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A CHARACTERIZATION OF THE CLOSABLE PARTS OF PRE-DIRICHLET FORMS BY HITTING DISTRIBUTIONS

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1. Introduction

Let X be a locally compact separable metric space with an extra point Δ such that $X_\Delta \equiv X \cup \{\Delta\}$ is a one point compactification and let m be a positive Radon measure with supp $[m]=X$. When X is compact, Δ is adjoined as an isolated point. For a subset B of X, we denote $B_\Delta = B \cup \{\Delta\}$. We consider a C_0-regular Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(X, m)$ having a nice core $\mathcal{C}$ (see Section 2) and $\boldsymbol{M}=(\Omega, \mathcal{F}_t, X_t, P_x, x\in X)$ the associated m-symmetric Hunt process. We say that a subset B of X is $\mathcal{E}_1$-polar if it is of zero capacity. Let $\{T_t, t\geqq 0\}$ be the L^2-semigroup associated with $(\mathcal{E}, \mathcal{F})$. We say that a Borel set B of X is T_t-invariant if $T_t(I_B u)=I_B T_t u$ for any $u\in L^2(X, m)$, and $t>0$. $(\mathcal{E}, \mathcal{F})$ is called irreducible if for any T_t-invariant set B, B or $X-B$ is m-negligible. A Borel set B of X is $\boldsymbol{M}$-invariant if $P_x(X_t\in B_\Delta, X_{t-}\in B_\Delta$, for any $t>0)=1$, for any $x\in B$. M. Fukushima-K. Sato-S. Taniguchi [10] investigated the closable part of general symmetric bilinear form on a real Hilbert space. They characterized the closable part of a pre-Dirichlet form under the changes of underlying measures and gave a necessary and sufficient condition for the closability. They used the analytic characterization of the time changed Dirichlet space formulated in K. Kuwae-S. Nakao [12]. In these mentioned articles assumed is that $(\mathcal{E}, \mathcal{F})$ is either transient or irreducible in order to make a reduction to the transient case, but the irreducibility is not easily checked.

In this paper, we will not assume the irreducibility of $(\mathcal{E}, \mathcal{F})$ nor its transience. In Section 2 and Section 3 we prepare some quasi-notions and decomposition theorems of the state space X. In particular, we give a decomposition

$$X = X^{(c)} + X^{(d)} + N,$$

where $X^{(c)}$ (resp. $X^{(d)}$) is an $\boldsymbol{M}$-invariant conservative (resp. dissipative) part of X, and N is a properly exceptional set. In Section 4 we give a characterization of the regular Dirichlet space associated with the time changed process using the above decomposition. In Section 5 we fix a closed set Y and consider the space $\mathcal{C}|_Y = \{u\in C_0(Y); u=\bar{u}|_Y$, for some $\bar{u}\in\mathcal{C}\}$. We then introduce, for each choice of a finely closed Borel set F with $F\subset Y$, a pre-Dirichlet form $\mathcal{A}_F$ with domain $\mathcal{C}|_Y$ defined by

$$\mathcal{A}_F(u, u) = \mathcal{E}(H_F \bar{u}, H_F \bar{u}),\ u\in\mathcal{C}|_Y,$$

where $\bar{u}$ is a function appearing in the definition of $\mathcal{C}|_Y$ and $H_F\bar{u}(x)=E_x[\bar{u}(X_{\sigma_F})]$, σ_F being the hitting time of F. Suppose μ is a positive Radon measure on X and $Y=\text{supp}[\mu]$. Using the characterization of time changed Dirichlet space in Section 4, we prove that the closable part of $(\mathcal{A}_Y, \mathcal{C}|_Y)$ on $L^2(Y; \mu)$ is $(\mathcal{A}_{\tilde{Y}_0}, \mathcal{C}|_Y)$ where $\tilde{Y}_0$ is the quasi-support of the smooth part of μ, generalizing a result of [10]. As a consequence, we can generalize the closability criterion of [10] (Theorem 5.4).

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2. Quasi-notions

As in Section 1, let X be a locally compact separable metric space with an extra point Δ such that X_Δ is a one point compactification and m be a positive Radon measure with supp $[m]=X$. For a Borel measure γ on X and Borel functions f and g on X, we denote $(f, g)_\gamma=\int_X f(x)\,g(x)\,\gamma(dx)$ if this integral makes sense. Let $C_0(X)$ be the family of continuous functions with compact support. Consider a dense subalegbra $\mathcal{C}$ of $C_0(X)$ satisfying the following two properties:

($\mathcal{C}$. 1) For any compact set K and relatively compact open set G with $K\subset G\subset X$, there exists $f\in\mathcal{C}$ such that $0\leqq f\leqq 1$ and $f=1$ on K and $f=0$ on $X-G$.

($\mathcal{C}$. 2) For any $\varepsilon>0$ there exists a real function $\varphi_\varepsilon(t)$ satisfying that $\varphi_\varepsilon(t)=t$ for any $t\in[0, 1]$, $-\varepsilon\leqq\varphi_\varepsilon(t)\leqq 1+\varepsilon$ for any t, and $0\leqq\varphi_\varepsilon(t)-\varphi_\varepsilon(s)\leqq t-s$ for $s\leqq t$, and $\varphi_\varepsilon(f)\in\mathcal{C}$ whenever $f\in\mathcal{C}$.

Let $(\mathcal{E}, \mathcal{F})$ be a C_0-regular Dirichlet space on $L^2(X, m)$ possessing $\mathcal{C}$ as its core, namely $\mathcal{C}$ is dense in $(\mathcal{E}_1, \mathcal{F})$, where $\mathcal{E}_1$ is defined by

$$\mathcal{E}_1(u, v) = \mathcal{E}(u, v)+(u, v)_m,\quad u, v\in\mathcal{F}.$$

Let $\boldsymbol{M}=(\Omega, \mathcal{F}_t, X_t, P_x, x\in X)$ be the m-symmetric Hunt process associated with $(\mathcal{E}, \mathcal{F})$. The capacity associated with $(\mathcal{E}, \mathcal{F})$ will be called the $\mathcal{E}_1$-capacity; for any open set G,

$$\mathcal{E}_1\text{-Cap}(G) = \inf\{\varepsilon_1(u, u);\ u\in\mathcal{F},\ u\geq 1\ m\text{-a.e. on } G\} \tag{2.1}$$

and, for any subset A of X,

$$\mathcal{E}_1\text{-Cap}(A) = \inf\{\mathcal{E}_1\text{-Cap}(G);\ A\subset G,\ \text{open}\}. \tag{2.2}$$

It is well-known that for any compact set K,

$$\mathcal{E}_1\text{-Cap}(K) = \inf\{\mathcal{E}_1(u, u);\ u\in\mathcal{C},\ u\geqq 1 \text{ on } K\}. \tag{2.3}$$

A set $B \subset X$ is called $\mathcal{E}_1$-polar if $\mathcal{E}_1$-Cap$(B)=0$. A statement Γ depending on $x \in A$ is said to hold $\mathcal{E}_1$-q.e. on A (abbreviated to q.e. on A) if there exists an $\mathcal{E}_1$-polar set N such that Γ is true for $x \in A-N$. A function $f: X \to [-\infty, \infty]$ is called $\mathcal{E}_1$-quasi-continuous (abbreviated to quasi-continuous) if for any $\varepsilon > 0$ there exists an open set G such that $\mathcal{E}_1$-Cap$(G) < \varepsilon$ and $f|_{X-G}$ is continuous. An increasing sequence of closed sets $\{F_n\}$ is called $\mathcal{E}_1$-nest (abbreviated to nest) if $\lim_{n \to +\infty} \mathcal{E}_1$-Cap$(X-F_n)=0$. Let $\mathcal{M}$ be the space of positive Radon measures on X and let $\mathcal{M}_0 = \{\nu \in \mathcal{M};\ \nu$ charges no $\mathcal{E}_1$-polar set$\}$. As in [9], we use following notations: For set $A, B \subset X$, we denote

$$A \subset B \text{ q.e.} \quad (\text{resp. } A=B \text{ q.e.})$$

if the set $A-B$(resp. $A \triangle B$) is $\mathcal{E}_1$-polar. Here $A \triangle B$ is the symmetric difference. Similarly we can define $A \subset B$ ν-a.e. if $\nu(A-B)=0$ for $\nu \in \mathcal{M}$. We say that a set A is a q.e. (resp. ν-a.e.) version of a set B or A is q.e. (resp. ν-a.e.) equivalent to B if $A=B$ q.e. (resp. ν-a.e). We call a set $E \subset X$ quasi-open if

$$\inf \{\mathcal{E}_1\text{-Cap}(E \triangle G);\ G \text{ open}\} = 0$$

and a set F is called quasi-closed if $X-F$ is quasi-open. It is easy to see that the notion of quasi-open (resp.-closed) is stable under q.e. equivlaence and a set E is quasi-open (resp.-closed) if and only if there exists a nest $\{F_n\}$ such that $E \cap F_n$ is an open (resp. a closed) subset of F_n with respect to relative topology of F_n. Any countable union and finite intersection of quasi-open sets are quasi-open and any countable intersection and finite union of quasi-closed sets are quasi-closed. A function $f: X \to [-\infty, \infty]$ is quasi-continuous if and only if for any open set $I \subset [-\infty, \infty]$, $f^{-1}(I)$ is quasi-open. In particular, for a quasi-open and quasi-closed set B, the indicator function I_B is quasi-continuous (B. Fuglede [4]). For two outer capacities $C^{(1)}$ and $C^{(2)}$ on X, we write $C^{(1)} < C^{(2)}$ if for any decreasing sequence of relatively compact open sets $\{A_n\}$

$$\lim_{n \to \infty} C^{(2)}(A_n) = 0 \text{ implies } \lim_{n \to \infty} C^{(1)}(A_n) = 0 .$$

Then $C^{(2)}$-polarity, $C^{(2)}$-quasi-open set, $C^{(2)}$-quasi-continuity are inherited to the corresponding notions relative to $C^{(1)}$. We say that $C^{(2)}$ is qeuivalent to $C^{(1)}$ if $C^{(2)} < C^{(1)}$ and $C^{(1)} < C^{(2)}$.

For $\nu \in \mathcal{M}_0$, a set $\tilde{Y} \subset X$ is called a quasi-support of ν if $\tilde{Y}$ is a quasi-closed ν-a.e. version of X and $\tilde{Y} \subset \hat{Y}$ q.e. for any quasi-closed $\hat{Y}$ which is a ν-a.e. version of X. Let $Y=\text{supp}\,[\nu]$ be the topological support of ν. Then $\tilde{Y} \subset Y$ q.e.. The existence of quasi-support of $\nu \in \mathcal{M}_0$ up to $\mathcal{E}_1$-polar set is guaranteed ([4], [10]). For $\nu \in \mathcal{M}_0$, denote by q-supp$[\nu]$ the quasi-support of ν. We let $\mathcal{M}_{00} = \{\nu \in \mathcal{M}_0;\ \mathcal{E}_1\text{-Cap}(X - \text{q-supp}\,[\nu])=0\}$. For $\nu \in \mathcal{M}_0$, there exists a unique (up to an $\mathcal{E}_1$-polar set) positive continuous additive functional (abbreviated to

PCAF) A_t of $\boldsymbol{M}$ characterized by

$$\langle\nu, f\rangle=\lim_{t\to 0}\frac{1}{t}E_m[\int_0^t f(X_s)\,dA_s]\,,\quad f\in\mathcal{B}^+(X)\,,$$

where $\mathcal{B}^+(X)$ denotes the family of all non-negative Borel functions on X and $\langle\nu, f\rangle$ stands for $\int_X f(x)\,\nu(dx)$. E_γ denotes integration by $P_\gamma(d\omega)=\int_X P_x(d\omega)$ $\gamma(dx)$ for a Borel measure γ on X. ν is called Revuz measure of A_t. We put $Y_A=\{x\in X-N_A;\ P_x(A_t>0 \text{ for any } t>0)=1\}$, where N_A is the defining exceptional set for A_t. Y_A is called the support of A_t. In [9], Fukushima and LeJan proved that the support of PCAF associated with $\nu\in\mathcal{M}_0$ is a quasi-support of ν.

A set $B\subset X_\Delta$ is called nearly Borel measurable if for any porbability measure ν on X_Δ there exist Borel sets B_1, $B_2\subset X_\Delta$ with $B_1\subset B\subset B_2$ such that $P_\nu(X_t\in B_2-B_1$ for some $t\geqq 0)=0$. A set $E\subset X$ is called finely open if for each $x\in E$ there exists nearly Borel set $B=B(x)$ with $X-E\subset B\subset X$ such that $P_x(\sigma_B>0)=1$. Here $\sigma_B=\inf\{t>0;\ X_t\in B\}$. A set F is finely closed if $X-F$ is finely open. For a set A we denote $A^r=\{x\in X;\ P_x(\sigma_A=0)=1\}$ the regular set for A. A nearly Borel set F is finely closed if and only if $F^r\subset F$. We say that a set E is q.e. finely open (resp. q.e. finely closed) if there exists a finely open (resp. finely closed) nearly Borel set $\tilde{E}$ with $E=\tilde{E}$ q.e. A function u: $X\to[-, \infty, \infty]$ is called finely continuous q.e. if there exists an $\mathcal{E}_1$-polar finely closed set N such that u is finely continuous and nearly Borel measurable on $X-N$. A set N is called properly exceptional if N is m-negligible Borel set and $X-N$ is $\boldsymbol{M}$-invariant. A function u: $X\to[-\infty, \infty]$ is finely continuous q.e. if and only if there exists a properly exceptional set $\tilde{N}$ such that u is finely continuous an Borel measurable on $X-\tilde{N}$ (Lemma 4.2.6 in [6]). We collect generalizations of some assertions in [6].

Lemma 2.1. (i) *For a quasi-open set E and a quasi-continuous function* u: $X\to[-\infty, \infty]$,

$$u\geqq 0\quad m\text{-a.e. on } E\quad \text{if and only if}\quad u\geqq 0\quad \text{q.e. on } E\,.$$

(ii) *For a quasi-open set E,*

$$\mathcal{E}_1\text{-Cap}(E)=\inf_{u\in\mathcal{L}_E}\mathcal{E}_1(u, u),\quad \text{where}\quad \mathcal{L}_E=\{u\in\mathcal{F};\ u\geqq 1\ m\text{-a,e. on } E\}\,.$$

(iii) *A quasi-open m-negrigible set E is* $\mathcal{E}_1$*-polar.*

Proof. (i) The "if" part is trivial. We show the "only if" part. Let $\{\tilde{F}_k\}$ and $\{F'_k\}$ be nests such that $E\cap\tilde{F}_k$ is open in $\tilde{F}_k$ and $u|_{F'_k}$ is continuous. We put $F_k=\text{supp}[m|_{\tilde{F}_k\cap F'_k}]$. Then $\{F_k\}$ is an m-regular nest, namely $m(U(x)\cap F_k)>0$, for any $x\in F_k$ and any open neighbourhood $U(x)$ of x. The rest of the proof is the same as in Lemma 3.1.3 in [6].

(ii) By (i) and Theorem 3.3.1 in [6], (ii) is clear in case $\mathcal{E}_1$-Cap$(E)<\infty$.

We show that $\mathcal{E}_1$-Cap$(E)=\infty$ implies $\mathcal{L}_E=\phi$. Suppose $\mathcal{L}_E\neq\phi$ and $\mathcal{E}_1$-Cap$(E)=\infty$. Then there exists unique element $e_E\in\mathcal{L}_E$ which attains the infimum. Let $\{G_n\}$ be an increasing sequence of relatively compact open sets such that $X=\bigcup_{n=1}^{\infty}G_n$. Then there exists unique element $e_{E\cap G_n}\in\mathcal{L}_{E\cap G_n}$ satisfying $\mathcal{E}_1$-Cap$(E\cap G_n)=\mathcal{E}_1(e_{E\cap G_n}, e_{E\cap G_n})$, because $\mathcal{E}_1$-Cap$(E\cap G_n)<\mathcal{E}_1$-Cap$(G_n)<\infty$. Since $\mathcal{E}_1$-Cap is a Choquet capacity, $\mathcal{E}_1(e_{E\cap G_n}, e_{E\cap G_n})\nearrow\mathcal{E}_1$-Cap$(E)=\infty$ as $n\to\infty$. On the other hand $\mathcal{E}_1(e_E, e_E)=\mathcal{E}_1(e_E-e_{E\cap G_n}, e_E-e_{E\cap G_n})+\mathcal{E}_1(e_{E\cap G_n}, e_{E\cap G_n})$, because $\mathcal{E}_1(e_{E\cap G_n}, v)=\mathcal{E}_1(e_{E\cap G_n}, e_{E\cap G_n})$ for any $v\in\mathcal{F}$, $\tilde{v}=1$ q.e. on $E\cap G_n$, where $\tilde{v}$ is an m-a.e. quasi-continuous version of v. This is a contradiction. (iii) is a trivial consequence of (ii). The proof is complete.

Theorem 2.2. (i) *A set E is quasi-open if and only if E is q.e. finely open.*
(ii) *A function u: $X\to[-\infty,\infty]$ is quasi-continuous if and only if u is finely continuous q.e.*

Proof. By Theorem 4.3.2 in [6], (ii) follows from (i). We show (i). Suppose that E is quasi-open and $\{F_n\}$ is a nest such that $E\cap F_n$ is open in F_n for each n. There exists a properly exceptional set $N\supset\bigcap_{n=1}^{\infty}(X-F_n)$ satisfying

$$P_x(\lim_{n\to\infty}\sigma_{Y-F_n}=\infty)=1 \quad \text{for any} \quad x\in X-N\,,$$

by (4.3.5) in [6], $E-N$ is then finely open and Borel measurable. Conversly suppose E is q.e. finely open. Then there exists a finely open and nearly Borel set $\tilde{E}$ with $E=\tilde{E}$ q.e.. For a strictly positive bounded $f\in L^2(X;m)$, we put

$$v(x)=E_x[\int_0^{\sigma_{X-\tilde{E}}}e^{-t}f(X_t)\,dt]\,.$$

Then $v\in\mathcal{F}$ and quasi-continuous by Theorem 4.3.2 in [6]. Further $v>0$ on $\tilde{E}$ and $v=0$ q.e. on $X-\tilde{E}$. Hence we get $\tilde{E}=v^{-1}(0,\infty))$ q.e. which implies that E is quasi-open. The proof is complete.

A universally measurable function h: $X\to[0,\infty]$ is said to be α-excessive if $e^{-\alpha t}p_t\,h(x)\nearrow h(x)$, $t\searrow 0$, $x\in X\,(\alpha\geqq 0)$. It is known that α-excessive function $(\alpha\geqq 0)$ is nearly Borel measurable and finely continuous.

Corollary 2.3. *For each $\alpha\geqq 0$, α-excessive function is quasi-continuous.*

3. Ergodic decomposition into M-invariant sets

As in Section 2, $(\mathcal{E},\mathcal{F})$ is a C_0-regular Dirichlet space possessing $\mathcal{C}$ as its core. Let $\{T_t, t\geqq 0\}$ be the L^2-semigroup associated with $(\mathcal{E},\mathcal{F})$. In this section we give a relation of T_t-invariant set and $\boldsymbol{M}$-invarinat set.

Lemma 3.1. *If a nearly Borel set B is T_t-invariant and simultaneously* *quasi-open and quasi-closed, then there exists a properly exceptional set N such that both $B-N$ and $X-B-N$ are $\boldsymbol{M}$-invariant and quasi-open.*

Proof. Denote by p_t the transition kernel of $\boldsymbol{M}$. Since I_B is a quasi-continuous function, we get

$$p_t I_B u = I_B p_t u \text{ q.e. for } u \in \mathcal{B}^+(X) \cap L^2(X; m) \text{ for each } t>0 ,$$

where $\mathcal{B}^+(X)$ is the family of positive Borel functions on X. Approximating 1 by $h_n \in \mathcal{B}^+(X) \cap L^2(X; m)$ with $h_n \nearrow 1$, we have

$$p_t I_B = I_B p_t 1 \text{ q.e. for each } t>0 ,$$

or equivalently

$$p_t I_B = 0 \text{ q.e. on } X-B \text{ and } p_t I_{X-B}=0 \text{ q.e. on } B \text{ for each } t>0 .$$

Since I_B is quasi-continuous, the map $t \mapsto I_B(X_t)$ is right continuous and has left limit $I_B(X_{t-})$ P_x-a.s. for q.e. $x \in X$. Thus we have

$$\text{(3.1)} \quad P_x(X_t \in B_\Delta \text{ for any } t \geqq 0,\ X_{t-} \in B_\Delta \text{ for any } t>0) = 1 \text{ q.e. } x \in B .$$

Similarly

$$\text{(3.2)} \quad \begin{aligned} P_x(X_t \in (X-B)_\Delta & \text{ for any } t \geqq 0,\ X_{t-} \in (X-B)_\Delta \\ & \text{for any } t>0) = 1 \text{ q.e. } x \in X-B . \end{aligned}$$

By Theorem 4.2.1 in [6] there exists an appropriate properly exceptional set N such that $B_1=B-N$ and $B_2=X-B-N$ are $\boldsymbol{M}$-invariant. Since quasi-notions are invariant under q.e. equivalence, B_1 and B_2 are also quasi-open and quasi-closed sets. The proof is complete.

The next Corollary was proven in [7] under the local property.

Corollary 3.2. *A Borel set B is T_1-invariant if and only if there exists a quasi-open and quasi-closed set B_1 (resp. B_2) which is an $\boldsymbol{M}$-invariant m-a.e. version of B (resp. $X-B$) and a properly exceptional set N such that $X=B_1+B_2+N$.*

Proof. The "if" part is trivial. We only show the "only if" part. Suppose that B is T_t-invariant. Then there exists an m-a.e. version $\tilde{B}$ of B such that $I_{\tilde{B}}$ is quasi-continuous (implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(vi)$\Rightarrow$(v) of Theorem 2 in [7]). Since $\tilde{B}$ is also T_t-invariant, we have the assertion by Lemma 3.1. The proof is complete.

For a strictly positive $f \in L^1X; m)$, the sets $C_f=\{x \in X; Gf(x)=\infty\}$ and $D_f=\{x \in X; Gf(x)<\infty\}$ are T_t-invariant (Theorem 1.5.8 in [14]). Here $Gf=\int_0^\infty T_t f\,dt$. Further C_f and D_f are indepdndent of the choice of f up to m-negligible sets. Hence by Corollary 3.2 the whole space X admits a decomposition $X=X^{(c)}+X^{(d)}+N$, where $X^{(c)}$ (resp. $X^{(d)}$) is an $\boldsymbol{M}$-invariant m-a.e. version of C_f (resp. D_f) and N is a properly exceptional set.

Lemma 3.3. *Let h be an excessive function. Then $p_t h=h$ q.e. on $X^{(c)}$ for each $t>0$.*

Proof. This lemma follows from Corollary 2.3 and Theorem 1 in [1]. For the convenience of readers we give a direct proof. Suppose that $f\in L^1(X;m)$ is m-a.e. strictly positive on X. Then $Rf(x)=E_x[\int_0^\infty f(X_t)\,dt]=\infty$ q.e. on $X^{(c)}$ by Lemma 2.1 (i) and Corollary 2.3. Put $h_n=h\wedge n$. Then h_n is an excessive function. By resolvent equation $R_p h_n-R_q h_n+(p-q)R_p R_q h_n=0$, we get

$$\begin{aligned}(h_n-qR_q h_n, Rf)_{m|X^{(c)}} &\leqq \lim_{p\searrow 0}(h_n-qR_q h_n, R_p f)_m\\ &= \lim_{p\searrow 0}(h_n-pR_p h_n, R_q f)_m\\ &\leqq (h_n, R_q f)_m\\ &\leqq \frac{n}{q}\langle m,f\rangle<\infty\,.\end{aligned}$$

Hence we have $qR_q h_n=h_n$ q.e. on $X^{(c)}$. Letting $n\to\infty$, we have $qR_q h=h$ q.e. on $X^{(c)}$. The proof is complete.

We say that the Dirichlet space $(\mathcal{E},\mathcal{F})$ is transient if there eixsts a bounded $g\in L^1(X;m)$ with $g>0$ m-a.e. such that $Gg<\infty$ m-a.e. and $(\mathcal{E},\mathcal{F})$ is recurrent if it is non-transient and irreducible ([8], [14]). The restricted process $\boldsymbol{M}|_{X^{(d)}}$ (resp. $\boldsymbol{M}|_{X^{(c)}}$) is transient (resp. conservative). $(\mathcal{E},\mathcal{F})$ is transient if and only if $m(X^{(c)})=0$. If $(\mathcal{E},\mathcal{F})$ is irreducible then $m(X^{(c)})=0$ or $m(X^{(d)})=0$, namely $(\mathcal{E},\mathcal{F})$ is transinet or recurrent. $X^{(c)}$ (resp. $X^{(d)}$) is called the conservative (resp. dissipative) part of $\boldsymbol{M}$ ([1], [3], [5], [11]).

Without loss of generality, we shall assume that the space $\mathcal{F}$ consists of $\mathcal{E}_1$-quasi-continuous functions, two functions which equal $\mathcal{E}_1$-q.e. being identified. For each non-trivial $\nu\in\mathcal{M}_0$, the symmetric form $(\mathcal{E}^\nu,\mathcal{F}^\nu)$ on $L^2(X;m)$ defined by

$$(3.3)\qquad \begin{cases}\mathcal{F}^\nu=\mathcal{F}\cap L^2(X;\nu)\,,\\ \mathcal{E}^\nu(u,v)=\mathcal{E}(u,v)+(u,v)_\nu\end{cases}$$

is a C_0-regular Dirichlet form having C as its core (see the proof of Lemma 3.1 in [10]). $(\mathcal{E}^\nu,\mathcal{F}^\nu)$ is called ν-killed Dirichlet space. Denote by $\boldsymbol{M}^\nu$ the m-symmetric Hunt process associated with $(\mathcal{E}^\nu,\mathcal{F}^\nu)$. Let A_t^ν be the PCAF associated with ν. The set $C_f^\nu=\{x\in X;\ E_x[\int_0^\infty e^{-A_t^\nu}f(X_t)\,dt]=\infty\}$ and $D_f^\nu=\{x\in X;\ E_x[\int_0^\infty e^{-A_t^\nu}f/X_t)\,dt]<\infty\}$ are T_t^ν-invariant set for $f\in L^1(X;m), f>0$ m-

a.e. on X. Since $\mathcal{E}_1^{\nu}$-Cap is equivalent to $\mathcal{E}_1$-Cap (Lemma 2.3 in [12]), we can denote by $X^{\nu(c)}$, $X^{\nu(d)}$ the $\boldsymbol{M}^{\nu}$-invariant $\mathcal{E}_1$-quasi-open and $\mathcal{E}_1$-quasi-closed m-a.e. version of C_f^{ν}, D_f^{ν} respectively. Put $B^{\nu}=\{x\in X; P_x(A_{\infty}^{\nu}>0)>0\}$.

Proposition 3.4. (i) *For $\nu\in\mathcal{M}_0$, $X^{\nu(c)}\subset B^{\nu}$ q.e. if and only if the ν-killed Dirichlet space $(\mathcal{E}^{\nu}, \mathcal{F}^{\nu})$ on $L^2(X; m)$ is transient.*

(ii) *In the above case the ν-killed extended Dirichlet space $\mathcal{F}_e^{\nu}$ is complete by $\mathcal{E}^{\nu}$-norm. $\mathcal{E}^{\nu}$-capacity is equivalent to $\mathcal{E}_1$-capacity.*

Proof. The proof of (ii) is the same as in Lemma 2.3 in [12]. We show (i). The "if" part is trivial. We show the "only if" part. Applying Lemma 3.3 to $\boldsymbol{M}^{\nu}$ with $h=1$, we have

$$E_x[e^{-A_t^{\nu}}] = 1 \text{ q.e. } x\in X^{\nu(c)} \text{ for each } t>0\,,$$

namely

$$P_x(A_{\infty}^{\nu} = 0) = 1 \text{ q.e. } x\in X^{\nu(c)}\,. \tag{3.4}$$

which, combined with the assumtion $X^{\nu(c)}\subset B^{\nu}$ q.e., implies $X^{\nu(c)}=\phi$ q.e.. The proof is complete.

Corollary 3.5. *If $(\mathcal{E}, \mathcal{F})$ is irreducible, $(\mathcal{E}^{\nu}, \mathcal{F}^{\nu})$ is transient.*

Proof. Suppose $(\mathcal{E}, \mathcal{F})$ is irreducible. Then $(\mathcal{E}^{\nu}, \mathcal{F}^{\nu})$ is irreducible. Hence $X^{\nu(c)}=\phi$ q.e. or $X^{\nu(c)}=X$ q.e.. Suppose $X^{\nu(c)}=X$ q.e.. Then by (3.4)

$$P_x(A_{\infty}^{\nu} = 0) = 1 \text{ q.e. } x\,,$$

which contradicts the non-triviality of ν. The proof is complete.

Corollary 3.6. *If $\nu\in\mathcal{M}_{00}$, $(\mathcal{E}^{\nu}, \mathcal{F}^{\nu})$ is transient.*

Proof. By Corollary 3.5 in [9], q-supp$[\nu]\subset B^{\nu}$ q.e.. Hence $B^{\nu}=X$ q.e.. The proof is complete.

4. Time changed Dirichlet space

In this section we give a characterization of the time changed Dirichlet space without irreducibility as in Fitzsimmons [2]. Fix $\mu\in\mathcal{M}_0$. Let A_t^{μ} be the associated PCAF with μ. Put $B^{\mu}=\{x\in X; P_x(A_{\infty}^{\mu}>0)>0\}$ and $\tilde{Y}=$q-supp $[\mu]$.

Lemma 4.1. *B^{μ} and $X-B^{\mu}$ have $\boldsymbol{M}$-invariant q.e. versions.*

Proof. It is easy to see that the function $u(x)=P_x(A_{\infty}^{\mu}>0)$ is excessive and hence B^{μ} is finely open and nearly Borel. Put $B_n^{\mu}=\{x\in X; P_x(A_{\infty}^{\mu}>0)\geq\frac{1}{n}\}$. Then B_n^{μ} is a finely closed and nearly Borel set. For each n and $x\in X-B^{\mu}$, we have

$$\begin{aligned}1 &= P_x(A^\mu_\infty = 0)\\ &= P_x(A^\mu_\infty = 0;\ \sigma_{B^\mu_n} < \infty) + P_x(A^\mu_\infty = 0;\ \sigma_{B^\mu_n} = \infty)\\ &= P_x(A^\mu_\infty(\theta_{\sigma_{B^\mu_n}}) = 0;\ \sigma_{B^\mu_n} < \infty) + P_x(A^\mu_\infty = 0;\ \sigma_{B^\mu_n} = \infty)\\ &= E_x[P_{X_{\sigma_{B^\mu_n}}}(A^\mu_\infty = 0;\ \sigma_{B^\mu_n} < \infty] + P_x(A^\mu_\infty = 0;\ \sigma_{B^\mu_n} = \infty)\\ &\leqq (1-\frac{1}{n})\, P_x(\sigma_{B^\mu_n} < \infty) + P_x(\sigma_{B^\mu_n} = \infty)\\ &= 1 - \frac{1}{n}\, P_x(\sigma_{B^\mu_n} < \infty)\,.\end{aligned}$$

Letting $n \nearrow \infty$, we get $P_x(\sigma_{B^\mu} < \infty) = 0$ for any $x \in X - B^\mu$. In particular, $X - B^\mu$ is T_t-invariant and finely open. Since B^μ is also finely open, we can find by Theorme 2.2 and Lemma 3.1 a properly exceptional set N such that $X - B^\mu - N$ and $B^\mu - N$ are $\boldsymbol{M}$-invariant. The proof is complete.

By the above lemma we may asume that $X^{(c)} - B^\mu$ and $X^{(c)} \cap B^\mu$ are $\boldsymbol{M}$-invariant. For each $\alpha > 0$, we let $\nu_\alpha = \alpha\mu + I_{X^{(c)}-B^\mu}\, m$. Then $\nu_\alpha \in \mathcal{M}_0$, $A^{\nu_\alpha}_t = \alpha A^\mu_t + \int_0^t I_{X^{(c)}-B^\mu}(X_s)\, ds$ and $X^{\nu_\alpha(c)} \subset X^{(c)} \subset B^{\nu_\alpha}$ q.e.. By Proposition 3.4, we see that the extended Dirichlet space $(\mathcal{E}^{\nu_\alpha}, \mathcal{F}^{\nu_\alpha}_e)$ can be defined as the $\mathcal{E}^{\nu_\alpha}$-completion of $\mathcal{F}^{\nu_\alpha}$ and that $\mathcal{E}^{\nu_\alpha}$-capacity is equivalent to $\mathcal{E}_1$-capacity. Note that the spaces $\mathcal{F}^{\nu_\alpha}$ and $\mathcal{F}^{\nu_\alpha}_e$ is independent of $\alpha > 0$. We denote $\mathcal{F}^\nu$ (resp. $\mathcal{F}^\nu_e$) instead of $\mathcal{F}^{\nu_\alpha}$ (resp. $\mathcal{F}^{\nu_\alpha}_e$). Without loss of generality, we shall assume that every element of $\mathcal{F}^\nu_e$ is $\mathcal{E}_1$-quasi-continuous. We let $\mathcal{F}^\nu_{eX-\tilde{Y}} = \{u \in \mathcal{F}^\nu_e;\ u = 0 \text{ q.e. on } \tilde{Y}\}$. This is a closed subspace of $\mathcal{F}^\nu_e$ and the Hilbert space $(\mathcal{E}^{\nu_\alpha}, \mathcal{F}^\nu_e)$ admits the orthogonal decomposition

$$\mathcal{F}^\nu_e = \mathcal{F}^\nu_{eX-\tilde{X}} \oplus \mathcal{H}^{\nu_\alpha}_{\tilde{Y}}\,,$$

where $\mathcal{H}^{\nu_\alpha}_{\tilde{Y}}$ is the orthogonal complement of $\mathcal{F}^\nu_{eX-\tilde{Y}}$ with respect to $\mathcal{E}^{\nu_\alpha}$. Denote by $\mathcal{P}^{\nu_\alpha}$ the orthogonal projection on $\mathcal{H}^{\nu_\alpha}_{\tilde{Y}}$. Note that the space $\mathcal{H}^{\nu_\alpha}_{\tilde{Y}}$ and the projection $\mathcal{P}^{\nu_\alpha}$ are independent of $\alpha > 0$. Indeed for any $u \in \mathcal{H}^{\nu_\alpha}_{\tilde{Y}}$ and $\beta > 0$,

$$\mathcal{E}^{\nu_\beta}(u, v) = \mathcal{E}^{\nu_\alpha}(u, v) + (\beta - \alpha)\,(u, v)_\mu = 0\,, \quad v \in \mathcal{F}^\nu_{eX-\tilde{Y}}\,,$$

because $\mu(X - \tilde{Y}) = 0$ ([6]). Hence $u \in \mathcal{H}^{\nu_\beta}_{\tilde{Y}}$. Conseqeuntly $\mathcal{P}^{\nu_\alpha}$ is also independent of $\alpha > 0$. We may omit the index α from ν_α. We notice that, for $f, g \in \mathcal{F}^\nu_e$, $\mathcal{P}^\nu f = \mathcal{P}^\nu g$ if and only if $f = g$ q.e. on $\tilde{Y}$.

We assume that μ is non-trivial. Put $Y = \text{supp}\,[\mu]$. Define a symmetric bilinear form on $L^2(Y;\mu)$ by

$$(4.1)\qquad \begin{cases} \mathcal{F}^\mu_Y = \{u \in L^2(Y;\mu);\ u = v|_Y\ \mu\text{-a.e. on } Y \text{ for some } v \in \mathcal{F}^\nu_e\} \\ \mathcal{E}^\mu_Y(u, u) = \mathcal{E}^{\nu-\alpha\mu}(\mathcal{P}^\nu v, \mathcal{P}^\nu v), \text{ for } u \in \mathcal{F}^\mu_Y, v \in \mathcal{F}^\nu_e\ s.t. u = v|_Y\ \mu\text{-a.e.}\,, \end{cases}$$

where $v|_Y$ is the restriction of function v to Y and $\mathcal{E}^{\nu-\alpha\mu}(v, v)=\mathcal{E}^{\nu^\alpha}(v, v)-(v, v)_{\alpha\mu}$ for $v\in\mathcal{F}_e^\nu$. $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ is a well defined closed symmetric form on $L^2(Y; \mu)$.

Theorem 4.2. *$(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ is the Dirichlet space on $L^2(Y; \mu)$ associated with the time changed process $\boldsymbol{M}^t=(X_{\tau_t}, P_x)_{x\in\tilde{Y}}$. Here $\tau_t=inf\{s>0; A_s^\mu>t\}$. $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$ is C_0-regular and has the core $\mathcal{C}|_Y=\{u\in C_0(Y); for\ some\ v\in\mathcal{C}, u=v|_Y\}$.*

Proof. First we show that $\mathcal{C}|_Y$ is a core of $(\mathcal{E}_Y^\mu, \mathcal{F}_Y^\mu)$. For $u\in\mathcal{F}_Y^\mu$, there exists $v\in\mathcal{F}_e^\nu$ such that $u=v|_Y$ μ.a.e.. Since $\mathcal{C}$ is a core of $(\mathcal{E}^\nu, \mathcal{F}_e^\nu)$, there exists $\{v_n\}\subset\mathcal{C}$ such that $\lim_{n\to\infty}\mathcal{E}^\nu(v_n-v, v_n-v)=0$. By (4.1) we get

$$\lim_{n\to\infty}\mathcal{E}_{Y\alpha}^\mu(u-v_n|_Y, u-v_n|_Y) = \lim_{n\to\infty}\mathcal{E}^\nu(\mathcal{P}^\nu(v-v_n), \mathcal{P}^\nu(v-v_n))$$
$$\leqq \lim_{n\to\infty}\mathcal{E}^\nu(v-v_n, v-v_n) = 0 .$$

For $u\in C_0(Y)$, there exists $w\in C_0(X)$ such that $u=w|_Y$. Since w is uniformly approximated by an element of $\mathcal{C}$, u is uniformly approximated by an element of $\mathcal{C}|_Y$.

Next we show that, for $u\in\mathcal{B}_b(Y)\cap L^2(Y; \mu)$ and $v\in\mathcal{F}_Y^\mu$,

$$\begin{cases} \check{R}_\alpha u\in\mathcal{F}_Y^\mu \\ \mathcal{E}_{Y\alpha}^\mu(\check{R}_\alpha u, v) = (u, v)_\mu , \end{cases} \tag{4.2}$$

where $\check{R}_\alpha u(x)=E_x[\int_0^\infty e^{-\alpha A_t^\mu} u(X_t)\, dA_t^\mu]$, $x\in\tilde{Y}$, is the reslovent kernel for $\boldsymbol{M}^t$. We introduce the kernel V_α on X by

$$V_\alpha f(x) = E_x[\int_0^\infty e^{-\alpha A_t^\mu} f(X_t)\, dA_t^\mu],\ x\in X, f\in\mathcal{B}_b(X) . \tag{4.3}$$

Take now $u\in\mathcal{B}_b(Y)\cap L^2(Y; \mu)$ and let $\bar{u}$ be any bounded Borel extension of u to X. Then $\check{R}_\alpha u=V_\alpha\bar{u}|_{\tilde{Y}}$. Applying Theorem 2.4 and Corollary 2.7 in [12] to A_t^ν and A_t^μ, $E_x[\int_0^\infty e^{-A_t^\nu}\bar{u}(X_t)\, dA_t^\mu]$, $x\in X$ is seen to be a quasi-continuous version of 0-order potential $U^\nu(\bar{u}\mu)$ with respect to $(\mathcal{E}^\nu, \mathcal{F}^\nu)$. Note that only the transience of $(\mathcal{E}^\nu, \mathcal{F}^\nu)$ is used and the irreducibility condition is irrelevant in the proof of Theorem 2.4 and Corollary 2.7 in [12]. By Lemma 4.1 and the identity $P_x(A_t^\mu=0$, for any $t>0)=1$ q.e. $x\in X-B^\mu$, we conclude that $V_\alpha\bar{u}$ is a quasi-continuous version of $U^\nu(\bar{u}\mu)$, and accordingly $\check{R}_\alpha u\in\mathcal{F}_Y^\mu$ and moreover $V_\alpha\bar{u}=\mathcal{P}^\nu V_\alpha\bar{u}\in\mathcal{H}_{\tilde{Y}}^\nu$. Let $\bar{v}$ be an element of $\mathcal{F}_e^\nu$ such that $v=\bar{v}|_Y$ μ-a.e.. Noting that $\mathcal{P}^\nu f=f$ μ-a.e. on Y for each $f\in\mathcal{F}_e^\nu$, we have

$$\begin{aligned}\mathcal{E}_{Y\alpha}^\mu(\check{R}_\alpha u, v) &= \mathcal{E}_Y^\mu(\check{R}_\alpha u, v)+\alpha(\check{R}_\alpha u, v)_\mu \\ &= \mathcal{E}^{\nu-\alpha\mu}(V_\alpha\bar{u}, \mathcal{P}^\nu\bar{v})+\alpha(\mathcal{P}^\nu V_\alpha\bar{u}, \mathcal{P}^\nu\bar{v})_\mu \\ &= \mathcal{E}^\nu(V_\alpha\bar{u}, \mathcal{P}^\nu\bar{v}) = \mathcal{E}^\nu(U^\nu(\bar{u}\mu), \mathcal{P}^\nu\bar{v})\end{aligned}$$

$$= (\tilde{u}, \mathcal{P}^{\nu} \tilde{v})_{\mu} = (u, v)_{\mu} .$$

The proof is complete.

For each $u \in \mathcal{B}_{+}(X)$, we denote $H_{\tilde{Y}}\, u(x) = E_x[u(X\sigma_{\tilde{Y}})]$.

Corollary 4.3. *$H_{\tilde{Y}}\tilde{v}$ is a quasi-continuous version of $\mathcal{P}^{\nu} v$ for each $v \in \mathcal{F}^{\nu}_{e}$ and the time changed Dirichlet space $(\mathcal{F}^{\mu}_{Y}, \mathcal{E}^{\mu}_{Y})$ is given by*

$$\begin{cases} \mathcal{F}^{\mu}_{Y} = \{u \in L^2(Y; \mu); u = v|_Y \ \mu\text{-a.e. on } Y \text{ for some } v \in \mathcal{F}^{\nu}_{e}\} \\ \mathcal{E}^{\mu}_{Y}(u, u) = \mathcal{E}(H_{\tilde{Y}}v, H_{\tilde{Y}}v), \text{ for } u \in \mathcal{E}^{\mu}_{Y}, v \in \mathcal{F}^{\nu}_{e} \text{ s.t. } u = v|_Y \ \mu\text{-a.e..} \end{cases}$$

Proof. Since $\tilde{Y} \subset B^{\mu}$ q.e., we get $H_{\tilde{Y}}v(x) = E_x[v(X_{\infty})] = 0$ q.e. $x \in X - B^{\mu}$. Therefore the latter assertion holds. Next we show the first assertion. We may assume that $v \in \mathcal{F}^{\nu}_{e}$ is non-negative. Put $v_n = v \wedge n$. Noting that $\sigma_{\tilde{Y}}(\omega) = \inf \{t > 0; A^{\mu}_{t}(\omega) > 0\}$, we get from (4.3)

$$H_{\tilde{Y}}v_n(x) = \lim_{m \to \infty} mV_m\, v_n(x) .$$

On the other hand $mV_m\, v_n = \mathcal{P}^{\nu}\, mV_m\, v_n$ is $\mathcal{E}^{\nu}$-convergent to $\mathcal{P}^{\nu}\, v_n \in \mathcal{F}^{\nu}_{e}$ as $m \to \infty$ because $m\tilde{R}_m(v_n|_Y)$ is $\mathcal{E}^{\mu}_{Y\alpha}$-convergent to $v_n|_Y \in \mathcal{F}^{\mu}_{Y}$ as $m \to \infty$. We get $H_{\tilde{Y}}v_n = \mathcal{P}^{\nu} v_n$ q.e.. Since $\mathcal{P}^{\nu} v_n$ is $\mathcal{E}^{\nu}$-convergent to $\mathcal{P}^{\nu} v \in \mathcal{H}^{\nu}_{\tilde{Y}}$ as $n \to \infty$, we have

$$\begin{aligned} H_{\tilde{Y}}\, v &= \lim_{n \to \infty} H_{\tilde{Y}}\, v_n \\ &= \lim_{n \to \infty} \mathcal{P}^{\nu} v_n = \mathcal{P}^{\nu} v \text{ q.e..} \end{aligned}$$

The proof is complete.

By Theorem 4.2 we can get next result in the similar manner as in Section 4 in [12].

Theorem 4.4. (i) *For a Borel set $B \subset Y$,*

$$\mathcal{E}^{\mu}_{Y\alpha}\text{-}Cap(B \cap \tilde{Y}) = 0 \text{ if and only if } \mathcal{E}_1\text{-}Cap(B \cap \tilde{Y}) = 0 .$$

(ii) *For any decreasing sequence of open sets A_n, $\mathcal{E}_1$-$Cap(A_n) \searrow 0$ implies $\mathcal{E}^{\mu}_{Y\alpha}$-$Cap(A_n \cap Y) \searrow 0$. In case $\mu \in \mathcal{M}_{00}$ $\mathcal{E}_1$-Cap is equivalent to $\mathcal{E}^{\mu}_{Y\alpha}$-Cap.*

(iii) *$\mathcal{E}^{\mu}_{Y1}$-$Cap(Y - \tilde{Y}) = 0$.*

(iii) *There exists a Borel set $\tilde{N}$ with $\mu(\tilde{N}) = 0$ such that $Y - \tilde{Y} \subset \tilde{N}$ and $\tilde{Y} - \tilde{N}$ is M^t-invariant. And further the restricted process $M^t|_{\tilde{Y}-\tilde{N}}$ of the time changed process M^t is a Hunt process on $\tilde{Y}$-$\tilde{N}$ associated with the rgeular Dirichlet space $(\mathcal{E}^{\mu}_{Y}, \mathcal{F}^{\mu}_{Y})$.*

5. Closable part of a pre-Dirichlet form on $C|_Y$

A non-negative definite symmetric bilinear form $\mathcal{A}$ on C is called a pre-

Dirichlet form if there exists a function φ_ε satisfying condition $(\mathcal{C}.2)$ and $\mathcal{A}(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) \leqq \mathcal{A}(u, u)$ for any $u \in \mathcal{C}$. For a closed set Y, $\mathcal{C}|_Y = \{u \in \mathcal{C}_0(Y); u = \bar{u}|_Y$ for some $\bar{u} \in \mathcal{C}\}$ satisfies $(\mathcal{C}.2)$ and $(\mathcal{C}.1)$ with respect to the relative topology on Y. A pre-Dirichlet form $(\mathcal{A}, \mathcal{C}|_Y)$ is said to be closable on $L^2(Y; \mu)$ for a positive Radon measure μ on Y with $Y = \mathrm{supp}[\mu]$ if $\mathcal{A}(u_n, u_n) \to 0$, $n \to \infty$ whenever $\{u_n\} \subset \mathcal{C}|_Y$ is $\mathcal{A}$-Cauchy and $u_n \to 0$ in $L^2(Y; \mu)$. A pre-Dirichlet form $(\mathcal{A}^0, \mathcal{C}|_Y)$ is said to be the closable part of $(\mathcal{A}, \mathcal{C}|_Y)$ on $L^2(Y; \mu)$ if $(\mathcal{A}^0, \mathcal{C}|_Y)$ is closable on $L^2(Y; \mu)$ and $\mathcal{A}^0(u, u) \leqq \mathcal{A}(u, u)$, $u \in \mathcal{C}|_Y$, and $\mathcal{B}(u, u) \leqq \mathcal{A}^0(u, u)$, $u \in \mathcal{C}|_Y$ for any other pre-Dirichlet form $(\mathcal{B}, \mathcal{C}|_Y)$ which is closable on $L^2(Y; \mu)$ and satisfies $\mathcal{B}(u, u) \leqq \mathcal{A}(u, u)$, $u \in \mathcal{C}|_Y$. In this section we study the closable part of a pre-Dirichlet form on $\mathcal{C}|_Y$ when Y is the support of a measure $\mu \in \mathcal{M}$.

Let $(\mathcal{E}, \mathcal{F})$ ba a C_0-regular Dirichlet space as in Section 2. In general, a function u defined m-a.e. is said to belong to the extended Dirichlet space $\mathcal{F}_e$ if there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\} \subset \mathcal{F}$ such that $u_n \to u$, m-a.e. as $n \to \infty$. In this case we define $\mathcal{E}(u, u) = \lim_{n\to\infty} \mathcal{E}(u_n, u_n)$. $\mathcal{E}(u, u)$ does not depend on the choice of $\{u_n\}$ ([16]). It is easy to see that $u \in \mathcal{F}_e$ if and only if there exists an $\mathcal{E}$-Cauchy sequence $\{v_n\} \subset \mathcal{C}$ such that $v_n \to u$, m-a.e. as $n \to \infty$, and that $\mathcal{E}(u, u) = \lim_{n\to\infty} \mathcal{E}(v_n, v_n)$ in this case.

Lemma 5.1. (i) *$u \in \mathcal{F}_e$ has quasi-continuous version $\tilde{u}$.*

(ii) *Every normal contraction operates on $(\mathcal{F}_e, \mathcal{E})$.*

(iii) *For a Borel set B, let $H_B \tilde{u}(x) = E_x[\tilde{u}(X_{\sigma_B})]$. Then $H_B \tilde{u} \in \mathcal{F}_e$ for any $u \in \mathcal{F}_e$. Furthermore*

$$(5.1) \quad \mathcal{E}(u, v) = \mathcal{E}(H_B \tilde{u}, H_B \tilde{v}) + \mathcal{E}((I - H_B)\tilde{u}, (I - H_B)\tilde{v}), \text{ for any } u, v \in \mathcal{F}_e .$$

Proof. For each $g \in L^1(X; m)$ with $g > 0$ m-a.e., the finite measure gm belongs to $\mathcal{M}_{00}$. Hence the gm-killed Dirichlet space $(\mathcal{E}^{gm}, \mathcal{F}^{gm})$ is transient by Corollary 3.6. Denote by $\mathcal{F}_e^{gm}$ its extended Dirichlet space. By (4.1) the time changed Dirichlet space $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(X; gm)$ associated with the time changed process M^t by the PCAF $A_t^g = \int_0^t g(X_t)\, dt$ is given by

$$(5.2) \qquad \begin{cases} \tilde{\mathcal{F}} = \mathcal{F}_e^{gm} \\ \tilde{\mathcal{E}}(u, v) = \mathcal{E}(u, v), \quad \text{for any} \quad u, v \in \tilde{\mathcal{F}} \end{cases}$$

and $\mathcal{C}$ is a core of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. Now the extended Dirichlet space $\tilde{\mathcal{F}}_e$ of this time changed Dirichlet space coincides with $\mathcal{F}_e$. We therefore get $\mathcal{F}_e \cap L^2(X; gm) = \tilde{\mathcal{F}}_e \cap L^2(X; gm) = \tilde{\mathcal{F}} = \mathcal{F}_e^{gm}$ by [16]. For each $u \in \mathcal{F}_e$ choose $g \in L^1(X; m)$, $g > 0$ m-a.e. such that $u \in L^2(X; gm)$. Then $u \in \tilde{\mathcal{F}} = \mathcal{F}_e^{gm}$ with this choice of g. Thus (i) follows from C_0-regularity of $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ and (ii) follows from that every normal contraction operates on $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$.

Next we show (iii). For each Borel set B, we denote $\tilde{\mathcal{F}}_{X-B}=\{u\in\tilde{\mathcal{F}};\ \tilde{u}=0$ q.e. on $B\}$. Then $\tilde{\mathcal{F}}$ admits the orthogonal decomposition as follows: For each $p>0$,

$$\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{X-B}\oplus\tilde{\mathcal{H}}_B^p\ ,$$

where $\tilde{\mathcal{H}}_B^p$ is the orghogonal complement of $\tilde{\mathcal{F}}_{X-B}$ with respect to $\tilde{\mathcal{E}}_p=\tilde{\mathcal{E}}+p(\cdot,\cdot)_{gm}$. For each $u\in\mathcal{F}_e^{gm}$ we denote $H_B^p\,\tilde{u}(x)=E_x[e^{-pA_{\sigma_B}^g}\,\tilde{u}(X_{\sigma_B})]$. Letting $\boldsymbol{M}^t=(Y_t,P_x)$ and denoting by $\hat{\sigma}_B$ its hitting time, we see that $H_B^p\,\tilde{u}(x)=E_x[e^{-p\hat{\sigma}_B}\,\tilde{u}(Y_{\hat{\sigma}_B})]$ and hence $H_B^p\,\tilde{u}$ is the quasi-continuous version of $\boldsymbol{P}_{\tilde{\mathcal{H}}_B^p}\,u$, where $\boldsymbol{P}_{\tilde{\mathcal{H}}_B^p}$ is the projection to $\tilde{\mathcal{H}}_B^p$ ([6]). Hence we have

$$\tilde{\mathcal{E}}_p(u,v) = \tilde{\mathcal{E}}_p(H_B^p\,\tilde{u},H_B^p\,\tilde{v})+\tilde{\mathcal{E}}_p((I-H_B^p)\,\tilde{u},(I-H_B^p)\,\tilde{v}),\quad \text{for any}\quad u,v\in\mathcal{F}_e^{gm}\ .$$

Fix non-negative $u,v\in\mathcal{F}_e$. Choose $g\in L^1(X;m)$, $g>0$, m-a.e. such that $u,v\in\mathcal{F}_e^{gm}$. Consider the time changed Dirichlet space $(\tilde{\mathcal{E}},\tilde{\mathcal{F}})$ with this choice of g. Put $u_n=u\wedge n$, $v_n=v\wedge n$. Then u_n, $v_n\in\tilde{\mathcal{F}}$ and $u_n\to u$, $v_1\to v$, $n\to\infty$ in $\tilde{\mathcal{E}}_1$. Since $B-B^r$ is $\mathcal{E}_1$-polar, $H_B^p\,u_n-H_B^q\,u_n\in\tilde{\mathcal{F}}_{X-B}$. Hence we have

$$\begin{aligned}
&\tilde{\mathcal{E}}(H_B^p\,\tilde{u}_n-H_B^q\,\tilde{u}_n,H_B^p\,\tilde{u}_n-H_B^q\,\tilde{u}_n)\leqq\tilde{\mathcal{E}}_p(H_B^p\,\tilde{u}_n-H_B^q\,\tilde{u}_n,H_B^p\,\tilde{u}_n-H_B^q\,\tilde{u}_n)\\
&\quad=\tilde{\mathcal{E}}_p(H_B^p\,\tilde{u}_n,H_B^p\,\tilde{u}_n-H_B^q\,\tilde{u}_n)-\tilde{\mathcal{E}}_q(H_B^q\,\tilde{u}_n,H_B^p\,\tilde{u}_n-H_B^q\,\tilde{u}_n)\\
&\qquad+(p-q)\,(H_B^q\,\tilde{u}_n,H_B^p\,\tilde{u}_n-H_B^q\,\tilde{u}_n)_{pm}\\
&\quad=(q-p)\,(H_B^q\,\tilde{u}_n,H_B^p\,\tilde{u}_n-H_B^q\,\tilde{u}_n)_{gm}\to 0,\ p,q\to 0\ ,
\end{aligned}$$

namely, $H_B^p\,\tilde{u}_n$ is $\tilde{\mathcal{E}}_1$-Cauchy. We have $H_B\,\tilde{u}_n\in\tilde{\mathcal{F}}$ and

$$\tilde{\mathcal{E}}(u_n,v_n) = \tilde{\mathcal{E}}(H_B\,\tilde{u}_n,H_B\,\tilde{v}_n)+\tilde{\mathcal{E}}((I-H_B)\,\tilde{u}_n,(I-H_B)\,\tilde{v}_n)$$

Since u_n and v_n are $\tilde{\mathcal{E}}_1$-convergent to u, v as $n\to\infty$, we arrive at (5.1). The proof is complete.

For a finely closed Borel set F and a closed set Y with $F\subset Y\subset X$, we introduce a symmetric bilinear form $(\mathcal{A}_F,\mathcal{C}\,|_Y)$ by

$$\mathcal{A}_F(u,v) = \mathcal{E}(H_F\,\bar{u},H_F\,\bar{v})\ u,v\in\mathcal{C}\,|_Y,\ \bar{u},\bar{v}\in\mathcal{C},\ u=\bar{u}\,|_Y, v=\bar{v}\,|_Y\ .$$

Suppose $u_1,u_2\in\mathcal{C}$ and $u_1=u_2$ on Y. Then $H_F\,u_1(x)=E_x[u_1(X_{\sigma_F})]=E_x[u_2(X_{\sigma_F})]=H_F\,u_2(x)$. Hence $(\mathcal{A}_F,\mathcal{C}\,|_Y)$ is well-defined.

Lemma 5.2.

$$\mathcal{A}_F(u,u) = \inf\{\mathcal{E}(v,v);\ v\in\mathcal{F}_e, u=\tilde{v}\ q.e.\ on\ F\},\ u\in\mathcal{C}\,|_Y\ .$$

Proof. For each $u\in\mathcal{C}\,|_Y$, we take $v\in\mathcal{F}_e$ such that $u=\tilde{v}$ q.e. on F. Then there exists a properly exceptional set N auch that $u(x)=\tilde{v}(x)$ for $x\in F-N$. Since $F-N$ is again finely closed Borel set of $\boldsymbol{M}\,|_{X-N}$, we have $H_F\,\bar{u}(x)=$

$E_x[\tilde{u}(X_{\sigma_{F-N}})]=E_x[\tilde{v}(X_{\sigma_{F-N}})]=H_F\,\tilde{v}(x)$ for any $x\in X-N$. Hence we get $\mathcal{A}_F(u,u)=\mathcal{E}(H_F\,\tilde{v}, H_F\,\tilde{v})\leqq\mathcal{E}(v,v)$. Moreover $H_F\,\tilde{u}\in\mathcal{F}_e$ attains the infimum, becase $H_F\,\tilde{u}$ is a bounded quasi-continuous function by virtue of Corollary 2.3. The proof is complete.

Theorem 5.3. *$(\mathcal{A}_F, \mathcal{C}|_Y)$ is a pre-Dirichlet form.*

Proof. Let φ_ε be the function described in $(\mathcal{C}.\,2)$. It suffices to show that

$$\mathcal{A}_F(\varphi_\varepsilon(u), \varphi_\varepsilon(u))\leqq\mathcal{A}_F(u,u)\,, \quad \text{for any} \quad u\in\mathcal{C}|_Y\,.$$

For each $u\in\mathcal{C}|_Y$,

$$\begin{aligned}\mathcal{A}_F(\varphi_\varepsilon(u), \varphi_\varepsilon(u)) &= \inf\{\mathcal{E}(v,v);\, v\in\mathcal{F}_e,\, \varphi_\varepsilon(u)=\tilde{v} \text{ q.e. on } F\}\\ &\leqq \inf\{\mathcal{E}(\varphi_\varepsilon(w),\varphi_\varepsilon(w));\, w\in\mathcal{F}_e,\, \varphi_\varepsilon(u)=\varphi_\varepsilon(\tilde{w}) \text{ q.e. on } F\}\\ &\leqq \inf\{\mathcal{E}(\varphi_\varepsilon(w),\varphi_\varepsilon(w));\, w\in\mathcal{F}_e,\, u=\tilde{w} \text{ q.e. on } F\}\\ &\leqq \inf\{\mathcal{E}(w,w);\, w\in\mathcal{F}_e,\, u=\tilde{w} \text{ q.e. on } F\}\\ &= \mathcal{A}_F(u,u)\,.\end{aligned}$$

The proof is complete.

Each $\mu\in\mathcal{M}$ is uniquely decomposed as follows:

$$\mu=\mu_0+\mu_1 \quad \mu_0\in\mathcal{M}_0,\ \mu_1=I_N\mu \quad \text{for some } \mathcal{E}_1\text{-polar set } N\,.$$

μ_0 is called the smooth part of μ, (cf. Fukushima-Sato-Taniguchi [10]). We let $Y=\mathrm{supp}[\mu]$, $Y_0=\mathrm{supp}[\mu_0]$ and $\tilde{Y}_0=\text{q-supp}[\mu_0]$. The $\mathcal{E}_1$-polar set N is unique upto a μ-negligible set. We may assume that $N\subset Y$. Hence $Y_0\cup N\subset Y$. We state the main theorem in this section.

Theorem 5.4. (i) *$(\mathcal{A}_{\tilde{Y}_0}, \mathcal{C}|_Y)$ is the closable part of $(\mathcal{A}_Y, \mathcal{C}|_Y)$ on $L^2(Y;\mu)$.*

(ii) *Suppose that $\mathcal{E}_1$-Cap$(Y-\tilde{Y}_0)=0$. Then $(\mathcal{A}_Y, \mathcal{C}|_Y)$ is closable on $L^2(Y;\mu)$ and $Y\cap(X^{(c)}-B^{\mu_0})=\phi$ q.e.*

(iii) *Suppose $(\mathcal{A}_Y, \mathcal{C}|_Y)$ is closable on $L^2(Y;\mu)$ and $X^{(c)}-B^{\mu_0}=\phi$ q.e.. Then $\mathcal{E}_1$-Cap$(Y-\tilde{Y}_0)=0$.*

(iv) *The closure $(\mathcal{A}_{\tilde{Y}_0}, \mathcal{C}|_Y)$ on $L^2(Y;\mu)$ is associated with the Hunt process $\boldsymbol{M}^\mu=(X^\mu_t, P^\mu_x)_{x\in Y}$ such that*

(*a*) *"the law of $X^\mu_\cdot$ under P^μ_x"="the law of $\hat{X}^{\mu_0}_\cdot$ under $\hat{P}^{\mu_0}_x$" for any $x\in Y_0-N$,*

(*b*) *$P^\mu_x(X^\mu_t=x$, for any $t\geqq0)=1$, for any $x\in N$,*

(*c*) *$Y-Y_0-N$ is an exceptional set for $\boldsymbol{M}^\mu$,*

where $\boldsymbol{M}^{\mu_0}_{Y_0}=(\hat{X}^{\mu_0}_t, \hat{P}^{\mu_0}_x)$ is the Hunt process associated with the time changed regular Dirichlet space $(\mathcal{F}^{\mu_0}_{Y_0}, \mathcal{F}^{\mu_0}_{Y_0})$ on $L^2(Y_0;\mu_0)$.

REMARK. *By Theorem 4.4 the condition (a) and (c) can be replaced by* (a′) *"the law of* $X^{\mu}_{\cdot}$ *under* P^{μ}_x*"="the law of* $X\tau^{\mu_0}_{\cdot}$ *under* P_x*" for any* $x\in\tilde{Y}_0-\tilde{N}_0-N$,

(c′) $Y-\tilde{Y}_0-\tilde{N}_0-N$ *is an exceptional set of* M^{μ},

where $M^t=(X\tau^{\mu_0}_t, P_x)_{x\in\tilde{Y}_0}$ *is the time changed process by the PCAF* $A^{\mu_0}_t$ *and* $\tilde{N}_0$ *is a properly exceptional set of* $M^{\mu_0}_{\tilde{Y}_0}$.

To prove this theorem we prepare several lemmas as in [10].

Lemma 5.5. *For a closed set* $\hat{X}\subset X$, *we let* $\hat{m}\in\mathcal{M}$ *with* $\hat{X}=supp[\hat{m}]$ *and* $(\hat{\mathcal{E}},\hat{\mathcal{F}})$ *be another Dirichlet form on* $L^2(\hat{X};\hat{m})$ *with* $C|_{\hat{X}}\subset\hat{\mathcal{F}}$. *Assume that* $\hat{\mathcal{E}}(u,u)\leqq\mathcal{E}(\bar{u},\bar{u})$, $u\in C|_{\hat{X}}$, $\bar{u}\in C$, $u=\bar{u}|_{\hat{X}}$. *Then for any* $\mathcal{E}_1$*-polar set* N',

$$\hat{G}_{\alpha}(I_{N'\cap\hat{X}}u)=\frac{1}{\alpha}I_{N'\cap\hat{X}}u\,,\quad \hat{m}\text{-a.e. on } \hat{X} \text{ for any } u\in L^2(\hat{X};\hat{m})\,,$$

where $\hat{G}_{\alpha}$ *is the resolvent on* $L^2(\hat{X};\hat{m})$ *associated with* $\hat{\mathcal{E}}$.

Proof. The proof is the same as in Lemma 4.1 in [10].

Lemma 5.6. *Let* $(\mathcal{B}, C|_Y)$ *be a closable pre-Dirichlet form on* $L^2(Y;\mu)$ *such that* $\mathcal{B}(u,u)\leqq\mathcal{E}(\bar{u},\bar{u})$, $u\in C|_Y$, $\bar{u}\in C$, $u=\bar{u}|_Y$. *Then* $(\mathcal{B}, C|_Y)$ *is well-defined on* $L^2(Y_0;\mu_0)$ *and closable on* $L^2(Y_0;\mu_0)$.

Proof. The proof is same as in Lemma 4.2 in [10].

Lemma 5.7. $(\mathcal{A}_{\tilde{Y}_0}, C|_Y)$ *is the closable part of* $(\mathcal{A}_Y, C|_Y)$ *on* $L^2(Y;\mu)$.

Proof. This follows from the description of Corollary 4.3 of the time changed Dirichlet space as the proof of Lemma 4.3 in [10]. We give the proof for completeness. We let $\nu_0=\mu_0+I_{X(c)-B^{\mu_0}}m$. Then the ν_0-killed Dirichlet space $(\mathcal{F}^{\nu_0},\mathcal{E}^{\nu_0})$ is transient. Let $\mathcal{F}^{\nu_0}_e$ be the extended Dirichlet space of $(\mathcal{F}^{\nu_0},\mathcal{E}^{\nu_0})$. We let $\mathcal{F}^{\nu_0}_{eX-\tilde{Y}_0}=\{u\in\mathcal{F}^{\nu_0}_e; u=0 \text{ q.e. on } \tilde{Y}_0\}$. Let $\mathcal{P}^{\nu_0}$ be the projection operator on the orthogonal complement of $\mathcal{F}^{\nu_0}_{eX-\tilde{Y}_0}$ with respect to $\mathcal{E}^{\nu_0}$. Since $\mathcal{E}^{\nu_0}$-Cauchy sequence is an $\mathcal{E}$-Cauchy sequence, $\mathcal{P}^{\nu_0}u\in\mathcal{F}_e$ for any $u\in\mathcal{F}^{\nu_0}_e$. Note that

$$\mathcal{A}_{\tilde{Y}_0}(u,u)=\mathcal{E}(\mathcal{P}^{\nu_0}\bar{u},\mathcal{P}^{\nu_0}\bar{u}),\ u\in C|_Y, \bar{u}\in C, u=\bar{u}|_Y\,. \tag{5.3}$$

Indeed if μ_0 is non-trivial, (5.3) follows from Corollary 4.3. Suppose that μ_0 is trivial. Then $\tilde{Y}_0=\phi$ q.e.. We have $\mathcal{F}^{\nu_0}_{eX-\tilde{Y}_0}=\mathcal{F}^{\nu_0}_e$ and $\mathcal{E}^{\nu_0}(\mathcal{P}^{\nu_0}\bar{u},\mathcal{P}^{\nu_0}\bar{u})=0$. On the other hand, $P_x(\sigma_{\tilde{Y}_0}=\infty)=1$ q.e. $x\in X$. We get $H_{\tilde{Y}_0}u=0$ q.e.. Thus we have (5.3).

If μ_0 is trivial, the closability of $(\mathcal{A}_{\tilde{Y}_0}, C|_Y)$ on $L^2(Y;\mu)$ is clear. If μ_0 is non-trivial, the closability follows from (5.3) and Theorem 4.2. The inequality $\mathcal{A}_{\tilde{Y}_0}(u,u)\leq\mathcal{A}_Y(u,u)$, $u\in C|_Y$ follows from (5.1) and $H_{\tilde{Y}_0}H_Y u=H_{\tilde{Y}_0}u$, $u\in C|_Y$. Let $(\mathcal{B}, C|_Y)$ is a closable pre-Dirichlet form with $\mathcal{B}(u,u)\leqq\mathcal{A}_Y(u,u)$ for $u\in C|_Y$.

Fix an $f\in\mathcal{C}|_Y$. Then there exists $\bar{f}\in\mathcal{C}$ such that $f=\bar{f}|_Y$. Since $\mathcal{C}$ is dense in $\mathcal{F}_e^{\nu_0}$, there exists a sequence $\{f_n\}\subset\mathcal{C}$ such that

$$\lim_{n\to\infty}\mathcal{E}^{\nu_0}(f_n-\mathcal{P}^{\nu_0}\bar{f}, f_n-\mathcal{P}^{\nu_0}\bar{f})=0\,.$$

We have

(5.4) $\{f_n\}$ is an $\mathcal{E}$-Cauchy sequence and $f_n\to f$ in $L^2(Y_0;\mu_0)$.

By (5.3), we see that

$$\mathcal{A}_{\tilde{Y}_0}(f,f)=\mathcal{E}(\mathcal{P}^{\nu_0}\bar{f},\mathcal{P}^{\nu_0}\bar{f})=\lim_{n\to\infty}\mathcal{E}(f_n,f_n)\,.$$

It follows from (5.3) and (5.1) that $\{f_n|_Y-f\}\subset\mathcal{C}|_Y$ is an $\mathcal{B}$-Cauchy sequence and $f_n-f\to 0$ in $L^2(Y_0;\mu_0)$. By Lemma 5.6, we have that $\mathcal{B}(f_n|_Y-f, f_n|_Y-f)\to 0$. Therefore it holds that

$$\mathcal{B}(f,f)=\lim_{n\to\infty}\mathcal{B}(f_n|_Y,f_n|_Y)\leqq\lim_{n\to\infty}\mathcal{E}(f_n,f_n)=\mathcal{A}_{\tilde{Y}_0}(f,f)\,.$$

The proof is complete.

Lemma 5.8. *Suppose $(\mathcal{A}_Y,\mathcal{C}|_Y)$ is closable on $L^2(Y;\mu)$. Then*

$$\mathcal{E}(H_Y u-H_{\tilde{Y}_0}u, H_Y \tilde{u}-H_{\tilde{Y}_0}u)=0\,,\quad \textit{for any}\quad u\in\mathcal{C}\,.$$

Proof. By Lemma 5.7 we have

$$\mathcal{E}(H_Y u, H_Y u)\leqq\mathcal{E}(H_{\tilde{Y}_0}u, H_{\tilde{Y}_0}u)\quad\text{for any}\quad u\in\mathcal{C}\,.$$

Hence by (5.1)

$$\begin{aligned}&\mathcal{E}(H_Y u-H_{\tilde{Y}_0}u, H_Y u-H_{\tilde{Y}_0}u)\\&\quad=\mathcal{E}(H_Y u, H_Y u)-2\mathcal{E}(H_Y u, H_{\tilde{Y}_0}u)+\mathcal{E}(H_{\tilde{Y}_0}u, H_{\tilde{Y}_0}u)\\&\quad=\mathcal{E}(H_Y u, H_Y u)-\mathcal{E}(H_{\tilde{Y}_0}u, H_{\tilde{Y}_0}u)\leqq 0\,.\end{aligned}$$

Lemma 5.9. *Denote the closure of $(\mathcal{A}_{\tilde{Y}_0},\mathcal{C}|_Y)$ on $L^2(Y;\mu)$ by $(\bar{\mathcal{A}}_{\tilde{Y}_0},\bar{\mathcal{C}}|_Y)$. Let $\{G_\alpha^{\mathcal{A}_{\tilde{Y}_r}},\alpha>0\}$ (resp. $\{\check{G}_\alpha^{\mu_0},\alpha>0\}$) be the resolvent on $L^2(Y;\mu)$ (resp. $L^2(Y_0;\mu_0)$) associated with $(\bar{\mathcal{A}}_{\tilde{Y}_0},\bar{\mathcal{C}}|_Y)$ (resp. $(\mathcal{E}_{Y_0}^{\mu_0},\mathcal{F}_{Y_0}^{\mu_0})$). Then*

(i) $G_\alpha^{\mathcal{A}_{\tilde{Y}_0}}(I_N u)=\dfrac{1}{\alpha}I_N u$, *$\mu$-a.e. for any $u\in L^2(Y;\mu)$.*

(ii) $G_\alpha^{\mathcal{A}_{\tilde{Y}_0}}u=\check{G}_\alpha^{\mu_0}u$, *$\mu_0$-a.e. on Y_0 for any $u\in L^2(Y;\mu)$.*

(iii) $\mathcal{A}_{\tilde{Y}_0 1}$-$\mathrm{Cap}(Y-Y_0-N)=0$.

Proof. (i) follows from Lemma 5.5. The proof of (ii) is same as in Lemma 4.5 in [10]. For compact set $K\subset Y-Y_0$ in Y, there exists a relatively compact open set G in Y and $f\in\mathcal{C}|_Y$ such that $G\subset Y-Y_0$ and $0\leqq f\leqq 1$, $f=1$ on

K, $f=0$ on $Y-G$. Then $\mathcal{A}_{\tilde{Y}_0}(f,f)=0$. By Lemma 5.7 we have

$$\mathcal{A}_{\tilde{Y}_0 1}\text{-Cap}(K) = \inf\{\mathcal{A}_{\tilde{Y}_0 1}(u,u);\, u\in\mathcal{C}|_Y,\, u\geqq 1 \text{ on } K\} \leqq (f,f)_\mu \leqq \mu(G).$$

Hence we can get $\mathcal{A}_{\tilde{Y}_0 1}$-Cap$(B)\leqq\mu(B)$ for any Borel set $B\subset(Y-Y_0)$, which implies (iii). The proof is complete.

Proof of Theorem 5.4. (i) follows from Lemma 5.7 (iv) follows from Lemma 5.9. We show (ii). Suppose $\mathcal{E}_1$-Cap$(Y-\tilde{Y}_0)=0$. Then $(\mathcal{A}_Y, \mathcal{C}|_Y)=(\mathcal{A}_{\tilde{Y}_0}, \mathcal{C}|_Y)$. Hence $(\mathcal{A}_Y, \mathcal{C}|_Y)$ is closable on $L^2(Y;\mu)$ and $Y\cap(X^{(c)}-B^{\mu_0})=\tilde{Y}_0\cap(X^{(c)}-B^{\mu_0})=\phi$ q.e., because $\tilde{Y}_0\subset B^{\mu_0}$. Next we show (iii). Suppose $X^{(c)}-B^{\mu_0}=\phi$ q.e. and $(\mathcal{A}_Y, \mathcal{C}|_Y)$ is closable on $L^2(Y;\mu)$. Then $\nu_0=\mu_0$. We get $H_Y\tilde{u}=H_{\tilde{Y}_0}\tilde{u}$, ν_0-a.e.. By Lemma 5.8 we have $\mathcal{E}^{\nu_0}(H_Y\tilde{u}-H_{\tilde{Y}_0}\tilde{u}, H_Y\tilde{u}-H_{\tilde{Y}_0}\tilde{u})=0$ for any $\tilde{u}\in\mathcal{C}$, namely $H_Y\tilde{u}=H_{\tilde{Y}_0}\tilde{u}$ q.e. for any $\tilde{u}\in\mathcal{C}$. Hence we have that $Y-\tilde{Y}_0$ is $\mathcal{E}_1$-polar. The proof is complete.

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