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REPRESENTATION OF THE SCATTERING KERNEL FOR THE ELASTIC WAVE EQUATION AND SINGULARITIES OF THE BACK-SCATTERING

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1. Introduction and main results

By Yamamoto [15], Shibata and Soga [7], etc., we know that we can construct the scattering theory for the elastic wave equation corresponding to the theory for the scalar-valued wave equation formulated by Lax and Phillips [3, 4]. Employing Lax and Phillips' theory, Majda [5] obtained a representation of the scattering kernel (operator), which was very useful for investigation on the inverse scattering problems (cf. Majda [5], Soga [8, 10], etc.). In the present paper we shall give the similar representation of the scattering kernel for the elastic wave equation considered in Shibata and Soga [7], and examine the singular support of that kernel.

Let Ω be an exterior domain in $\boldsymbol{R}^n_x$ $(x=(x_1, \cdots, x_n))$ whose boundary $\partial\Omega$ is a compact C^∞ hypersurface, and consider the elastic wave equation

$$
(1.1) \qquad \begin{cases} (\partial_t^2 - \sum_{i,j=1}^n a_{ij}\partial_{x_i}\partial_{x_j})u(t, x) = 0 & \text{in} \quad \boldsymbol{R}\times\Omega, \\ Bu(t, x) = 0 \quad \text{on} \quad \boldsymbol{R}\times\partial\Omega, & \\ u(0, x) = f_1(x), \quad \partial_t u(0, x) = f_2(x) & \text{on} \quad \Omega. \end{cases}
$$

Here, $u={}^t(u_1, \cdots, u_n)$ is the displacement vector, a_{ij} are constant $n\times n$ matrices whose (p, q)-components a_{ipjq} satisfy

$$
(\text{A.1}) \qquad a_{ipjq} = a_{pijq} = a_{jqip}, \quad i, j, p, q = 1, 2, \cdots, n,
$$

$$
(\text{A.2}) \qquad \sum_{i,p,j,q=1}^n a_{ipjq}\varepsilon_{jq}\bar{\varepsilon}_{ip} \geqq \delta \sum_{i,p=1}^n |\varepsilon_{ip}|^2 \text{ for every symmetric matrices } (\varepsilon_{ij}),
$$

and the boundary operator B is of the form

$$
Bu = u|_{\partial\Omega} \quad \text{or} \quad \sum_{i,j=1}^n \nu_i(x)\, a_{ij}\partial_{x_j}u|_{\partial\Omega},
$$

where $\nu=(\nu_1, \cdots, \nu_n)$ is the unit outer vector normal to $\partial\Omega$. We denote by $U(t)$ the mapping: $f=(f_1, f_2)\rightarrow(u(t, \cdot), \partial_t u(t, \cdot))$ associated with (1.1), and by $U_0(t)$

the one associated with the equation in the free space ($\Omega=\boldsymbol{R}^n$).

Shibata and Soga [7] show that under the assumptions (A.1) and (A.2) $U(t)$ (resp. $U_0(t)$) becomes a group of unitary operators on the Hilbert space H (resp. H_0) equipped with the energy norm

$$\|f\|_{E,\Omega}=\{\frac{1}{2}\int_\Omega(\sum_{i,j,p,q=1}^n a_{ipjq}\partial_{x_j}f_{1q}\overline{\partial_{x_i}f_{1p}}+|f_2|^2)dx\}^{1/2}$$

(resp. $\|f\|_{E,\boldsymbol{R}^n}$) (cf. §1 of [7]). Furthermore, adding the assumption

(A.3) $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j$ has eigenvalues of constant multiplicity for any $\xi={}^t(\xi_1,\cdots,\xi_n)$ $\in\boldsymbol{R}^n-\{0\}$,

they show that the wave operators $W_\pm=\lim_{t\to\pm\infty}U(-t)U_0(t)$ are well defined and complete (cf. §3 of [7]).

As is shown in §2 of Shibata and Soga [7], if (A.1)$\sim$(A.3) are satisfied, we can construct concretely the translation representations corresponding to Lax and Phillips' [3, 4] by means of the Radon transformation: $g(x)\to\tilde{g}(s,\omega)=\int_{x\cdot\omega=s}g(x)dS_x$ $((s,\omega)\in\boldsymbol{R}\times S^{n-1})$. Let $\{\lambda_j(\xi)\}_{j=1,\cdots,d}$ $(\lambda_1<\cdots<\lambda_d)$ be the distinct positive eigenvalues of $\sum_{i,j=1}^n a_{ij}\xi_i\xi_j$, and let $P_j(\xi)$ be the projection into the eigenspace of $\lambda_j(\xi)$. Then the translation representations of the data $f=(f_1,f_2)$ $(\in\mathcal{S})$ in the free space are defined by

$$T_0^\pm f(s,\omega)=\sum_{j=1}^d\lambda_j(\omega)^{1/4}P_j(\omega)J_\pm(-\lambda_j(\omega))^{1/2}\partial_s\tilde{f}_1+\tilde{f}_2)(\lambda_j(\omega)^{1/2}s,\omega),\quad(s,\omega)\in\boldsymbol{R}\times S^{n-1},$$

where $J_\pm=(-\partial_s)^{(n-1)/2}$ for odd n and $J_\pm$ for even n mean the operators $F^{-1}[(-i\sigma)_\pm^{(n-1)/2}F\cdot]$ (F being the Fourier transformation in s) whose symbols $(-i\sigma)_+^{(n-1)/2}$ and $(-i\sigma)_-^{(n-1)/2}$ denote the branches of $(-i\sigma)^{(n-1)/2}$ continued analytically into the upper and lower half complex plane respectively (cf. §2 in Shibata and Soga [7]).

We define the scattering operator S by $S=T_0^+(W_+)^{-1}W_-(T_0^-)^{-1}$, as Lax and Phillips [3, 4] did. S is a unitary operator from $L^2(\boldsymbol{R}\times S^{n-1})$ $(=\{L^2(\boldsymbol{R}\times S^{n-1})\}^n)$ to itself, and is expressed with the distribution kernel $S(s,\theta,\omega)$ (called the scattering kernel):

$$(Sk)(s,\theta)=\iint S(s-t,\theta,\omega)k(t,\omega)dtd\omega\,.$$

Let $v_l(t,x;\omega)$ be the solution of the equation

$$\begin{cases} \partial_t^2 v - \sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j} v = 0 & \text{in } \boldsymbol{R}\times\Omega, \\ Bv = -2^{-1}(-2\pi i)^{1-n}\lambda_l(\omega)^{-n/4} B\{\delta(t-\lambda_l(\omega)^{-1/2}\omega\cdot x)P_l(\omega)\} & \text{on } \boldsymbol{R}\times\partial\Omega, \\ v = 0 & \text{if } t \ll 0. \end{cases}$$

$v_l(t, x; \omega)$ is an $n\times n$ matrix of C^∞ functions of x and ω with the value of the distribution in t.

Theorem 1. *We assume* (A.1)~(A.3) *and the following*

(A.4) *every slowness hypersurface* $\Sigma_i=\{\xi: \lambda_i(\xi)=1\}$ $(i=1, \cdots, d)$ *is strictly convex (i.e. the Gaussian curvature does not vanish anywhere).*

Then the scattering kernel is represented in the following way:

$$S(s, \theta, \omega) = \sum_{i,j=1}^{n} \lambda_i(\theta)^{-n/4}\int_{\partial\Omega} \{P_i(\theta)\ (\partial_t^{n-2}Nv_j)(\lambda_i(\theta)^{-1/2}\theta\cdot x-s, x; \omega) \\ -\lambda_i(\theta)^{-1/2}P_i(\theta)({}^tN\theta\cdot x)(\partial_t^{n-1}v_j)(\lambda_i(\theta)^{-1/2}\theta\cdot x-s, x; \omega)\}\, dS_x \\ (\theta\neq\omega),$$

where $N=\sum_{i,j=1}^{n} \nu_i(x)a_{ij}\partial_{x_j}$.

It has been announced in Soga [12] that we can obtain this representation in the space of the odd dimension n. Improving the methods in [12], we shall prove it in §3 inclusive of the even dimension. We need much more precise analysis when n is even than when n is odd.

In the proof of Theorem 1 there are two difficulties which we did not encounter in the scalar-valued wave equation treated in Majda [5], Soga [8, 9], etc., although the idea of the procedures is similar. The first difficulty is to verify the following theorem. This is one of the bases for the proof of Theorem 1.

Theorem 2. *For* $\theta\in S^{n-1}$ *and* $j=1, \cdots, d$ *set*

$$\kappa_j(\theta) = \pi^{(n-1)/2}K_j(\theta)^{1/2}|\partial_\xi\lambda_j(\theta)|^{(n+1)/2}\lambda_j(\theta)^{-(2n+1)/4},$$
$$\eta_j(\theta) = 2^{-1}\lambda_j(\theta)^{-1/2}\partial_\xi\lambda_j(\theta),$$
$$\zeta_j(\theta) = \lambda_j(\theta)^{1/2}\theta,$$

where $K_j(\theta)$ *denotes the Gaussian curvature of* Σ_j *at* $\lambda_j(\theta)^{-1/2}\theta$ $(\in\Sigma_j)$. *Then, for any* f *with* $T_0^+f\in\mathcal{S}(\boldsymbol{R}\times S^{n-1})$ *we have*

$$T_0^+f(s, \theta) = \lim_{t\to\infty} t^{(n-1)/2}\sum_{j=1}^{d} \kappa_j(\theta)P_j(\theta)(U_0(t)f)_2(t\eta_j(\theta)+s\zeta_j(\theta)).$$

In the scalar-valued wave equation Lax and Phillips [3] obtained similar formula (see Theorem 2.4 in Ch. IV of [3]), but their methods do not work well in our case.

In the proof of Theorem 1 we use the fundamental solution of the wave equation (1.1) as Majda [5] and Soga [8, 9] did. The fundamental solution of the d'Alembert equation can be expressed concretely, e.g., $4\pi t^{-1}\delta(|x|-t)$ $(n=3)$, etc. They in [5, 8, 9] utilized this concrete expression skilfully. In the present case, however, we cannot expect that the fundamental solution is written in such a concrete form. This is the second difficulty. We can obtain the representation of the scattering kernel without the concrete expression of the fundamental solution, by examining the integrals $\int (J_{\pm}k)(t\varphi(\omega), \omega)d\omega$ as $|t|\to\infty$ ($\varphi(\omega)$ being some real-valued function on S^{n-1}). The proof of Theorem 2 is also reduced to examination of the same integrals. In Appendix (§5) we show some properties of those integrals needed in the proofs, which are similar to the well-known results for the method of stationary phases; however, our results can not obviously be derived by this method. Especially, this derivation for even n is much more difficult than for odd n. This is because $J_{\pm}$ are not differential operators when n is even.

Using the representation in Theorem 1, we obtain the following theorem about the (singular) support of the back-scattering $S(s, -\omega, \omega)$ corresponding to the one in Majda [5] and Soga [10]:

Theorem 3. *For any fixed $\omega\in S^{n-1}$ we have*
(i) $\operatorname{supp}[P_i(-\omega)\, S(\cdot, -\omega, \omega)\, P_j(\omega)]\subset$
$$(-\infty, -(\lambda_i(\omega)^{-1/2}+\lambda_j(\omega)^{-1/2})r(\omega)] \quad (i, j=1, \cdots, d),$$
(ii) $P_j(-\omega)\, S(s, -\omega, \omega)\, P_j(\omega)$ *is singular (not C^∞) at* $s=-2\lambda_j(\omega)^{-1/2}r(\omega)$ $(j=1, \cdots, d)$,
where $r(\omega)=\min_{x\in\partial\Omega} x\cdot\omega$.

The above results are reasonable since $P_i(-\omega)S(\lambda_i(\omega)^{-1/2}\omega\cdot x-t, -\omega, \omega)$ $P_j(\omega)$ means the $\lambda_i(\omega)$-mode wave scattered in the direction $-\omega$ for the incident $\lambda_j(\omega)$-mode wave in the direction ω. Theorem 3 was announced in Soga [12] under some more restricted assumptions. When (1.1) is the isotropic equation (i.e. $a_{ipjq}=\mu(\delta_{pq}\delta_{ij}+\delta_{iq}\delta_{jp})+\lambda\delta_{ip}\delta_{jq}$, $\mu>0$, $\mu+2\lambda/3>0$) with the Dirichlet boundary condition in the case of $n=3$, Yamamoto [16] has also obtained the same results as in Theorem 3. But it seems difficult to apply his methods to our case.

We can obtain Theorem 3 by the same procedures as in Majda [5], Soga [10], etc.: Employing the asymptotic solutions of (1.1) (constructed in Soga [13]), we expand asymptotically the Fourier transform of $\alpha(s)P_i(-\omega)S(s, -\omega, \omega)$ $P_j(\omega)$ in the form $\sum_{k=0}^{\infty}\int_{R^{n-1}} e^{-i\sigma\varphi(y)}\, \beta_k(y)(i\sigma)^{-k}\, dy$ (as $|\sigma|\to\infty$), and apply Theorem 2 in Soga [10] to this form (which deals with those oscillatory integals).

It is an interesting problem whether or not $P_i(-\omega)S(s, -\omega, \omega)P_j(\omega)$ with

$i\neq j$ is singular at $s=-(\lambda_i(\omega)^{-1/2}+\lambda_j(\omega)^{-1/2})r(\omega)$. To solve this problem, however, we need more precise analysis than for Theorem 3. It is examined in the paper Kawashita and Soga [1].

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2. Properties of the translation representations

In this section we assume (A.1)$\sim$(A.4) in Introduction, and show some properties of the translation representations $T_0^{\pm}$ in the free space and the ones $T^{\pm}(=T_0^{\pm}(W^{\pm})^{-1})$ for the mixed problem (1.1). Theorem 2 in §1 follows immediately from Theorem 2.1 below. The notations in §1 are used also in this section.

We set $L=\sum_{i,j=1}^{n} a_{ij}\partial_{x_i}\partial_{x_j}$. Let A_0 be the operator on H_0 defined by $A_0=\begin{bmatrix} I & 0 \\ 0 & L\end{bmatrix}$, and denote the domain of A_0^m by $D(A_0^m)$. A_0 is the generator of $U_0(t)$. In the same way we denote by A the generator of $U(t)$. The properties of these operators are described precisely in Shibata and Soga [7]. We denote by $C_0^m(M)$ the space of C^m functions on M with compact support, and by $\mathcal{B}^\infty(M)$ the space of C^m functions $v(x)$ on M satisfying $\sum_{|\alpha|\leqq m}\sup_{x\in M}|\partial_x^\alpha v(x)|<\infty$ $(\mathcal{B}^\infty=\bigcap_{m\geqq 0}\mathcal{B}^m)$. $\mathcal{S}$ means the set of the Schwartz rapidly decreasing functions. We abbreviate the product $E\times\cdots\times E$ of a functional space E by E.

Theorem 2.1. *Let $\kappa_j(\theta)$, $\eta_j(\theta)$ and $\zeta_j(\theta)$ be the functions defined in Theorem 2.*

(i) *Let f be any data satisfying $f\in D(A_0^{n+3})$. Then, for any $h(s,\theta)\in C_0^\infty(\boldsymbol{R}\times S^{n-1})$ there exists a constant C independent of f and $t\in\boldsymbol{R}$ such that*

$$\left|\iint_{\boldsymbol{R}\times S^{n-1}} t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j(\theta)P_j(\theta)(U_0(t)f)_2(t\eta_j(\theta)+s\zeta_j(\theta))h(s,\theta)dsd\theta\right| \leqq C\sum_{i=0}^{n+3}\|A_0^i f\|_{E,\boldsymbol{R}^n}.$$

(ii) *If $f\in H_0$ satisfies $T_0^+f\in\mathcal{S}(\boldsymbol{R}\times S^{n-1})$, then for any (s,θ) we have*

$$T_0^+f(s,\theta)=\lim_{t\to\infty} t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j(\theta)P_j(\theta)(U_0(t)f)_2(t\eta_j(\theta)+s\zeta_j(\theta)).$$

If $f\in D(A_0^{n+3})$, then the above equality is valid in the sense of the distributions on $\boldsymbol{R}_s\times S_\theta^{n-1}$.

From the above theorem we obtain the following theorem concerning $T^+=T_0^+(W_+)^{-1}$, which is one of the bases for the proof of Theorem 1 in §1.

Theorem 2.2. *Let $\kappa_j(\theta)$, $\eta_j(\theta)$ and $\zeta_j(\theta)$ be the functions defined in Theo-*

rem 2. *If* $f\in D(A^{n+3})$, *then* $t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j(\theta)P_j(\theta)(U(t)f)_2(t\eta_j(\theta)+s\zeta_j(\theta))$ *converges to* $T^+f(s,\theta)$ *in the sense of the distributions as* $t\to\infty$; *i.e., for any* $h(s,\theta)\in C_0^\infty(\boldsymbol{R}\times S^{n-1})$ *we have*

$$\lim_{t\to\infty}\iint_{\boldsymbol{R}\times S^{n-1}}\{t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j(\theta)P_j(\theta)(U(t)f)_2(t\eta_j(\theta)+s\zeta_j(\theta)) -T^+f(s,\theta)\}h(s,\theta)dsd\theta=0\,.$$

The above theorems are the main results in this section and will be proved later.

The slowness surfaces Σ_i are $(n-1)$ dimensional C^∞ hypersurfaces expressed by $\{\lambda_i(\omega)^{-1/2}\omega\}_{\omega\in S^{n-1}}$. Each mapping: $\omega\to\lambda_i(\omega)^{-1/2}\omega$ is a diffeomorphism from S^{n-1} to Σ_i. It is easy to see

Lemma 2.3. *Fix* $\eta\in\boldsymbol{R}^n$ $(\neq 0)$ *arbitrarily, and set* $\varphi_i(\omega)=\lambda_i(\omega)^{-1/2}\omega\cdot\eta$ $(\omega\in S^{n-1})$.

(i) $\varphi_i(\omega)$ *is maximum at only one point* ω_i^+ *and minimum at only one point* ω_i^-; *furthermore,* $\omega_i^+=-\omega_i^-$.

(ii) *The gradient of* $\varphi_i(\omega)$ *does not vanish at any* ω *which is not equal to* ω_i^+ *or* ω_i^-.

(iii) $\varphi_i(\omega_1^-)<\cdots<\varphi_i(\omega_d^-)<0<\varphi_i(\omega_d^+)<\cdots<\varphi_i(\omega_1^+)$.

The following lemmas are key lemmas for the proof of Theorem 2.1.

Lemma 2.4. *Let* $\eta\in\boldsymbol{R}^n$ $(\neq 0)$ *and* $s_0\in\boldsymbol{R}$, *and set* $\varphi_l(\omega)=\lambda_l(\omega)^{-1/2}\omega\cdot\eta+s_0$ $(\omega\in S^{n-1})$. *Then, for any* $k(s,\omega)\in\mathcal{S}(\boldsymbol{R}\times S^{n-1})$ *we have*

$$\lim_{t\to\infty}t^{(n-1)/2}\int_{S^{n-1}}(J_+^*k)(t\varphi_l(\omega),\omega)d\omega =\begin{cases}0 \text{ if } \varphi_l(\omega_l^\mp)\neq 0\,,\\ \tilde{\kappa}_l(\omega_l^\dagger)k(0,\omega_l^\dagger)\text{ if }\varphi_l(\omega_l^\dagger)=0\,,\end{cases}$$

where $\omega_l^\mp$ *are the points in Lemma* 2.3 *and* $\tilde{\kappa}_l(\omega)=2(2\pi)^{(n-1)/2}|\eta|^{(1-n)/2}\lambda_l(\omega)^{(n+1)/2}K_l(\omega)^{-1/2}|\partial_\xi\lambda_l(\omega)|^{-1}$.

Lemma 2.5. *Let* μ *be any vector in* $\{\mu\in\boldsymbol{R}^n;\ \mu_0^{-1}\leqq|\mu|\leqq\mu_0\}$ $(\mu_0>1)$, *and let* $s_0\in\boldsymbol{R}$. *Set* $\psi_i(\theta)=\mu\cdot(\partial_\xi\lambda_i)(\lambda_i(\theta)^{-1/2}\theta)+s_0$ $(\theta\in S^{n-1})$. *Then, for any* $k(s,\theta)\in\mathcal{S}(\boldsymbol{R}\times S^{n-1})$ *we have*

$$|t^{(n-1)/2}\int_{S^{n-1}}(J_+^*k)(t\psi_i(\theta),\theta)d\theta|\leqq C\|k\|_{n+3,\boldsymbol{R}\times S^{n-1}}\,,$$

where $\|\cdot\|_{m,\boldsymbol{R}\times S^{n-1}}$ *denotes the Sobolev norm on* $\boldsymbol{R}\times S^{n-1}$ *of order m and C is a constant independent of* μ, k *and* t $(\in\boldsymbol{R})$.

Proof of Lemma 2.4. Take a prtition $\{\chi_j\}_{j\geq 0}$ of unity on S^{n-1} such that $\mathrm{supp}[\chi_0]\ni\omega_l^+\notin\mathrm{supp}[\chi_j]$ $(j\geqq 1)$ and $\mathrm{supp}[\chi_1]\ni\omega_l^-\notin\mathrm{supp}[\chi_j]$ $(j\geqq 2$ and $=0)$. For each χ_j we choose a local coordinate system $\omega=\omega(y)$ $(y\in\boldsymbol{R}^{n-1})$, and consider the integral

$$I_j(t)=t^{(n-1)/2}\int_{\boldsymbol{R}^{n-1}}(J_+^*k)\,(t\varphi_l(\omega(y)),\,\omega(y))\chi_j(\omega(y))\,|\frac{\partial\omega}{\partial y}|\,dy\,.$$

As is stated in Lemma 2.3. $|\partial_y\varphi_l(\omega(y))|$ does not vanish on $\mathrm{supp}[\chi_j]$ when $j\geqq 2$. Therefore, by Theorem 5.2 (and (i) of Remark below Theorem 5.4) in §5, we have $|I_j(t)|\leqq Ct^{-1}\to 0$ (as $t\to\infty$) when $j\geqq 2$. From the assumption (A.4) it is seen that Theorem 5.4 or 5.6 in §5 can be applied to $I_0(t)$ and $I_1(t)$ (see (ii) of Remark below Theorem 5.4 also). Therefore, if $\varphi_l(\omega_l^+)\neq 0$, then $\lim_{t\to\infty} I_j(t)=0$ for $j=0,1$ (by (ii) of Theorem 5.4). And if $\varphi_l(\omega_l^+)=0$, then, noting that $J_+^*=i^{n-1}J_-$ and $-\varphi_l(\omega)\geqq 0$ near ω_l^+, we have $\lim_{t\to\infty} I_0(t)=\tilde{\kappa}_l(\omega_l^+)k(0,\omega_l^+)$ (by (ii) of Theorem 5.6); $\lim_{t\to\infty} I_1(t)=0$ since $\varphi_l(\omega_l^-)\neq 0$. Hence the lemma is obtained.

Proof of Lemma 2.5. It is seen that $\psi_i(\theta)$ is maximum at $\theta_+=|\mu|^{-1}\mu$ and minimum at $\theta_-=-|\mu|^{-1}\mu$, and that the gradient of $\psi_i(\theta)$ does not vanish when $\theta\neq\theta_\pm$; furthermore the Hessian of $\psi_i(\theta)$ does not vanish near $\theta=\theta_\pm$. In the same way as in the proof of Lemma 2.4, taking the partition of unity, we can derive Lemma 2.5 from Theorem 5.2 and (i) of Theorem 5.6 in §5. The proof is complete.

Proof of Theorem 2.1. (i) Let us note that

$$(2.1)\qquad \begin{aligned}(U_0(t)f)_2(x)&=2^{-1}(2\pi)^{1-n}\int_{S^{n-1}}\sum_{l=1}^{d}\lambda_l(\varphi)^{-n/4}P_l(\omega)\\ &\qquad (J_+^*k)\,(\lambda_l(\omega)^{-1/2}\omega\cdot x-t,\,\omega)d\omega,\quad k=T_0^+f\end{aligned}$$

(cf. Theorem 2.1 and (2.14) in Shibata and Soga [7]). We see that this is valid if $f\in D(A_0^\infty)$. Therefore the integral $\Psi(t)\equiv\iint_{\boldsymbol{R}\times S^{n-1}} t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j(\theta)P_j(\theta)$ $(U_0(t)f)_2(t\eta_j(\theta)+s\zeta_j(\theta))h(s,\theta)dsd\theta$ is written in the form

$$\begin{aligned}\Psi(t)=2^{-1}(2\pi)^{1-n}\sum_{j,l=1}^{d}\int_{S^{n-1}}d\omega\int_{-\infty}^{\infty}ds\;\lambda_l(\omega)^{-n/4}t^{(n-1)/2}\int_{S^{n-1}}\kappa_j(\theta)P_j(\theta)P_l(\omega)\\ (J_+^*T_0^+f)\,(t\lambda_l(\omega)^{-1/2}\omega\cdot\eta_j(\theta)+s\lambda_l(\omega)^{-1/2}\omega\cdot\zeta_j(\theta)-t,\,\omega)h(s,\theta)d\theta\,.\end{aligned}$$

Applying Lemma 3.5 to the above integral $\int_{S^{n-1}}\kappa_jP_jP_l(J_+^*T_0^+f)hd\theta$ with $\mu=2^{-1}\lambda_l(\omega)^{-1/2}\omega$, we have

$$|\Psi(t)|\leqq C_1\int_{S^{n-1}}d\omega\int_{-\infty}^{\infty}ds\sum_{i=0}^{n+3}\|\partial_\tau^i T_0^+f(\tau,\omega)\|_{L_\tau^2}\sup_{\substack{\theta\in S^{n-1}\\|\alpha|\leqq n+3}}|\partial_\theta^\alpha h(s,\theta)|$$
$$\leqq C_2\sum_{i=0}^{n+3}\|A_0^i f\|_{E,R^n}\,.$$

Here we have used the equality $\|\partial_s^i T_0^+ f(s,\omega)\|_{L^2(R\times S^{n-1})}=\|A_0^i f\|_{E,R^n}$ (cf. §2 of Shibata and Soga [7]). Hence, (i) of Theorem 2.1 is proved.

(ii) By (2.1), $\Phi(t)\equiv t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j(\theta)P_j(\theta)(U_0(t)f)_2(t\eta_j(\theta)+s\zeta_j(\theta))$ is of the form

$$\Phi(t)=2^{-1}(2\pi)^{1-n}\sum_{j,l=1}^{d}\kappa_j(\theta)P_j(\theta)t^{(n-1)/2}\int_{S^{n-1}}\lambda_l(\omega)^{-n/4}P_l(\omega)$$
$$(J_+^*k)(t\lambda_l(\omega)^{-1/2}\omega\cdot\eta_j(\theta)+s\lambda_l(\omega)^{-1/2}\omega\cdot\zeta_j(\theta)-t,\omega)d\omega\,.$$

Let ω_{lj}^+ (resp. ω_{lj}^-) be the point at which $\varphi_{lj}(\omega)=\lambda_l(\omega)^{-1/2}\omega\cdot\eta_j(\theta)-1$ is maximum (resp. minimum) on S_ω^{n-1}. Then, since $\lambda_l(\theta)^{-1/2}\theta\cdot 2^{-1}\partial_\xi\lambda_l(\lambda_l(\theta)^{-1/2}\theta)=1$ (from the Euler equality) by Lemma 2.3 we have

$$\omega_{ll}^{\pm}=\pm\theta,\quad \varphi_{lj}(\omega_{lj}^-)<0,\quad \varphi_{lj}(\omega_{lj}^+)\neq\varphi_{ll}(\omega_{ll}^+)=0\quad\text{if}\quad l\neq j\,.$$

Therefore, applying Lemma 2.4, we obtain

$$\lim_{t\to\infty}\Phi(t)=\sum_{l=1}^{d}P_l(\theta)k(s,\theta)=T_0^+f(s,\theta)\,.$$

Thus Theorem 2.1 is proved.

Proof of Theorem 2.2. We take a C^∞ function $\chi(x)$ such that $\mathrm{supp}[\chi]\subset\Omega$ and $\chi(x)=1$ for $|x|\geqq r_0$ ($\partial\Omega\subset\{x\colon |x|<r_0\}$). The problem (1.1) has a finite propagation speed (cf. §3 of Shibata and Soga [7]). Let this speed be less than $s_{\max}$. Then it follows that

$$(U(t)f)(x)=(U_0(t-ct)\chi U(ct)f)(x)\quad\text{when}\ |x|\geqq s_{\max}(1-c)t+r_0\,,$$

where we choose the constant c so that $c=1-2^{-1}\min\{1,s_{\max}^{-1}|\eta_j(\theta)|\ (j=1,\cdots,d,\ \theta\in S^{n-1})\}$. Since $|\eta_j(\theta)|\geqq 2\,s_{\max}(1-c)$, there is a constant t_0 such that $|t\eta_j(\theta)+s\zeta_j(\theta)|\geqq s_{\max}(1-c)t+r_0$ for $t\geqq t_0$. Therefore we can write

$$\begin{aligned}&U(t)f(t\eta_j(\theta)+s\zeta_j(\theta))\\ &\quad=U_0(t)\{U_0(-ct)\chi U(ct)f-W_+^{-1}f\}(t\eta_j+s\zeta_j)\\ &\qquad+U_0(t)W_+^{-1}f(t\eta_j+s\zeta_j)\quad\text{when}\ t\geqq t_0\,.\end{aligned}\tag{2.2}$$

Since $f\in D(A^{n+3})$, we have $W_+^{-1}f\in D(A_0^{n+3})$ and

$$\lim_{t\to\infty}\|A_0^i\{U_0(-ct)\chi U(ct)f-W_+^{-1}f\}\|_{E,R^n}=0$$

for any integer i with $0\leqq i\leqq n+3$. This and (i) of Theorem 2.1 yield that

$$
\begin{aligned}
(2.3)\quad & \left|\iint t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j P_j[U_0(t)\{U_0(-ct)\chi U(ct)f-W_+^{-1}f\}]_2(t\eta_j+s\zeta_j)hdsd\theta\right| \\
& \leqq C\sum_{i=0}^{n+3}\|A_0^i\{U_0(-ct)\chi U(ct)f-W_+^{-1}f\}\|_{E,\boldsymbol{R}^n}\to 0 \text{ as } t\to\infty .
\end{aligned}
$$

For any $\delta>0$ there exists $g\in H_0$ such that $T_0^+g\in\mathcal{S}$ and g and $\sum_{i=0}^{n+3}\|A_0^i(g-W_+^{-1}f)\|_{E,\boldsymbol{R}^n}\ (=\sum_{i=0}^{n+3}\|\partial_s^i(T_0^+g-T^+f)\|_{L^2(\boldsymbol{R}\times S^{n-1})})<\delta$. By Theorem 2.1 and the Lebesgue convergence theorem we have

$$
\begin{aligned}
& \left|\iint \{t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j P_j(U_0(t)W_+^{-1}f)_2(t\eta_j+s\zeta_j)-T^+f\}hdsd\theta\right| \\
(2.4)\quad & \leqq C\sum_{i=0}^{n+3}\|A_0^i(W_+^{-1}f-g)\|_{E,\boldsymbol{R}^n} \\
& \quad +\left|\iint\{t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j P_j(U_0(t)g)_2(t\eta_j+s\zeta_j)-T_0^+g\}hdsd\theta\right| \\
& \quad\quad +\left|\iint(T_0^+g-T^+f)hdsd\theta\right| \\
& \leqq C\delta+\delta+\delta\|h\|_{L^2} \quad \text{if } t \text{ is large enough.}
\end{aligned}
$$

Combing (2.2), (2.3) and (2.4), for any $\delta'>0$ we obtain

$$
\begin{aligned}
& \left|\iint\{t^{(n-1)/2}\sum_{j=1}^{d}\kappa_j P_j(U(t)f)_2(t\eta_j+s\zeta_j)-T^+g\}hdsd\theta\right| \\
& \qquad\qquad \leqq\delta' \quad \text{if } t \text{ is large enough.}
\end{aligned}
$$

Thus Thoerem 2.2 is proved.

3. The representation of the scattering kernel

At first, we state a theorem (Theorem 3.1) basic for proof of the representation of the scattering kernel. And next we prove this representation, that is, Theorem 1 in §1 and more precise results (Theorem 3.3). We use the notations in the previous sections also in this section.

Theorem 3.1. *Let $\kappa_j(\theta)$, $\eta_j(\theta)$ and $\zeta_j(\theta)$ be the functions defined in Theorem 2. Assume that $w(t, x)$ is a function $\in\mathcal{B}^\infty(\boldsymbol{R}\times\Omega)$ satisfying*

$$
(3.1)\quad \begin{cases} (\partial_t^2-L)w(t,x)=0 & \text{in } \boldsymbol{R}\times\Omega, \\ w(t,x)=0 & \text{if } t<r_1 \end{cases}
$$

(where $L=\sum_{i,l=1}^{n}a_{il}\partial_{x_i}\partial_{x_l}$). Then we have

(i) *there exists a constant C independent of $(\tilde{t}, s, \theta)\in\boldsymbol{R}_+\times\boldsymbol{R}\times S^{n-1}$ ($\boldsymbol{R}_+=$*

(0, ∞)) *such that*

$$|\tilde{t}^{(n-1)/2}\kappa_j(\theta)(J_+^*w)(\tilde{t}, \tilde{t}\eta_j(\theta)+s\zeta_j(\theta))| \leqq C;$$

(ii) *if* $\lim_{t\to\infty} \partial_t^k w(t, x) = \lim_{t\to\infty} \partial_t^k \partial_x w(t, x) = 0$ *for every* $x \in \partial\Omega$ *and every non-negative integer* $k \leqq 3(n+1)/2$, *then for any* $(s, \theta) \in \boldsymbol{R} \times S^{n-1}$ *we have*

$$\begin{aligned}&\lim_{\tilde{t}\to\infty} \tilde{t}^{(n-1)/2}\kappa_j(\theta)(J_+^*w)(\tilde{t}, \tilde{t}\eta_j(\theta)+s\zeta_j(\theta))\\ &= \lambda_j(\theta)^{-n/4}\int_{\partial\Omega} \{P_j(\theta)\partial_t^{n-2}Nw(\lambda_j(\theta)^{-1/2}\theta\cdot x-s, x)\\ &\qquad -\lambda_j(\theta)^{-1/2}P_j(\theta)({}^tN\theta\cdot x)\partial_t^{n-1}w(\lambda_j(\theta)^{-1/2}\theta\cdot x-s, x)\}\, dS_x\,.\end{aligned}$$

We know that the solutions of the equation $(\partial_t^2-L)u(t, x)=0$ have a finite propagation speed (cf. §3 of Shibata and Soga [7]); we denote the maximum of this speed by $s_{\max}$. Let us note that if the function $u(t, x)$ satisfies $(\partial_t^2-L)u(t, x)=0$ in $\boldsymbol{R}\times\boldsymbol{R}^n$ and $u(0, x)=\partial_t u(0, x)=0$ for $|x| \geqq x_0$ (>0), then for any point $\tilde{x}$ with $|\tilde{x}| \geqq x_0$ we have $u(t, \tilde{x})=0$ when $|t| \leqq (|\tilde{x}|-x_0)s_{\max}^{-1}$.

To prove Theorem 3.1, we verify

Lemma 3.2. *Take* $\tilde{t} \in \boldsymbol{R}$ *and* $\tilde{x} \in \boldsymbol{R}^n$ *with* $|\tilde{x}| > r_0+1$, *and let* $w(t, x)$ *be a function* $\in \mathcal{B}^\infty(\boldsymbol{R}\times\bar{\Omega})$ *satisfying* (3.1). *Then we have*

$$\begin{aligned}w(\tilde{t}, \tilde{x}) = (2\pi)^{1-n}(2i)^{-1}\sum_{l=1}^{d}\iint_{\partial\Omega\times S^{n-1}} &\{\lambda_l(\omega)^{-n/2}P_l(\omega)\\ &N\Lambda_{n-2}(\chi w)(\tilde{t}+\lambda_l(\omega)^{-1/2}\omega\cdot(x-\tilde{x}), x)\\ -\lambda_l(\omega)^{-(n+1)/2}P_l(\omega)({}^tN\omega\cdot x)&\\ &i\Lambda_{n-1}(\chi w)(\tilde{t}+\lambda_l(\omega)^{-1/2}\omega\cdot(x-\tilde{x}, x)\}\, dS_x d\omega\,,\end{aligned} \tag{3.2}$$

where $\Lambda_m(=\Lambda_m(D_t)) = D_t|D_t|^{m-1}$ *and* $\chi=\chi(t)$ *is an arbitrary* C^∞ *function such that* $\chi(t)=1$ *for* $t \leqq \tilde{t}$ *and* $\chi(t)=0$ *for* $t \geqq \tilde{t}+(|\tilde{x}|-r_0-1)\, s_{\max}^{-1}$.

Proof. Take $\rho(x) \in C^\infty(\boldsymbol{R}^n)$ such that $\int \rho(x)dx=1$ and $\mathrm{supp}[\rho] \subset \{x\colon |x|<1\}$, and set $\delta_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ $(0<\varepsilon\leqq 1)$. Let $E_\varepsilon(t, x)$ be the solution of the equation

$$\begin{cases}(\partial_t^2-L)E_\varepsilon(t, x) = 0 & \text{in } \boldsymbol{R}\times\boldsymbol{R}^n,\\ E_\varepsilon(0, x) = 0, \quad \partial_t E_\varepsilon(0, x) = \delta_\varepsilon(x)I & \text{on } \boldsymbol{R}^n\end{cases}$$

(where I is the $n\times n$ identity matrix). By intgeration by parts we have

$$\begin{aligned}&\int_\Omega \delta_\varepsilon(x-\tilde{x})w(\tilde{t}, x)dx\\ &= \int_{\partial\Omega} dS_x\int_{-\infty}^{\tilde{t}} dt\, {}^tE_\varepsilon(\tilde{t}-t, x-\tilde{x})Nw(t, x)\end{aligned} \tag{3.3}$$

$$- \int_{\partial\Omega} dS_x \int_{-\infty}^{\tilde{t}} dt(\tilde{N}^t E_\varepsilon)(\tilde{t}-t, x-\tilde{x})w(t, x)$$
$$- \int_{\Omega} dx \int_{-\infty}^{\tilde{t}} dt[(\partial_t^2-\tilde{L})^t E_\varepsilon](\tilde{t}-t, x-\tilde{x})w(t, x)$$
$$\equiv I_1(\varepsilon)-I_2(\varepsilon)-I_3(\varepsilon),$$

where $\tilde{N}^t E_\varepsilon = \sum_{i,j=1}^{n} \nu_j (\partial_{x_i} {}^t E_\varepsilon)^t a_{ij}$ and $\tilde{L}^t E_\varepsilon = \sum_{i,j=1}^{n} (\partial_{x_i}\partial_{x_j} {}^t E_\varepsilon)^t a_{ij}$. In view of $\tilde{L}^t E_\varepsilon = {}^t(LE_\varepsilon)$, we have $I_3(\varepsilon)=0$. Since ${}^t E_\varepsilon(t, x) = \sum_{l=1}^{d} \mathcal{F}^{-1}[\lambda_l(\xi)^{-1/2} \sin \lambda_l(\xi)^{1/2} t)\hat{\delta}_\varepsilon(\xi) P_l(\xi)](x)$ ($\hat{\delta}_\varepsilon = \mathcal{F}[\delta_\varepsilon]$) and $E_\varepsilon(t, x)=0$ when $|t| \leqq (|x|-1)s_{\max}^{-1}$, it follows that

$$I_1(\varepsilon) = \sum_{l=1}^{d} \int_{\partial\Omega} dS_x \int_{-\infty}^{\infty} dt \int_{S^{n-1}} d\omega \, (2\pi)^{-n} \int_0^{\infty} d\sigma e^{i\sigma\omega\cdot(x-\tilde{x})}$$
$$\sin(\lambda_l(\omega)^{1/2}\sigma(\tilde{t}-t)) \, \hat{\delta}_\varepsilon(\sigma\omega)\sigma^{n-2}\lambda_l(\omega)^{-1/2}P_l(\omega)N\chi w(t, x)$$

(where $\chi(t)$ is the function in (3.2)). We see that $\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} dt \int_0^{\infty} d\sigma e^{i\sigma s} e^{\pm i\sigma t} \hat{\delta}_\varepsilon(\sigma\omega)\sigma^{n-2} v(t) = \int_0^{\infty} e^{i\sigma s} F[v](\mp\sigma)\sigma^{n-2} d\sigma$ for any $v(t) \in C_0^\infty(\boldsymbol{R})$. Therefore, noting that $\sin\theta = (2i)^{-1}(e^{i\theta}-e^{-i\theta})$, we obtain

$$\lim_{\varepsilon\to 0} I_1(\varepsilon) = \sum_{l=1}^{d} \int_{\partial\Omega} dS_x (2\pi)^{-n}(2i)^{-1} \int_{S^{n-1}} d\omega \int_0^{\infty} d\sigma \sum_{\tau=+,-} \tau\lambda_l(\omega)^{-1/2}P_l(\omega)$$
$$(3.4) \qquad \exp\{i\sigma\omega\cdot(x-\tilde{x})+\tau i\sigma\tilde{t}\lambda_l(\omega)^{1/2}\} F[N\chi w](\tau\lambda_l(\omega)^{1/2}\sigma, x)\sigma^{n-2}$$
$$= (2\pi)^{1-n}(2i)^{-1} \sum_{l=1}^{d} \iint_{\partial\Omega\times S^{n-1}} \lambda_l(\omega)^{-n/2}P_l(\omega)$$
$$N\Lambda_{n-2}(\chi w)(\tilde{t}+\lambda_l(\omega)^{-1/2}\omega\cdot(x-\tilde{x}), x)dS_x d\omega .$$

In the same way we get

$$(3.5) \qquad \lim_{\varepsilon\to 0} I_2(\varepsilon) = (2\pi)^{1-n}(2i)^{-1} \sum_{l=1}^{d} \iint_{\partial\Omega\times S^{n-1}} \lambda_l(\omega)^{-(n+1)/2}P_l(\omega)({}^tN\omega\cdot x)$$
$$i\Lambda_{n-1}(\chi w)(\tilde{t}+\lambda_l(\omega)^{-1/2}\omega\cdot(x-\tilde{x}), x)dS_x d\omega .$$

Combining (3.5) with (3.3) and (3.4) yields (3.2). The proof is complete.

Proof of Theorem 3.1. Choose a C^∞ function $\chi_0(t)$ such that $\chi_0(t)=1$ for $t<0$ and $\chi_0(t)=0$ for $t>1$, and set

$$\chi_{\tilde{t}}(t) = \chi_0\left(\frac{t-\tilde{t}}{\gamma\tilde{t}}\right), \quad \gamma = (2\, s_{\max})^{-1} \min_{\substack{\theta\in S^{n-1}\\ 1\leqq l\leqq d}} |\eta_l(\theta)| .$$

Then we can apply Lemma 3.2 with $\tilde{x} = \tilde{t}\eta_j(\theta)+s\zeta_j(\theta)$ and $\chi(t)=\chi_{\tilde{t}}(t)$ if $\tilde{t}$

is large enough. Therefore, when $\tilde{t}$ is large enough, it follows that

$$\tilde{t}^{(n-1)/2}\kappa_j(\theta)(J_+^*w)(\tilde{t}, \tilde{t}\eta_j(\theta)+s\zeta_j(\theta))$$
$$=\sum_{\varepsilon=0,1}(2\pi)^{1-n}(2i)^{-1}\kappa_j(\theta)\sum_{l=1}^{d}\iint_{\partial\Omega\times S^{n-1}}\Big(\lambda_l(\omega)^{-n/2}P_l(\omega)$$
$$\tilde{t}^{(n-1)/2}[J_+^*\Lambda_{n-2}\{(\varepsilon\chi_{\tilde{t}}-2\varepsilon+1)Nw\}]\,(\tilde{t}+\lambda_l(\omega)^{-1/2}\omega\cdot(x-\tilde{x}), x)$$
$$-\lambda_l(\omega)^{-(n+1)/2}P_l(\omega)({}^tN\omega\cdot x)$$
$$i\tilde{t}^{(n-1)/2}[J_+^*\Lambda_{n-1}\{(\varepsilon\chi_{\tilde{t}}-2\varepsilon+1)w\}]\,(\tilde{t}+\lambda_l(\omega)^{-1/2}\omega\cdot(x-\tilde{x}), x)\Big)\,dS_x d\omega\,,$$
$$\equiv\sum_{\varepsilon=0,1}\Phi_\varepsilon(\tilde{t})\,,$$

where $\tilde{x}=\tilde{t}\eta_j(\theta)+s\zeta_j(\theta)$.

The first step is to prove that $\lim_{\tilde{t}\to\infty}\Phi_1(\tilde{t})=0$. The function $\{1-\lambda_l(\omega)^{-1/2}\omega\cdot\eta_j(\theta)\}$ (on S_ω^{n-1}) is maximum at only one point ω_+ and minimum at only another point ω_-; furthermore there are no stationary points in S_ω^{n-1} except $\omega_\pm$ (cf. Lemma 2.3). We take a partition $\{\alpha_i(\omega)\}_{i=0,1,\dots}$ of unity on S^{n-1} such that $\omega_+\in\text{supp}[\alpha_0]\not\ni\omega_-\in\text{supp}[\alpha_1]\not\ni\omega_+$ and $\omega_\pm\notin\text{supp}[\alpha_i]$ $(i\geq 2)$, and miltiply the integrand in $\Phi_1(\tilde{t})$ by $\alpha_i(\omega)$. Introduce local coordinates $\omega=\omega(y)$ $(y\in \boldsymbol{R}^{n-1})$ in a neighborhood of each supp$[\alpha_i]$ and apply Theorem 5.2 and (i) of Theorem 5.4 in §5 with $\varphi(y)=1-\lambda_l(\omega(y))^{-1/2}\omega(y)\cdot\eta_j(\theta)$ when $i\geqq 2$ and $i=0, 1$ respectively. Here, let us note that the Hessian of $\varphi(y)$ is non-degenerate on supp$[\alpha_i(\omega(y))]$ when $i=0$ and 1 (from the assumption (A.4)). Therefore we have

$$|\tilde{t}^{(n-1)/2}\int_{S^{n-1}}p(D_t)\{(\chi_{\tilde{t}}-1)v\}\,(\tilde{t}+\lambda_l(\omega)^{-1/2}\omega\cdot(x-\tilde{x}), x)d\omega|$$
$$\leqq C_1\sum_{k=0}^{m}\sup_{t\in\boldsymbol{R}}|\partial_t^k\{(\chi_{\tilde{t}}-1)v\}\,(t, x)|\,,$$

where $p(D_s)=J_+^*\Lambda_{n-2}$ or $iJ_+^*\Lambda_{n-1}$, $v=Nw$ or w and $m=$ the maximum integer $<3(n+1)/2$. $\sup_{t\in\boldsymbol{R}}|\partial_t^k\{(\chi_{\tilde{t}}-1)v\}(t, x)|$ is bounded in $\tilde{t}>0$ and $x\in\partial\Omega$, and converges to 0 as $\tilde{t}\to\infty$ (for any fixed $x\in\partial\Omega$)) since $\lim_{t\to\infty}\partial_t^k v\,(t, x)=0$ $(0\leqq k\leqq m)$. Therefore, by the Lebesgue convergence theorem, we get $\lim_{t\to\infty}\Phi_1(\tilde{t})=0$.

The second step is to examine the behavior of $\Phi_0(\tilde{t})$ as $\tilde{t}\to\infty$. We employ the same reduction with the partition $\{\alpha_i(\omega)\}_{i=0,1,\dots}$ as in the first step. Then there are no stationary points of $1-\lambda_l(\omega)^{-1/2}\omega\cdot\eta_j$ in supp$[\alpha_i(\omega)]$ if $i\geqq 2$. And therefore, in the same way as in the first step we see from Theorem 5.2 in §5 that the absolute values of the terms with α_i $(i\geqq 2)$ are of the order less than $\tilde{t}^{2-n}$ (as $\tilde{t}\to\infty$) and tend to 0 as $\tilde{t}\to\infty$. Consequently it suffices to examine only the terms containing α_0 or α_1.

We take local coordinates $\omega=\omega(y)$ $(y\in\boldsymbol{R}^{n-1})$ available in a neighborhood

U_1 of supp$[\alpha_1]$ such that $\varphi(y)=1-\lambda_l(\omega(y))^{-1/2}\omega(y)\cdot\eta_j(\theta)$ is minimum at $y=0$, i.e., $\varphi(0)\leqq\varphi(y)$ on $\tilde{U}_1=\{y: \omega(y)\in U_1\}$. Then, from the assumption (A.4) we can assume that the Hessian of $\varphi(y)$ is not degenerate on $\tilde{U}_1$. Furthermore, since $\lim_{t\to\infty}\partial_t^k w(t, x)=\lim_{t\to\infty}\partial_x\partial_t^k w(t, x)=0$ for any non-negative integer $k\leqq 3(n+1)/2$, the function

$$k(t, y; x)=\alpha_1(\omega)\lambda_l(\omega)^{-n/2}P_l(\omega)v(t+\lambda_l(\omega)^{-1/2}\omega\cdot(x-s\zeta_j), x)\,|_{\omega=\omega(y)}$$
$$(v=i\partial_t^{n-1}w \quad\text{or}\quad i\partial_t^{n-2}Nw)$$

satisfies $\int_0^t k(r, y; x)dr\in\mathcal{B}^\infty(\boldsymbol{R}\times\tilde{U}_1\times\partial\Omega)$ and $\lim_{t\to\infty}\sup_{\substack{|\alpha|\leqq(n+5)/2\\ y\in\tilde{U}_1}}|\partial_y^\alpha k(\pm t, y; x)|=0$ for any $x\in\partial\Omega$. In view of the equalities that $J_+^*\Lambda_{n-2}=iJ_+\partial_t^{n-2}$ and $iJ_+^*\Lambda_{n-1}=iJ_+\partial_t^{n-1}$, we see that the term (with α_1) to be examined is sum of the forms $\tilde{t}^{(n-1)/2}\iint_{\partial\Omega\times\tilde{U}_1}J_+k(\tilde{t}\varphi(y), y; x)dydS_x$. Noting that $\varphi(0)\neq 0$ if $l\neq j$ and $\varphi(0)=0$ if $l=j$, we apply Theorem 5.4 or Theorem 5.6 to the integral $\int J_+k(\tilde{t}\varphi(y), y; x)dy$ (see (ii) of Remark below Theorem 5.4 also). Then it is seen in the same way as in the proof of Lemma 2.4 that as $\tilde{t}\to\infty$ each part with α_1 tends to 0 if $l\neq j$ and to $\lambda_j(\theta)^{-n/4}\int_{\partial\Omega}\{P_j(\theta)\partial_t^{n-n}Nw(\lambda_j(\theta)^{-1/2}\theta\cdot x-s, x)-\lambda_j(\theta)^{-1/2}P_j(\theta)({}^tN\theta\cdot x)\partial_t^{n-1}w(\lambda_j(\theta)^{-1/2}\theta\cdot x-s, x)\}dS_x$ if $l=j$. Similarly we can know that the other term containing $\alpha_0(\omega)$ converges to 0 as $\tilde{t}\to\infty$ since the maximum of $1-\lambda_l(\omega)^{-1/2}\omega\cdot\eta_j(\theta)$ on supp$[\alpha_0(\omega)]$ is not equal to 0. Thus it is proved that as $\tilde{t}\to\infty$ $\Phi_1(\tilde{t})$ converges to the limit of the term containing α_1 and satisfying $l=j$. The proof is complete.

Regard the Dirac function $\delta(t-\lambda_j(\omega)^{-1/2}\omega\cdot x)$ as a distribution-valued C^∞ funation on $\boldsymbol{R}_x^n\times S_\omega^{n-1}$, and consider the equation

$$(3.6)\quad\begin{cases}(\partial_t^2-L)v(t, x)=0 & \text{in } \boldsymbol{R}\times\Omega,\\ Bv=-2^{-1}(-2\pi i)^{1-n}\lambda_j(\omega)^{-n/4}B\{\delta(t-\lambda_j(\omega)^{-1/2}\omega\cdot x)P_j(\omega)\} & \text{on } \boldsymbol{R}\times\partial\Omega,\\ v=0 & \text{for } t\ll 0.\end{cases}$$

Then there exists a unique solution $v_j(t, x; \omega)$ which is an $n\times n$ matrix of distribution-valued C^∞ functions on $\boldsymbol{R}_x^n\times S_\omega^{n-1}$ (cf. Soga [14]).

The scattering operator S is represented by means of v_j $(j=1, \cdots, d)$ in the following way:

Theorem 3.3. *Let the assumptions* (A.1)~(A.4) *be satisfied, and set*

$$S_0(s, \theta, \omega)=\sum_{i,j=1}^d\lambda_i(\theta)^{-n/4}\int_{\partial\Omega}\{P_i(\theta)\,(\partial_t^{n-2}Nv_j)\,(\lambda_i(\theta)^{-1/2}\theta\cdot x-s, x; \omega)$$
$$-\lambda_i(\theta)^{-1/2}P_i(\theta)({}^tN\theta\cdot x)\partial_t^{n-1}v_j)(\lambda_i(\theta)^{-1/2}\theta\cdot x-s, x; \omega)\}dS_x,$$

where $N=\sum_{k,l=1}^{n} \nu_k(x)a_{kl}\partial_{x_l}$. *Then we have*

$$(Sk)(s, \theta) = \iint_{R\times S^{n-1}} S_0(s-t, \theta, \omega)\, k(t, \omega)dtd\omega + Kk(s, \theta)\,,$$
$$k(s, \omega)\in C_0^\infty(\boldsymbol{R}\times S^{n-1})\,,$$

where $Kk(s)=F^{-1}[(\text{sgn}\,\sigma)^{n-1}[Fk]\,(\sigma)](s)$.

In the above theorem, the integral means the Riemann sum with the value of the distributions on $\boldsymbol{R}$. Theorem 2 in §1 follows from the above theorem since the kernel of K is equal to 0 when $\theta\neq\omega$. This theorem will be derived later from Theorem 2.2, Theorem 3.1 and the following lemma.

Lemma 3.4. *Let the data f in* (1.1) *satisfy* $T_0^- W_-^{-1}f(s, \omega)\in C_0^\infty(\boldsymbol{R}\times S^{n-1})$, *and set* $k=T_0^-\, W_-^{-1}f$. *Then we have*

$$(3.7)\qquad \begin{aligned}(U(t)f)_2(x) &= (U_0(t)\, W_-^{-1}f)_2\,(x)\\ &\quad +(-i)^{n-1}\sum_{l=1}^{d}\iint_{R\times S^{n-1}} v_l(t+s, x; \omega)\, J_-^* k(s, \omega)\, ds\, d\omega\,.\end{aligned}$$

Proof. Denote the left side and the right one of the equality (3.7) by $u_t(t, x)$ and $\tilde{u}_t(t, x)$ respectively. Then it is obvious that $(\partial_t^2-L)u_t=(\partial_t^2-L)\tilde{u}_s=0$ in $\boldsymbol{R}\times\Omega$. From the assumption it follows that $k(s, \omega)=0$ when $s>s_0$ (for some constant s_0), which implies that $U(t)f=U_0(t)\,W_-^{-1}f$ when $t<t_0$ (t_0 being a constant determined by s_0) (cf. §3 of Shibata and Soga [7]). Furthermore, $\sum_{l=1}^{d}\iint_{R\times S^{n-1}} v_l(t+s, x; \omega)J_-^* k(s, \omega)dsd\omega=0$ if t is small enough, because $J_-^* k(s, \omega)=0$ for $s>s_0$ (cf. Lemma 2.1 of Soga [8]). Therefore we have $u_t(t, x)=\tilde{u}_t(t, x)$ if t is small enough.

We know that $(U_0(t)W_-^{-1}f)_2(x) = 2^{-1}(2\pi)^{1-n}\sum_{l=1}^{d}\int_{S^{n-1}} \lambda_l(\omega)^{-n/4}\, P_l(\omega)\,(J_-^* k)(\lambda_l(\omega)^{-1/2}\,\omega\cdot x-t, \omega)d\omega$ (cf. Theorem 2.1 (and 2.14)) in Shibata and Soga [7]; this is valid if $W_-^{-1}f\in D(A_0^\infty)$). Therefore we have $B(U_0(t)\, W_-^{-1}f)_2=2^{-1}(2\pi)^{1-n}\sum_{l=1}^{d}\int_{S^{n-1}}\lambda_l(\omega)^{-n/4}B\,\{P_l(\omega)(J_-^* k)(\lambda_l(\omega)^{-1/2}\omega\cdot x-t)\}\,d\omega$. This is equal to $-(-i)^{n-1} B\sum_{l=1}^{d}\iint_{R\times S^{n-1}} v_l(t+s, x; \omega)J_-^* k(s, \omega)dsd\omega$, which is seen from the form of the boundary value Bv_l (cf. (3.6)). Hence we have $Bu_t=B\tilde{u}_t=0$ on $\boldsymbol{R}\times\partial\Omega$. Thus, from the uniqueness of the solutions, it follows that $u_t(t, x)=\tilde{u}_t(t, x)$ on $\boldsymbol{R}\times\Omega$. The proof is complete.

Proof of Theorem 3.3. Let $k(s, \omega)\in C_0^\infty(\boldsymbol{R}\times S^{n-1})$, and set $f= W_-(T_0^-)^{-1}k$. Then $f\in D(A^\infty)$ and $U(t)f(x)\in C^\infty$. Theorem 2.2 means that

$$Sk(s, \theta) = \lim_{t\to\infty} t^{(n-1)/2} \sum_{j=1}^{d} \kappa_j(\theta)\, P_j(\theta)\, (U(t)f)_2(t\eta_j(\theta)+s\zeta_j(\theta)),$$

where the above convergence is in the sense of the distributions on $\boldsymbol{R}\times S^{n-1}$. In view of Lemma 3.4, we have to examine the limits of

$$t^{(n-1)/2} \sum_{j=1}^{d} \kappa_j(\theta)\, P_j(\theta)\, (U_0(t)\, W_-^{-1}f)_2(t\eta_j(\theta)+s\zeta_j(\theta)), \tag{3.8}$$

$$\sum_{j=1}^{d} \kappa_j(\theta)\, P_j(\theta)\, t^{(n-1)/2}(-i)^{n-1} \sum_{l=1}^{d} \iint_{\boldsymbol{R}\times S^{n-1}} v_l(t+s, t\eta_j(\theta)+s\zeta_j(\theta); \omega)\, J_-^* k(s, \omega)ds d\omega \tag{3.9}$$

as $t\to\infty$. Theorem 2.1 implies that (3.8) converges to $Kk(s, \theta)$ since $T_0^+=KT_0^-$. Therefore we have only to show that (3.9) converges to $\iint_{\boldsymbol{R}\times S^{n-1}} S_0(s-\tilde{s}, \theta, \omega)\, k(\tilde{s}, \omega)\, d\tilde{s}\, d\omega$ as $t\to\infty$.

Set

$$w(t, x) = \sum_{l=1}^{d} \iint_{\boldsymbol{R}\times S^{n-1}} v_l(t+s, x; \omega)\, k(s, \omega)\, ds\, d\omega\,. \tag{3.10}$$

Then w is a C^∞ function on $\boldsymbol{R}\times \bar{\Omega}$ and satisfies

$$\begin{cases} (\partial_t^2-L)\, w(t, x) = 0 & \text{in} \quad \boldsymbol{R}\times\Omega\,, \\ Bw = g(t, x') & \text{on} \quad \boldsymbol{R}\times \partial\Omega\,, \\ w = 0 & \text{if} \quad t\ll 0\,, \end{cases}$$

where g is of the form

$$g(t, x') = -2^{-1}(-2\pi i)^{1-n} \sum_{l=1}^{d} \int_{S^{n-1}} \lambda_l(\omega)^{-n/4} B\{P_l(\omega)\, k(\lambda_l(\omega)^{-1/2}\, \omega\cdot x'-t, \omega)\}\, d\omega\,.$$

Since $g(t, x')\in C_0^\infty(\boldsymbol{R}\times \partial\Omega)$, ${}^t(w(t, \cdot), \partial_t w(t, \cdot))$ belongs to $D(A^\infty)$ for any large t. Therefore, from the decay property of the mixed problem (1.1) (cf. §3 of Shibata and Soga [7]), it follows that $\lim_{t\to\infty} \partial_t^i\, w(t, x') = \lim_{t\to\infty} \partial_t^i\, \partial_x\, w(t, x') = 0$ for any noo-negative integer $i\leqq 3(n+1)/2$ and for any $x'\in\partial\Omega$. Hence we can apply Theorem 3.1 to the $w(t, x)$ defined by (3.10). Furthermore we see that $J_+^*\, w(t, x) = (-i)^{n-1} \sum_{l=1}^{d} \iint_{\boldsymbol{R}\times S^{n-1}} v_l(t+s, x; \omega) J_-^* k(s, \omega)ds d\omega$. Thus it is seen that (3.9) converges to $\iint S_0(s-\tilde{s}, \theta, \omega)\, k(\tilde{s}, \omega)\, d\tilde{s}\, d\omega$ in the sense of the distributions on $\boldsymbol{R}_s\times S_\theta^{n-1}$ as

$t\to\infty$. The proof is complete.

4. Singularities of the scattering kernel

In this section, using Theorem 1 in §1, we examine the support and the singular support of the back-scattering $S(s, -\omega, \omega)$, and prove Theorem 3 in §1. The notations in the previous sections are used also in this section. We denote by $H^{-\infty}(M)$ the dual space of the Sobolev space of order ∞ on M.

At first we solve (3.6) approximately by means of the asymptotic solutions obtained in Soga [13]. Namely we construct the Poisson operator for the non-glancing boundary data:

Theorem 4.1. *We have an operator* P: $H^{-\infty}(\boldsymbol{R}\times\partial\Omega)\to H^{-\infty}(\boldsymbol{R}\times\Omega)$ *with the following properties: There exist a neighborhood* U *of* $\partial\Omega$ *(in* $\boldsymbol{R}^n$*) and an open conic set* Γ *in* $T^*(\boldsymbol{R}\times\partial\Omega)=\boldsymbol{R}_t\times\boldsymbol{R}_\sigma\times T^*(\partial\Omega)$ *containing* $\boldsymbol{R}_t\times(\boldsymbol{R}_\sigma-\{0\})\times\partial\Omega\times\{0\}$ *such that*

(i) $(\partial_t^2-L)Ph\in C^\infty(\boldsymbol{R}\times(\bar{\Omega}\cap U))$, $h(t, x')\in H^{-\infty}(\boldsymbol{R}\times\partial\Omega)$;

(ii) $Ph(t, x)\in C^\infty((-\infty, r_0)\times\bar{\Omega})$ *if* $h(t, x')$ *is* C^∞ *smooth when* $t<r_0$;

(iii) $BPh(t, x')$ *is well defined for any* $h(t, x')\in H^{-\infty}(\boldsymbol{R}\times\partial\Omega)$, *and furthermore*

$$BPh(t, x')-h(t, x')\in C^\infty(\boldsymbol{R}\times\partial\Omega)$$

if the wave front set of h *is contained in* Γ;

(iv) *the mapping* T: $h\to NPh|_{\boldsymbol{R}\times\partial\Omega}$ *(resp.* $h\to Ph|_{\boldsymbol{R}\times\partial\Omega}$*) is a pseudo-differential operator of order* 1 *(resp.* -1*) on* $\boldsymbol{R}\times\partial\Omega$ *independent of* t *in the case of* $Bu=u|_{\boldsymbol{R}\times\partial\Omega}$ *(resp.* $Bu=Nu|_{\boldsymbol{R}\times\partial\Omega}$*); furthermore, each local symbol* $T(y;\sigma,\eta)$ *(*(y,η) *being local coordinates of* $T^*(\partial\Omega)$*) has homogeneous asymptotic expansion* $\sum_{j=0}^{\infty} T_j(y;\sigma,\eta)$ *such that every* $i^{j+1}T_j(y;\sigma,\eta)$ *is real-valued and homogeneous of order* $1-j$ *(resp.* $-1-j$*) in* (σ,η) *and that*

$$T_0(y;\sigma,0)=i\sigma\sum_{l=1}^{d}\lambda_l(\nu(x'(y)))^{1/2}P_l(\nu(x'(y)))$$

$$(resp. =(i\sigma)^{-1}\sum_{l=1}^{d}\lambda_l(\nu(x'(y)))^{-1/2}P_l(\nu(x'(y))))$$

when $Bu=u|_{\boldsymbol{R}\times\partial\Omega}$ *(resp.* $Bu=Nu|_{\boldsymbol{R}\times\partial\Omega}$*).*

Proof. We take a fine partition $\{\chi_k\}_{k=1,2,\cdots}$ of unity on $\partial\Omega$, and carry out the following analysis in each neighborhood W_k ($\subset\boldsymbol{R}^n$) of supp $[\chi_k]$. We introduce local coordinates $x'=x'(y)$ on $\partial\Omega\cap W_k$ ($y\in\boldsymbol{R}^{n-1}$).

If $|\eta|$ ($\eta\in\boldsymbol{R}^{n-1}$) is small enough, we have the solution $\varphi^l(x;\eta)$ ($l=1,\cdots,d$) of the equation

$$\begin{cases}\lambda_l(\partial_x\varphi^l)=1 \quad \text{in} \quad \Omega\cap W_k, \\ \varphi^l(x';\eta)=\eta\cdot y(x'),\ \dfrac{\partial\varphi^l}{\partial\nu}(x';\eta)<0 \quad \text{on} \quad \partial\Omega\cap W_k\end{cases}$$

by the Hamilton-Jacobi method. Here, we choose W_k so that $\varphi^l(x;\eta)$ is defined on the whole $\Omega\cap W_k$. Using the asymptotic solutions in Soga [13] (cf. Theorem 1.1 of [13]), in $\Omega\cap W_k$ and for $\sigma\in\boldsymbol{R}$ and small $|\eta|$ we can construct C^∞ functions $p^l(x;\sigma,\eta)$ $(l=1,\cdots,d)$ with asymptotic expansions $\sum_{j=0}^{\infty}p_j^l(x;\eta)(i\sigma)^{-j+\varepsilon}$ (as $|\sigma|\to\infty$) such that $i^{j+1}p_j^l(x;\eta)$ are real-valued and that

$$\begin{cases}(\partial_t^2-L)(e^{i\sigma t+i\sigma\varphi^l(x;\eta)}p^l(x;\sigma,\eta))=0(|\sigma|^{-\infty}) \quad\text{in}\quad \Omega\cap W_k,\\ B\{\sum_{l=1}^{d}e^{i\sigma t+i\sigma\varphi^l(x;\eta)}p^l(x;\sigma,\eta)\}|_{x=x'(y)}-e^{i\sigma t+i\sigma\eta\cdot y}I\\ \quad=0(|\sigma|^{-\infty}) \quad\text{on}\quad \partial\Omega\cap W_k,\end{cases}$$

where $\varepsilon=0$ (resp. -1) when $Bu=u|_{\boldsymbol{R}\times\partial\Omega}$ (resp. $Bu=Nu|_{\boldsymbol{R}\times\partial\Omega}$) and I is the $n\times n$ identity matrix.

Let $\chi_k^0(x;\eta)$ be a C^∞ function such that $\chi_k^0(x;\eta)=1$ in a neighborhood of supp $[\chi_k]\times\{\eta=0\}$ and supp $[\chi_k^0]$ is in a place where φ^l and p^l are defined. We set

$$\begin{aligned}P_k h(t,x)=(2\pi)^{-n}\sum_{l=1}^{d}\iint e^{i\sigma t+i\sigma\varphi_l(x;\eta)}p^l(x;\sigma,\eta)\\ \chi_k^0(x,\eta)\mathcal{F}[\chi_k h](\sigma,\sigma\eta)\sigma^{n-1}d\sigma d\eta,\quad h\in C_0^\infty(\boldsymbol{R}\times\partial\Omega),\end{aligned}$$

where $\mathcal{F}[\chi_k h](\sigma,\eta)=\iint e^{-i\sigma t-i\eta y}\chi_k(x'(y))h(t,x'(y))\,dt\,dy$. Then it is seen that the operator $P=\sum_k P_k$ has the all properties stated in Theorem 4.1. Hnece the theorem is obtained.

Proof of Theorem 3. (i) Let $v_j(t,x;\omega)$ $(j=1,\cdots,d)$ be the solutions of (3.6) in §3. Then $u=v_j P_l(\omega)$ satisfies the equations $(\partial_t^2-L)u=0$ in $\boldsymbol{R}\times\Omega$ and $Bu=\delta_{jl}Bv_j$ on $\boldsymbol{R}\times\partial\Omega$, which implies that $v_j(t,x;\omega)=\delta_{jl}v_j(t,x;\omega)P_l(\omega)$ in $\boldsymbol{R}\times\Omega$ (from the uniqueness of the solutions). Combining these equalities with Theorem 1, we have

$$\begin{aligned}&P_i(-\omega)S(s,-\omega,\omega)P_j(\omega)\\ &=\lambda_i(\omega)^{-n/4}P_i(\omega)\int_{\partial\Omega}\partial_t^{n-2}Nv_j(-\lambda_i(\omega)^{-1/2}\omega\cdot x-s,x;\omega)\,dS_x\\ &+\lambda_i(\omega)^{-(n+2)/4}P_i(\omega)\int_{\partial\Omega}N(\omega\cdot x)\\ &\quad(\partial_t^{n-1}v_j)(-\lambda_i(\omega)^{-1/2}\omega\cdot x-s,x;\omega)\,dS_x.\end{aligned}\tag{4.1}$$

Since $v_j(t,x;\omega)=0$ if $t<\lambda_j(\omega)^{-1/2}r(\omega)$, it follows that $v_j(-\lambda_i(\omega)^{-1/2}\omega\cdot x-s,x;\omega)=0$ if $s>-(\lambda_i(\omega)^{-1/2}+\lambda_j(\omega)^{-1/2})r(\omega)$. Therefore the right side of (4.1) is equal to 0 if $s>-(\lambda_i(\omega)^{-1/2}+\lambda_j(\omega)^{-1/2})r(\omega)$. which proves (i) of Theorem 3.

(ii) Choose $\psi(t)\in C_0^\infty(\boldsymbol{R})$ satisfying $\psi(t)=1$ in a neighborhood of $t=\lambda_j(\omega)^{-1/2}r(\omega)$, and multiply the boundary data in (3.6) (with $l=j$) by $\psi(t)$.

Let $h(t, x')$ be this multiplied data. Then the wave front set (cf. §3 in Ch. 10 of Kumano-go [2], etc.) of h is contained in the set Γ stated in Theorem 4.1 if supp $[\psi]$ is small enough. From Theorem 4.1 it follows that $B(Ph(t, x)-v_j(t, x; \omega))$ are C^∞ smooth when $t<\lambda_j(\omega)^{-1/2} r(\omega)+\delta$ (δ is some positive constant) and that $(\partial_t^2-L)(Ph(t, x)-v_j(t, x; \omega))\in C^\infty(\boldsymbol{R}\times(\bar{\Omega}\cap U))$. This implies that $v_j(t, x; \omega)$ can be approximated by $Ph(t, x)$ mod C^∞ in $(-\infty, \lambda_j(\omega)^{-1/2} r(\omega)+\delta)\times(\bar{\Omega}\cap U)$, since the mixed problem (1.1) is C^∞ solvable and has a finite propagation speed (cf. Shibata and Soga [7]). Therefore, when $s<2\lambda_j(\omega)^{-1/2} r(\omega)+2\delta$, we can change $Nv_j(-\lambda_j(\omega)^{-1/2}\omega\cdot x'-s, x'; \omega)$ (resp. $v_j(-\lambda_j(\omega)^{-1/2}\omega\cdot x'-s, x'; \omega)$) for $Th(-\lambda_j(\omega)^{-1/2}\omega\cdot x'-s, x')$ mod C^∞ in the case of $Bu=u|_{\partial\Omega}$(resp. $Bu=Nu|_{\partial\Omega}$). Combining this with (4.1) yields that

$$
\begin{aligned}
&P_j(-\omega)\,\alpha(s)\,S(s, -\omega, \omega)\,P_j(\omega)\\
&\quad=\lambda_j(\omega)^{-n/4} P_j(\omega)\int_{\partial\Omega}\alpha(s)\,\partial_t^{n-2}\,Th(-\lambda_j(\omega)^{-1/2}\omega\cdot x'-s, x')\,\chi(x')\,dS_{x'}\\
&\qquad+\lambda_j(\omega)^{(n+2)/4} P_j(\omega)\int_{\partial\Omega} N(\omega\cdot x)\,\alpha(s)\\
&\qquad\qquad\partial_t^{n-1} h(-\lambda_j(\omega)^{-1/2}\omega\cdot x'-s, x')\,\chi(x')\,dS_{x'}\quad \text{mod } C^\infty\\
&\quad[\text{resp.} =\lambda_j(\omega)^{-n/4} P_j(\omega)\int_{\partial\Omega}\alpha(s)\,\partial_t^{n-2} h(-\lambda_j(\omega)^{-1/2}\omega\cdot x'-s, x')\,\chi(x')\,dS_{x'}\\
&\qquad+\lambda_j(\omega)^{(n+2)/4} P_j(\omega)\int_{\partial\Omega} N(\omega\cdot x)\,\alpha(s)\\
&\qquad\qquad(\partial_t^{n-1}\,Th)(-\lambda_j(\omega)^{-1/2}\omega\cdot x'-s, x')\,\chi(x')\,dS_{x'}\quad \text{mod } C^\infty]
\end{aligned}
\tag{4.2}
$$

when $Bu=u|_{\partial\Omega}$ (resp. $Bu=Nu|_{\partial\Omega}$). Here, $\alpha(s)$ is an arbitrary cutoff C^∞ function with sufficently small support such that $\alpha(-2\lambda_j(\omega)^{-1/2} r(\omega))\neq 0$, and $\chi(x')$ is a C^∞ function on $\partial\Omega$ such that $\chi(x')=1$ on the set $W=\{x'\in\partial\Omega;\ (s, x')\in \text{supp}\,[\alpha(s)\,h(-\lambda_j(\omega)^{-1/2}\omega\cdot x'-s, x')]$ for some $s\}$ and that supp $[\chi]$ is sufficiently small.

Let (y, z) $(\in\boldsymbol{R}^{n-1}\times\boldsymbol{R})$ be an orthonormal system of coordinates such that the plane $\{x\colon \omega\cdot x=s\}$ is expressed by $z=s$. Since $\partial\Omega\cap\text{supp}\,[\chi]$ is close to the plane $\{x\colon \omega\cdot x=r(\omega)\}$, $\partial\Omega$ is represented near supp $[\chi]$ by $z=2^{-1}\lambda_j(\omega)^{1/2}\varphi(y)$ for some C^∞ function $\varphi(y)$. Introducing the local coordinates (z, y) and combining (4.2) with (iv) of Theorem 4.1, we see in the same way as in §4 of Soga [8] that there exists a matrix $\beta_k(y)$ of real-valued C^∞ functions ($k=0, 1, \cdots$) with supp $[\beta_k(y)]\subset$ supp $[\chi(x'(y))]$ such that for any positive integer m

$$
\begin{aligned}
&\bar{F}[P_j(-\omega)\,\alpha(s)\,S(s, -\omega, \omega)\,P_j(\omega)]\\
&\quad=-2(-2\pi i)^{1-n}\lambda_j(\omega)^{-n/2}\int_{\boldsymbol{R}^{n-1}} e^{-i\sigma\varphi(y)}\sum_{k=0}^{m-1}\beta_k(y)\,(i\sigma)^{n-1-k}\,dy\\
&\qquad+0(|\sigma|^{-m+n-1})\quad\text{as}\quad |\sigma|\to\infty\ (\bar{F}=2\pi F^{-1}).
\end{aligned}
\tag{4.3}
$$

Furthermore β_0 satisfies

$$\beta_0(y) = \alpha(-2\lambda_j(\omega)^{-1/2} r(\omega))\, P_j(\omega)\, \{-iT_0(y; 1, 0) + \lambda_j(\omega)^{-1/2} \sum_{k,l=1}^{n} a_{kl}\, \omega_k\, \omega_l\}\, P_j(\omega)$$

$$(\text{resp. } \alpha(-2\lambda_j(\omega)^{-1/2} r(\omega))\, \lambda_j(\omega)^{1/2} P_j(\omega) \{-I - i\lambda_j(\omega)^{-1/2} \sum a_{kl}\, \omega_k\, \omega_l\, T_0(y; 1, 0)\}\, P_j(\omega))$$

$$\text{for } y \text{ with } \quad \omega \cdot x'(y) = r(\omega)$$

when $Bu = u|_{\partial\Omega}$ (resp. when $Bu = Nu|_{\partial\Omega}$). The function $\varphi(y)$ is minimum when $\omega \cdot x'(y) = r(\omega)$, and then, by (iv) of Theorem 4.1, we have $\beta_0(y) = \varepsilon 2\alpha(-2\lambda_j(\omega)^{-1/2} r(\omega))\lambda_j(\omega)^{1/2} P_j(\omega) \neq 0$ ($\varepsilon = +$ (resp. $-$) when $Bu = u|_{\partial\Omega}$ (resp. $Bu = Nu|_{\partial\Omega}$)). Therefore we can apply Theorem 2 in Soga [10] to the oscillatory integral in (4.3). Hence it follows that $(1+|\sigma|^{\kappa})\, F[P_j(-\omega)\, \alpha(s)\, S(s, -\omega, \omega)\, P_j(\omega)]\, (\sigma) \notin L^2(\boldsymbol{R})$ for some $\kappa \in \boldsymbol{R}$, which proves (ii) of Theorem 3.

REMARK. The above methods (for the proof of (ii) of Theorem 3) do not work well when examining the singularities of $P_i(-\omega)\, S(s, -\omega, \omega)\, P_j(\omega)$ in the case of $i \neq i$. For in this case the $\beta_0(y)$ is of the form $\varepsilon 2\alpha(-2\lambda_j(\omega)^{-1/2} r(\omega))\, \lambda_j(\omega)^{1/2} P_i(\omega)\, P_j(\omega)$ for y with $\omega \cdot x'(y) = r(\omega)$, and so $\beta_0(y) = 0$ for these y. Therefore, in order to examine these singularities, we need a new analysis. Kawashita and Soga [1] deal with the case of $i \neq j$ under some assumptions.

5. Appendix

For a function $p(\sigma)$ on $\boldsymbol{R}$ we denote by $p(D_s)$ (or p) the pseudo-differential operator: $h(s) \to F^{-1}[p(\sigma)\, F[h]\, (\sigma)]\, (s)$. It is known that $p(D_s)$ becomes a continuous operator from $\mathcal{B}^\infty(\boldsymbol{R})$ to itself if $p(\sigma)$ is a C^∞ function satisfying $\sup_{\sigma \in \boldsymbol{R}} (1+|\sigma|)^{-\kappa+1} |\partial_\sigma^i p(\sigma)| < \infty$ for all non-negative integers i (cf. §1 in Ch. 2 of Kumano-go [2]). We denote by $|\cdot|_m$ or $|\cdot|_{m,M}$ the norm of $\mathcal{B}^m(M)$, and by $\|\cdot\|_m$ or $\|\cdot\|_{m,M}$ the norm of the Sobolev space on M of order m.

Lemma 5.1. *Let $p(\sigma)$ be a function on $\boldsymbol{R}$ homogeneous of order $\kappa > 0$. Then $p(D_s)$ becomes a bounded operator from $\mathcal{B}^m(\boldsymbol{R})$ to $\mathcal{B}^{\tilde{m}}(\boldsymbol{R})$ (m and $\tilde{m}$ being any integers with $0 \leqq \tilde{m} \leqq m - \kappa - 2$). Furthermore, for any δ with $0 \leqq \delta < \kappa$ we have the estimate*

$$|(1+s^2)^{\delta/2} ph(s)|_{\tilde{m}} \leqq C\, |(1+s^2)^{\delta/2} h(s)|_m,$$

where $(1+s^2)^{\delta/2} h(s) \in \mathcal{B}^\infty$ and C is a constant independent of h.

Let us note that Lemma 5.1 implies that if $p(\sigma)$ is homogeneous of order $\kappa > 0$, $p(D_s)$ becomes a continuous operator from $\mathcal{B}^\infty(\boldsymbol{R})$ to itself.

Let U be an open ball $\{y: |y| < \varepsilon\}$ in $\boldsymbol{R}^l$ ($l \geqq 2$). Assume that $\varphi(y)$ is a real-valued function $\in \mathcal{B}^\infty(U)$, and denote by $H_\varphi(y)$ the Hesse matrix $\{\partial_{y_i}\partial_{y_j}\varphi(y)\}_{i,j=1,\cdots,l}$. From now on, we examine the behavior of the integral

$\int_U p(D_s)\, k(t\varphi(y), y)\, dy$ $(k(s, y) \in \mathscr{B}^\infty(\boldsymbol{R}\times U))$ as the parameter t tends to ∞ or $-\infty$.

Theorem 5.2. *Let $p(\sigma)$ be homogeneous of order $\kappa > 1$. Assume that $\partial_y \varphi(y) \neq 0$ on $\bar{U}$, and that $k(s, y)$ is any function $\in \mathscr{B}^\infty(\boldsymbol{R}\times U)$ with* supp $[k] \subset$ *$\boldsymbol{R}\times U$. Then there is a constant C independent of k and $t \in \boldsymbol{R}$ such that*

$$|t^\kappa \int_U (p(D_s)\, k)\, (t\varphi(y), y)\, dy| \leqq C \sup_{\substack{|\alpha|<\kappa+3 \\ (s,y)\in \boldsymbol{R}\times U}} |\partial_y^\alpha k(s, y)|\,.$$

If $p(D_s)$ is a differential operator, the above estimate is valid for $\kappa = 1$.

This theorem is reduced to Lemma 5.1 and

Lemma 5.3. *Let $p(\sigma)$ be homogeneous of order $\kappa > 1$, and let $h(s, \tilde{s})$ be any function with $(1+\tilde{s}^2)\, h(s, \tilde{s}) \in \mathscr{B}^\infty(\boldsymbol{R}\times\boldsymbol{R})$. Then $(p(D_s)\, h)\, (s', s')$ and $(p(-D_{\tilde{s}})\, h)(s', s')$ are integrable on $\boldsymbol{R}_{s'}$, and satisfy*

$$\int_{\boldsymbol{R}} (p(D_s)\, h)\, (s', s')\, ds' = \int_{\boldsymbol{R}} (p(-D_{\tilde{s}})\, h)\, (s', s')\, ds'\,.$$

Proof of Theorem 5.2. Introducing a partition of unity on U, we can assume that the support of $k(s, y)$ in y is contained in a small open ball $\tilde{U}$ and that $\partial_{y_i} \varphi(y) \neq 0$ on the closure of $\tilde{U}$ for some i. Let $i=1$ and $y=(y_1, y')$. Transforming the variable y_1 into $s=\varphi(y)$, we have

$$\int_U (p(D_s)\, k)\, (t\varphi(y), y)\, dy$$
$$= \int \{\int_{\boldsymbol{R}} (p(D_s)\, k)\, (ts, y_1(s, y'), y') \left|\frac{\partial y_1}{\partial s}\right| ds\}\, dy'\,.$$

It is seen that $(p(D_s)\, h)(ts) = t^{-\kappa}[p(D_r)\, (h(tr))]\, (s)$. Therefore, by Lemma 5.3 we obtain

$$\int_U (p(D_s)\, k)\, (t\varphi(y), y)\, dy$$
$$= t^{-\kappa} \int \{\int_{\boldsymbol{R}} (p(-D_{\tilde{s}})\, \tilde{k})\, (ts', s', y')\, ds'\}\, dy'$$

where $\tilde{k}(s, \tilde{s}, y') = k(s, y_1(\tilde{s}, y'), y') \left|\frac{\partial y_1}{\partial s}(\tilde{s}, y')\right|$. Lemma 5.1 yields that $|(p(-D_{\tilde{s}})\tilde{k})(ts', s', y')| \leqq C_1(1+|s'|)^{-1-\delta} \sup_{\substack{0\leq i<\kappa+3 \\ s,\tilde{s}\in \boldsymbol{R}}} (1+|\tilde{s}|)^{-1-\delta} |\partial_{\tilde{s}}^i \tilde{k}(s, \tilde{s}, y')| \leqq C_2$ $(1+|s'|)^{-1-\delta} \sup_{\substack{|\alpha|<\kappa+3 \\ (s,y)\in \boldsymbol{R}\times U}} |\partial_y^\alpha k(s, y)|$ $(0<\delta<\kappa-1)$. Hence Theorem 5.2 is obtained.

Hereafter we consider the case

$$\varphi(0)\leqq\varphi(y) \quad \text{and} \quad \det H_\varphi(y)\neq 0 \text{ on } \bar{U}. \tag{5.1}$$

Theorem 5.4. *Let* (5.1) *be satisfied, and assume that* $p(\sigma)$ *is homogeneous of order* κ *with* $[\kappa]>l/2$ ($[\kappa]=$ *the maximum integer* $<\kappa$). *Then we have the following properties* (i) *and* (ii) *for any function* $k(s,y)\in\mathcal{B}^\infty(\boldsymbol{R}\times U)$ *with supp* $[k]\subset \boldsymbol{R}\times \boldsymbol{U}$.

(i) *There is a constant* C *independent of* k *and* $t\in\boldsymbol{R}$ *such that*

$$|t^{l/2}\int_U (p(D_s)\,k)\,(t\varphi(y),y)\,dy|\leqq C\,|k|_{[\kappa]+3,\boldsymbol{R}\times U}\,.$$

(ii) *If* $\varphi(0)\neq 0$ *and* $\lim_{|s|\to\infty}\sup_{\substack{|\alpha|<\kappa+3\\ y\in U}}|\partial_y^\alpha k(s,y)|=0$, *then it follows that*

$$\lim_{|t|\to\infty} t^{l/2}\int_U (p(D_s)\,k)\,(t\varphi(y),y)\,dy=0\,.$$

If $p(D_s)$ *is a differential operator, the above statements* (i) *and* (ii) *are valid also when* $\kappa>l/2$.

REMARK. (i) In Theorem 5.2, let $h(s,y)=\int_0^s k(r,y)\,dr\in\mathcal{B}^\infty(\boldsymbol{R}\times U)$, which is satisfied if, e.g., $(1+s^2)\,k(s,y)\in\mathcal{B}^\infty(\boldsymbol{R}\times U)$ or $k(s,y)=\partial_s\tilde{k}(s,y)$ for some $\tilde{k}(s,y)\in\mathcal{B}^\infty(\boldsymbol{R}\times U)$. Then, noting that $p(D_s)\,k=p(D_s)\,\partial_s h$ for such functions k, we obtain the estimate $|t^{\kappa+1}\int_U pk(t\varphi(y),y)\,dy|\leqq C_1$ (for a constant C_1 independent of t).

(ii) The same assertion is correct also in Theorem 5.4: For any $k(s,y)$ with $\int_0^s k(r,y)dr\in\mathcal{B}^\infty(\boldsymbol{R}\times U)$, we have the estimate $|t^{l/2}\int_U pk(t\varphi(y),y)\,dy|\leqq C_2$ (for a constant C_2 independent of t) and the property (ii) in Theorem 5.4 so long as $[\kappa]>2^{-1}l-1$; furthermore, if $p(D_s)$ is a differential operator, they are valid when $\kappa>2^{-1}l-1$.

Theorem 5.4 is reduced to

Lemma 5.5. *Let* $\chi(s)\in C_0^\infty(\boldsymbol{R})$, *and let* r_0 *be any real constant. Assume that* $p(\sigma)$ *is homogeneous of order* $\kappa>1$ *and that* $h(s,r)$ *is any* C^∞ *function on* $\boldsymbol{R}\times\boldsymbol{R}_+$ ($\boldsymbol{R}_+=(0,\infty)$) *satisfying* $\sup_{(s,r)\in\boldsymbol{R}\times\boldsymbol{R}_+}|r^i\,\partial_r^i\,\partial_s^j\,h(s,r)|<\infty$ *fo revery* $i,j=0,1,\cdots$. *Then for any constant* m *with* $0<m<[\kappa]$ ($[\kappa]=$ *the maximum integer* $<\kappa$) *we have the following properties* (i) *and* (ii):

(i) *There is a constant* C *independent of* h *and* $t\in\boldsymbol{R}$ *such that*

$$|t^m\int_0^\infty (p(D_s)\,h)\,(tr+tr_0,r)\,r^{m-1}\,\chi(r)\,dr|$$
$$\leqq C\sum_{i,j=0}^{[\kappa]+3}\sup_{(s,r)\in\boldsymbol{R}\times\boldsymbol{R}_+}|r^i\,\partial_r^i\,\partial_s^j\,h(s,r)|\,.$$

(ii) *If* $r_0 \neq 0$ *and* $\lim_{|s|\to\infty} \sum_{i=0}^{[\kappa]+3} \sup_{r\in I_+} |r^i \partial_r^i h(s, r)| = 0$ $(I_+ = \mathrm{supp}\,[\chi] \cap \boldsymbol{R}_+)$, *then*

$$\lim_{|t|\to\infty} t^m \int_0^\infty (p(D_s)\, h)\, (tr + tr_0, r)\, r^{m-1} \chi(r)\, dr = 0\,.$$

If $p(D_s)$ *is a differential operator, the above statements* (i) *and* (ii) *are valid when* $1 \leqq \kappa$ *and* $0 < m < \kappa$.

Proof. Let $\tilde{\chi}(r)$ be a C^∞ function on $\boldsymbol{R}$ such that $\tilde{\chi}(r) = 1$ when $r \leqq 1$ and $\tilde{\chi}(r) = 0$ when $r \geqq 2$, and set $\chi_\varepsilon^0(r) = \tilde{\chi}(r/\varepsilon)$, $\chi_\varepsilon^1(r) = \tilde{\chi}(\varepsilon r) - \tilde{\chi}(r/\varepsilon)$ and $\chi_\varepsilon^2(r) = 1 - \tilde{\chi}(\varepsilon r)$, where ε is a positive constant determined later. Then we have

$$\begin{aligned} &|t|^m \int_0^\infty (p(D_s)\, h)\, (tr + tr_0, r)\, r^{m-1} \chi(r)\, dr \\ &= \sum_{i=0}^2 \int_0^\infty (p(D_s)\, h)\, ((\mathrm{sgn}\, t)\, r + tr_0, r/|t|)\, r^{m-1} \chi(r/|t|)\, \chi_\varepsilon^i(r)\, dr \\ &\equiv \sum_{i=0}^2 I_\varepsilon^i(t)\,. \end{aligned}$$

By Lemma 5.1 we get

$$|I_\varepsilon^i(t)| \leqq C_1 \sum_{j=0}^{[\kappa]+3} \sup_{(s,\rho)\in \boldsymbol{R}\times I_+} |\partial_s^j h(s, \rho)| \int_0^\infty r^{m-1} |\chi_\varepsilon^i|\, dr \quad (i = 0, 1) \tag{5.2}$$

for a constant C_1 independent of t, ε and h. By integration by parts, we have

$$\begin{aligned} I_\varepsilon^2(t) = \int_0^\infty \sum_{i_1+i_2=[\kappa]} c_{i_1 i_2} (\partial_s^{-[\kappa]} p \partial_r^{i_1} h)\, ((\mathrm{sgn}\, t)\, r + tr_0, r/|t|) \\ |t|^{-i_1} \partial_r^{i_2} \{r^{m-1} \chi(r/|t|)\, \chi_\varepsilon^2(r)\}\, dr \end{aligned} \tag{5.3}$$

for some constants $c_{i_1 i_2}$ depending only on $[\kappa]$. There are constants C_2 and C_2 independent of ε, t and h such that

$$\begin{aligned} &|\partial_r^{i_2} \{r^{m-1} \chi(r/|t|)\, \chi_\varepsilon^2(r)\}| \leqq C_2\, r^{m-1-i_2}\,, \\ &|(\partial_s^{-[\kappa]} p \partial_r^{i_1} h)\, ((\mathrm{sgn}\, t)\, r + tr_0, r/|t|)\, |t|^{-i_1}| \\ &\qquad \leqq C_3 \sum_{j=0}^{3} \sup_{(s,\rho)\in \boldsymbol{R}\times I_+} |\rho^{i_1} \partial_\rho^{i_1} \partial_s^j h(s, \rho)|\, r^{-i_1}\,. \end{aligned}$$

Here, Lemma 5.1 is used in the above second inequality. Therefore we have

$$|I_\varepsilon^2(t)| \leqq C_4 \sum_{\substack{i=0,\cdots,[\kappa] \\ j=0,\cdots,3}} \sup_{(s,r)\in \boldsymbol{R}\times I_+} |r^i \partial_r^i \partial_s^j h(s, r)|\, \varepsilon^{[\kappa]-m}\,, \tag{5.4}$$

where the constant C_4 is independent of ε, t and h.

Combining (5.4) with (5.2) yields (i) of the lemma. (5.2) (with $i=0$) and (5.4) imply that for any $\delta > 0$ we can choose ε so that $|I_\varepsilon^i(t)| < \delta$ for $i = 0$ and 2 (uniformly in t). After fixing ε in this way, we let $|t|$ tend to ∞. Then we get $|I_\varepsilon^1(t)| < \delta$ (as $|t| \to \infty$) also. In fact, it follows from Lemma 5.3 that

$$I_\varepsilon^1(t) = \int_{-\infty}^{\infty} p(-D_r)\{h(s, r/t)\,|r|^{m-1}\,\chi(r/t)\,\chi_\varepsilon^1((\operatorname{sgn} t)\, r)\}\,|_{s=r+tr_0}\, dr\,.$$

By Lemma 5.1 we see (in the above integral) that $|p(-D_r)\{\cdots\}| \leqq C_1(1+r^2)^{-(1+\gamma)/2}$ $\sum_{i=0}^{[\kappa]+3} \sup_{\rho\in\boldsymbol{R}} |\partial_\rho^i\{(1+\rho^2)^{(1+\gamma)/2} h(s, \rho/t)\,|\rho|^{m-1}\,\chi(\rho/t)\chi_\varepsilon^1((\operatorname{sgn} t)\,\rho)\}| \leqq C_2(\varepsilon)(1+r^2)^{-(1+\gamma)/2}$ $\sum_{i=0}^{[\kappa]+3} \sup_{\rho\in\bar{I}_+} |\rho^i\, \partial_\rho^i h(s, \rho)|$ $(0<\gamma<\kappa-1)$, where the constants C_1 and $C_2(\varepsilon)$ do not depend on t. Therefore, by the Lebesgue convergence theorem, we get $\lim_{|t|\to\infty} I_\varepsilon^1(t) =0$. Hence (ii) of the lemma is also obtained.

Since $\partial_s^{-\kappa} p(D_s)$ is a constant if $p(D_s)$ is a differential operator, we can change $[\kappa]$ in (5.3) for κ and carry out the above arguments with $m<\kappa$ and $\kappa\geqq 1$. Hence, in this case, (i) and (ii) in the lemma are valid also when $m<\kappa$ and $\kappa\geqq 1$. Thus Lemma 5.5 is proved.

Proof of Theorem 5.4. Since $\varphi(y)$ is minimum at $y=0$, the eigenvalues μ_i of $H_\varphi(0)$ are all positive. By means of Morse's lemma (e.g., cf. the proof of Theorem 4.1 in Mastumura [6]), we see that there exist local coordinates $y(\tilde{y})$ available in a neighborhood U_0 of the origin $y=0$ such that $\varphi(y(\tilde{y}))=\varphi(0)+2^{-1} \sum_{i=1}^{l} \mu_i\, \tilde{y}_i^2$ $(\tilde{y}=(\tilde{y}_1, \cdots, \tilde{y}_l))$ and that $\dfrac{\partial y}{\partial \tilde{y}}(0)$ is an orthogonal matrix. Therefore, introducing the variables $z={}^t(\mu_1^{1/2}\,\tilde{y}_1, \cdots, \mu_l^{1/2}\,\tilde{y}_l)$, we have

$$\varphi(y(z)) = \varphi(0)+2^{-1}\,|z|^2\,, \quad \det \frac{\partial y}{\partial z}(0) = (\det H_\varphi(0))^{-1/2}\,.$$

Assume that the support of $k(s, y)$ is contained in $\boldsymbol{R}\times U_0$. Then, using the variables $r=2^{-1}|z|^2$ and $\zeta=|z|^{-1}z$ yields that

$$\begin{aligned} & t^{l/2}\int_U (P(D_s)\,k)\,(t\varphi(y), y)\,dy \\ (5.5)\qquad & = 2^{(l-2)/2}\int_{S^{l-1}} \{t^{l/2}\int_0^\infty (p(D_s)\,k)\,(tr+t\varphi(0), y(\sqrt{2r}\,\zeta)) \\ & \qquad r^{(l-2)/2}\,|\det \frac{\partial y}{\partial z}\,|dr\}\, d\zeta\,. \end{aligned}$$

The function $\tilde{k}(s, r, \zeta)=k(s, y(\sqrt{2r}\,\zeta))\,|\det \dfrac{\partial y}{\partial z}(\sqrt{2r}\,\zeta)|$ is a C^∞ function on $\boldsymbol{R}\times \boldsymbol{R}_+\times S^{l-1}$ satisfying $\sup_{(s,r,\zeta)\in\boldsymbol{R}\times\boldsymbol{R}_+\times S^{l-1}} |r^i\,\partial_r^i\,\partial_s^j\,\tilde{k}(s, r, \zeta)|<\infty$ for every $i, j= 0, 1, \cdots$ and $\tilde{k}(s, r, \zeta)=0$ when $r>r_1$ (for some constrnt r_1). Therefore we can apply Lemma 5.5 to integral $t^{l/2}\int p(D_s)k\, r^{(l-2)/2}\,|\det \dfrac{\partial y}{\partial z}|dr$, and from (i) of Lemma 5.5 derive

$$|t^{l/2}\int_U (p(D_s)\,k)\,(t\varphi(y), y)\,dy|$$

$$\leqq C_1 \sum_{i,j=0}^{[\kappa]+3} \sup_{(s,r,\zeta)\in \boldsymbol{R}\times\boldsymbol{R}_+\times S^{l-1}} |r^i \partial_r^i \partial_s^j \tilde{k}(s,r,\zeta)|$$

for a constant C_1 independent of t and k. This yields (i) of Theorem 5.4. $\lim_{|s|\to\infty} \sum_{i=0}^{[\kappa]+3} \sup_{(r,\zeta)\in\boldsymbol{R}_+\times S^{l-1}} |r^i \partial_r^i \tilde{k}(s,r,\zeta)|=0$ follows from $\lim_{|s|\to\infty} \sup_{|\alpha|<\kappa+3,\, y\in U} |\partial_y^\alpha k(s,y)|=0$. Hence, if $\varphi(0)\neq 0$ and $\lim_{|\alpha|\to\infty} \sup_{|\alpha|<\kappa+3,\, y\in U} |\partial_y^\alpha k(s,y)|=0$, by (ii) of Lemma 5.5 and the Lebesgue convergence theorem we get $\lim_{|t|\to\infty} t^{l/2}\int (p(D_s)k)(t\varphi(y),y)dy=0$. Thus the requirements are obtained if $\mathrm{supp}[k]\subset \boldsymbol{R}\times U_0$.

If $\mathrm{supp}[k]$ is not contained in $\boldsymbol{R}\times U_0$, we take a function $\chi(y)\in C_0^\infty(U_0)$ with $\mathrm{supp}\,[1-\chi]\not\ni 0$, and apply the above methods to $\int_U (P(D_s)k)(t\varphi(y),y)\chi(y)\,dy$. For the other term $\int_U (p(D_s)k)(t\varphi(y),y)(1-\chi(y))\,dy$, we can use Theorem 5.2 since $\partial_\varphi \varphi(y)\neq 0$ on $\mathrm{supp}\,[1-\chi]$. Hence it follows that $|t^{l/2}\int_U (p(D_s)k)(t\varphi(y),y)(1-\chi(y))\,dy| \leqq C_2|t|^{2^{-1}l-\kappa}\to 0$ (as $|t|\to\infty$), which yields the required properties. Thus (i) and (ii) of Theorem 5.4 in the general case are proved. It is easy to verify the note in the case where $p(D)$ is a differential operator (by Lemma 5.5). The proof is complete.

Theorem 5.4 is not valid if $p(D_s)=J_\pm(=(-\partial_s)_\pm^{l/2}$ defined in §1), but we obtain similar results:

Theorem 5.6. *Let* (5.1) *be satisfied, and assume that* $k(s,y)$ *is any function* $\in\mathcal{B}^\infty(\boldsymbol{R}\times U)$ *such that* $\mathrm{supp}\,[k]\subset\boldsymbol{R}\times U$. *Then we have the following properties* (i) *and* (ii):

(i) *There is a constant* C *independent of* k *and* $t\in\boldsymbol{R}$ *such that*

$$|t^{l/2}\int_U (J_\pm k)(t\varphi(y),y)\,dy| \leqq C\|k\|_{l+4,\boldsymbol{R}\times U}\,.$$

(ii) *If* $\varphi(0)=0$ *and* $\int_0^s k(\tau,y)\,d\tau\in\mathcal{B}^\infty(\boldsymbol{R}\times U)$, *then we have*

$$\lim_{t\to\infty} t^{l/2}\int_U (J_\pm k)(\pm t\varphi(y),y)\,dy = c_\pm^l(2\pi)^{l/2}(\det H_\varphi(0))^{-1/2}k(0,0) \quad (c_+=1,\ c_-=-i)\,.$$

Proof. We give the proof only when l is odd since we can get the theorem by much easier methods when l is even.

(i) We can obtain (i) of Theorem 5.6 in the same way as (i) of Theorem 5.4 if we have the following estimate similar to the one in (i) of Lemma 5.5:

$$(5.6)\qquad |t^{l/2}\int_0^\infty (J_\pm h)(tr+tr_0,r)\,r^{(l-2)/2}\chi(r)\,dr|$$

$$\leqq C_1 \sum_{i,j=0}^{m} \| \sup_{r \in \boldsymbol{R}_+} |r^i \partial_r^i \partial_s^j h(s, r)| \|_{L_s^2},$$

where h and χ are the functions of the same types as in (i) of Lemma 5.5 and m is the integer with $(l+6)/2 \leqq m \leqq (l+7)/2$. Define $I_\varepsilon^i(t)$ $(i=0, 1, 2)$ in the same way as in the proof of Lemma 5.5. Then it is easy to see that $|I_\varepsilon^i(t)| \leqq C_2 \sum_{j=0}^{[l/2]+4} \| \sup_{r \in \boldsymbol{R}_+} |\partial_s^j h(s, r)| \|_{L_s^2}$ for $i=0$ and 1.

When l is odd, $\partial_s^{-(l+1)/2} J_\pm$ is written in the form $\partial_s^{-(l+1)/2} J_\pm h(s) = \int_{-\infty}^{\infty} \chi_\pm(\tau) h(s-\tau)\, d\tau$ for some functions $\chi_\pm(\tau)$ homogeneous of order $-1/2$ (cf. the proof of Theorem 1.2 in Soga [8]). We take $\psi_1(s) \in C^\infty(\boldsymbol{R})$ such that $|\psi_1| \leqq 1$, $\psi_1(s) = 1$ for $|s| < 1$ and $=0$ for $|s| > 2$, and set $\psi_2(s) = 1 - \psi_1(s)$. By the same calculation as in (5.3) (with $p = J_\pm$ and $m = l/2$), we have

$$\begin{aligned} I^2(t) = & \sum_{i_1+i_2=\kappa} c_{i_1 i_2} \Big[\int_{1/\varepsilon}^{\infty} \Big\{ \int_{|\tau| \leq 1/\varepsilon} \chi_\pm(\tau)\, h_{i_1}((\mathrm{sgn}\, t) r - \tau, r)\, d\tau \Big\}\, \alpha_{i_1 i_2}(r)\, dr \\ & + \sum_{j=1,2} \int_{1/\varepsilon}^{\infty} \Big\{ \int_{|\tau| > 1/\varepsilon} \chi_\pm(r)\, \psi_j(\tau/r)\, h_{i_1}((\mathrm{sgn}\, t) r - \tau, r)\, d\tau \Big\}\, \alpha_{i_1 i_2}(r)\, dr \Big] \\ \equiv & \sum_{i_1+i_2=\kappa} c_{i_1 i_2} \Big[\Phi_{i_1 i_2}^0(t) + \sum_{j=1,2} \Phi_{i_1 i_2}^j(t) \Big] \qquad (\kappa = (l+1)/2)), \end{aligned}$$

where $h_{i_1}(s, r) = \partial_r^{i_1} h(s + t r_0, r/|t|)\, r^{i_1} |t|^{-i_1}$ and $\alpha_{i_1 i_2}(r) = r^{-i_1} \partial_r^{i_2} \{ r^{(l-2)/2} \chi(r/|t|)\, \chi_\varepsilon^2(r) \}$. Since $|\alpha_{i_1 i_2}(r)| \leqq C_3 (1+|r|)^{-3/2}$ (for a constant C_3 independent of t and r), it follows that

$$\begin{aligned} |\Phi_{i_1 i_2}^0(t)| & \leqq C_3 \|(1+|r|)^{-3/2}\|_{L^2} \int_{|\tau| \leq 1/\varepsilon} |\chi_\pm(\tau)|\, d\tau\, \| \sup_{r \in \boldsymbol{R}_+} |h_{i_1}(s, r)| \|_{L_s^2}, \\ |\Phi_{i_1 i_2}^1(t)| & \leqq C_3 \int_{1/\varepsilon}^{\infty} dr \int_{1/\varepsilon < |\tau| < 2r} d\tau\, |\chi_\pm(\tau)\, \tau^{-\delta}\, h_{i_1}((\mathrm{sgn}\, t) r - \tau, r)|\, (1+|r|)^{-3/2} (2r)^\delta \\ & \leqq C_4 \| \sup_{r \in \boldsymbol{R}_+} |h_{i_1}(s, r)| \|_{L_s^2} \quad (0 < \delta < 1/2). \end{aligned}$$

The function $\tilde{h}_{i_1 i_2}(s, r) \equiv \int_\infty^r h_{i_1}(s, \rho)\, \alpha_{i_1 i_2}(\rho)\, d\rho$ satisfies $|\tilde{h}_{i_1 i_2}(s, r)| \leqq C_5 \sup_{\rho \in \boldsymbol{R}_+} |h_{i_1}(s, \rho)| (1+|r|)^{-1/2}$. On the hand, by integration by parts we have

$$\begin{aligned} \Phi_{i_1 i_2}^2(t) = \int_{1/\varepsilon}^{\infty} dr \int_{r < |\tau|} d\tau \Big\{ & -(\mathrm{sgn}\, t)\, \chi_\pm'(\tau)\, \psi_2(\tau/r) - (\mathrm{sgn}\, t)\, \chi_\pm(\tau) \frac{1}{r} \psi_2'(\tau/r) \\ & + \chi_\pm(\tau) \frac{\tau}{r^2} \psi_2'(\tau/r) \Big\}\, \tilde{h}_{i_1 i_2}((\mathrm{sgn}\, t) r - \tau, r). \end{aligned}$$

Therefore it follows that

$$|\Phi_{i_1 i_2}^2(t)| \leqq C_6 \int_{1/\varepsilon}^{\infty} r^{-1-\delta}\, dr\, \|(1+|\tau|)^{-1+\delta}\|_{L^2} \| \sup_{\rho \in \boldsymbol{R}_+} |h_{i_1}(\tau, \rho)| \|_{L_\tau^2} \qquad (0 < \delta < 1/2).$$

Thus (5.6) is obtained.

(ii) Next let us prove (ii) of Theorem 5.6, In veiw of (5.5) we have

$$
\begin{aligned}
&t^{l/2}\int_U (J_\pm k)(\pm t\varphi(y), y)\, dy \\
&= 2^{(l-2)/2}\int_{S^{l-1}} d\zeta\, t^{l/2}\int_0^\infty (J_\pm k)(\pm tr, 0)\,|\det \frac{\partial y}{\partial z}(0)|\,\chi(r)\, r^{(l-2)/2}\, dr \\
&\qquad + t^{-1/4}\int_{S^{l-1}} \{t^{(2l+1)/4}\int_0^\infty (J_\pm \tilde{k})(\pm tr, r, \zeta)\, r^{(2l+1)/4-1}\, dr\}\, d\zeta \\
&\equiv I_1(t)+I_2(t)\,,
\end{aligned}
$$

where $\chi(r)$ is a C^∞ function satisfying $\chi(r)=1$ for $r\leqq 1$ and $\chi(r)=0$ for $r\geqq 2$, and $\tilde{k}(s, r, \zeta)$ is of the form

$$
\begin{aligned}
\tilde{k}(s, r, \zeta) = 2^{(l-2)/2}\, &\{k(s, y(\sqrt{2r}\,\zeta))\,|\det \frac{\partial y}{\partial z}(\sqrt{2r}\,\zeta)| \\
&-k(s, 0)\,|\det \frac{\partial y}{\partial z}(0)|\,\chi(r)\}\, r^{-1/4}\,.
\end{aligned}
$$

It is seen that $h(s, r, \zeta)=\int_0^s \tilde{k}(\tau, r, \zeta)\, d\tau$ satisfies $\sup_{(s,r,\zeta)\in \boldsymbol{R}\times\boldsymbol{R}_+\times S^{l-1}} |r^i\, \partial_r^i\, \partial_s^j\, h(s, r, \zeta)| < \infty$ for every $i, j=1, 2, \cdots$, and that $J_\pm \tilde{k} = J_\pm \partial_s h$. Applying Lemma 5.5 with $p(D_s)=J_\pm \partial_s$ and $m=(2l+1)/4$, we get $\sup_{\substack{t\geqq 1\\ \zeta\in S^{l-1}}} |t^{(2l+1)/4}\int_0^\infty (J_\pm \tilde{k})(\pm tr, r, \zeta)\, r^{((2l+1)/4)-1}\, dr| < \infty$, which yields that $\lim_{t\to\infty} I_2(t)=0$.

When l is odd, $J_\pm$ are of the form $(-\partial_s)^{(l-1)/2}\lambda_\pm(D_s)$, where $\lambda_\pm(\sigma)$ are defined by $\lambda_\pm(\sigma)=2^{-1/2}(1-i)\sigma^{1/2}$ for $\sigma\geqq 0$ and $=\pm 2^{-1/2}(1+i)|\sigma|^{1/2}$ for $\sigma<0$. By integration by parts, we have

$$
\begin{aligned}
I_1(t) = (\pm 1)^{(l-1)/2}\, &2^{(l-2)/2}\left(\frac{l-2}{2}\,\frac{l-4}{2}\cdots\frac{1}{2}\right)|\det \frac{\partial y}{\partial z}(0)| \\
&\int_{S^{l-1}} d\zeta\, t^{1/2}\int_0^\infty (\lambda_\pm k)(\pm tr, 0)\, r^{-1/2}\chi(r)\, dr \\
&+(\pm 1)^{(l+1)/2}\, 2^{(l-2)/2}\,|\det \frac{\partial y}{\partial z}(0)|\int_{S^{l-1}} d\zeta\, t^{-1/2}\int_0^\infty (\lambda_\pm h)(\pm tr, 0) \\
&\partial_r\{\partial_r^{(l-1)/2}(r^{(l-2)/2}\chi(r))-(\partial_r^{(l-1)/2} r^{(l-2)/2})\chi(r)\}\, dr\,,
\end{aligned}
$$

where $h(s)=\int_0^s k(\tau, 0)\, d\tau$. Since $\lambda_\pm h(s)\in\mathscr{B}^0(\boldsymbol{R})$, the second term of the right side in the above equality converges to 0 as $|t|\to\infty$. Therefore, by means of Lemma 5.7 below, we obtain $\lim_{t\to\infty} I_1(t)=c_\pm^l(2\pi)^{l/2}(\det H_\varphi(0))^{-1/2}K(0, 0)$. Thus (ii) of Theorem 5.6 is also proved.

Lemma 5.7. *Let $\chi(r)$ be a C^∞ function on $\boldsymbol{R}$ such that $\chi(r)=1$ for $r\leqq 1$*

and $\chi(r)=0$ *for* $r\geqq 2$. *Then, for any* C^∞ *function* $h(r)$ *with* $\int_0^r h(s)\,ds\in\mathscr{B}^0(\boldsymbol{R})$, *we obtain*

$$\begin{aligned}&\left|t^{1/2}\int_0^\infty (\lambda_\pm h)(\pm tr)^{-1/2}\chi(r)\,dr-c_\pm\sqrt{\pi}\,h(0)\right|\\&\leqq Ct^{-1/2}\left|\int_0^r h(s)\,ds\right|_3, \qquad t\geqq 1,\end{aligned}$$

where $c_+=1$, $c_-=-i$ *and the constant* C *does not depend on* h *or* t.

We can derive the above lemma from the equalities that $\int_{-\infty}^0 (\lambda_+k)(s-r)(-r)^{-1/2}\,dr=\sqrt{\pi}\,k(s)$ and $\int_0^\infty (\lambda_-k)(s-r)\,r^{-1/2}\,dr=-i\sqrt{\pi}\,k(s)$ for $k(s)\in C_0^\infty(\boldsymbol{R})$ (cf. 1.13) of Soga [8]).

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