n-1}\Omega^{2t}N$ (mod projectives).

From Theorem 2.1 and Lemmas 2.2 and 2.5, we have;

Lemma 2.6. *Assume (♯). Let Q be a proper subgroup of P. Let M be an indecomposable kG-module whose k-dimension is not divisible by p. Suppose that $N\oplus\Omega^2N\not| M\downarrow_Q$ and $N\oplus\Omega^{-2}N\not| M\downarrow_Q$ for some non-projective indecomposable direct summand N of $M\downarrow_Q$. Then M lies at the end of a $\boldsymbol{Z}A_\infty$-component.*

Corollary 2.7. *Assume (♯). Let M be an indecomposable kG-module with vertex P and S a P-source of M.*

(1) *Suppose that p is odd and* $\dim_k S=2$. *Then M lies at the end of a $\boldsymbol{Z}A_\infty$-component.*

(2) *Suppose that $p\neq 3$ and* $\dim_k S=3$. *Then M lies at the end of a $\boldsymbol{Z}A_\infty$-component.*

(3) *Suppose that $p\neq 5$ and* $\dim_k S=5$. *Then M lies at the end of a $\boldsymbol{Z}A_\infty$-component.*

Proof. There exists an element x of P such that x does not act on S trivially. Let $Q=\langle x\rangle$. Then $S\downarrow_Q$ satisfies the assumption in Lemma 2.6. Therefore S lies at the end of a $\boldsymbol{Z}A_\infty$-component, and M lies at the end of a $\boldsymbol{Z}A_\infty$-component by Corollary 2.4.

REMARK. In [8], Erdmann proved that there are infinitely many kP-modules of dimension 2 or 3 lying at the ends of $\boldsymbol{Z}A_\infty$-components under the hypothesis (♯) ([8, Propositions 4.2 and 4.4]). Consequently she showed that for a block B over an algebraically closed field, the stable Auslander-Reiten quiver $\Gamma_s(B)$ has infinitely many components isomorphic to $\boldsymbol{Z}A_\infty$ if a defect group of B is not cyclic, dihedral, semidihedral or generalized quaternion ([8, Theorem 5.1]).

3. Remarks on Tensoring with a Certain Module

Suppose that M is an indecomposable kG-module such that $p\not|\dim_k M$, and let Θ be the connected component of $\Gamma_s(kG)$ containing M. Let Δ_0 be the connected component of $\Gamma_s(kG)$ containing the trivial kG-module k. In this section we consider tensoring modules in Δ_0 with M under the same hypothesis as in Section 2:

(♯) k is an algebraically closed field of characteristic $p>0$ and a Sylow

p-subgroup P of G is not cyclic, dihedral, semidihedral or generalized quaternion.

Thus both Θ and Δ_0 are isomorphic to $\boldsymbol{Z}A_\infty$ by Theorem 2.1. We fix some notation: T_0: $k=L_1\leftarrow L_2\leftarrow L_3\leftarrow\cdots\leftarrow L_n\leftarrow$ is a tree in Δ_0 with $\Delta_0\cong\boldsymbol{Z}T_0$.

Proposition 3.1. *Assume (♯). Suppose that M is an indecomposable kG-module such that $p\nmid\dim_k M$ and M lies at the end of its component Θ. Let S be a P-source of M. Let Ξ and Λ_0 be the connected components of $\Gamma_s(kP)$ containing S and the trivial kP-module k respectively. Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ if and only if tensoring with S induces a graph isomorphism from Λ_0 onto Ξ.*

REMARK. The assumption in Proposition 3.1 implise that both Λ_0 and Ξ are isomorphic to $\boldsymbol{Z}A_\infty$ and S lies at the end of Ξ by Theorem 2.1 and Corollary 2.4.

Proof of Proposition 3.1. Let T: $M=M_1\leftarrow M_2\leftarrow\cdots\leftarrow M_n\leftarrow\cdots$ be a tree in Θ with $\Theta\cong\boldsymbol{Z}T$. Then we have P-sources S_n of M_n $(n\geq 1)$ and a tree T': $S=S_1\leftarrow S_2\leftarrow\cdots\leftarrow S_n\leftarrow\cdots$ with $\Xi\cong\boldsymbol{Z}T'$ by Proposition 2.3. Let T'': $k=H_1\leftarrow H_2\leftarrow H_3\leftarrow\cdots\leftarrow H_n\leftarrow\cdots$ be a tree in Λ_0 with $\Lambda_0\cong\boldsymbol{Z}T''$.

Suppose that the tensoring with S_1 induces a graph isomorphism from Λ_0 onto Ξ. This means that $H_n\otimes S_1\cong S_n$ (mod projectives) and $\mathcal{A}(H_n)\otimes S_1$ is the AR-sequence $\mathcal{A}(S_n)$ modulo projectives for $n\geq 1$. We show that $L_n\otimes M_1\cong M_n$ (mod projectives) for all $n\geq 1$ by induction on n. Clearly $L_1\otimes M_1=k\otimes M_1\cong M_1$. By Theorem 1.3, $\mathcal{A}(k)\otimes M_1$ is the AR-sequence $\mathcal{A}(M_1)$ modulo projectives. Hence $L_2\otimes M_1\cong M_2$ (mod projectives). Suppose then that $L_i\otimes M_1\cong M_i$ (mod projectives) for all i with $1\leq i\leq n-1$. We claim that $\mathcal{A}(L_{n-1})\otimes M_1$ is the AR-sequence $\mathcal{A}(M_{n-1})$ modulo projectives: Since $L_{n-1}\,|\,L_{n-1}\otimes M_1\otimes M_1^*$ by Theorem 1.4, we have $0\neq(L_{n-1}\otimes M_1\otimes M_1^*,\ [\mathcal{A}(L_{n-1})])=(L_{n-1}\otimes M_1,\ [\mathcal{A}(L_{n-1})\otimes M_1])$. This implies that $\mathcal{A}(L_{n-1})\otimes M_1$ does not split. Thus in order to show that $\mathcal{A}(L_{n-1})\otimes M_1$ is the AR-sequence $\mathcal{A}(M_{n-1})$ modulo projectives, it is enough to show that $(m(M_{n-1}),\ [\mathcal{A}(L_{n-1})\otimes M_1])=0$ by Lemma 1.7(2). From Proposition 2.3, we have $m(M_{n-1})\,|\,m(S_{n-1})\uparrow^G$ and $M_1\,|\,S_1\uparrow^G$. Thus it follows that $(m(S_{n-1})\uparrow^G,\ [\mathcal{A}(L_{n-1})\otimes(S_1\uparrow^G)])\geq(m(M_{n-1}),\ [\mathcal{A}(L_{n-1})\otimes M_1])\geq 0$. Now we have $(m(S_{n-1})\uparrow^G,\ [\mathcal{A}(L_{n-1})\otimes(S_1\uparrow^G)])=(m(S_{n-1}),\ [\mathcal{A}(L_{n-1})\downarrow_P\otimes(S_1\uparrow^G)\downarrow_P])$ from the Frobenius reciprocity. By the Mackey decomposition theorem, we have $(S_1\uparrow^G)\downarrow_P=\bigoplus_{g\in P\backslash G/P}(S_1^g\downarrow_{P\cap P^g})\uparrow^P$. Since $[\mathcal{A}(L_{n-1})\downarrow_P]=[\mathcal{A}(H_{n-1})]$ as elements of the Green ring $a(kP)$ by Lemma 1.6(1), we get $[\mathcal{A}(L_{n-1})\downarrow_P\otimes(S_1\uparrow^G)\downarrow_P]=\sum_{g\in N_G(P)/P}[\mathcal{A}(S_{n-1}^g)]$ by our assumption. Since $S_{n-1}^g\not\cong m(S_{n-1})$ for any g in $N_G(P)$, we get $(m(S_{n-1}),\ [\mathcal{A}(L_{n-1})\downarrow_P\otimes(S_1\uparrow^G)\downarrow_P])=0$. Thus we obtain $(m(M_{n-1}),\ [\mathcal{A}(L_{n-1})\otimes M_1])=0$ as desired. Therefore $\mathcal{A}(L_{n-1})\otimes M_1$: $0\rightarrow\Omega^2L_{n-1}\otimes M_1\rightarrow(\Omega^2L_{n-2}\oplus L_n)\otimes M_1\rightarrow L_{n-1}\otimes M_1\rightarrow 0$ is the AR-sequence $\mathcal{A}(M_{n-1})$ modulo projectives. This implies that $L_n \otimes M_1 \cong M_n$ (mod projectives).

Conversely suppose that the tensoring with M_1 induces a graph isomorphism from Δ_0 onto Θ. We show that $H_n \otimes S_1 \cong S_n$ (mod projectives) for all $n \geq 1$ by induction on n. Clearly $H_1 \otimes S_1 = k \otimes S_1 \cong S_1$. By Theorem 1.3, $\mathcal{A}(k) \otimes S_1$ is the AR-seqeunce $\mathcal{A}(S_1)$ modulo projectives. Hence $H_2 \otimes S_1 \cong S_2$ (mod projectives). Suppose then that $H_i \otimes S_1 \cong S_i$ (mod projectives) for all i with $1 \leq i \leq n-1$. We claim that $\mathcal{A}(H_{n-1}) \otimes S_1$ is the AR-sequence $\mathcal{A}(S_{n-1})$ modulo projectives: Since $H_{n-1} \otimes S_1 \cong S_{n-1}$ (mod projectives) and $\Omega^2 H_{n-1} \otimes S_1 \cong \Omega^2 S_{n-1}$ (mod projectives), it is enough to show that $(m(S_{n-1}), [\mathcal{A}(H_{n-1}) \otimes S_1]) = 0$ by Lemma 1.7(2). From Lemma 1.6(1), we have $m(S_{n-1}) \mid m(M_{n-1}){\downarrow_P}$, $S_1 \mid M_1{\downarrow_P}$ and $[\mathcal{A}(H_{n-1})] = [\mathcal{A}(L_{n-1}){\downarrow_P}]$. Hence it follows that $(m(S_{n-1}), [\mathcal{A}(L_{n-1}){\downarrow_P} \otimes (M_1{\downarrow_P})]) \geq (m(S_{n-1}), [\mathcal{A}(H_{n-1}) \otimes S_1]) \geq 0$. Using the Frobenius reciprocity, we have $(m(S_{n-1}), [\mathcal{A}(L_{n-1}){\downarrow_P} \otimes (M_1{\downarrow_P})]) = (m(S_{n-1}){\uparrow^G}, [\mathcal{A}(L_{n-1}) \otimes M_1]) = (m(S_{n-1}){\uparrow^G}, [\mathcal{A}(M_{n-1})])$, which is zero since $m(S_{n-1}) = S_n \oplus \Omega^2 S_{n-2}$ yields $M_{n-1} \nmid m(S_{n-1}){\uparrow^G}$. This implies that $(m(S_{n-1}), [\mathcal{A}(H_{n-1}) \otimes S_1]) = 0$ as desired. Therefore $\mathcal{A}(H_{n-1}) \otimes S_1$: $0 \to \Omega^2 H_{n-1} \otimes S_1 \to (\Omega^2 H_{n-2} \oplus H_n) \otimes S_1 \to H_{n-1} \otimes S_1 \to 0$ is the AR-sequence $\mathcal{A}(S_{n-1})$ modulo projectives. This implies that $H_n \otimes S_1 \cong S_n$ (mod projectives).

Corollary 3.2. *Let M be a trivial source module with vertex P. Let Θ be the connected component of $\Gamma_s(kG)$ containing M. Then Θ is isomorphic to $\boldsymbol{Z}A_\infty$ and M lies at the end of Θ. Moreover tensoring with M induces a graph isomorphism from Δ_0 onto Θ.*

Proof. Proposition 2.3 and Corollary 2.4 imply that Θ is isomorphic to $\boldsymbol{Z}A_\infty$ and M lies at the end of Θ. The second statement follows by Proposition 3.1.

In the following, we give some conditions each of which implies that tensoring an indecomposable kG-module M induces a graph isomorphism from Δ_0 onto a component isomorphic to $\boldsymbol{Z}A_\infty$.

Proposition 3.3. *Assume ($\sharp$). Let M be an indecomposable kG-module such that $p \nmid \dim_k M$, and let Θ be the connected component of $\Gamma_s(kG)$ containing M. Let Q be a proper subgroup of P. Suppose that M satisfies the following conditions (with respect to Q).*

(1) *The trivial kQ-module k is a direct summand of $(M \otimes M^*){\downarrow_Q}$ with multiplicity one;*

(2) *If Q is generalized quaternion, then $\Omega^2 k \nmid (M \otimes M^*){\downarrow_Q}$.*

Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ.

REMARK. (i) From Theorem 1.4, the above condition (1) is equivalent to the following condition:

(1′) We have an indecomposable direct sum decomposition $N \oplus (\oplus_i W_i)$ of

$M\downarrow_Q$, where $p\nmid \dim_k N$ and $p\mid \dim_k W_t$ for all t.

(ii) Θ is isomorphic to $\boldsymbol{Z}A_\infty$ by Theorem 2.1. Moreover M lies at the end of Θ by Lemma 2.6.

In ordre to prove Proposition 3.3, we need the following.

Lemma 3.4. *Under the same assumption as in Proposition* 3.3, L_n *is a direct summand of* $L_n\otimes M\otimes M^*$ *with multiplicity one for all* $n\geq 1$.

Proof. Note that L_n is a direct summand of $L_n\otimes M\otimes M^*$ since $k\mid M\otimes M^*$. From Lemma 2.5, we have $L_n\downarrow_Q=\oplus_{t=0}^{n-1}\Omega^{2t}k$ (mod projectives). Since the multiplicity of k in $(M\otimes M^*)\downarrow_Q$ is one (and $\Omega^2 k$ is not a direct summand of $(M\otimes M^*)\downarrow_Q$ if Q is generalized quaternion), it follows that $2(\oplus_{i=0}^{n-1}\Omega^{2t}k)\nmid (L_n\otimes M\otimes M^*)\downarrow_Q$. This implies that the multiplicity of L_n in $L_n\otimes M\otimes M^*$ is one.

Proof of Proposition 3.3. Let $T\colon M=M_1\leftarrow M_2\leftarrow M_3\leftarrow\cdots\leftarrow M_n\leftarrow\cdots$ be a tree in Θ with $\Theta\cong \boldsymbol{Z}T$. We show that $L_n\otimes M\cong M_n$ (mod proejctives) for all $n\geq 1$ by induction on n.

Clearly $L_1\otimes M=k\otimes M_1\cong M_1$. Let $\mathcal{A}(k)\colon 0\to\Omega^2k\to L_2\oplus U\to k\to 0$ be the AR-sequence terminating at k, where U is projective or 0. Then the tensor sequence $\mathcal{A}(k)\otimes M$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives by Theorem 1.3. Hence $L_2\otimes M\cong M_2$ (mod projectives).

Suppose then that $L_i\otimes M\cong M_i$ (mod projectives) for all i with $1\leq i\leq n-1$. We claim that $\mathcal{A}(L_{n-1})\otimes M$ is the AR-sequence $\mathcal{A}(M_{n-1})$ modulo projectives: By lemma 1.7(1), it suffices to show that $(M_{n-1}, [\mathcal{A}(L_{n-1})\otimes M])=1$. Since L_{n-1} is a direct summand of $L_{n-1}\otimes M\otimes M^*$ with multiplicity one by Lemma 3.4, we have $(M_{n-1}, [\mathcal{A}(L_{n-1})\otimes M])=(L_{n-1}\otimes M, [\mathcal{A}(L_{n-1})\otimes M])=(L_{n-1}\otimes M\otimes M^*, [\mathcal{A}(L_{n-1})])=1$ as desired.

Now $\mathcal{A}(L_{n-1})\otimes M\colon 0\to\Omega^2L_{n-1}\otimes M\to(\Omega^2L_{n-2}\oplus L_n\oplus U')\otimes M\to L_{n-1}\otimes M\to 0$ is the AR-sequence $\mathcal{A}(M_{n-1})$ modulo projectives, where U' is projective or 0. Thus we get $L_n\otimes M\cong M_n$ (mod projectives).

Corollary 3.5. (1) *Suppose that p is odd. Let M be an indecomposable kG-module with vertex P and S a P-source of M. Suppose that* $\dim_k S=2$. *Then tensoring with M induces a graph isomorphism from Δ_0 onto the connected component containing M.*

(2) *Suppose that $p=2$. Let M be an indecomposable kG-module with vertex P and S a P-source of M. Suppose that* $\dim_k S=3$. *Then tensoring with M induces a graph isomorphism from Δ_0 onto the connected component containing M.*

Proof. The result follows from Corollary 2.7 and Propositions 3.1 and 3.3.

Proposition 3.6. *Assume* (♯). *Let M be an indecomposable kG-module with $p\nmid \dim_k M$, and let Θ be the connected component containing M. Suppose* *that M satisfies the following conditions.*

(1) *M lies at the end of Θ.*

(2) *$M\otimes M^*\cong k\oplus(\oplus_t W_t)$, where each W_t is indecomposable and $p\,|\dim_k W_t$.*
Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ.

In order to prove Proposition 3.6, we need the following.

Lemma 3.7([22, p.16, Konstruktionslemma]). *Let M and N be non-projective indecomposable kG-modules and*

$$\begin{tikzcd}[column sep=small] & & N \arrow[dr, "\beta"] & & \\ \mathcal{E}\colon 0 \arrow[r] & \Omega^2 M \arrow[ur, "\alpha"] \arrow[dr] & & M \arrow[r] & 0 \\ & & N' \arrow[ur] & & \end{tikzcd}$$

an exact sequence. Suppose that $\alpha\colon \Omega^2M\to N$ and $\beta\colon N\to M$ are irreducible maps and $N \not| N'$. Then $\mathcal{E}$ is the AR-sequence $\mathcal{A}(M)$.

Proof of Proposition 3.6. Let $T\colon M=M_1\leftarrow M_2\leftarrow M_3\leftarrow\cdots\leftarrow M_n\leftarrow\cdots$ be a tree in Θ with $\Theta\cong \boldsymbol{Z}T$. We will show that $L_n\otimes M\cong M_n$ (mod projectives) and the tensor sequence $\mathcal{A}(L_n)\otimes M$ is the AR-sequence $\mathcal{A}(M_n)$ modulo projectives for all $n\geq 1$ by induction on n. Clearly $L_1\otimes M=k\otimes M_1\cong M_1$. By Theorem 1.3, the tensor sequence $\mathcal{A}(k)\otimes M$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives. Hence $L_2\otimes M\cong M_2$ (mod projectives).

Suppose then that $L_i\otimes M\cong M_i$ (mod projectives) for all i with $1\leq i\leq n-1$ and the tensor sequence $\mathcal{A}(L_i)\otimes M$ is the AR-sequence $\mathcal{A}(M_i)$ modulo projectives for all i with $1\leq i\leq n-2$. We will show that the tensor sequence $\mathcal{A}(L_{n-1})\otimes M$ is the AR-sequence $\mathcal{A}(M_{n-1})$ modulo projectives.

Now $\mathcal{A}(L_{n-2})\otimes M\colon 0\to\Omega^2L_{n-2}\otimes M\to\Omega^2L_{n-3}\otimes M\oplus L_{n-1}\otimes M\to L_{n-2}\otimes M\to 0$ and $\mathcal{A}(\Omega^2L_{n-2})\otimes M\colon 0\to\Omega^4L_{n-2}\otimes M\to\Omega^4L_{n-3}\otimes M\oplus\Omega^2L_{n-1}\otimes M\to\Omega^2L_{n-2}\otimes M\to 0$ are the AR-sequences $\mathcal{A}(M_{n-2})$ modulo projectives and $\mathcal{A}(\Omega^2M_{n-2})$ modulo projectives respectively. Let $\alpha\colon\Omega^2L_{n-1}\to\Omega^2L_{n-2}$ and $\beta\colon\Omega^2L_{n-2}\to L_{n-1}$ be irreducible maps. Then $\alpha\otimes id_M\colon\Omega^2L_{n-1}\otimes M\to\Omega^2L_{n-2}\otimes M$ is an irredicible map $\Omega^2M_{n-1}\to\Omega^2M_{n-2}$ plus some split map from the projective part of $\Omega^2L_{n-1}\otimes M$ to the projective part of $\Omega^2L_{n-2}\otimes M$, and $\beta\otimes id_M\colon\Omega^2L_{n-2}\otimes M\to L_{n-1}\otimes M$ is an irreducible map $\Omega^2M_{n-2}\to M_{n-1}$ plus some split map from the projective part of $\Omega^2L_{n-2}\otimes M$ to the projective part of $L_{n-1}\otimes M$.

Consider the tensor sequence $\mathcal{A}(L_{n-1})\otimes M$:

$$\begin{tikzcd}[column sep=small] & & \Omega^2L_{n-2}\otimes M \arrow[dr, "\beta\otimes id_M"] & & \\ 0 \arrow[r] & \Omega^2L_{n-1}\otimes M \arrow[ur, "\alpha\otimes id_M"] \arrow[dr] & & L_{n-1}\otimes M \arrow[r] & 0\,. \\ & & L_n\otimes M \arrow[ur] & & \end{tikzcd}$$

Here $\Omega^2 M_{n-2} \nmid L_n \otimes M$: Assume not. Then $\Omega^2 M_{n-2} | L_n \otimes M$ and $\Omega^2 M_{n-2} \otimes M^* | L_n \otimes M \otimes M^*$. Now by the inductive hypothesis $L_{n-2} \otimes M \cong M_{n-2}$ (mod projectives) and $\Omega^2 L_{n-2} \otimes M \cong \Omega^2 M_{n-2}$ (mod projectives). Thus the condition (2) implies that $\Omega^2 M_{n-2} \otimes M^* \cong \Omega^2 L_{n-2} \oplus (\oplus_t W'_t)$, where each W'_t is indecomposable and $p | \dim_k W'_t$. Also the condition (2) implies that $L_n \otimes M \otimes M^* \cong L_n \oplus (\oplus_t W''_t)$, where each W''_t is indecomposable and $p | \dim_k W''_t$. This implies that $L_n \cong \Omega^2 L_{n-2}$, a contradiction.

Now the tensor sequence $\mathcal{A}(L_{n-1}) \otimes M$ satisfies the assumption in Lemma 3.7. Thus $\mathcal{A}(L_{n-1}) \otimes M$ is the AR-sequence $\mathcal{A}(M_{n-1})$ modulo projectives. This implies that $L_n \otimes M \cong M_n$ (mod projectives).

Corollary 3.8. *Assume (♯). Suppose that M is an endotrivial kG-module. Let Θ be the connected component containing M. Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ.*

Proof. Let $\mathcal{A}(k)\colon 0 \to \Omega^2 k \to L_2 \oplus U \to k \to 0$ be the AR-sequence. Here L_2 is non-projective indecomposable and U is projective or 0 by our assumption. By Theorem 1.3, the tensor sequence $\mathcal{A}(k) \otimes M$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives. Since tensoring with an endotrivial module preserves the number of non-projective indecomposable direct summands, the projective-free part of $L_2 \otimes M$ is indecomposable. This implies that M lies at the end of Θ. Hence M satisfies the conditions in Proposition 3.6 and the result follows.

REMARK. In [6], Bessenrodt studied endotrivial modules in the Auslander-Reiten quiver. She showed that without the hypothesis (♯), if M is an endotrivial kG-module, then tensoring with M induces a graph isomorphism from the connected component containing the trivial kG-module k onto the connected component containing M ([6, Theorem 2.3]).

4. $\boldsymbol{ZA}_\infty^\infty$-Components of Dihedral 2-Groups

Throughout this section we assume that

k is a field of characteristic 2 and a Sylow 2-subgroup P of G is dihedral of order at least 8.

Let Δ_0 be the connected component containing the trivial kG-module k. Then Δ_0 is isomorphic to $\boldsymbol{Z}A_\infty^\infty$ by Theorem 1.2. It is known that all modules in Δ_0 are endotrivial kG-modules (see, e.g., [6]).

Proposition 4.1. *Let M be an odd dimensional indecomposable kG-module. Let Θ be the connected component of $\Gamma_s(kG)$ containing M and Δ_0 the connected* *component containing k. Then Θ is isomorphic to $\boldsymbol{Z}A_\infty^\infty$ and tensoring with M induces a graph isomorphism from Δ_0 onto Θ.*

Proof. Let T_0: $\cdots\to V_n\to\cdots\to V_2\to k\leftarrow L_2\leftarrow L_3\leftarrow\cdots\leftarrow L_n\leftarrow\cdots$ be a tree in Δ_0 with $\Delta_0\cong\boldsymbol{Z}T_0$. Since tensoring with an endotrivial module preserves the number of non-projective indecomposable direct summands, the projective-free part M_n (resp. W_n) of $L_n\otimes M$ (resp. $V_n\otimes M$) is indecomposable and odd dimensional. Therefore the tensor seqences $\mathcal{A}(L_n)\otimes M$ and $\mathcal{A}(V_n)\otimes M$ are the AR-sequences $\mathcal{A}(M_n)$ and $\mathcal{A}(W_n)$ modulo projectives respectively by Lemma 1.5. Thus we obtain a tree T: $\cdots\to W_n\to\cdots\to W_2\to M\leftarrow M_2\leftarrow M_3\leftarrow\cdots\leftarrow M_n\leftarrow\cdots$ with $\Theta\cong\boldsymbol{Z}T$.

Corollary 4.2. *Let M be an odd dimensional indecomposable kG-module and Θ the connected component containing M. Then all modules in Θ have the same vertex P.*

Proof. By Proposition 4.1, the tree class of Θ is A_∞^∞. Therefore all modules in Θ are odd dimensional by Lemma 1.9(1). This implies the result.

5. $\boldsymbol{Z}D_\infty$-Components of Semidihedral 2-Groups

Throughout this section, we assume that

k is an algebraically closed field of characteristic 2 and a Sylow 2-subgroup P of G is semidihedral.

Let Δ_0 be the connected component of $\Gamma_s(kG)$ containing the trivial kG-module k. Then Δ_0 is isomorphic to $\boldsymbol{Z}D_\infty$ by Theorem 1.2 (see [7, p 76 II. 10.7 Remark]). Thus a part of Δ_0 is as follows for some indecomposable kG-modules L_2, L_3 and I.

$$
\begin{tikzcd}
 & \Omega^2 k \arrow[dr] & & k \arrow[dr] & & {} \\
\Omega^2 L_2 \arrow[ur] \arrow[r] \arrow[dr] & \Omega^2 I \arrow[r] & L_2 \arrow[ur] \arrow[r] \arrow[dr] & I \arrow[r] & \Omega^{-2} L_2 \arrow[ur] \arrow[r] \arrow[dr] & {} \\
 & L_3 \arrow[ur] \arrow[dr] & & \Omega^{-2} L_3 \arrow[ur] \arrow[dr] & & {} \\
{} \arrow[ur] & & {} \arrow[ur] & & {} &
\end{tikzcd}
$$

Let $P=\langle x,y;\ x^2=y^{2^{n-1}}=1,\ y^x=y^{-1+2^{n-2}}\rangle$ and $\mathfrak{X}=\{\langle x\rangle\}$. Let $0\to\Omega_{\mathfrak{X}}k\to U\to k\to 0$ be an $\mathfrak{X}$-projective cover resolution of the trivial kG-module k. Concerning some basic facts on relative projective cover, we refer to [15], [19] and [18]. The following result is due to Okuyama.

Theorem 5.1([18]). *With the same assumption and notation as above,*

(1) $I \cong \Omega(\Omega_{\mathfrak{X}} k)$ *and* I *is an endotrivial* kG*-module.*

(2) I *is self-dual and odd dimensional.*

(3) *If* I' *is self-dual, odd dimensional and indecomposable, then* $I' \cong k$ *or* I.

Lemma 5.2. *Let* M *be an odd dimensional indecomposable* kG*-module. Then* $M \nmid M \otimes I$.

Proof. Assume contrary that $M \mid M \otimes I$. Then $M \otimes I \cong M$ (mod projectives), since tensoring with an endotrivial module preserves the number of non-projective indecomposable direct summands. Moreover it follows by Theorem 1.4 that $k \mid M \otimes M^* \mid (M \otimes M^*) \otimes I$. This implies that $I \mid M \otimes M^*$.

Since $2 \nmid \dim_k M$, k is a direct summand of $M \otimes M^*$ with multiplicity one. If an indecomposable kG-module W is a direct summand of $M \otimes M^*$, then W^* is also a direct summand of $M \otimes M^*$. Let $M \otimes M^* \cong k \oplus I \oplus (\oplus_i (W_i \oplus W_i^*)) \oplus (\oplus_j T_j)$ be an indecomposable direct sum decomposition, where W_i is not self-dual and T_j is self-dual. Since $M \otimes M^*$ is odd dimensional, some T_j is odd dimensional. By Theorem 5.1(3), this T_j must be isomorphic to I. Hence we get $I \oplus I \mid M \otimes M^*$ and $k \oplus k \mid (I \oplus I) \otimes I \mid (M \otimes M^*) \otimes I \cong M \otimes M^*$ (mod projectives). But this contradicts that the multiplicity of k in $M \otimes M^*$ is one.

Theorem 5.3. *Let* M *be an odd dimensional indecomposable* kG*-module and* Θ *the connected component of* $\Gamma_s(kG)$ *containing* M. *Then* Θ *is isomorphic to* $\boldsymbol{Z}D_\infty$ *and* M *lies at the end of* Θ.

Proof. We continue to use the same notation as above.

Let $\mathcal{A}(k): 0 \to \Omega^2 k \to m(k) \to k \to 0$ and $\mathcal{A}(I): 0 \to \Omega^2 I \to m(I) \to I \to 0$ be the AR-sequences terminating at k and I respectively. Note that $L_2 \cong m(k) \cong m(I)$ (mod projectives). By Theorem 1.3, the tensor sequence $\mathcal{A}(k) \otimes M$ is the AR-sequence $\mathcal{A}(M)$ modulo projectives. Since I is an endotrivial kG-module, the projective-free part M' of $I \otimes M$ is indecomposable. Hence by Lemma 1.5, the tensor sequence $\mathcal{A}(I) \otimes M$ is the AR-sequence $\mathcal{A}(M')$ modulo projectives. Note that M' is not isomorphic to M by Lemma 5.2.

We claim that the projective-free part M_2 of $L_2 \otimes M$ is indecomposable: Assume not. Then we have $X_1 \oplus X_2 \mid L_2 \otimes M$ for some non-projective indecomposable kG-modules X_1 and X_2. Note that X_1 is not isomorphic to X_2 by Theorem 1.1. Since $X_1 \oplus X_2 \mid m(M)$ and $X_1 \oplus X_2 \mid m(M')$, where $m(M)$ and $m(M')$ are the middle terms of $\mathcal{A}(M)$ and $\mathcal{A}(M')$ respectively, we get a part of Θ as follows.

$$\begin{tikzcd}[column sep=small]
\Omega^2 M \arrow[dr] & & M \\
& X_1 \arrow[ur] \arrow[dr] & \\
\Omega^2 M' \arrow[ur] \arrow[dr] & & M' \\
& X_2 \arrow[ur] \arrow[dr] & \\
\Omega^2 M \arrow[ur] & & M
\end{tikzcd}$$

But this is a contradiction since Θ can not have such a subquiver by Theorem 1.1.

Consequently we have $m(M)\cong M_2$ (mod projectives) and $m(M')\cong M_2$ (mod projectives). This implies that $\Theta\cong \boldsymbol{Z}D_\infty$ and M lies at the end.

Lemma 5.4. *Let M be an odd dimensional indecomposable kG-module and Θ the connected component containing M. Then all modules in Θ have the same vertex P.*

Proof. By Theorem 5.3 and Lemma 1.9(2), Θ is isomorphic to $\boldsymbol{Z}D_\infty$ and M lies at the end of Θ. Since M is odd dimensional, a Sylow 2-subgroup P of G is a vertex of M. The result follows from [20, Theorem 4.3].

Lemma 5.5. *Let M be an odd dimensional indecomposable kG-module and Θ the connected component of $\Gamma_s(kG)$ containing M. Let*
$$T\colon \begin{array}{l} M\leftarrow M_2\leftarrow M_3\leftarrow M_4\leftarrow\cdots\leftarrow M_n\leftarrow\cdots \\ \qquad\ \ \downarrow \\ \qquad\ \ M' \end{array}$$
be a tree in Θ with $\Theta\cong\boldsymbol{Z}T$. Let S be a P-source of M and Ξ the connected component of $\Gamma_s(kP)$ containing S. Then we have P-sources S' and S_n of M' and M_n $(n\geq 2)$ respectively and a tree
$$T'\colon \begin{array}{l} S\leftarrow S_2\leftarrow S_3\leftarrow\cdots\leftarrow S_n\leftarrow\cdots \\ \qquad\ \downarrow \\ \qquad\ S' \end{array}$$
with $\Xi\cong\boldsymbol{Z}T'$.

Proof. All modules in Θ have the same vertex P by Lemma 5.4. Thus applying the similar argument in the proof of Proposition 2.3, Steps 1 and 2, we may assume that P is a normal subgroup of G and G is the inertial group of Ξ. Since the order of G/P is odd and Ξ is isomorphic to $\boldsymbol{Z}D_\infty$, G acts on Ξ trivially. Therefore we may also assume that every module in Ξ is G-invariant. Applying Lemma 1.11, we get the result.

In the rest we consider tensoring Δ_0 with an odd dimensional indecomposable kG-module.

Proposition 5.6. *Let S be an odd dimensional indecomposable kP-module and Ξ the connected component of $\Gamma_s(kP)$ containing S. Let Λ_0 be the connect-*

ed component of $\Gamma_s(kP)$ containing the trivial kP-module k. Then tensoring with S induces a graph isomorphism from Λ_0 onto Ξ.

In order to prove Proposition 5.6, we need the following Lemmas 5.7 and 5.8. Let T_0:
$$\begin{array}{l} k \leftarrow H_2 \leftarrow H_3 \leftarrow \cdots \leftarrow H_n \leftarrow \cdots \\ \qquad\downarrow \\ \qquad I_0 \end{array}$$
be a tree in Λ_0 with $\Lambda_0 \cong \boldsymbol{Z}T_0$. Let $P=\langle x,y;\, x^2=y^{2^{n-1}}=1,\, y^x=y^{-1+2^{n-2}}\rangle$.

Lemma 5.7. *$H_n\downarrow_{\langle x\rangle}\cong k\oplus k$ (mod projectives) for all $n\geq 2$.*

Proof. Use induction on n. Since all modules in Λ_0 have the same vertex P, the AR-sequences $\mathscr{A}(k)$, $\mathscr{A}(I_0)$ and $\mathscr{A}(H_n)$ split on restriction to $\langle x\rangle$. Hence $(k\oplus\Omega^2k)\downarrow_{\langle x\rangle}\cong m(k)\downarrow_{\langle x\rangle}\cong H_2\downarrow_{\langle x\rangle}\cong m(I_0)\downarrow_{\langle x\rangle}\cong(I_0\oplus\Omega^2I_0)\downarrow_{\langle x\rangle}$. Thus we get $I_0\downarrow_{\langle x\rangle}\cong k$ (mod projectives), $\Omega^2I_0\downarrow_{\langle x\rangle}\cong k$ (mod projectives) and $H_2\downarrow_{\langle x\rangle}\cong k\oplus k$ (mod projectives). Also $\mathscr{A}(H_2)$: $0\to\Omega^2H_2\to H_3\oplus\Omega^2k\oplus\Omega^2I_0\to H_2\to 0$ splits on restrictio to $\langle x\rangle$. So we have $(H_3\oplus\Omega^2k\oplus\Omega^2I_0)\downarrow_{\langle x\rangle}\cong(\Omega^2H_2\oplus H_2)\downarrow_{\langle x\rangle}$ and $H_3\downarrow_{\langle x\rangle}\cong k\oplus k$ (mod projectives).

Suppose then that $H_i\downarrow_{\langle x\rangle}\cong k\oplus k$ (mod projectives) for all i with $2\leq i\leq n-1$. Since $\mathscr{A}(H_{n-1})$: $0\to\Omega^2H_{n-1}\to H_n\oplus\Omega^2H_{n-2}\to H_{n-1}\to 0$ splits on restriction to $\langle x\rangle$, we have $(H_n\oplus\Omega^2H_{n-2})\downarrow_{\langle x\rangle}\cong(\Omega^2H_{n-1}\oplus H_{n-1})\downarrow_{\langle x\rangle}$. This implies that $H_n\downarrow_{\langle x\rangle}\cong k\oplus k$ (mod projectives).

Lemma 5.8. *Let S be an odd dimensional kP-module.*

(1) *The trivial $k\langle x\rangle$-module k is a direct summand of $S\downarrow_{\langle x\rangle}$ with multiplicity one.*

(2) *H_n is a direct summand of $H_n\otimes S\otimes S^*$ with multiplicity one for all $n\geq 2$.*

Proof. (1) The statement follows from [7, p 73. Lemma II 10.5].

(2) From (1) we have $(S\otimes S^*)\downarrow_{\langle x\rangle}\cong k$ (mod projectives). Hence $(H_n\otimes S\otimes S^*)\downarrow_{\langle x\rangle}\cong k\oplus k$ (mod projectives) from Lemm 5.7. Thus we have $2H_n\downarrow_{\langle x\rangle}\not|\,(H_n\otimes S\otimes S^*)\downarrow_{\langle x\rangle}$, which implies the result.

Proof of Proposition 5.6. Let T:
$$\begin{array}{l} S \leftarrow S_2 \leftarrow S_3 \leftarrow S_4 \leftarrow \cdots \leftarrow S_n \leftarrow \cdots \\ \qquad\downarrow \\ \qquad S' \end{array}$$
be a tree in Ξ with $\Xi\cong\boldsymbol{Z}T$. Since $k\otimes S\cong S$ and $I_0\otimes S\cong S'$, it suffices to show that $H_n\otimes S\cong S_n$ (mod projectives) for all $n\geq 2$. We proceed by induction on n.

From the argument in the proof of Theorem 5.3, we have $H_2\otimes S\cong S_2$ (mod proejctives) and $\Omega^2H_2\otimes S\cong\Omega^2S_2$ (mod projectives). Also we have $(S_2, [\mathscr{A}(H_2)\otimes S])=(H_2\otimes S, [\mathscr{A}(H_2)\otimes S])=(H_2\otimes S\otimes S^*, [\mathscr{A}(H_2)])=1$ since the multiplicity of H_2 in $H_2\otimes S\otimes S^*$ is one by Lemma 5.8(2). This implies that the tensor sequence $\mathscr{A}(H_2)\otimes S$: $0\to\Omega^2H_2\otimes S\to(H_3\oplus\Omega^2k\oplus\Omega^2I_0)\otimes S\to H_2\otimes S\to 0$ is the AR-sequence $\mathscr{A}(S_2)$ modulo projectives by Lemma 1.7(1). Thus we get $H_3\otimes S\cong S_3$

(mod projectives).

Suppose then that $H_i \otimes S \cong S_i$ (mod projectives) for all i with $2 \leq i \leq n-1$. Using Lemma 5.8(2) again, we have $(S_{n-1}, [\mathcal{A}(H_{n-1}) \otimes S]) = (H_{n-1} \otimes S \otimes S^*, [\mathcal{A}(H_{n-1})]) = 1$. Thus the tensor sequence $\mathcal{A}(H_{n-1}) \otimes S$: $0 \to \Omega^2 H_{n-1} \otimes S \to (H_n \oplus \Omega^2 H_{n-2}) \otimes S \to H_{n-1} \otimes S \to 0$ is the AR-sequence $\mathcal{A}(S_{n-1})$ modulo projectives. Therefore we get $H_n \otimes S \cong S_n$ (mod projectives).

Proposition 5.9. *Let M be an odd dimensional indecomposable kG-module and Θ the connected component containing M. Let Δ_0 be the connected component containing the trivial kG-module k. Then tensoring with M induces a graph isomorphism from Δ_0 onto Θ.*

Proof. Let S be a P-source of M. Let Ξ and Λ_0 be the connected components of $\Gamma_s(kP)$ containing S and k respectively. Then tensoring with S induces a graph isomorphism from Λ_0 onto Ξ by Proposition 5.6. Using an argument similar to the one in the proof of Proposition 3.1 (use Lemma 5.5 in place of Proposition 2.3), we get the result.

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