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SPECTRAL AND SCATTERING THEORY FOR 3-PARTICLE HAMILTONIAN WITH STARK EFFECT: NON-EXISTENCE OF BOUND STATES AND RESOLVENT ESTIMATE

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Introduction

The present work is a continuation to [16], in which the author has proved the asymptotic completeness of wave operators for three-particle Stark Hamiltonians. In the proof there, the following two results about the spectral properties of two-particle subsystem Hamiltonians have played a central role: (1) non-existence of bound states; (2) uniform resolvent estimate at high energies. We here consider these two problems for three-particle systems and apply the obtained results to prove the asymptotic completeness for four-particle Stark Hamiltonians under the main assumption that any subsystem Hamiltonian does not have zero reduced charge.

1. Non-existence of bound states

The first half of this work is devoted to proving the non-existence of bound states for three-particle Stark Hamiltonians. We consider a system of three particles moving in a uniform electric field $\mathcal{E}\in \boldsymbol{R}^3$. The total energy Hamiltonian for such a system has the form

$$-\sum_{j=1}^{3}(\Delta/2m_j+e_j\langle\mathcal{E},r_j\rangle)+\sum_{1\le j<k\le 3}V_{jk}(r_j-r_k)\,.$$

Here m_j, e_j and $r_j\in\boldsymbol{R}^3$, $1\le j\le 3$, are the mass, charge and position vector of the j-th particle, while $-e_j\langle\mathcal{E},r_j\rangle$, $\langle,\rangle$ being the usual scalar product in the Euclidean space, is the energy of interaction with the electric field and the real function V_{jk} is the pair potential between the j-th and k-th particles. For notational brevity, we assume that the three particles have the identical masses

$$m_j=1,\quad 1\le j\le 3\,.$$

For the three-particle system with identical masses, the configuration space X in the center-of-mass frame is described as

$$X = \{r = (r_1, r_2, r_3) \in \boldsymbol{R}^{3\times 3} : \sum_{j=1}^{3} r_j = 0\}.$$

Let $E_X \in X$ be the projection onto X of $(e_1\mathcal{E}, e_2\mathcal{E}, e_3\mathcal{E}) \in \boldsymbol{R}^{3\times 3}$. Then the energy Hamiltonian H takes the following form in the center-of-mass frame:

$$H = -\frac{1}{2}\Delta - \langle E_X, r\rangle + V \quad \text{on } L^2(X),$$

where $V = V(r)$ is given as the sum of pair potentials V_{jk}

$$V(r) = \sum_{1\le j<k\le 3} V_{jk}(r_j - r_k).$$

We assume that $E_X \neq 0$ and write it as

$$E_X = E_0\omega, \quad E_0 = |E_X| > 0,$$

for $\omega \in S_X$, S_X being the unit sphere in X. For a generic point $x \in X$, we also write $x = z\omega + z_\perp$ with $z_\perp \in \Pi_\omega$, Π_ω being the hyperplane orthogonal to ω. According to this notation, the Hamiltonian H is represented as

$$H = -\frac{1}{2}\Delta - E_0 z + V \quad \text{on } L^2(X).$$

As stated above, one of our goals is to prove that H has no bound states.

Let us proceed to the precise formulation of the obtained result. We begin by making the assumptions on the pair potentials V_{jk}.

$(\mathrm{A})_\rho$ $V_{jk}(y)$, $y \in \boldsymbol{R}^3$, is a real C^1-smooth function and has the following decay property as $|y| \to \infty$:

$$|V_{jk}(y)| + |\nabla_y V_{jk}(y)| = O(|y|^{-\rho}) \quad \text{for some } \rho > 1/2.$$

Under this assumption, the Hamiltonian H formally defined above is essentially self-adjoint on the Schwartz space $\mathcal{S}(X)$. We denote by the same notation H this self-adjoint realization in $L^2(X)$. Then the first main theorem is formulated as follows.

Theorem 1.1 (Non-Existence of Bound State). *Let the notations be as above. Assume that* $(\mathrm{A})_\rho$ *with* $\rho > 1/2$ *is satisfied. Then* H *has no bound states.*

Since the work by Kato [6], many articles have been devoted to the study on the non-existence of eigenvalues imbedded in continuous spectrum for two-body Schrödinger operators in case of the absence of uniform electric fields. For related references, see Eastham-Kalf [2] and Reed-Simon [12]. A similar problem has been also studied for N-body Schrödinger operators. Froese-Herbst [3] have first proved the non-existence of positive eigenvalues for a large class of N-body Schrödinger operators. On the other hand, the non-existence problem has been also studied for Schrödinger operators in case of the presence of uniform electric fields. For example, this has been proved by Titchmarch [17] (one dimensional case) and by Avron-Herbst [1] (n dimensional case) for two-particle Stark Hamiltonians. Recently Sigal [13] has dealt with the case of N-particle systems. For one example, the result obtained there applies to the following three-particle Hamiltonian. Take the masses m_j, $1\leq j\leq 3$, as $m_1=m_2=1$ and $m_3=\infty$ and the charges e_j as $e_1=e_2=e_3=e>0$. Then the energy Hamiltonian for such a system takes the form

$$-\frac{1}{2}\Delta-e\langle\mathcal{E}, r_1\rangle-e\langle\mathcal{E}, r_2\rangle+V_{12}(r_1-r_2)+V_{13}(r_1)+V_{23}(r_2)$$

in the center-of-mass frame. If the pair potential V_{12} is repulsive

$$\langle\mathcal{E}, y\rangle\langle\mathcal{E}, \nabla_y V_{12}(y)\rangle\leq 0\,,$$

in the direction $\mathcal{E}\in\boldsymbol{R}^3$, then the operator above is shown to have no bound states under mild assumptions on the smoothness of pair potentials. Roughly speaking, our theorem above asserts that under somewhat restrictive smoothness assumptions on pair potentials, three-paritcle Stark Hamiltonians have no bound states, even if pair potentials are not necessarily repulsive along the direction of electric fields in the above sense.

We conclude this section by making a brief review on the results obtained in [16]. In the previous work [16], we have considered the class of pair potentials satisfying the following assumptions.

(V)$_\rho$ $V_{jk}(y)$, $y\in\boldsymbol{R}^3$, is a real C^2-smooth function and has the decay properties as $|y|\to\infty$: (V.0) $V_{jk}(y)=O(|y|^{-\rho})$, $\rho>1/2$; (V.1) $\partial_y^\alpha V_{jk}(y)=o(1)$, $|\alpha|=1$; (V.2) $\partial_y^\alpha V_{jk}(y)=O(1)$, $|\alpha|=2$.
Under these assumptions, we have proved that: (1) The set $\sigma_p(H)$ of point spectrum is discrete with possible accumulating points $\pm\infty$. (2) For $\lambda\notin\sigma_p(H)$, the resolvents

$$R(\lambda\pm i\kappa; H)=(H-\lambda\mp i\kappa)^{-1}\colon L_\nu^2(X)\to L_{-\nu}^2(X)\,,\quad \nu>1/4\,,$$

are bounded uniformly in κ, $0<\kappa\leq 1$, when considered as an operator from the weighted L^2 space $L_\nu^2(X)=L^2(X;\langle x\rangle^{2\nu}\,dx)$, $\langle x\rangle=(1+|x|^2)^{1/2}$, into $L_{-\nu}^2(X)$. (3) The boundary values $R(\lambda\pm i0; H)$ to the real axis exist in the topology above, the convergence being locally uniform in $\lambda\in\boldsymbol{R}^1\backslash\sigma_p(H)$.

We now combine these results with Theorem 1.1 to obtain the following

Theorem 1.2. *Let the pair potential V_{jk} be a real C^2-smooth function. Assume, in addition to* (A)$_\rho$ *with $\rho>1/2$, that V_{jk} satisfies* (V.2). *Fix a compact interval $I\subset\boldsymbol{R}^1$ arbitrarily. Then one has*

$$\sup_{\lambda\in I,\, 0<\kappa\leq 1} \|\langle x\rangle^{-\nu}R(\lambda\pm i\kappa; H)\langle x\rangle^{-\nu}\|\leq C_I, \quad \nu>1/4,$$

where $\|\ \|$ *denotes the operator norm when considered as an operator from* $L^2(X)$ *into itself. Furthermore, the boundary values* $R(\lambda\pm i0; H)$ *to the real axis exist in the topology above.*

As stated above, the other aim of this work is to study the resolvent estimate at high energies for three-particle Stark Hamiltonians. In section 3, the bound above will be proved to remain ture for all $\lambda\in \boldsymbol{R}^1$ under the assumption that any two-paritlce subsystem Hamiltonian does not have zero reduced charge. This result plays an important role in proving the asymptotic completeness for four-particle Stark Hamiltonians.

2. Proof of Theorem 1.1

Throughout this section, the same notations as in the previous section are kept and $(A)_\rho$ with $\rho>1/2$ is always assumed to be satisfied. We also use the constant ρ with the meaning ascribed there and assume, without loss of generality, that $1/2<\rho<1$.

Let $\psi(x)\in L^2(X)$ be the eigenstate associated with eigenvalue $E\in R^1$;

$$H\psi = E\psi, \quad \psi\in L^3(X). \tag{2.1}$$

The proof of Theorem 1.1 is based on a modification of the positive commutator method in [3]. We analyze the two commutators $i[H, A]$ and $i[H, A_1]$, where

$$A = \frac{1}{2i}(\langle x, \nabla_x\rangle + \langle \nabla_x, x\rangle), \tag{2.2}$$

$$A_i = \frac{1}{i}\langle \omega, \nabla_x\rangle. \tag{2.3}$$

By use of these commutators, we prove that the eigenstate ψ has the polynomial and exponential decay properties at infinity and finally we conclude that ψ vanishes identically. In the work [3], only the commutator $i[H, A]$ with the generator A of dilation unitary group has been used to prove the non-existence of positive eigenvalues for N-body Schrödinger operators without uniform electric fileds (see also [13]). The following proposition, which has been established in [16] under assumptions (V.0) and (V.1) (Proposition 5.1), plays a central role in proving these decay properties.

Proposition 2.1. *Assume* $(A)_\rho$ *with* $\rho>1/2$. *Let* A_1 *be defined by* (2.3). *Fix* $\lambda\in \boldsymbol{R}^1$ *arbitrarily. Let* $f\in C_0^\infty(\boldsymbol{R}^1)$ *be a non-negative smooth function supported in a small neighborhood around* λ. *Then, for any* δ, $0<\delta\ll 1$, *small enough, one can take the support of* f *so small that for a compact operator* $K=K_\delta$

acting on $L^2(X)$,

$$f(H)i[H, A_1]f(H) \geq (E_0-\delta)f(H)^2+K$$

in the form sense.

The local commutator estimate as above is called the Mourre estimate ([10]), which has made major progress in the spectral and scattering theory for many-body Schrödinger operators during the last decade.

2.1. Polynomial decay property. We begin by proving the polynomial decay property of the eigenstate $\psi \in L^2(X)$ in (2.1).

Proposition 2.2 *For any* $k>0$, $\langle x\rangle^k \psi \in L^2(X)$.

Proof. The proposition is verified by contradiction and the proof is divided into several steps.

(1) Assume that there exists $k>0$ such that

$$\langle x\rangle^k \psi \notin L^2(X)\,. \tag{2.4}$$

For ε, $0<\varepsilon\ll 1$, small enough, we define the function $F=F(|x|;\varepsilon)$ by

$$F = k(\log\langle x\rangle - \log(1+\varepsilon\langle x\rangle))$$

with $k>0$ as above and set

$$\psi_\varepsilon = \langle x\rangle^k(1+\varepsilon\langle x\rangle)^{-k}\psi \in L^2(X)\,,$$

so that ψ_ε is represented as $\psi_\varepsilon = e^F\psi$. This function obeys the equation

$$H_F\psi_\varepsilon = E\psi_\varepsilon\,, \tag{2.5}$$

where

$$H_F = e^F H e^{-F} = -\frac{1}{2}(\nabla_x - \nabla_x F)^2 - E_0 z + V\,.$$

The operator H_F can be also written as

$$H_F = H + B_F - \frac{1}{2}|\nabla_x F|^2$$

with

$$B_F = \frac{1}{2}(\langle \nabla_x F, \nabla_x\rangle + \langle \nabla_x, \nabla_x F\rangle)\,.$$

Let A be defined by (2.2). We calculate $\nabla_x F$ as $\nabla_x F = xG$, where

$$G = G(|x|;\varepsilon) = |x|^{-1}\partial F/\partial|x| = k\langle x\rangle^{-2}(1+\varepsilon\langle x\rangle)^{-1} > 0$$

and it behaves like $G=O(|x|^{-2})$, $|x|\to\infty$, uniformly in ε. Hence we have

$$B_F = iGA+\frac{1}{2}|x|\partial G/\partial|x|. \tag{2.6}$$

(2) We now normalize $\psi_\varepsilon\in L^2(X)$ as

$$\varphi_\varepsilon(x) = \psi_\varepsilon/\|\psi_\varepsilon\|_{L^2(X)}, \quad \|\varphi_\varepsilon\|_{L^2(X)} = 1 .$$

By assumption (2.4), it follows that

$$\varphi_\varepsilon\to 0 \text{ weakly in } L^2(X) \text{ as } \varepsilon\to 0 . \tag{2.7}$$

By (2.5) and (2.6), the function φ_ε satisfies the equation

$$H\varphi_\varepsilon = E\varphi_\varepsilon - iGA\varphi_\varepsilon + J\varphi_\varepsilon , \tag{2.8}$$

where $J=J(|x|;\varepsilon)$ is defined by

$$J = \frac{1}{2}(|\nabla_\varepsilon F|^2 - |x|\partial G/\partial|x|) = O(|x|^{-2}), \quad |x|\to\infty.$$

Hence we see that $\langle x\rangle^{-1/2}\varphi_\varepsilon$, and $\langle x\rangle^{-1}\varphi_\varepsilon$ are bounded in the Sobolev spaces $H^1(X)$ and $H^2(X)$ uniformly in ε, respectively. This, together with (2.7), implies that both the terms $GA\varphi_\varepsilon$ and $J\varphi_\varepsilon$ converge to zero strongly in $L^2(X)$ as $\varepsilon\to 0$. Thus we have

$$\|(H-E)\varphi_\varepsilon\|_{L^2(X)}\to 0 , \quad \varepsilon\to 0 . \tag{2.9}$$

(3) We now use Proposition 2.1 (Mourre estimate). For $\delta>0$ small enough, we take $f\in C_0^\infty(\boldsymbol{R}^1)$, $0\le f\le 1$, to satisfy that f is supported in a small interval $(E-2\delta, E+2\delta)$ around E and that $f=1$ on $[E-\delta, E+\delta]$. Then it follows from Proposition 2.1 and (2.7) that

$$\liminf_{\varepsilon\to 0} \langle f(H)i[H, A_1]f(H)\varphi_\varepsilon, \varphi_\varepsilon\rangle_{L^2(X)} \ge d \liminf_{\varepsilon\to 0} \|f(H)\varphi_\varepsilon\|^2_{L^2(X)}$$

for some $d>0$. By assumption $(A)_\rho$, the commutator $[H, A_1]\colon L^2(X)\to L^2(X)$ is bounded and also we obtain from (2.9) that

$$\|(Id-f(H))\varphi_\varepsilon\|_{L^2(X)}\to 0 , \quad \varepsilon\to 0 .$$

Thus we have

$$\liminf_{\varepsilon\to 0} \langle i[H, A_1]\varphi_\varepsilon, \varphi_\varepsilon\rangle_{L^2(X)} \ge d\|\varphi_\varepsilon\|^2_{L^2(X)} = d>0 . \tag{2.10}$$

(4) Next we calculate the term $\langle i[H, A_1]\varphi_\varepsilon, \varphi_\varepsilon\rangle_{L^2(X)}$ by use of relation (2.8). Let $\chi_R(x)\in C_0^\infty(X)$ be a non-negative function such that χ_R is supported in $|x|< 2R$ and $\chi_R=1$ on $|x|\le R$. We approximate φ_ε by $\varphi_\varepsilon^R=\chi_R\varphi_\varepsilon$; $\varphi_\varepsilon^R\to\varphi_\varepsilon$, $R\to\infty$, strongly in $L^2(X)$. Then we have

$$\langle i[H, A_1]\varphi_\varepsilon, \varphi_\varepsilon\rangle_{L^2(X)} = \lim_{R\to\infty} i(\langle A_1\varphi_\varepsilon^R, H\varphi_\varepsilon^R\rangle_{L^2(X)} - \langle H\varphi_\varepsilon^R, A_1\varphi_\varepsilon^R\rangle_{L^2(X)}) .$$

We here make use of relation (2.8) to obtain that

$$(2.11) \qquad \begin{aligned} &\langle i[H, A_1]\varphi_\varepsilon, \varphi_\varepsilon\rangle_{L^2(X)} = \\ &\qquad -2 \operatorname{Re} \langle A_1\varphi_\varepsilon, GA\varphi_\varepsilon\rangle_{L^2(X)} - \langle i[A_1, J]\varphi_\varepsilon, \varphi_\varepsilon\rangle_{L^2(X)} . \end{aligned}$$

The second term on the right side converges to zero as $\varepsilon\to 0$.

Lemma 2.3.

$$\|\langle x\rangle^{(\rho-1)/2} G^{1/2} A\varphi_\varepsilon\|_{L^2(X)} = O(1) , \quad \varepsilon\to 0 .$$

We accept this lemma as proved. The proof of the lemma is given after completing the proof of the proposition. It follows immediately from Lemma 2.3 that

$$\lim_{\varepsilon\to 0} \|\langle x\rangle^{1/2} GA\varphi_\varepsilon\|_{L^2(X)} = 0$$

and hence the first term on the right side of (2.11) also converges to zero as $\varepsilon\to 0$. Thus we have

$$\limsup_{\varepsilon\to 0} \langle i[H, A_1]\varphi_\varepsilon, \varphi_\varepsilon\rangle_{L^2(X)} = 0 .$$

This contradicts (2.10) and completes the proof. □

We now prove Lemma 2.3, which has played a basic role in proving Proposition 2.2.

Proof of Lemma 2.3. For $\delta>0$ small enough, we define the function $\theta_\delta(x)$ as

$$\theta_\delta = \langle x\rangle^{1-\rho}\langle \delta x\rangle^{\rho+1/2}$$

and the operator A_δ as

$$A_\delta = \frac{1}{2i}(\langle x/\theta_\delta, \nabla_x\rangle + \langle \nabla_x, x/\theta_\delta\rangle) .$$

Throughout the proof, we denote by b_k the multiplication operator by $b_k(x; \delta)$ with bound $|b_k(x; \delta)| \le C\langle x\rangle^k$ uniformly in δ. According to this notation, the operator A_δ defined above is related to A through the relation

$$A_\delta = \theta_\delta^{-1} A + b_{\rho-1} .$$

To prove the lemma, we analyze the term

$$I_{\delta\varepsilon} = \langle i[H, A_\delta]\varphi_\varepsilon, \varphi_\varepsilon\rangle_{L^2(X)} .$$

It should be noted that $A_\delta\varphi_\varepsilon \in L^2(X)$ for $\delta>0$.

We first calculate the commutator $i[H, A_\delta]$ in the above term as

$$(2.12)\qquad i[H, A_\delta] = i[-\Delta/2, A_\delta]+i[-E_0 z, A_\delta]+i[V, A_\delta]\,.$$

The second operator on the right side is equal to

$$(2.13)\qquad i[-E_0 z, A_\delta] = \theta_\delta^{-1}E_0 z = \theta_\delta^{-1}(-H-\Delta/2+V)\,.$$

The first operator is represented in the form

$$i[-\Delta/2, A_\delta] = i[-\Delta/2, \theta_\delta^{-1}]A+\theta_\delta^{-1}i[-\Delta/2, A]+b_{\rho-2}\nabla_x+b_{\rho-3}\,.$$

Define the operator D_0 as

$$D_0 = \frac{1}{i}\langle x/\langle x\rangle, \nabla_x\rangle\,.$$

Then a simple calculation yields that $i[-\Delta/2, A]=-\Delta$ and that

$$i[-\Delta/2, \theta_\delta^{-1}] = \theta_\delta^{-1}\{(\rho-1)\langle x\rangle^{-1}-(\rho+1/2)\delta^2\langle\delta x\rangle^{-2}\langle x\rangle\}D_0+b_{\rho-3}\,.$$

Hence the first operator takes the form

$$(2.14)\quad i[-\Delta/2, A_\delta] = \theta_\delta^{-1}\{((\rho-1)-(\rho+1/2)\delta^2\langle\delta x\rangle^{-2}\langle x\rangle^2)D_0^* D_0-\Delta\}+B$$

with $B=b_{\rho-2}\nabla_x+b_{\rho-3}$. By assumption $(A)_\rho$, $1/2<\rho<1$, the third operator on the right side of (2.12) has the form

$$(2.15)\qquad i[V, A_\delta] = b_0\,.$$

We now evaluate the term $I_{\delta\varepsilon}$ in question from below. We combine (2.13) ~(2.15) to obtain that

$$i[H, A_\delta] = B_1+B_2+B_3+\theta^{-1}(-H+V)+b_{\rho-2}\nabla_x+b_0$$

where

$$B_1 = (1-\rho)\theta_\delta^{-1/2}(-\Delta-D_0^* D_0)\theta_\delta^{-1/2}\,,$$
$$B_2 = (\rho+1/2)\theta_\delta^{-1/2}\delta\langle\delta x\rangle^{-1}\langle x\rangle(-\Delta-D_0^* D_0)\langle x\rangle\langle\delta x\rangle^{-1}\delta\theta_\delta^{-1/2}\,,$$
$$B_3 = (\rho+1/2)\theta_\delta^{-1/2}p_\delta^{1/2}(-\Delta)p_\delta^{1/2}\theta_\delta^{-1/2}$$

with $p_\delta(x)=1-\delta^2\langle\delta x\rangle^{-2}\langle x\rangle^2>0$. The first three operators B_j, $1\leq j\leq 3$, on the right side are non-negative and also it follows from relation (2.8) that

$$\langle H\varphi_\varepsilon, \theta_\delta^{-1}\varphi_\varepsilon\rangle_{L^2(X)} = O(1)$$

uniformly in δ and ε. Thus the term $I_{\delta\varepsilon}$ is evaluated from below as

$$(2.16)\qquad I_{\delta\varepsilon}\geq -d$$

for some $d>0$ independent of δ and ε.

Next we evaluate the term $I_{\delta\varepsilon}$ from above. We write it as

$$I_{\delta\varepsilon} = i\{\langle A_\delta\varphi_\varepsilon, H\varphi_\varepsilon\rangle_{L^2(X)} - \langle H\varphi_\varepsilon, A_\delta\varphi_\varepsilon\rangle_{L^2(X)}\}.$$

Then it follows from (2.8) that

$$I_{\delta\varepsilon} = -2\|\theta_\delta^{-1/2}G^{1/2}A\varphi_\varepsilon\|_{L^2(X)}^2 + O(1)$$

uniformly in δ and ε. This, together with (2.16), proves the lemma. □

2.2. Exponential decay property. Next we prove the exponential decay property of the eigenstate $\psi\in L^2(X)$ in (2.1).

Proposition 2.4. *For any* $k>0$, $\exp(k\langle x\rangle)\psi\in L^2(X)$.

Proof. This proposition is also verified by contradiction. The proof is done by repeated use of the same arguments as in the proof of Propisition 2.2.

(1) Define $k_0\geq 0$ as

$$k_0 = \sup\{k\geq 0\colon \exp(k\langle x\rangle)\psi\in L^2(X)\}.$$

We deny the statement of the proposition and assume that $k_0<\infty$. Fix k, $0<k<k_0$, close enough to k_0, if $k_0>0$. If $k_0=0$, then we take k as $k=0$. By the assumption $k_0<\infty$, we can choose $\gamma>0$, $0<\gamma\ll 1$, small enough to satisfy $k+\gamma>k_0$ and hence

$$\exp((k+\gamma)\langle x\rangle)\psi\notin L^2(X). \tag{2.17}$$

For $\lambda\gg 1$ large enough, we set

$$\psi_\lambda = (1+\gamma\lambda^{-1}\langle x\rangle)^\lambda \exp(k\langle x\rangle)\psi\in L^2(X).$$

with $\gamma>0$ as above. We should note that $\psi_\lambda\in L^2(X)$, even if $k=0$, which follows from Proposition 2.2 at once.

We write ψ_λ as $\psi_\lambda=e^F\psi$ with

$$F = F(|x|;\lambda) = k\langle x\rangle+\lambda\log(1+\gamma\lambda^{-1}\langle x\rangle)$$

and normalize $\psi_\lambda\in L^2(X)$ as

$$\varphi_\lambda = \psi_\lambda/\|\psi_\lambda\|_{L^2(X)}, \quad \|\varphi_\lambda\|_{L^2(X)} = 1.$$

By (2.17), φ_λ converges to zero weakly in $L^2(X)$ as $\lambda\to\infty$ and also by the same calculation as in the proof of Proposition 2.2, we see that φ_λ obeys the equation

$$H\varphi_\lambda = E\varphi_\lambda - iGA\varphi_\lambda + J\varphi_\lambda, \tag{2.18}$$

where $G=G(|x|;\lambda)$ and $J=J(|x|;\lambda)$ are defined by

$$G = |x|^{-1}\partial F/\partial|x| = \langle x\rangle^{-1}\{k+\gamma\lambda(\lambda+\gamma\langle x\rangle)^{-1}\}>0,$$
$$J = (|\nabla_x F|^2 - |x|\partial G/\partial|x|)/2.$$

These functions behave like $G=O(|x|^{-1})$ and $J=O(1)$ as $|x|\to\infty$ uniformly in $\lambda\gg1$. Hence it follows from equation (2.18) that $\langle x\rangle^{-1/2}\varphi_\lambda$ and $\langle x\rangle^{-1}\varphi_\lambda$ are bounded in the Sobolev spaces $H^1(X)$ and $H^2(X)$ uniformly in $\lambda\gg1$, respectively. Furthermore, J satisfies the estimate

$$|J(|x|;\lambda)-k^2/2|\le C(\gamma+\langle x\rangle^{-1})$$

uniformly in λ, so that we may write

$$H\varphi_\lambda=(E+k^2/2)\varphi_\lambda-iGA\varphi_\lambda+J_1\varphi_\lambda\,, \tag{2.19}$$

where

$$J_1=J_1(|x|;\lambda)=J-k^2/2=O(\gamma)+O(\langle x\rangle^{-1})\,.$$

(2) We accept the following lemma as porved, the proof of which is given after completing the proof of this proposition.

Lemma 2.5. *As* $\lambda\to\infty$, *one has:*
(i) $\|\langle x\rangle^{(\rho-1)/2}G^{1/2}A\varphi_\lambda\|_{L^2(X)}=O(1)$.
(ii) $\|\langle x\rangle^{(\rho-1)/2}\nabla_x\varphi_\lambda\|_{L^2(X)}=O(1)$.
It follws immediately from this lemma that

$$\lim_{\lambda\to\infty}\|GA\varphi_\lambda\|_{L^2(X)}=0\,.$$

Hence we have by (2.19) that

$$\limsup_{\lambda\to\infty}\|(H-E-k^2/2)\varphi_\lambda\|_{L^2(X)}=O(\gamma)\,.$$

We here again use Proposition 2.1. Then, by repeating the same arguments as in the proof of Proposition 2.1, we obtain

$$\liminf_{\lambda\to\infty}\langle i[H,A_1]\varphi_\lambda,\varphi_\lambda\rangle_{L^2(X)}\ge d>0 \tag{2.20}$$

for some d independent of $\gamma>0$ small enough.

(3) We calculate the term on the left side of (2.20) by making use of relation (2.18). Since $\langle x\rangle^N\varphi_\lambda$ belongs to $H^2(X)$ for any $N\gg1$, we have

$$\langle i[H,A_1]\varphi_\lambda,\varphi_\lambda\rangle_{L^2(X)}=-2\mathrm{Re}\langle A_1\varphi_\lambda,GA\varphi_\lambda\rangle_{L^2(X)}-\langle i[J,A_1]\varphi_\lambda,\varphi_\lambda\rangle_{L^2(X)}\,.$$

The second term on the right side converges to zero as $\lambda\to\infty$, because $[J,A_1]=O(|x|^{-1})$, $|x|\to\infty$, uniformly in $\lambda\gg1$. We shall show that the first term also converges to zero as $\lambda\to\infty$. To see this, we write this term as

$$\langle A_1\varphi_\lambda,GA\varphi_\lambda\rangle_{L^2(X)}=\langle\langle x\rangle^\nu G^{1/2}\langle x\rangle^\nu\langle x\rangle^{-\nu}A_1\varphi_\lambda,\langle x\rangle^{-\nu}G^{1/2}A\varphi_\lambda\rangle_{L^2(X)}$$

with $\nu=(1-\rho)/2$. By the assumption $\rho>1/2$, $q^\nu G^{1/2}q^\nu=o(1)$ as $|x|\to\infty$. Hence, by Lemma 2.5, the first term is also convergent to zero as $\lambda\to\infty$. Thus we have

$$\limsup_{\lambda\to\infty} \langle i[H, A_1]\varphi_\lambda, \varphi_\lambda\rangle_{L^2(X)} = 0,$$

which condradicts (2.20) and the proof is complete. □

Proof of Lemma 2.5. The idea of proof is almost the same as in the proof of Lemma 2.3, so we give only a sketch for the proof. We again denote by b_k the multiplication operator by $b_k(x)$ with bound $|b_k(x)| \le C\langle x\rangle^k$.

To prove the lemma, we consider the term

$$I_\lambda = \langle i[H, A_\theta]\varphi_\lambda, \varphi_\lambda\rangle_{L^2(X)},$$

where the operator A_θ is defined by

$$A_\theta = \frac{1}{2i}(\langle x/\theta, \nabla_x\rangle + \langle \nabla_x, x/\theta\rangle)$$

with $\theta(x)=\langle x\rangle^{1-\rho}$. The commutator $i[H, A_\theta]$ in the above term takes the form

$$i[H, A_\theta] = Q_1+Q_2+\theta^{-1}(-H+V)+b_{\rho-2}\nabla_x+b_0,$$

where

$$Q_1 = (1-\rho)\theta^{-1/2}(-\Delta-D_0^*D_0)\theta^{-1/2},$$
$$Q_2 = (\rho+1/2)\theta^{-1/2}(-\Delta)\theta^{-1/2}.$$

The operators Q_1 and Q_2 are both non-negative. Hence it follows from (2.18) that

$$I_\lambda \ge d\|\theta^{-1/2}\nabla_x\varphi_\lambda\|^2_{L^2(X)} - 1/d \tag{2.21}$$

for some $d>0$ independent of λ. On the other hand, we again use (2.18) to obtain that

$$I_\lambda \le -\|\theta^{-1/2}G^{1/2}A\varphi_\lambda\|^2_{L^2(X)} + d$$

with another $d>0$. This, together with (2.21), proves the lemma. □

2.3. Completion of proof of Theorem 1.1. We complete the proof of Theorem 1.1 by showing that the eigenstate $\psi\in L^2(X)$ in (2.1) must vanish identically.

Proof of Theorem 1.1. We begin by recalling the notations in section 1. Let $\omega\in S_X$ be the direction for which $E_X=E_0\omega$ with $E_0=|E_X|>0$. We write $x\in X$ as $x=z\omega+z_\perp$ with $z_\perp\in\Pi_\omega$, Π_ω being the hyperplane orthogonal to ω.

Assume that $\psi\neq 0$ and set $\psi_k(x)=\exp(kz)\psi(x)$ for $k\gg 1$ large enough. By Proposition 2.4, $\psi_z\in L^2(X)$. We normalize ψ_k as

$$\varphi_k = \psi_k/\|\psi_k\|_{L^2(X)}, \quad \|\varphi_k\|_{L^2(X)} = 1.$$

As is easily seen, φ_k obeys the equation

$$H\varphi_k = (E+k^2/2)\varphi_k - ikA_1\varphi_k\,. \tag{2.22}$$

The commutator $i[H, A_1]$ is calculated as $i[H, A_1]=E_0+i[V, A_1]$. Therefore it follows from assumption $(A)_\rho$ that

$$\langle i[H, A_1]\varphi_k, \varphi_k\rangle_{L^2(X)} \geq 1/d - d\|A_1\varphi_k\|^2_{L^2(X)} \tag{2.23}$$

for some $d>1$ independent of $k\gg 1$. On the other hand, we have by (2.22) that

$$\langle i[H, A_1]\varphi_k, \varphi_k\rangle_{L^2(X)} = -2k\|A_1\varphi_k\|^2_{L^2(X)}\,,$$

which, together with (2.23), concludes that $\psi=0$. Thus the proof is now completed. □

3. Resolvent estimate at high energies

In this section we study the resolvent estimate at high energies for three-particle Stark Hamiltonians under the assumption that any two-particle subsystem Hamiltonian does not have zero reduced charge. Throughout the present section, we again keep the same notations as in section 1.

We further introduce several new notations which are required to formulate the obtained result. We use the letters α, β and γ to denote pairs (j, k), $1\leq j<k\leq 3$. For given pair $\alpha=(j, k)$, we define the two subspaces X^α and X_α of X as follows:

$$X^\alpha = \{r=(r_1, r_2, r_3)\in X : r_j+r_k=0\}\,,$$
$$X_\alpha = \{r=(r_1, r_2, r_3)\in X : r_j=r_k\}\,.$$

These two subspaces are mutually orthogonal with respect to the scalar product $\langle , \rangle$ and span X, $X=X^\alpha\oplus X_\alpha$, so that $L^2(X)$ is decomposed into

$$L^2(X) = L^2(X^\alpha)\otimes L^2(X_\alpha)\,. \tag{3.1}$$

Let $\pi^\alpha : X\to X^\alpha$ and $\pi_\alpha : X\to X_\alpha$ be the projections from X onto X^α and X_α, respectively. For a generic point $x\in X$, we write $x^\alpha=\pi^\alpha x$ and $x_\alpha=\pi_\alpha x$. For pair $\alpha=(j, k)$, the relative coordinates r_j-r_k is represented in terms of only the coordinates x^α over X^α. Hence we can write $V_\alpha(x^\alpha)$ for $V_{jk}(r_j-r_k)$. Let $E_X\neq 0$ be again the projection onto X of $E=(e_1\mathcal{E}, e_2\mathcal{E}, e_3\mathcal{E})$ and denote by $E^\alpha=\pi^\alpha E_X$ and $E_\alpha=\pi_\alpha E_X$ the projections of E_X onto X^α and X_α, respectively. We further define the cluster Hamiltonian H_α as

$$H_\alpha = -\frac{1}{2}\Delta - \langle E_X, x\rangle + V_\alpha(x^\alpha) \quad \text{on } L^2(X)\,. \tag{3.2}$$

According to (3.1), this operator is decomposed as

$$H_\alpha = H^\alpha \otimes Id + Id \otimes T_\alpha \quad \text{on} \quad L^2(X^\alpha) \otimes L^2(X_\alpha),$$

where the operators H^α and T_α are given as

$$H^\alpha = -\frac{1}{2}\Delta - \langle E^\alpha, x^\alpha \rangle + V_\alpha \quad \text{on} \quad L^2(X^\alpha), \tag{3.3}$$

$$T_\alpha = -\frac{1}{2}\Delta - \langle E_\alpha, x_\alpha \rangle \quad \text{on} \quad L^2(X_\alpha). \tag{3.4}$$

We here make the assumption that the two-particle subsystem Hamiltonian H^α defined above does not have zero reduced charge;

(C) $E^\alpha \neq 0$ for all pairs $\alpha = (j, k)$.

We are now considering only a system consisting of three particles with identical masses. For such a system, the assumption above means that all the charges e_j, $1 \leq j \leq 3$, are different from each other. Under this assumption, we can also say that all the pair potentials $V_\alpha(x^\alpha)$ decay along the direction $\omega \in S_X$ of uniform electric field E_X.

We are now in a position to formulate the main theorem in this section.

Theorem 3.1. *Assume that all the pair potentials $V_{jk}(y)$, $y \in \boldsymbol{R}^3$, are bounded and have the decay property $V_{jk}(y) = O(|y|^{-\rho})$, $|y| \to \infty$, for some $\rho > 1/2$ and that the non-zero reduced charge condition* (C) *is fulfilled. Then there exists $M \gg 1$ large enough such that*

$$\sup_{|\lambda| \geq M,\ 0 < \kappa \leq 1} \| \langle x^\alpha \rangle^{-\nu} R(\lambda \pm i\kappa; H) \langle x^\beta \rangle^{-\nu} \| < C_M$$

for $\nu > 1/4$. Furthermore, the boundary values $R(\lambda \pm i0; H)$ to the real axis exist for λ, $|\lambda| \geq M$, in the topology above.

Without loss of generality, it suffices to prove the theorem only for $\nu = \rho/2 > 1/4$. The proof is done on the basis of the Faddeev equation method. Throughout the discussion below, all the assumptions in the theorem are assumed to be fulfilled. Define the operators $R_{\alpha\beta}(\zeta; H)$ and $R_{\alpha\beta}(\zeta; H_\alpha)$, $\operatorname{Im} \zeta \neq 0$, by

$$R_{\alpha\beta}(\zeta; H) = \langle x^\alpha \rangle^{-\rho/2} R(\zeta; H) \langle x^\beta \rangle^{-\rho/2},$$
$$R_{\alpha\beta}(\zeta; H_\alpha) = \langle x^\alpha \rangle^{-\rho/2} R(\zeta; H_\alpha) \langle x^\beta \rangle^{-\rho/2}$$

and the multiplication operator M_α by

$$M_\alpha = \langle x^\alpha \rangle^{\rho/2} V_\alpha(x^\alpha) \langle x^\alpha \rangle^{\rho/2}.$$

Then the following relation can be easily derived by repeated use of the resolvent equation:

$$R_{\alpha\beta}(\zeta; H) = R_{\alpha\beta}(\zeta; H_\alpha) - \sum_{\gamma \neq \alpha} R_{\alpha\gamma}(\zeta; H_\alpha) M_\gamma R_{\gamma\beta}(\zeta; H),$$

where the summation is taken over all pairs γ with $\gamma \neq \alpha$. Hence Theorem 3.1 is obtained as an immediate consequence of the following

Lemma 3.2. *Let the notations be as above. If $\alpha \neq \beta$, then*

$$||R_{\alpha\beta}(\lambda \pm i\kappa; H_\alpha)|| \to 0, \quad |\lambda| \to \infty,$$

uniformly in κ, $0 < \kappa \leq 1$, and the limits $R_{\alpha\beta}(\lambda \pm i0; H_\alpha)$ exist for all $\lambda \in \boldsymbol{R}^1$ in the uniform topology.

We further continue the reduction. Let H_0 be the unperturbed Stark Hamiltonian defined by

$$H_0 = -\frac{1}{2}\Delta - \langle E_X, x \rangle \quad \text{on } L^2(X).$$

Then, by the resolvent equation again, we have

$$R_{\alpha\beta}(\zeta; H_\alpha) = R_{\alpha\beta}(\zeta; H_0) + R_{\alpha\alpha}(\zeta; H_\alpha) M_\alpha R_{\alpha\beta}(\zeta; H_0).$$

Hence Lemma 3.2 follows from this relation at once, if we have only to prove the following two lemmas.

Lemma 3.3. *The operators $R_{\alpha\alpha}(\lambda \pm i\kappa; H_\alpha): L^2(X) \to L^2(X)$ are bounded uniformly in $\lambda \in R^1$ and κ, $0 < \kappa \leq 1$, and the limits $R_{\alpha\alpha}(\lambda \pm i0; H_\alpha)$ exist for all $\lambda \in R^1$ in the uniform topology.*

Lemma 3.4. *If $\alpha \neq \beta$, then the operators $R_{\alpha\beta}(\lambda \pm i\kappa; H_0): L^2(X) \to L^2(X)$ are bounded uniformly in $\lambda \in R^1$ and κ, $0 < \kappa \leq 1$, and satisfy*

$$||R_{\alpha\beta}(\lambda \pm i\kappa; H_0)|| \to 0, \quad |\lambda| \to \infty, \tag{3.5}$$

uniformly in κ. Furthermore, the limits $R_{\alpha\beta}(\lambda \pm i0; H_0)$ exist for all $\lambda \in R^1$ in the uniform topology.

The proof of both the lemmas above is based on the resolvent estimate at high energies for two-particle subsystem Hamiltonians.

Lemma 3.5. *Let $T = -\Delta/2 - \langle \mathcal{E}_0, y \rangle + U$, $\mathcal{E}_0 \neq 0$, be a two-particle Stark Hamiltonian acting on $L^2(\boldsymbol{R}_y^3)$, where the real-valued potential $U = U(y)$ is assumed to satisfy $|U(y)| \leq C(1+|y|)^{-\rho}$ for some $\rho > 1/2$. Then, T has the following spectral properties:*

(i) *T has no bound states.*

(ii) *For $\nu > 1/4$, the operators $\langle y \rangle^{-\nu} R(\lambda \pm i\kappa; T) \langle y \rangle^{-\nu}: L^2(\boldsymbol{R}_y^3) \to L^2(\boldsymbol{R}_y^3)$ are bounded uniformly in $\lambda \in R^1$ and κ, $0 < \kappa \leq 1$, and satisfy*

$$||\langle y \rangle^{-\nu} R(\lambda \pm i\kappa; T) \langle y \rangle^{-\nu}|| \to 0, \quad |\lambda| \to \infty.$$

(iii) *The boundary values $R(\lambda \pm i0; T)$ to the real axis exist in the topology above.*

For two-particle Stark Hamiltonians, the non-existence of bound states and the principle of limiting absorption have been already established by [1, 5, 18] and the resolvent estimate at high energies has been also proved by [9, 16].

Proof of Lemma 3.3. Let H^α and T_α be defined by (3.3) and (3.4), respectively. Assume that $E_\alpha \neq 0$. Then, by use of the spectral representation for the unperturbed two-particle Stark Hamiltonian T_α, the operator $R_{\alpha\alpha}(\lambda \pm i\kappa; H_\alpha)$ in question can be expressed as the direct integral

$$R_{\alpha\alpha}(\lambda \pm i\kappa; H_\alpha) = \int_{-\infty}^{\infty} \oplus R_{\alpha\alpha}(\lambda - \theta \pm i\kappa; H^\alpha) d\theta ,$$

where

$$R_{\alpha\alpha}(\zeta; H^\alpha) = \langle x^\alpha \rangle^{-\rho/2} R(\zeta; H^\alpha) \langle x^\alpha \rangle^{-\rho/2}$$

is considered as an operator from $L^2(X^\alpha)$ into itself. If $E_\alpha = 0$, then we can get a similar direct integral representation by use of the Fourier transformation with respect to the variables x_α. Since $E^\alpha \neq 0$ by assumption (C), the lemma follows immediately from Lemma 3.5. □

We prove Lemma 3.4 for the $+$case only. The proof is rather long and is done through a series of lemmas. The following lemma is well known (see [12] for example).

Lemma 3.6. *Let $T_0 = -\Delta/2$ act on $L^2(\boldsymbol{R}^3_y)$ and let $f, g \in L^p(\boldsymbol{R}^3_y)$ with $p > 2$. Then*

$$\| f \exp(-itT_0) g \| \le C \| f \|_{L^p} \| g \|_{L^p} |t|^{-3/p} , \quad t \in R^1 ,$$

for C independent of f and g.

Lemma 3.7. *Let $T_1 = T_0 - \langle \mathcal{E}_0, y \rangle$, $\mathcal{E}_0 \neq 0$, be the unperturbed two-particle Stark Hamiltonian acting on $L^2(\boldsymbol{R}^3_y)$. Let f and g again belong to $L^p(\boldsymbol{R}^3_y)$ with $p > 2$. Then*

$$\| f \exp(-itT_1) g \| \le C \| f \|_{L^p} \| g \|_{L^p} |t|^{-3/p} , \quad t \in R^1 .$$

Proof. For notational brevity, we take $\mathcal{E}_0 = (1, 0, 0)$ and write $y \in \boldsymbol{R}^3$ as $y = (y_1, y_\perp) \in \boldsymbol{R}^1 \times \boldsymbol{R}^2$. Let $D_1 = -i\partial/\partial y_1$. Then the following relation is known to hold between $\exp(-itT_1)$ and $\exp(-itT_0)$ (see [11] for example):

$$\exp(-itT_1) = \exp(-it^3/6) \exp(ity_1) \exp(-it^2 D_1/2) \exp(-itT_0) .$$

Therefore, by making a change of variables, we have

$$\| f \exp(-itT_1) g\varphi \|_{L^2}^2 = \int | f(y_1 + t^2/2, y_\perp) \exp(-itT_0)(g\varphi)(y) |^2 \, dy .$$

This, together with Lemma 3.6 proves the lemma. □

Lemma 3.8. *Let H_0 be again the unperturbed three-particle Stark Hamiltonian acting on $L^2(X)$. If $\nu>1$, then there exists $d>1$ such that*

$$\|\langle x^\alpha\rangle^{-\nu}\exp(-itH_0)\langle x^\beta\rangle^{-\nu}\|\leq C(1+|t|)^{-d}, \quad t\in R^1,$$

where $\alpha\neq\beta$ is not necessarily assumed.

Proof. We may write $\exp(itH_0)$ as

$$\exp(-itH_0)=\exp(-itH_0^\beta)\otimes\exp(-itT_\beta) \quad \text{on } L^2(X^\beta)\otimes L^2(X_\beta),$$

where $H_0^\beta=-\Delta/2-\langle E^\beta, x^\beta\rangle$. The coordinates x^α are represented as $x^\alpha=\pi^\alpha\pi^\beta x^\beta+\pi^\alpha\pi_\beta x_\beta$. For three-particle systems with identical masses, we have that

(3.6) $$\Pi^{\alpha\beta}=\pi^\alpha\pi^\beta|_{X^\beta}: X^\beta\to X^\alpha \text{ is invertible.}$$

Hence, $\langle x^\alpha\rangle^{-\nu}$ belongs to $L^p(X^\beta)$ with some p, $2<p<3$, as a function of x^β. Thus the lemma can be easily obtained by applying Lemma 3.7 to the operator $\exp(-itH_0^\beta)$. □

We here make a brief comment on (3.6). This remains true also for three-particle systems with finite masses but if one of three particle take an inifinite mass, then it happens that $\Pi^{\alpha\beta}=0$ for some pairs α and β. Even in such a case, we can prove in a different way that

$$\|R_{\alpha\beta}(\lambda\pm i\kappa; H_0)\|\to 0, \quad |\lambda|\to\infty,$$

uniformly in κ, $0<\kappa\leq 1$, and that the boundary values $R_{\alpha\beta}(\lambda\pm i0; H_0)$ to the real axis exist in the strong topology but not necessarily in the uniform topology. However, we do not go into details about this problem here.

Let $\Sigma=\{\zeta\in \boldsymbol{C}: 0<\operatorname{Im}\zeta\leq 1\}$. The lemma below follows from Lemma 3.8 at once.

Lemma 3.9. *Assume again that $\nu>1$. Then there exists $p>0$ such that*

$$\|\langle x^\alpha\rangle^{-\nu}(R(\zeta_1; H_0)-R(\zeta_2; H_0))\langle x^\beta\rangle^{-\nu}\|\leq C|\zeta_1-\zeta_2|^p$$

for any $\zeta_1, \zeta_2\in\Sigma$.

By use of this lemma, we shall first show that:

(3.7) $R_{\alpha\beta}(\zeta; H_0)$ is bounded uniformly in $\zeta\in\Sigma$.

(3.8) $\|R_{\alpha\beta}(\zeta_1; H_0)-R_{\alpha\beta}(\zeta_2; H_0)\|\leq C|\zeta_1-\zeta_2|^p$ for some $p>0$.

(3.9) $R_{\alpha\beta}(\lambda+i0; H_0)$, $\lambda\in R^1$ exists in the uniform topology.

We follow the arguments due to Korotyaev [9] to prove the facts above. Let

$$F(\zeta; H_0)=\frac{1}{2\pi i}(R(\zeta; H_0)-R(\bar\zeta; H_0)), \quad \zeta\in\Sigma.$$

This operator can be rewritten as

$$F(\zeta; H_0) = \frac{\operatorname{Im}\zeta}{\pi} R(\zeta; H_0)R(\bar{\zeta}; H_0)$$

and also it follows from Lemma 3.9 that

$$\text{(3.10)} \qquad \|\langle x^\alpha\rangle^{-\nu}(F(\zeta_1; H_0)-F(\zeta_2; H_0))\langle x^\beta\rangle^{-\nu}\| \le C|\zeta_1-\zeta_2|^p$$

for $\nu>1$. We assert that

$$\text{(3.11)} \qquad \|\langle x^\alpha\rangle^{-\nu}F(\zeta; H_0)\langle x^\beta\rangle^{-\nu}\| \le C_\nu$$

for any $\nu>1/4$. To see this, it suffices to show that

$$2\operatorname{Im}\zeta\|\langle x^\alpha\rangle^{-\nu}R(\zeta; H_0)\|^2 = \|\langle x^\alpha\rangle^{-\nu}(R(\zeta; H_0)-R(\bar{\zeta}; H_0))\langle x^\alpha\rangle^{-\nu}\| \le C .$$

However, this can be verified by repeating the same arguments as used in the proof of Lemma 3.3. Thus, by interpolaiton, it follows from (3.10) and (3.11) that

$$\|\langle x^\alpha\rangle^{-p/2}(F(\zeta_1; H_0)-F(\zeta_2; H_0))\langle x^\beta\rangle^{-p/2}\| \le C|\zeta_1-\zeta_2|^p$$

with another $p>0$. This also shows that the boundary value $F(\lambda; H_0)$, $\lambda\in \boldsymbol{R}^1$, to the real axis exists in the topology above and that

$$\text{(3.12)} \qquad \|\langle x^\alpha\rangle^{-p/2}(F(\lambda_1; H_0)-F(\lambda_2; H_0))\langle x^\beta\rangle^{-p/2}\| \le C|\lambda_1-\lambda_2|_q .$$

We denote by $E(\lambda; H_0)$, $\lambda\in \boldsymbol{R}^1$, the spectral resolution associated with H_0. By the Stone formula, we have $dE(\lambda; H_0)=F(\lambda; H_0)d\lambda$ and hence

$$R(\zeta: H_0) = \int \frac{1}{\lambda-\zeta} F(\lambda; H_0)d\lambda .$$

Thus, by (3.12), the Privalov lemma implies (3.7)~(3.9).

To complete the proof of Lemma 3.4, it remains to prove (3.5) only. We now fix two real smooth functions $f^\alpha\in C_0^\infty(X^\alpha)$ and $f^\beta\in C_0^\infty(X^\beta)$, $\alpha\neq\beta$, with compact support. To prove (3.5), it suffices by (3.7)~(3.9) to show that

$$\text{(3.13)} \qquad \|f^\alpha R(\lambda+i\kappa; H_0)f^\beta\|\to 0 , \quad |\lambda|\to\infty ,$$

for each κ, $0<\kappa\le 1$. We do not necessarily have to prove the uniform convergence. For notational brevity, we prove (3.13) only for $\kappa=1$.

We first deal with the case $E_\alpha\neq 0$. The case $E_\alpha=0$ is much easier to deal with. Since $\alpha\neq\beta$ by assumption, $\Pi=\pi^\beta\pi_\alpha|_{X_\alpha}: X_\alpha\to X^\beta$ is invertible. Let $\Pi_1=\Pi^{-1}\pi^\beta\pi^\alpha: X^\alpha\to X_\alpha$. Then we can write

$$\text{(3.14)} \qquad x^\beta = \Pi(x_\alpha+\Pi_1 x^\alpha) .$$

Let $\chi\in C_0^\infty(\boldsymbol{R}^1)$ be a non-negative smooth cut-off function such that $\chi(s)=1$

for $|s|\leq 1$ and $\chi(s)=0$ for $|s|\geq 2$. We define $g_{\lambda M}\in C_0^\infty(\boldsymbol{R}^1)$ by

$$g_{\lambda M}(s) = \chi((s-\lambda)/M)$$

and $h_{\lambda N}^\alpha \in C^\infty(X^\alpha)$ by

$$h_{\lambda N}^\alpha(x^\alpha) = \chi((\langle F^\alpha, x^\alpha\rangle - \lambda)/N)$$

with

$$F^\alpha = \Pi_1^* E_\alpha = \pi^\alpha \pi^\beta (\Pi^{-1})^* E_\alpha \in X^\alpha\,, \tag{3.15}$$

where M and N are sufficiently large numbers to be determined later.

With these notations, we now decompose $R(\lambda+i; H_0)$ as follows:

$$R(\lambda+i; H_0) = \Gamma^M(\lambda)+\Gamma_1^{ME}(\lambda)+\Gamma_2^{MN}(\lambda)\,,$$

where

$$\begin{aligned}\Gamma^M &= (Id-g_{\lambda M}(T_\alpha))R(\lambda+i; H_0)\,,\\ \Gamma_1^{MN} &= g_{\lambda M}(T_\alpha)R(\lambda+i; H_0)(Id-h_{\lambda N}^\alpha)\,,\\ \Gamma_2^{MN} &= g_{\lambda M}(T_\alpha)R(\lambda+i; H_0)h_{\lambda N}^\alpha\,.\end{aligned}$$

Lemma 3.10. *For any $\varepsilon>0$ small enough, one can take M so large that*

$$\|f^\alpha \Gamma^M(\lambda) f^\beta\| < \varepsilon\,, \quad \lambda\in\boldsymbol{R}^1\,.$$

Proof. Let $H_0^\alpha = -\Delta/2 - \langle E^\alpha, x^\alpha\rangle$ act on $L^2(X^\alpha)$ and write H_0 as

$$H_0 = H_0^\alpha \otimes Id + Id\otimes T_\alpha \text{ on } L^2(X^\alpha)\otimes L^2(X_\alpha)\,.$$

Then we obtain by use of the spectral representation for T_α that

$$(Id-g_{\lambda M}(T_\alpha))R(\lambda+i; H_0) = \int_{-\infty}^{\infty} \oplus (1-g_{\lambda M}(\theta))R(\lambda-\theta+i; H_0^\alpha)\,d\theta\,.$$

Hence, by Lemma 3.5, we can take M so large that

$$\|f^\alpha R(\lambda-\theta+i; H_0^\alpha)\|^2 = \|f^\alpha((H_0^\alpha+(\theta-\lambda))^2+1)^{-1} f^\alpha\| < \varepsilon$$

uniformly in θ, $|\theta-\lambda|\geq M$. This proves the lemma. □

Lemma 3.11. *Let M, $M\gg 1$, be as in Lemma 3.10. Then one can take N so large that*

$$\|f^\alpha \Gamma_1^{MN}(\lambda) f^\beta\| < \varepsilon\,, \quad \lambda\in\boldsymbol{R}^1\,.$$

Proof. We write the operator under consideration as

$$\Gamma_1^{MN}(\lambda) = g_{\lambda M}(T_\alpha)(T_\alpha-\lambda-i)R(\lambda+i; H_0)R(\lambda+i; T_\alpha)(Id-h_{\lambda N}^\alpha)$$

and we regard $f^\beta(x^\beta)=f^\beta(\Pi(x_\alpha+\Pi_1 x^\alpha))$, Π and Π_1 being as in (3.14), as a function over X_α. To prove the lemma, it suffices to show that we can take N so large that if $x^\alpha \in \mathrm{supp}(1-h^\alpha_{\lambda N})$, then

$$\|R(\lambda+i;\, T_\alpha)f^\beta\| < \varepsilon\,, \quad \lambda \in R^1\,, \tag{3.16}$$

as an operator acting on $L^2(X_\alpha)$. Let $\varphi \in L^2(X_\alpha)$. Then, by a change of variables $x_\alpha + \Pi_1 x^\alpha \to x_\alpha$, we obtain that

$$\|R(\lambda+i;\, T_\alpha)f^\beta \varphi\|_{L^2(X_\alpha)} = \|R(\lambda - \langle F^\alpha, x^\alpha\rangle + i;\, T_\alpha)f_\alpha \varphi_\alpha\|_{L^2(X_\alpha)}\,,$$

where $f_\alpha = f^\beta(\Pi x_\alpha) \in C_0^\infty(X_\alpha)$, $\varphi_\alpha = \varphi(x_\alpha - \Pi_1 x^\alpha) \in L^2(X_\alpha)$, and $F^\alpha \in X^\alpha$ is defined by (3.15). This relation, together with Lemma 3.5, enables us to choose N so large that (3.16) holds. Thus the proof is complete. □

The lemma below, together with Lemmas 3.10 and 3.11, proves (3.13) and hence completes the proof of Lemma 3.4 in the case $E_\alpha \neq 0$.

Lemma 3.12. *Let M and N be fixed as above. Then*

$$\|f^\alpha \Gamma_2^{MN}(\lambda) f^\beta\| \to 0\,, \quad |\lambda| \to \infty\,.$$

Proof. We again use the spectral representation for T_α. We may assume that the spectral parameter θ ranges over $(\lambda-2M, \lambda+2M)$. Hence, to prove the lemma, it suffices to show that

$$\|h^\alpha_{\lambda N} R(\mu - i;\, H_0^\alpha) f^\alpha\| \to 0\,, \quad |\lambda| \to \infty\,, \tag{3.17}$$

uniformly in μ, $|\mu| < 2M$, when considered as an operator from $L^2(X^\alpha)$ into itself.

Let $F^\alpha \in X^\alpha$ be defined by (3.15). It can be easily seen that $E^\alpha \neq 0$ is related to F^α through the relation $E^\alpha = \sigma F^\alpha$ for some $\sigma \neq 0$. In fact, if $\alpha = (1, 2)$ for example, then both the vectors take the form $(e\mathcal{E}, -e\mathcal{E}, 0)$, $e \neq 0$. We introduce the auxiliary operator $T_0 - \sigma\lambda$, $T_0 = -\Delta/2$, to approximate $H_0^\alpha = T_0 - \langle E^\alpha, x^\alpha\rangle$ on the support of $h^\alpha_{\lambda N}$. Write

$$h^\alpha_{\lambda N} R(\mu - i;\, H_0^\alpha) = R(\mu + \sigma\lambda - i;\, T_0)(T_0 - \mu - \sigma\lambda - i) h^\alpha_{\lambda N} R(\mu - i;\, H_0^\alpha)\,.$$

Since $f^\alpha \in C_0^\infty(X^\alpha)$, we may assume that $h^\alpha_{\lambda N} f^\alpha = 0$ for $|\lambda| \gg 1$ large enough. Hence a simple commutator calculation yields that

$$h^\alpha_{\lambda N} R(\mu - i;\, H_0^\alpha) f^\alpha = R(\mu + \sigma\lambda - i;\, T_0)(\Lambda_1(\lambda) + \Lambda_2(\lambda)) f^\alpha\,,$$

where

$$\Lambda_1 = \sigma(\langle F^\alpha, x^\alpha\rangle - \lambda) h^\alpha_{\lambda N} R(\mu - i;\, H_0^\alpha)\,,$$
$$\Lambda_2 = [T_0,\, h^\alpha_{\lambda N}] R(\mu - i;\, H_0^\alpha)\,.$$

To evaluate the norm of these operators, we here prepare two lemmas. Both the lemmas are easy to prove, so we omit the proof.

Lemma 3.13. *For any $\nu>0$, one has*

$$\|R(\mu+\sigma\lambda-i;\,T_0)\langle x^{\alpha}\rangle^{-\nu}\|\to 0\,,\quad |\lambda|\to\infty\,,$$

uniformly in μ, $|\mu|<2M$.

Lemma 3.14. *For any $k\in\boldsymbol{R}^1$, one has:*

$$\|\langle x^{\alpha}\rangle^{-k}R(\mu-i;\,H_0^{\alpha})\langle x^{\alpha}\rangle^{k}\|\le C_k$$
$$\|\langle x^{\alpha}\rangle^{-k-1/2}\nabla R(\mu-i;\,H_0^{\alpha})\langle x^{\alpha}\rangle^{k}\|\le C_k$$

with C_k independent of μ, $|\mu|<2M$.

We can easily see that (3.17) follows immediately form the two lemmas above and hence the proof is complete. □

Finally we consider the case $E_\alpha=0$. In this case also, (3.13) can be verified in almost the same way as in the case $E_\alpha\neq 0$, but the proof is much easier. In fact, if $E_\alpha=0$, then the operator T_α becomes invariant under translation. Hence the proof can be done without using the cut-off function $h^{\alpha}_{\lambda N}$. We omit the detailed proof of (3.13) for the case $E_\alpha=0$.

4. Asymptotic completeness for four-particle systems

The remaining sections are devoted to proving the asymptotic completeness for four-particle Stark Hamiltonians. We again consider a system of four particles with identical masses $m_j=1$, $1\le j\le 4$, moving in a uniform electric field $\mathcal{E}\in R^3$. For such a four-particle system, the configuration space X in the center-of-mass frame is described as

$$X=\{r=(r_1,r_2,r_3,r_4)\in\boldsymbol{R}^{3\times 4}:\sum_{j=1}^{4}r_j=0\}$$

and also the energy Hamiltonian H takes the form

$$H=-\frac{1}{2}\Delta-\langle E_X,r\rangle+V\quad\text{on }L^2(X)\,,$$

where $E_X\in X$ is again defined as the projection onto X of $(e_1\mathcal{E}, e_2\mathcal{E}, e_3\mathcal{E}, e_4\mathcal{E})\in\boldsymbol{R}^{3\times 4}$, e_j, $1\le j\le 4$, being the charge of the j-th particle, and $V=V(r)$ is given by the sum of pair potentials V_{jk}

$$V(r)=\sum_{1\le j<k\le 4}V_{jk}(r_j-r_k)\,.$$

We also assume that $E_X\neq 0$ and further make the following assumptions on the pair potential V_{jk}.

$(\mathrm{B})_\rho$ $V_{jk}(y)$, $y\in\boldsymbol{R}^3$, is a real C^2-smooth function and has the decay property as $|y|\to\infty$;

$$|y|^\rho(|V_{jk}(y)|+|\nabla_y V_{jk}(y)|)+|\nabla_y\nabla_y V_{jk}| = O(1) \quad \text{for some } \rho>1/2.$$

Roughly speaking, the problem of asymptotic comleteness is to determine completely the asymptotic behavior as $t\to\pm\infty$ of solution $u(t)=\exp(-itH)\psi$ to the Schrödinger equation

$$i\partial_t u = Hu\,, \quad u(0) = \psi\in \text{Range}\,(Id-P_H)\,, \tag{4.1}$$

where $P_H: L^2(X)\to L^2(X)$ is the eigenprojection associated with H and Range T stands for the range of operator T.

We require several basic notations in many-particle scattering theory to formulate the obtained result precisely. We use the letter a to denote a cluster decomposition of the set $\{1, 2, 3, 4\}$ and denote by $\#(a)$ the number of clusters in a. Throughout the entire discussion, we consider only a cluster decomposition a with $2\leq\#(a)\leq 4$. When j and k, $j<k$, are in the same cluster in a, we denote by $j\,a\,k$ this relation.

For given cluster decomposition a, we define the two subspaces X^a and X_a of X as follows:

$$X^a = \{r\in X: \sum_{j\in C} r_j = 0 \text{ for all clusters } C \text{ in } a\}\,,$$

$$X_a = \{r\in X: r_j=r_k \text{ for pair } (j, k) \text{ with } j\,a\,k\}\,.$$

These two subspaces can be easily seen to be mutually orthogonal with respect to the scalar product $\langle\,,\,\rangle$ and to span the total space X; $X=X^a\oplus X_a$, so that $L^2(X)$ is decomposed as

$$L^2(X) = L^2(X^a)\otimes L^2(X_a)\,.$$

Let $\pi^a: X\to X^a$ and $\pi_a: X\to X_a$ be the projections onto X^a and X_a, respectively. For a generic point $x\in X$, we write $x^a=\pi^a x\in X^a$ and $x_a=\pi_a x\in X_a$. Let $E^a=\pi^a E_X\in X^a$ and $E_a=\pi_a E_X\in X_a$. We further define the cluster Hamiltonian H_a as

$$H_a = -\frac{1}{2}\Delta-\langle E_X, x\rangle+\sum_{j\,a\,k} V_{jk}(r_j-r_k) \quad \text{on } L^2(X)\,. \tag{4.2}$$

This operator is decomposed into

$$H_a = H^a\otimes Id+Id\otimes T_a \quad \text{on } L^2(X^a)\otimes L^2(X_a)\,,$$

where

$$H^a = -\frac{1}{2}\Delta-\langle E^a, x^a\rangle+\sum_{j\,a\,k} V_{jk} \quad \text{on } L^2(X^a)\,, \tag{4.3}$$

$$T_a = -\frac{1}{2}\Delta-\langle E_a, x_a\rangle \quad \text{on } L^2(X_a)\,. \tag{4.4}$$

If, in paritcular, $\sharp(a)=4$, then H^a is defined as the zero operator acting on $L^2(X^a)=C$ (scalar field), so that H_a becomes the unperturbed Stark Hamiltonian

$$H_0 = -\frac{1}{2}\Delta - \langle E_X, x\rangle \quad \text{on } L^2(X).$$

We assume, in addition to $(B)_\rho$, that any subsystem Hamiltonian H^a, $2\leq \sharp(a)\leq 3$, defined above does not have zero reduced charge;

(D) $E^a \neq 0$ for any cluster decomposition a with $2\leq\sharp(a)\leq 3$.

We here make a brief comment on the assumption above. This assumption means that all the pair potentials V_{jk} decay along the direction of uniform electric field E_X. Charged classical particles are scattered along this direction and hence the sloution $u(t)$ to the Schrödinger equation (4.1) is expected to behave like

$$u(t)\sim\exp(-itH_0)\psi^\pm, \quad t\to\infty, \tag{4.5}$$

for some $\psi^\pm\in L^2(X)$. If $E^a=0$ for some cluster decomposition a, then it happens that H^a has bound states. Hence the scattering channels associated with such bound states may arise even in systems with uniform electric fields as in systems without electric fields and also the solution $u(t)$ takes multiple asymptotic states. Thus the case in which $E^a=0$ for some a is more difficult to deal with. In proving the asymptotic completeness for four-particle systems, we make an essential use of resolvent estimate at high energies for three-particle subsystem Hamiltonian H^a, $\sharp(a)=2$, which has been obtained as Theorem 3.1 under the non-zero reduced charge condition (C). This is the reason why (D) is assumed here.

Finally we introduce the wave operators. We define $W_0^\pm: L^2(X)\to L^2(X)$ by

$$W_0^\pm = s-\lim_{t\to\pm\infty}\exp(itH)\exp(-itH_0).$$

By definition, it can be easily seen that the solution $u(t)=\exp(-itH)\psi$ to equation (4.1) with initial state $\psi\in\text{Range } W_0^\pm$ asymptotically behaves as in (4.5). The second main theorem stated below asserts that all the solutions to equation (4.1) with initial state $\psi\in L^2(X)$ behave as above.

Theorem 4.1. (Asymptotic completeness). *Let the notations be as above. Assume that all the pair potentials V_{jk} satisfy the assumption* $(B)_\rho$ *with $\rho>1/2$ and that the non-zero reduced charge condition* (D) *is fulfilled. Then:*

(i) *H has no bound states.*

(ii) *$W_0^\pm$ exist and are asymptotically complete;* Range $W_0^\pm=L^2(X)$.

REMARK. The existence of wave operators is proved under only the assumption that $V_{jk}(y)=O(|y|^{-\rho})$, $|y|\to\infty$, for some $\rho>1/2$.

The asymptotic completeness for many-particle scattering systems without electric fields was first proved by Sigal-Soffer [14] for a large class of short-range pair potentials and then alternative proofs have been given by [4, 8, 15]. Much attention is now paid to the long-range scattering systems which include the Coulomb system as an important example. On the other hand, in the case of presence of electric fields, the asymptotic completeness has been proved by [9,16] only for three-particle systems. In both the works, the non-zero reduced charge condition is not necessarily assumed. The proof in [9] uses the stationary method based on the Faddeev equation under the additional assumption that two-particle subsystem Hamiltonian with zero reduced charge does not have zero resonance energy. After this work, an alternative proof has been given by [16] without assuming such a zero resonance condition, althogh somewhat restrictive smoothness assumptions ((V.1) and (V.2) in section 1) are imposed on pair potentials V_{jk}. The proof there, which is, in principle, similar to that in [14], is based on the local commutator estimate and on the (non)-propagation estimate showing that a relative motion of particles is asymptotically concentrated on classical trajectories. The asymptotic completeness for four-particle systems (Theorem 4.1) can be also proved in almost the same way as in the previous work [16], once the non-existence of bound states (Theorem 1.1) and the resolvent estimate at high energies (Theorem 3.1) have been established for three-particle subsystem Hamiltonians.

5. Local commutator estimate and propagation estimate

As stated in the previous section, Theorem 4.1 is proved on the bais of local commutator estimate and of propagation estimate and also the proof is done in almost the same way as used by [16] in the case of three-particle systems, so we here mention only a modification to be made in the case of four-paritlce systems.

We first omit the proof for the existence of wave operators $W_0^{\pm}$. This is verified in the same way as in the case without electric fields (see [12, 16] for example),

To prove the non-existence of bound states, we have to establish the Mourre estimate (Proposition 2.1) for the four-particle Stark Hamiltonian H under consideration. Let $\omega \in S_X$, S_X being the unit sphere in X, be again the direction of uniform electric field E_X for which $E_X = E_0\omega$ with $E_0 = |E_X| > 0$ and define the operator A_1 as $A_1 = -i\langle \omega, \nabla_x \rangle$.

Proposition 5.1. *Assume that the same assumptions as in Theorem* 4.1 *are fulfilled. Let* $\lambda \in \boldsymbol{R}^1$ *be fixed arbitrarily and let* $f \in C_0^\infty(\boldsymbol{R}^1)$ *be a non-negative smooth function supported in a small neighborhood around* λ. *Then, for* $\delta > 0$ *small enougugh, one can take the support of f so small that*

$$f(H)i[H, A_1]f(H) \geq (E_0 - \delta)f(H)^2 + K$$

for some compact operator $K=K_\delta$ acting on $L^2(X)$.

This proposition enables us to repeat the same arguments as used in proving Theorem 1.1 and hence H is proved to have no bound states.

The following propagation properties of propagator $\exp(-itH)$, which are also obtained as a consequence of Proposition 5.1, play a central role in proving the asymptotic completeness.

Proposition 5.2. *Suppose that the same assumptions as in Theorem 4.1 are fulfilled. Let $\Lambda\subset \boldsymbol{R}^1$ be a bounded open interval and let $E(\Lambda)=E(\Lambda; H)$ be the spectral resolution onto Λ of H. Then:*

(i) *The multiplication operator by $\langle x\rangle^{-\nu}$, $\nu>1/4$, is H-smooth on Λ in Kato's sense* ([7])*;*

$$\int_{-\infty}^{\infty}\|\langle x\rangle^{-\nu}\exp(-itH)E(\Lambda)\psi\|_{L^2(X)}^2dt\leq C_\Lambda\|\psi\|_{L^2(X)}^2$$

for C_Λ independent of $\psi\in L^2(X)$.

(ii) *If $q(x)$ is a bounded function vanishing in a conical neighborhood of ω, $\omega\in S_X$ being the direction of uniform electric field E_X, then the multiplication operator by $q\langle x\rangle^{-1/4}$ is H-smooth on Λ in the sense as above.*

Once these propagation estimates are established, the asymptotic completeness of wave operators $W_0^\pm$ is porved by repeating the same arguments as in [16].

In general, Stark Hamiltonians take all real values, especially any negative value as possible energies. Let $H_a=H^a\otimes Id+Id\otimes T^a$ be defined by (4.2). Then the subsystem Hamiltonians H^a and T_a can take all real values as energies, even if the energy of Hamiltonian H_a is localized in a bounded interval. This is not the case for Hamiltonians without uniform electric fields, because such operators are bounded from below. This is one of main differences between the case with electric fields and without electric fields and also this difference makes it difficult to prove the local commutator estimate in Proposition 5.1.

We shall explain the modification to be made in the case of four-particle systems. For pair $\alpha=(j, k)$ with $1\leq j<k\leq 4$, we write $r^\alpha=r_j-r_k$. We now fix a non-negative smooth function $g\in C_0^\infty(\boldsymbol{R}^3)$ with compact support and denote by g^α the multiplication operator by $g(r^\alpha)$. The lemma below plays a crucial role in proving Propositions 5.1 and 5.2 and also makes it possible to prove these propositions by use of the same arguments as in [16].

Lemma 5.3. *Suppose that the same assumptions as in Theorem 4.1 are fulfilled. Let $f\in C_0^\infty(\boldsymbol{R}^1)$ be as in Proposition 5.1. Then, for any $\delta>0$ small enough, one can take the support of f so small that*

$$\|g^\alpha f(H_a)\|<\delta$$

for pair $\alpha=(j, k)$ with $j\,a\,k$.

Proof. The multiplication g^{α} can be regarded as an operator acting on $L^2(X^a)$. Let H^a and T_a be defined by (4.3) and (4.4), respectively. Assume that $E_a \neq 0$. Then the spectral representation for T_a yields

$$g^{\alpha} f(H_a)^2 g^{\alpha} = \int_{-\infty}^{\infty} \oplus g^{\alpha} f(\theta+H^a)^2 g^{\alpha}\, d\theta\,.$$

Even if $E_a=0$, we can obtain a similar direct integral representation by use of the Fourier transformation. Thus we consider only the case $E_a \neq 0$. To prove the lemma, it suffices to show that we can take the support of f so small that

$$\|g^{\alpha} f(\theta+H^a)\|<\delta \tag{5.1}$$

uniformly in $\theta\in R^1$, when considered as an operator from $L^2(X^a)$ into itself. We have only to prove this for the following three cases: (1) $a=\{(j, k), (l), (m)\}$ is a 3-cluster decomposition; (2) a is a 2-cluster decomposition of the form $a=\{(j, k), (l, m)\}$; (3) a is a 2-cluster decomposition of the form $a=\{(j, k, l), (m)\}$.

(1) Since $E^a \neq 0$ by assumption (D), we can apply Lemma 3.5 with $T=H^a$ to obtain that

$$F(\lambda; H^a) = \frac{1}{2\pi i} g^{\alpha}(R(\lambda+i0; H^a)-R(\lambda-i0; H^a))g^{\alpha}: L^2(X^a)\to L^2(X^a)$$

is bounded uniformly in $\lambda\in R^1$. Thus we have

$$g^{\alpha} f(\theta+H^a)^2 g^{\alpha} = \int_{-\infty}^{\infty} f(\theta+\mu)^2 F(\lambda; H^a)\, d\lambda\,,$$

which proves (5.1) in case (1).

(2) The proof uses Lemma 3.5 again. Let $r^{\beta}=r_l-r_m$. Then $L^2(X^a)=L^2(R^3; dr^{\alpha})\otimes L^2(R^3; dr^{\beta})$ and hence H^a is decomposed as $H^a=T^{\alpha}\otimes Id+Id\otimes T^{\beta}$. By assumption (D), both the operators T^{α} and T^{β} take a form similar to T in Lemma 3.5 and also become unitarily equivalent to the free Stark Hamiltonian. Thus we obtain the direct integral representation

$$g^{\alpha} f(\theta+H^a)^2 g^{\alpha} = \int_{-\infty}^{\infty} \oplus g^{\alpha} f(\theta+\mu+T^{\alpha})^2 g^{\alpha}\, d\mu$$

by use of the spectral representation for T^{β}. This, together with Lemma 3.5, proves (5.1) in case (2).

(3) The proof uses Theorems 1.2 and 3.1 and Lemma 3.5. In particular, Theorem 1.2 has been obtained as a consequence of Theorem 1.1 (non-eixstence of bound states).

First we can choose the support of f so small that (5.1) holds for $|\theta|\gg 1$ large enough, which follows immediately from Theorem 3.1. Thus we have only to consider the case in which θ ranges over a bounded interval $[-M, M]$.

Let $a=\{(j, k, l), (m)\}$, $j<k<l$, be the 2-cluster decomposition under consideration. For such an a, we can easily construct a non-negative smooth partition of unity $\{j_1, j_2\}$, $j_1+j_2=1$, over X^a with the following property (j):

$$|j_1 V_{jk}(r_j-r_k)|+|j_2 V_{jl}(r_j-r_l)|+|j_2 V_{kl}(r_k-r_l)| = O(|x^a|^{-\rho}), \quad |x^a|\to\infty .$$

By construction, $j_1(x^a)g(r^\alpha)$, $r^\alpha=r_j-r_k$, is of compact support as a function over X^a. Hence, by Theorem 3.1, we can take the support of f so small that

$$\|j_1 g^\alpha f(\theta+H^a)\|<\delta \tag{5.2}$$

uniformly in θ, $|\theta|\le M$.

Let $b=\{(j, k), (l), (m)\}$ be the 3-cluster decomposition obtained as a refinement of a. We define the subspace X_b^a of X^a by $X_b^a=\pi_b X^a$, so that $L^2(X^a)$ is decomposed as

$$L^2(X^a) = L^2(X^b)\otimes L^2(X_b^a) .$$

We further write $x_b^a=\pi_b x^a \in X_b^a$ and define the operator H_b^a as

$$H_b^a = H^b\otimes Id+Id\otimes T_b^a \quad \text{on } L^2(X^b)\otimes L^2(X_b^a) ,$$

where

$$T_b^a = T_b-T_a = -\frac{1}{2}\Delta-\langle E_b^a, x_b^a\rangle , \quad E_b^a = \pi_b E^a\in X_b^a .$$

Then it follows by property (j) above that $H^a-H_b^a=O(|x^a|^{-\rho})$, $|x^a|\to\infty$, on the support of j_2 and hence we have

$$\|j_2(f(\theta+H^a)-f(\theta+H_b^a))\langle x^a\rangle^\rho\|\le C \tag{5.3}$$

for C independent of θ, $|\theta|\le M$.

Let $f_1\in C_0^\infty(\boldsymbol{R}^1)$ be a function such that f_1 has the same properties as f and $f_1=1$ on the support of f. We decompose $f(\theta+H^a)$ into

$$f(\theta+H_b^a)f_1(\theta+H^a)+(f(\theta+H^a)-f(\theta+H_b^a))f_1(\theta+H^a) .$$

Then we can take the supports of f and f_1 so small that

$$\|g^\alpha f(\theta+H_b^a)\|<\delta ,$$
$$\|\langle x^a\rangle^{-\rho} f_1(\theta+H^a)\|<\delta$$

uniformly in θ, $|\theta|\le M$. The first estimate follows from Lemma 3.5 and the second one from Theorem 1.2. These estimates and (5.3) imply that

$$\|j_2 g^\alpha f(\theta+H^a)\|<\delta$$

uniformly in θ as above. This, together with (5.2), proves (5.1) in case (3).

Thus the proof of the lemma is now complete. □

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