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LIFSCHITZ TAILS FOR RANDOM SCHRÖDINGER OPERATORS ON NESTED FRACTALS

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1. Introduction

The nested fractal introduced by Lindstrøm [7] is a certain class of fractal possessing finite ramifiedness and some symmetry. The Sierpinski gasket, the snowflake fractal and the Pentakun are members of this class (see [6], [7]). In this paper, we are concerned with two types of random operators on nested fractals; *the Laplacian with Poisson obstacles* and *the random Schrödinger operator* both formulated presently.

Let (Ψ, E) be a unit nested fractal in R^D constructed by a family of α-similitudes $\Psi=\{\Psi_0, \cdots, \Psi_{N-1}\}$ with $\Psi_0(x)=\alpha^{-1}x$, $\alpha>1$ (see Definition (2.1) and (2.2)). We then consider the expanded nested fractals defined by $E^{\langle m\rangle}=\alpha^m E$ and $E^{\langle\infty\rangle}=\bigcup_m E^{\langle m\rangle}$. By the Laplacian Δ, we mean the generator of Lindstrøm's Brownian motion of E ([7]). The associated Dirichlet form has been identified by Kusuoka [5]. We formulate our random operators by perturbing the corresponding Dirichlet form $(\mathcal{F}^{\langle m\rangle}, \mathcal{E}^{\langle m\rangle})$ on $L^2(E^{\langle m\rangle}; \mu)$, where μ is a $\log N/\log\alpha$-dimensional Hausdorff measure on $E^{\langle\infty\rangle}$ with $\mu(E)=1$.

Let $\mathcal{N}_\omega$ be the support of the Poisson random measure on $E^{\langle\infty\rangle}$ with the intensity measure $\nu\mu$ (ν is a positive constant) and let $\Delta_\omega^{\langle m\rangle}$ be the self-adjoint operator on $L^2(E^{\langle m\rangle}; \mu)$ associated with the Dirichlet form $(\mathcal{F}_\omega^{\langle m\rangle}, \mathcal{E}^{\langle m\rangle})$, where

$$\mathcal{F}_\omega^{\langle m\rangle} = \{f\in\mathcal{F}^{\langle m\rangle}; f(p) = 0, p\in\mathcal{N}_\omega\cap E^{\langle m\rangle}\}.$$

$\Delta_\omega^{\langle m\rangle}$ is called the Laplacian with Poisson obstacles.

For another type of random operator, we first introduce a probability space $(\hat{\Omega}, \hat{\Sigma}, \hat{P})$ on the set $\hat{\Omega}$ of positive Radon measures on $E^{\langle\infty\rangle}$ so that restrictions of each measure to unit fractals constituting $E^{\langle\infty\rangle}$ behave as independent, identically distributed random variables (see (3.8)). The random Schrödinger operator is by definition the self-adjoint operator $H_{\hat{\omega}}^{\langle m\rangle}$ on $L^2(E^{\langle m\rangle}; \mu)$ associated with the Dirichlet form $(\mathcal{E}_{\hat{\omega}}^{\langle m\rangle}, \mathcal{F}^{\langle m\rangle})$, where

$$\mathcal{E}_{\hat{\omega}}^{\langle m\rangle}(u, v) = \mathcal{E}^{\langle m\rangle}(u, v)+ \int_{E^{\langle m\rangle}} u(x)v(x)\hat{\omega}(dx) \quad \text{for } \hat{\omega}\in\hat{\Omega}.$$

The spectrum of $-\Delta_\omega^{\langle m\rangle}$ consists only of non-negative eigenvalues with finite multiplicitiy accumulating only at ∞. We let

$$k_\omega^{\langle m\rangle}(\lambda)=\#\{\text{the eigenvalues of } -\Delta_\omega^{\langle m\rangle} \text{ no grater than } \lambda\}.$$

From $k_\omega^{\langle m\rangle}(\lambda)$ we can derive a non-random right continuous non-decreasing function $k(\lambda)$ such that, for almost all $\omega\in\Omega$, $\lim_{m\to\infty} k_\omega^{\langle m\rangle}(\lambda)/\mu(E^{\langle m\rangle})=k(\lambda)$ at all continuous points of $k(\lambda)$. $k(\lambda)$ is called *the integrated density of states* (abb. I.D.S.) for $-\Delta_\omega^{\langle m\rangle}$, $m=0, 1, \cdots$. In the same way, we can obtain the I.D.S. $\hat{k}(\lambda)$ for $H_{\hat{\omega}}^{\langle m\rangle}$ $m=0, 1, \cdots$.

In the deterministic case, Fukushima [2] has shown that the I.D.S. behaves near zero like a polynomial with order being the half of the spectral dimension $d_s=\log N^2/\log\frac{N}{1-c}$. The main objective of this paper is to derive the Lifschitz tail behavior in the present setting, that is to say, to show that $k(\lambda)$ and $\hat{k}(\lambda)$ behave like $\exp[-\text{const.}\ (\lambda-a)^{-d_s/2}]$ near the bottom a of the spectrum.

In the following, we summarize the results. As for the Laplacian with Poisson obstacles on nested fractals, we get the following estimate:
For $\lambda\in[0, C)$,

$$(1.1)\quad \frac{1}{N}\Big(\frac{\lambda}{\lambda_{0,1}}\Big)^{d_s/2}\exp\Big\{-\nu N\Big(\frac{\lambda}{\lambda_{0,1}}\Big)^{-d_s/2}\Big\}\le k(\lambda)\le N\Big(\frac{\lambda}{C}\Big)^{d_s/2}\exp\Big\{-\frac{\nu}{N}\Big(\frac{\lambda}{C}\Big)^{-d_s/2}\Big\},$$

where C is a positive constant defined in Lemma (2.10), *and $\lambda_{0,1}$ is the first eigenvalue of the Laplacian on E with the Dirichlet boundary condition.*

As for the random Schrödinger operator, we generally get the following under the assumption (4.5) of non-degeneracy of the distribution of $\hat{\omega}(E)$:

$$(1.2)\qquad \lim_{\lambda\downarrow 0}\frac{\log[-\log\hat{k}(\lambda)]}{\log\lambda}=-\frac{d_s}{2}.$$

Giving more specific structures to the probabililty space $(\hat{\Omega}, \hat{\Sigma}, \hat{P})$, we can obtain more precise information.

Poisson noise. Let $N(\cdot, \omega)$ be the Poisson random measures on $E^{\langle\infty\rangle}$ with the intensity measure $\nu\mu$ defined on a probability space (Ω, Σ, P). Define π: $\Omega\to\hat{\Omega}$ by

$$(1.3)\qquad \pi(\omega)(dx)=\kappa N(dx, \omega),$$

where κ is a positive constant. When the probability $\hat{P}$ is given as the image measure of P by π, we have that

$$(1.4)\qquad -\nu N\lambda_{0,1}^{d_s/2}\le\varliminf_{\lambda\downarrow 0}\lambda^{d_s/2}\log\hat{k}(\lambda)\le\varlimsup_{\lambda\downarrow 0}\lambda^{d_s/2}\log\hat{k}(\lambda)\le-\frac{\nu}{N}C^{d_s/2}.$$

Poisson integral potential. Let $\phi(x,y)$ be a non-negative, integrable function on $E^{\langle\infty\rangle}\times E^{\langle\infty\rangle}$ with certain assumptions (see (3.14)). We define $\pi\colon \Omega\to\hat{\Omega}$ as

$$(1.5)\qquad \pi(\omega)(dx)=\int_{E^{\langle\infty\rangle}}\phi(x,y)N(dy,\omega)\mu(dx)\,,$$

and introduce a probability $\hat{P}$ as above. Then we have that

$$(1.6)\qquad -\nu N\,\lambda_{0,1}^{d_s/2}\leq\varliminf_{\lambda\downarrow 0}\lambda^{d_s/2}\log\hat{k}(\lambda)\leq\varlimsup_{\lambda\downarrow 0}\lambda^{d_s/2}\log\hat{k}(\lambda)\leq-\frac{\nu\tau}{N}C^{d_s/2}\,,$$

where $\tau=\mu(\{y;\int_E\phi(x,y)\mu(dx)>0\})$.

Recently, Paluba [8] treated the same theme as for the Laplacian with Poisson obstacles on the Sierpinski gasket in R^2. In [8], the discussion began with the Brownian motion on the Sierpinski gasket constructed by Barlow and Perkins [1], and the Laplace transform of the I.D.S. was adapted along with the method due to Sznitman [11]. Then, besides the Lifschitz tail, a counterpart of the Donsker-Varadhan Wiener sausage estimate in R^d was also derived. In the present paper, we start with the Dirichlet form due to Kusuoka [5] and treat the I.D.S. directly. To get the result, the method by Kirsch and Martinelli [4] and Simon [10] is adapted. In carrying out this method, the scaling property of the Dirichlet form observed by Fukushima [2] works effectively.

The organization of the present paper is as follows. In §2, we first give a nested fractal and expanded ones in general. To handle the expanded nested fractals some notions and relations are prepared. Then we introduce Dirichlet forms on them and collect several facts about the Dirichlet forms from [2], [5]. In §3, the construction of the I.D.S. $k(\lambda)$ and $\hat{k}(\lambda)$ is carried out in accordance with [3]. At the same time, the Dirichlet-Neumann bracketing, which plays an important role in the following section, is derived. Furthermore it turns out that, although the I.D.S. can be alternatively constructed by imposing the Dirichlet boundary condition on the above random operators, $k(\lambda)$ and $\hat{k}(\lambda)$ are determined independently of the boundary conditions. §4 is devoted to the Lifschitz tail behaviors of $k(\lambda)$ and $\hat{k}(\lambda)$ stated above. To get the lower bound, we estimate the upper bound of the 1st eigenvalue of the random operators with the Dirichlet boundary condition when the influence of randomness is negligible, namely, when $\mathcal{N}_\omega\cap E^{\langle m\rangle}=\emptyset$, and this is not difficult because this case amounts to the deterministic one. As for the upper bound, we estimate the lower bound of the 1st eigenvalue of random operators with the Neumann boundary condition when the influence of randomness persists, namely, when $\mathcal{N}_\omega\cap E^{\langle m\rangle}\neq\emptyset$. In doing so, Kirsch and Martinelli [4] used Thirring's inequality, and Simon [10] did Temple's. We instead use the specific bound (2.10) by the Dirichlet norm due to Kusuoka [5].

Finally, we remark that the argument in the present paper is also applicable to the model on the pre-nested fractal (e.g. the Sierpinski pre-gasket)- the right analogue to the classical Anderson model on Z^d.

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2. Preliminaries

(2.1) DEFINITION. Let $\alpha>1$. We say that $\Psi: R^D \to R^D$ is an α-similitude, if

$$|\Psi(x)-\Psi(y)|=\alpha^{-1}|x-y| \quad \text{for any } x, y\in R^D .$$

First, we introduce a nested fractal (Ψ, E) in R^D, where $\Psi=\{\Psi_0, \cdots, \Psi_{N-1}\}$ is a family of α-similitudes. A self-similar fractal with the open set condition is called a nested fractal if it satisfies three axioms (axioms of connectivity, symmetry and nesting), and possesses no less than two essential fixed points. By the selfsimilarity, E is a unique compact set satisfying $E=\bigcup_{i=0}^{N-1}\Psi_i(E)$. We refer the reader to Lindstrøm [7], Kusuoka [5] for the above condition and axioms, and for the geometrical features of nested fractals.

(2.2) DEFINITION. Let F_0 be the set of fixed points of Ψ_i's, $i\in\{0, \cdots, N-1\}$. We call $p\in F_0$ an essential fixed point, if there are $i, j\in\{0, \cdots, N-1\}$, $i\neq j$ and $q\in F_0$ for which $\Psi_i(p)=\Psi_j(q)$.

We denote by F the set of essential fixed points, and write M for $\# F$. Note that $2\leq M\leq \# F_0(=N)$. Let

$$I_0^n = \{0, \cdots, N-1\}^{\{1,\cdots,n\}} . \tag{2.3}$$

For $A\subset R^D$ and $(i_1, \cdots, i_n)\in I_0^n$,

$$\begin{aligned} \Psi_{(i_1,\cdots,i_n)} &= \Psi_{i_1}\circ\cdots\circ\Psi_{i_n} , \\ A_{(i_1,\cdots,i_n)} &= \Psi_{(i_1,\cdots,i_n)}(A) , \\ A^{(n)} &= \bigcup_{i\in I_0^n} A_i , \quad A^{(0)} = A , \\ A^{(*)} &= \bigcup_{n=0}^{\infty} A^{(n)} . \end{aligned}$$

The sequence of finite sets $\{F^{(n)}\}$ is increasing, and approximats E, i.e. $\overline{F^{(*)}}=E$.

The snowflake fractal is a typical example of nested fractals. It is constructed by seven 3-similitudes $\{\Psi_0, \cdots, \Psi_6\}$ in R^2, where

$$\Psi_k(x) = \frac{1}{3}(x-a_k)+a_k \quad \text{and}$$

$$a_0 = 0 \quad a_k = (\cos(\frac{k\pi}{3}), \sin(\frac{k\pi}{3})), \quad k = 1, \cdots, 6.$$

a_0 is a fixed point but not an essential fixed point. The essential fixed points consist of a_k, $k=1, \cdots, 6$.

In the following, a nested fractal (Ψ, E) in R^D is fixed. According to [2], [5], we expand this nested fractal and define the Laplacian on the expanded ones. It can be assumed that $\Psi_0(x)=\alpha^{-1}x$, $x\in R^D$ without loss of generality. For $n\geq 0$, let

$$I^n = \{\boldsymbol{i}\in \{0, \cdots, N-1\}^{\{\cdots,-1,0,1,\cdots,n\}}; \text{ there is an } m \text{ such that } \boldsymbol{i}(k) = 0 \text{ for } k\leq m\},$$

and $I=\bigcup_{n\geq 0} I^n$. For $\boldsymbol{i}\in I^n$, define $\mathrm{S}\boldsymbol{i}\in I^{n+1}$ and $\mathrm{P}\boldsymbol{i}\in I_0^n$ by

$$\mathrm{S}\boldsymbol{i}(j) = \boldsymbol{i}(j-1), \quad \mathrm{P}\boldsymbol{i}(j) = \boldsymbol{i}(j), \quad (n\geq j\geq 1).$$

If m is sufficiently large, $\alpha^m\Psi_{\mathrm{PS}^m\boldsymbol{i}}(\cdot)$ is independent of m for each $\boldsymbol{i}\in I$. Thus we denote this mapping from R^D to R^D by $\Phi_{\boldsymbol{i}}$.

For $m\geq 0$, let

$$I_m^n = \{\boldsymbol{i}\in I^n; \boldsymbol{i}(k) = 0 \text{ for } k\leq -m\}.$$

When $m=0$, this set is identified with the I_0^n in (2.3) in usual way, and we see that $\Phi_{\boldsymbol{i}}=\Psi_{\boldsymbol{i}}$ for $\boldsymbol{i}\in I_0^n$. Thus the definition of $A_{\boldsymbol{i}}$ is extended as follows:

$$A_{\boldsymbol{i}} = \Phi_{\boldsymbol{i}}(A), \quad \text{for } \boldsymbol{i}\in I,\ A\subset R^D.$$

Note that $A_{\boldsymbol{i}}$ for $\boldsymbol{i}\in I^0$ is congruent to A. For $A\subset R^D$, we set

$$A^{\langle m,n\rangle} = \bigcup_{\boldsymbol{i}\in I_m^n} A_{\boldsymbol{i}},$$

$$A^{\langle m,*\rangle} = \bigcup_{n=0}^{\infty} A^{\langle m,n\rangle}, \quad A^{\langle \infty,n\rangle} = \bigcup_{m=0}^{\infty} A^{\langle m,n\rangle},$$

$$A^{\langle m\rangle} = A^{\langle m,0\rangle} \quad (m=0, 1, \cdots, \infty),$$

and note that $A^{\langle 0,n\rangle}=A^{(n)}(n\geq 0 \text{ or}=*)$ and that, for $m, n=0, 1, \cdots$,

$$\begin{aligned} E^{\langle m\rangle} &= E^{\langle m,n\rangle} = \alpha^m E, \\ E^{\langle\infty\rangle} &= \bigcup_{\boldsymbol{i}\in I^n} E_{\boldsymbol{i}} = \overline{F^{\langle\infty,*\rangle}}. \end{aligned} \tag{2.4}$$

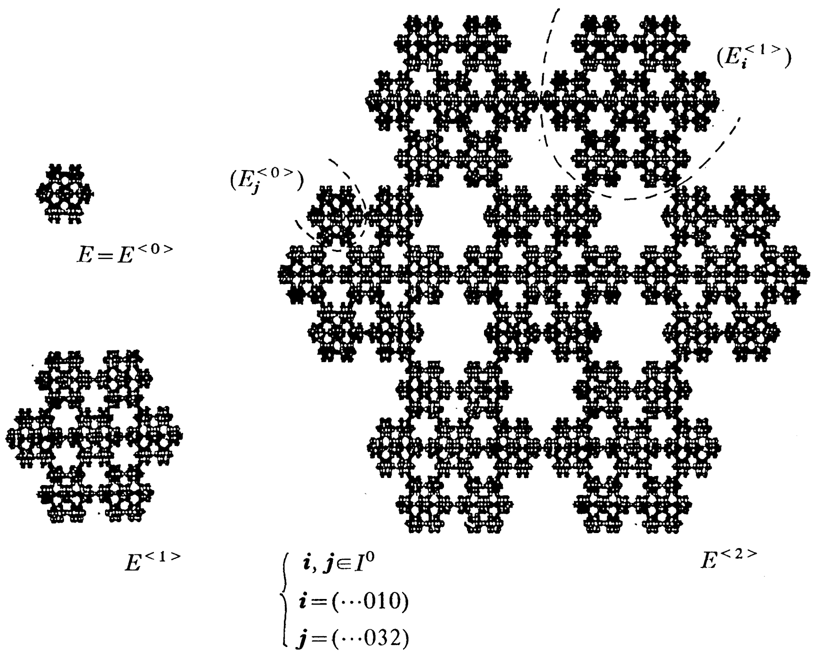
$\alpha^m F$ is regarded as the boundary of $E^{\langle m\rangle}$ so we write $\partial E^{\langle m\rangle}$ for $\alpha^m F$, and further $\overset{o}{E}{}^{\langle m\rangle}$ for $E^{\langle m\rangle}\backslash\partial E^{\langle m\rangle}$. Finally let, for $n\geq 0$, $m\geq l\geq 0$,

$$I_{m,l}^n = \{\boldsymbol{i}\in I^n; \boldsymbol{i}(k) = 0 \text{ for } k\leq -m,\ k>-l\}.$$

Then, for $m>l$ and $\boldsymbol{j}, \boldsymbol{k}\in I_{m,l}^0$ $\boldsymbol{j}\neq\boldsymbol{k}$,

$$\begin{aligned} E^{\langle m\rangle} &= \bigcup_{\boldsymbol{i}\in I_{m,l}^0} (E^{\langle l\rangle})_{\boldsymbol{i}}, \\ (E^{\langle l\rangle})_{\boldsymbol{j}}\cap(E^{\langle l\rangle})_{\boldsymbol{k}} &= (\partial E^{\langle l\rangle})_{\boldsymbol{j}}\cap(\partial E^{\langle l\rangle})_{\boldsymbol{k}}. \end{aligned} \tag{2.5}$$

In the following we write $\partial(E^{\langle l\rangle})_j$ for $(\partial E^{\langle l\rangle})_j$. Note that, in this case, $(E^{\langle l\rangle})_j$ is congruent to $E^{\langle l\rangle}$. We call $E^{\langle m\rangle}$ the m-th expanded nested fractal. In accordance with the terminology of [7], $F_{\boldsymbol{i}}$ for $\boldsymbol{i}\in I_m^n$ may be called an n-cell in the expanded fractal $E^{\langle m\rangle}$, and $(E^{\langle l\rangle})_{\boldsymbol{i}}$ for $\boldsymbol{i}\in I_{m,l}^0$ $(m>l)$ an $m-l$ complex.



The expanded snowflake fractal (by T. Kumagai)

Let μ be the $\log N/\log\alpha$-dimensional Hausdorff measure defined on $E^{\langle\infty\rangle}$ with $\mu(E)=1$, and let $l(A)$ (resp. $C(A)$) be the set of real valued (resp. continuous) functions defined on $A\subset E^{\langle\infty\rangle}$. Now we construct a Dirichlet form on $L^2(E^{\langle m\rangle};\mu)$. To do this we first give a symmetric form on $l(F^{\langle m,n\rangle})$: for $f, g\in l(F^{\langle m,n\rangle})$,

$$(2.6)\quad \mathcal{E}^{\langle m,n\rangle}(f,g)=\frac{1}{2(1-c)^n}\sum_{\substack{\xi,\eta\in F_{\boldsymbol{i}}\\ \boldsymbol{i}\in I_m^n}}(f(\xi)-f(\eta))(g(\xi)-g(\eta))\pi_{\Phi_{\boldsymbol{i}}^{-1}(\xi)\Phi_{\boldsymbol{i}}^{-1}(\eta)}\,,$$

where c and $\pi_{\xi\eta}$, $(\xi, \eta\in F, \xi\neq\eta)$ are some positive constants such that $c\in(0, 1)$, and $\sum_\eta \pi_{\xi\eta}=1$, $\pi_{\xi\eta}=\pi_{\eta\xi}$. Here $\{\pi_{\xi\eta}\}$ represents Lindstrøm's invariant probability on F, and c is a returning probability of the corresponding random walk on $F^{(1)}$. See [5], [7] for details. It is known in [5] that $\mathcal{E}^{\langle m,n\rangle}(f,f)$ is non-decreasing in n for any $f\in l(F^{\langle m,*\rangle})$.† We let

† For simplicity of the notation, we write $\mathcal{E}^{\langle m,n\rangle}(f,f)$ for $\mathcal{E}^{\langle m,n\rangle}(f|_{F^{\langle m,n\rangle}}, f|_{F^{\langle m,n\rangle}})$.

$$\mathcal{F}^{\langle m\rangle} = \{f \in \mathcal{l}(F^{\langle m,*\rangle});\ \sup_n \mathcal{E}^{\langle m,n\rangle}(f,f) < \infty\},$$

$$\mathcal{E}^{\langle m\rangle}(f,g) = \lim_{n\to\infty} \mathcal{E}^{\langle m,n\rangle}(f,g), \quad f,g \in \mathcal{F}^{\langle m\rangle}.$$

The next theorem is shown in [5].

Theorem (2.7).

(1) $(\mathcal{F}^{\langle m\rangle}, \mathcal{E}^{\langle m\rangle})$ *is a regular local Dirichlet form on* $L^2(E^{\langle m\rangle}; \mu)$.

(2) *Any function of* $\mathcal{F}^{\langle m\rangle}$ *can be uniquely extended to a continuous function on* $E^{\langle m\rangle}$, *that is,* $\mathcal{F}^{\langle m\rangle} \subset C(E^{\langle m\rangle})$.

Let $\mathcal{F}_0^{\langle m\rangle} = \{f \in \mathcal{F}^{\langle m\rangle};\ f|_{\partial E^{\langle m\rangle}} = 0\}$. We also know that $(\mathcal{F}_0^{\langle m\rangle}, \mathcal{E}^{\langle m\rangle})$ is a regular local Dirichlet form on $L^2(\overset{o}{E}{}^{\langle m\rangle}; \mu)$. Denote by $\Delta^{\langle m\rangle}$ (resp. $\Delta_0^{\langle m\rangle}$) the self-adjoint operator on $L^2(E^{\langle m\rangle}; \mu)$ associated with $(\mathcal{F}^{\langle m\rangle}, \mathcal{E}^{\langle m\rangle})$ (resp. $(\mathcal{F}_0^{\langle m\rangle}, \mathcal{E}^{\langle m\rangle})$):

$$\mathcal{E}^{\langle m\rangle}(f,g) = -(\Delta_\sharp^{\langle m\rangle} f, g), \quad f \in \mathcal{D}(\Delta_\sharp^{\langle m\rangle}) \subset \mathcal{F}_\sharp^{\langle m\rangle} \quad g \in \mathcal{F}_\sharp^{\langle m\rangle},$$

where $(\Delta_\sharp^{\langle m\rangle}, \mathcal{F}_\sharp^{\langle m\rangle})$ stands for either $(\Delta^{\langle m\rangle}, \mathcal{F}^{\langle m\rangle})$ or $(\Delta_0^{\langle m\rangle}, \mathcal{F}_0^{\langle m\rangle})$. We call $\Delta_0^{\langle m\rangle}$ (resp. $\Delta^{\langle m\rangle}$) the Laplacian on $E^{\langle m\rangle}$ with the Dirichlet (resp. Neumann) boundary condition. In the following we omit the superscript $\langle m\rangle$ when $m=0$, and often use the subscript $\sharp$ to express notations about the Laplacian with any one of the two boundary conditions.

The Laplacian $\Delta_\sharp^{\langle m\rangle}$ is of compact resolvent (see [2], [5]), so the spectrum of $-\Delta_\sharp^{\langle m\rangle}$ consists only of non-negative eigenvalues with finite multiplicity accumulating only at ∞. We write for the eigenvalues taking multiplicities into account

$$0 \le \lambda_{\sharp,1}^{\langle m\rangle} \le \lambda_{\sharp,2}^{\langle m\rangle} \le \cdots \le \lambda_{\sharp,k}^{\langle m\rangle} \le \cdots,$$

and define the eigenvalue counting function $k_\sharp^{\langle m\rangle}(\lambda)$ by

$$k_\sharp^{\langle m\rangle}(\lambda) = \sum_{\{i\,:\,\lambda_{\sharp,i}^{\langle m\rangle} \le \lambda\}} 1.$$

Furthermore $\mathrm{Pr}_{\sharp,B}^{\langle m\rangle}$ (B is a Borel subset of R) denotes the spectral projection of $-\Delta_\sharp^{\langle m\rangle}$ on $L^2(E^{\langle m\rangle}; \mu)$

Since $(E^{\langle m\rangle})_{\boldsymbol{i}}$ is congruent with $E^{\langle m\rangle}$ for $\boldsymbol{i} \in I^0$, the Dirichlet form $(\mathcal{F}_\sharp^{\langle m\rangle,\boldsymbol{i}}, \mathcal{E}^{\langle m\rangle,\boldsymbol{i}})$ on $L^2((E^{\langle m\rangle})_{\boldsymbol{i}}; \mu)$ is defined in the same manner as $(\mathcal{F}_\sharp^{\langle m\rangle}, \mathcal{E}^{\langle m\rangle})$, and the self-adjoint operator on $L^2((E^{\langle m\rangle})_{\boldsymbol{i}}; \mu)$ associated with the Dirichlet form is denoted by $\Delta_\sharp^{\langle m\rangle,\boldsymbol{i}}$. The corresponding notions are disguised by the superscript $\boldsymbol{i}$ for example; $\lambda_{\sharp,j}^{\langle m\rangle,\boldsymbol{i}}$, $k_\sharp^{\langle m\rangle,\boldsymbol{i}}(\lambda)$, $\cdots$.

Finally we state several facts for later use. Let σ_m be the mapping from $\mathcal{l}(E^{\langle m\rangle})$ to $\mathcal{l}(E)$ defined as follows:

$$(\sigma_m f)(x) = f(\alpha^m x), \quad x \in E.$$

Lemma (2.8). (Fukushima [2])
(1) *For any measurable function* $f \in l(E^{\langle m \rangle})$,

$$\int_{E^{\langle m \rangle}} f\, d\mu = N^m \int_E (\sigma_m f) d\mu\,.$$

(2) *For any* $f \in \mathcal{F}^{\langle m \rangle}$,

$$\mathcal{E}^{\langle m \rangle}(f, f) = (1-c)^m \mathcal{E}(\sigma_m f, \sigma_m f)\,.$$

Noting $\mathcal{F}_{\sharp}^{\langle m \rangle} = \sigma_m^{-1}\mathcal{F}_{\sharp}$, we obtain the following from the above lemma:

Corollary (2.9). *κ is an eigenvalue of $-\Delta_{\sharp}$ if and only if $\left(\frac{1-c}{N}\right)^m \kappa$ is an eigenvalue of $-\Delta_{\sharp}^{\langle m \rangle}$.*

Lemma (2.10). (Kusuoka [5])

$$0 < C = \inf_{\substack{f \in \mathcal{F} \\ \Pr_{(-\infty, 0]} f = 0}} \frac{\mathcal{E}(f, f)}{\max_{x, y \in E}\{|f(x) - f(y)|^2\}} < \infty\,.$$

Since $\lambda_1 = 0$ with the eigenspace consisting of constant functions, $\Pr_{(-\infty,0]} f = 0$ means simply $\int_E f(x)\mu(dx) = 0$. By the scaling property (2.8), we have

Corollary (2.11).

$$0 < (1-c)^m C = \inf_{\substack{f \in \mathcal{F}^{\langle m \rangle} \\ \Pr^{\langle m \rangle}_{(-\infty,0]} f = 0}} \frac{\mathcal{E}^{\langle m \rangle}(f, f)}{\max_{x, y \in E^{\langle m \rangle}}\{|f(x) - f(y)|^2\}} < \infty\,.$$

3. Existence of the integrated density of states

3-1. The Laplacian with Poisson obstacles

We first define random operators on the m-th expanded nested fractal $E^{\langle m \rangle}$. Let $\{N(B, \omega);\ B \in \mathcal{B}(E^{\langle \infty \rangle})\}$† be a Poisson random measure with the intensity measure $\nu\mu$ (ν is a positive constant) defined on a probability space (Ω, Σ, P). Define $\mathcal{N}_\omega$ and $\mathcal{F}_{\sharp,\omega}^{\langle m \rangle}$ by

$$\mathcal{N}_\omega = \{x;\ N(\{x\}, \omega) = 1\}\,,$$
$$\mathcal{F}_{\sharp,\omega}^{\langle m \rangle} = \{f \in \mathcal{F}_{\sharp}^{\langle m \rangle};\ f(p) = 0,\ p \in \mathcal{N}_\omega \cap E^{\langle m \rangle}\}\,.$$

By (2.11) it is easy to see that the symmetric form $(\mathcal{F}_{\sharp,\omega}^{\langle m \rangle}, \mathcal{E}^{\langle m \rangle})$ is closed on $L^2(E^{\langle m \rangle};\ \mu)$, and we get the following proposition:

Proposition (3.1). *$(\mathcal{F}_{\sharp,\omega}^{\langle m \rangle}, \mathcal{E}^{\langle m \rangle})$ is a regular local Dirichlet form on* $L^2(E^{\langle m \rangle} \backslash$

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† $\mathcal{B}(E^{\langle \infty \rangle})$ is a topological σ-field on $E^{\langle \infty \rangle}$

$\mathcal{N}_\omega; \mu)(L^2(\overset{o}{E}{}^{\langle m\rangle}\backslash\mathcal{N}_\omega; \mu)$, *if* $\sharp=0$).

We denote by $\Delta_{\sharp,\omega}^{\langle m\rangle}$ the self-adjoint operator on $L^2(E^{\langle m\rangle}; \mu)$ associated with the Dirichlet form $(\mathcal{F}_{\sharp,\omega}^{\langle m\rangle}, \mathcal{E}^{\langle m\rangle})$. The structure of the spectrum of these random operators is like that of the Laplacian defined in §2. So we use the same notions defined in §2 but add the subscript ω, that is, $\lambda_{\sharp,\omega,j}^{\langle m\rangle}$, $k_{\sharp,\omega}^{\langle m\rangle}(\lambda)$, $\cdots$.

Notice that

$$I_{m,m-1}^0 = \{(\cdots, 0, \overset{-(m-1)}{\overset{\vee}{i}}, 0, \cdots, 0); i = 0, 1, \cdots, N-1\}.$$

Thus when $\boldsymbol{i}=(\cdots, 0, \overset{-(m-1)}{\overset{\vee}{i}}, 0, \cdots, 0)$, we simply write $E^{\langle m-1\rangle,i}$ for $(E^{\langle m-1\rangle})_{\boldsymbol{i}}$, and $k_\omega^{\langle m-1\rangle,i}(\lambda)$ for $k_\omega^{\langle m-1\rangle,\boldsymbol{i}}(\lambda)$ and write similarly for other notions. From (2.5), it follows that

$$\text{(3.2)}\qquad \begin{gathered} E^{\langle m\rangle} = \bigcup_{i=0}^{N-1} E^{\langle m-1\rangle,i}, \\ E^{\langle m-1\rangle,i} \cap E^{\langle m-1\rangle,j} = \partial E^{\langle m-1\rangle,i} \cap \partial E^{\langle m-1\rangle,j}, \quad \text{for } i \neq j. \end{gathered}$$

We start with the following lemma:

Lemma (3.3). *For any* $\omega\in\Omega$ *and* m,

$$\text{(1)}\qquad 0\leq k_\omega^{\langle m\rangle}(\lambda)-k_{0,\omega}^{\langle m\rangle}(\lambda)\leq M.$$

$$\text{(2)}\qquad \sum_{i=0}^{N-1} k_{0,\omega}^{\langle m-1\rangle,i}(\lambda)\leq k_{0,\omega}^{\langle m\rangle}(\lambda).$$

$$\text{(3)}\qquad \sum_{i=0}^{N-1} k_\omega^{\langle m-1\rangle,i}(\lambda)\geq k_\omega^{\langle m\rangle}(\lambda).$$

Proof. (1) of this lemma will be proved in §3-3. As for (2), we first define a quadratic form $(\tilde{\mathcal{F}}_0, \tilde{\mathcal{E}}_0)$ as follows:

$$\begin{gathered} \tilde{\mathcal{F}}_0 = \{u\in\mathcal{F}_{0,\omega}^{\langle m\rangle}; u(p) = 0, p\in \textstyle\bigcup_{i=0}^{N-1} \partial E^{\langle m-1\rangle,i}\}, \\ \tilde{\mathcal{E}}_0(u, v) = \displaystyle\sum_{i=0}^{N-1} \mathcal{E}^{\langle m-1\rangle,i}(u|_{E^{\langle m-1\rangle,i}}, v|_{E^{\langle m-1\rangle,i}}), \quad \text{for } u, v\in\tilde{\mathcal{F}}_0. \end{gathered}$$

The quadratic form $(\tilde{\mathcal{F}}_0, \tilde{\mathcal{E}}_0)$ is a regular local Dirichlet form on $L^2(\bigcup_{i=0}^{N-1} \overset{o}{E}{}^{\langle m-1\rangle,i}\backslash \mathcal{N}_\omega; \mu)$. Let $\tilde{\Delta}_0$ be the self-adjoint operator on $L^2(E^{\langle m\rangle}; \mu)$ associated with $(\tilde{\mathcal{F}}_0, \tilde{\mathcal{E}}_0)$, and $\tilde{k}_0(\lambda)$ the eigenvalue counting function of $-\tilde{\Delta}_0$. Because of (3.2), it can be seen that the eigenvalues of $-\tilde{\Delta}_0$: $\tilde{\lambda}_{0,1}\leq\tilde{\lambda}_{0,2}\leq \cdots \leq\tilde{\lambda}_{0,n}\leq \cdots$ are obtained by collecting the eigenvalues of $-\Delta_{0,\omega}^{\langle m-1\rangle,i}$, $(i=0, 1, \cdots, N-1)$ and rearranging them in ascending order. Thus we get

$$\tilde{k}_0(\lambda) = \sum_{i=0}^{N-1} k_{0,\omega}^{\langle m-1\rangle,i}(\lambda).$$

On the other hand, since $\tilde{\mathcal{F}}_0 \subset \mathcal{F}_{0,\omega}^{\langle m\rangle}$ and $\tilde{\mathcal{E}}_0(u,u)=\mathcal{E}^{\langle m\rangle}(u,u)$ on $\tilde{\mathcal{F}}_0$, the min-max principle (see [9]) leads us to

$$\tilde{\lambda}_{0,n} \geq \lambda_{0,\omega,n}^{\langle m\rangle}.$$

Therefore we have $\sum_{i=0}^{N-1} k_{0,\omega}^{\langle m-1\rangle,i}(\lambda) \leq k_{0,\omega}^{\langle m\rangle}(\lambda)$.

As for (3), let $\tilde{\mathcal{F}}$ be the totality of the function $u \in l(E^{\langle m\rangle})$ such that, for any $i=0, \cdots, N-1$, there exists $u_i \in \mathcal{F}_{\omega}^{\langle m-1\rangle,i}$, whose restriction to $\overset{o}{E}{}^{\langle m-1\rangle,i}$ coincides with $u|_{\overset{o}{E}{}^{\langle m-1\rangle,i}}$, and let

$$\tilde{\mathcal{E}}(u,v) = \sum_{i=0}^{N-1} \mathcal{E}^{\langle m-1\rangle,i}(u_i, v_i) \quad \text{for } u, v \in \tilde{\mathcal{F}}.$$

Then this quadratic form $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$ is a Dirichlet form on $L^2(E^{\langle m\rangle}; \mu)$. Let $\tilde{k}(\lambda)$ be the eigenvalue counting function associated with $(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$. By (3.2), we also have $\tilde{k}(\lambda)=\sum_{i=0}^{N-1} k_{\omega}^{\langle m-1\rangle,i}(\lambda)$. Since $\tilde{\mathcal{F}} \supset \mathcal{F}_{\omega}^{\langle m\rangle}$ and $\mathcal{E}(u,u)=\mathcal{E}^{\langle m\rangle}(u,u)$ on $\mathcal{F}_{\omega}^{\langle m\rangle}$, we get $\tilde{k}(\lambda) \geq k_{\omega}^{\langle m\rangle}(\lambda)$. ■

Theorem (3.4). *There exists a non-random right continuous non-decreasing function $k(\lambda)$ such that for almost all $\omega \in \Omega$,*

$$k(\lambda) = \lim_{n\to\infty} \frac{k_{\omega}^{\langle n\rangle}(\lambda)}{N^n} = \lim_{n\to\infty} \frac{k_{0,\omega}^{\langle n\rangle}(\lambda)}{N^n} \quad \textit{for all continuous points of } k(\lambda).$$

We further have the Dirichlet-Neumann bracketing, that is:

$$\text{(3.5)} \qquad \frac{1}{N^m} \boldsymbol{E}[k_{0,\omega}^{\langle m\rangle}(\lambda)] \leq k(\lambda) \leq \frac{1}{N^m} \boldsymbol{E}[k_{\omega}^{\langle m\rangle}(\lambda)] \quad \textit{for any } m,$$

where $\boldsymbol{E}$ stands for the expectation with respect to the probability $\boldsymbol{P}$.

Proof. By using (2) and (3) of Lemma (3.3) repeatedly, we have for $n>m$

$$\sum_{i \in I_{n,m}^0} k_{0,\omega}^{\langle m\rangle,i}(\lambda) \leq k_{0,\omega}^{\langle n\rangle}(\lambda) \leq k_{\omega}^{\langle n\rangle}(\lambda) \leq \sum_{i \in I_{n,m}^0} k_{\omega}^{\langle m\rangle,i}(\lambda).$$

By the law of large numbers,

$$\frac{1}{N^{n-m}} \sum_{i \in I_{n,m}^0} k_{\sharp,\omega}^{\langle m\rangle,i}(\lambda) \to \boldsymbol{E}[k_{\sharp,\omega}^{\langle m\rangle}(\lambda)] \quad \text{a.s.}$$

as $n \to \infty$. Hence we have, almost surely

$$\text{(3.6)} \qquad \frac{\boldsymbol{E}[k_{0,\omega}^{\langle m\rangle}(\lambda)]}{N^m} \leq \varliminf_{n\to\infty} \frac{k_{\sharp,\omega}^{\langle n\rangle}(\lambda)}{N^n} \leq \varlimsup_{n\to\infty} \frac{k_{\sharp,\omega}^{\langle n\rangle}(\lambda)}{N^n} \leq \frac{\boldsymbol{E}[k_{\omega}^{\langle m\rangle}(\lambda)]}{N^m},$$

for any m. Furthermore by (1) of Lemma (3.3),

$$(3.7)\qquad \frac{E[k_{0,\omega}^{\langle m\rangle}(\lambda)]}{N^m}\leq \lim_{n\to\infty}\frac{k_{\sharp,\omega}^{\langle n\rangle}(\lambda)}{N^n}\leq \overline{\lim_{n\to\infty}}\frac{k_{\sharp,\omega}^{\langle n\rangle}(\lambda)}{N^n}\leq\frac{E[k_{0,\omega}^{\langle m\rangle}(\lambda)]}{N^m}+\frac{M}{N^m}.$$

On the other hand, by (2) of Lemma (3.3),

$$E[k_{0,\omega}^{\langle m-1\rangle}(\lambda)]=\frac{1}{N}\sum_{i=0}^{N-1}E[k_{0,\omega}^{\langle m-1\rangle,i}(\lambda)]$$
$$\leq\frac{1}{N}E[k_{0,\omega}^{\langle m\rangle}(\lambda)].$$

Therefore $\{\frac{E[k_{0,\omega}^{\langle m\rangle}(\lambda)]}{N^m}\}$ is non-decreasing in m. Letting m tend to infinity in (3.7), we have

$$\lim_{m\to\infty}\frac{E[k_{0,\omega}^{\langle m\rangle}(\lambda)]}{N^m}=\lim_{n\to\infty}\frac{k_{\sharp,\omega}^{\langle n\rangle}(\lambda)}{N^n},\qquad \text{a.s..}$$

Let $\tilde{k}(\lambda)$ denote the left hand side of the above equality and let

$$k(\lambda)=\lim_{\substack{\lambda'\in Q\\ \lambda'\downarrow\lambda}}\tilde{k}(\lambda').$$

This is what we want. The inequality (3.5) follows from (3.6). ■

3-2. Random Schrödinger operators

In §3-1, we gave the random operators on the nested fractal by randomizing the domains of the Dirichlet forms. In this section we consider another type of Dirichlet forms with the random killing measures. Let $\hat{\Omega}$ be the set of positive Radon measures on $E^{\langle\infty\rangle}$, and $\hat{\Sigma}$ the smallest σ-field on $\hat{\Omega}$ making all $\hat{\omega}(B)$, $B\in\mathcal{B}(E^{\langle\infty\rangle})$, measurable. We shall then consider a probability measure $\hat{P}$ on $(\hat{\Omega},\hat{\Sigma})$ satisfying below.

(3.8) ASSUMPTION. *σ-fields $\sigma[\hat{\omega}|_{E_i}]$, $i\in I^0$ are independent. For $B\in\mathcal{B}(E)$, random variables $\hat{\omega}(B_i)$, $i\in I^0$ are integrable and identically distributed.*

(3.9) DEFINITION. For $f,g\in\mathcal{F}_{\sharp}^{\langle m\rangle}$, $\hat{\omega}\in\hat{\Omega}$, we define a symmetric form $(\mathcal{F}_{\sharp}^{\langle m\rangle}, \mathcal{E}_{\hat{\omega}}^{\langle m\rangle})$ by

$$\mathcal{E}_{\hat{\omega}}^{\langle m\rangle}(f,g)=\mathcal{E}^{\langle m\rangle}(f,g)+\int_{E^{\langle m\rangle}}f(x)g(x)\hat{\omega}(dx).$$

The symmetric form $(\mathcal{F}_{\sharp}^{\langle m\rangle},\mathcal{E}_{\hat{\omega}}^{\langle m\rangle})$ is a regular local Dirichlet form on $L^2(E^{\langle m\rangle};\mu)$ $(L^2(\mathring{E}^{\langle m\rangle};\mu)$, if $\sharp=0)$. The self-adjoint operator on $L^2(E^{\langle m\rangle};\mu)$ associated with $(\mathcal{F}_{\sharp}^{\langle m\rangle},\mathcal{E}_{\hat{\omega}}^{\langle m\rangle})$, which is of compact reslovent, is denoted by $H_{\sharp,\hat{\omega}}^{\langle m\rangle}$. Other related notions are distinguished by the subscript $\hat{\omega}$ as in §3-1. For $i\in I^0$ we define a Dirichlet form $(\mathcal{F}_{\sharp}^{\langle m\rangle,i},\mathcal{E}_{\hat{\omega}}^{\langle m\rangle,i})$ on $L^2(E^{\langle m\rangle,i};\mu)$ by

$$\mathcal{E}^{\langle m\rangle,i}_{\hat{\omega}}(u,v)=\mathcal{E}^{\langle m\rangle,i}(u,v)+\int_{(E^{\langle m\rangle})_i}u(x)v(x)\hat{\omega}(dx),\quad \text{for } u,v\in\mathcal{F}^{\langle m\rangle,i}_{*},$$

and denote the associated eigenvalue counting function by $k^{\langle m\rangle,i}_{*,\hat{\omega}}(\lambda)$.

Lemma (3.10). *For any $\hat{\omega}\in\hat{\Omega}$ and m,*

(1) $$0\le k^{\langle m\rangle}_{\hat{\omega}}(\lambda)-k^{\langle m\rangle}_{0,\hat{\omega}}(\lambda)\le M.$$

(2) $$\sum_{i=0}^{N-1}k^{\langle m-1\rangle,i}_{0,\hat{\omega}}(\lambda)\le k^{\langle m\rangle}_{0,\hat{\omega}}(\lambda).$$

(3) $$\sum_{i=0}^{N-1}k^{\langle m-1\rangle,i}_{\hat{\omega}}(\lambda)\ge k^{\langle m\rangle}_{\hat{\omega}}(\lambda).$$

We shall prove (1) of this lemma in §3-3. Other assertions of the lemma and the following theorem are proved in the same way as Lemma (3.3) and Theorem (3.4) respectively.

Theorem (3.11). *There exists a non-random right continuous and non-decreasing function $\hat{k}(\lambda)$ such that for almost all $\hat{\omega}\in\hat{\Omega}$,*

$$\hat{k}(\lambda)=\lim_{n\to\infty}\frac{k^{\langle n\rangle}_{\hat{\omega}}(\lambda)}{N^n}=\lim_{n\to\infty}\frac{k^{\langle n\rangle}_{0,\hat{\omega}}(\lambda)}{N^n}\quad \textit{for all continuous points of } \hat{k}(\lambda).$$

We further have the Dirichlet-Neumann bracketing:

(3.12) $$\frac{1}{N^m}\hat{E}[k^{\langle m\rangle}_{0,\hat{\omega}}(\lambda)]\le\hat{k}(\lambda)\le\frac{1}{N^m}\hat{E}[k^{\langle m\rangle}_{\hat{\omega}}(\lambda)]\quad \textit{for any } m,$$

where $\hat{E}$ stands for the expectation with respect to the probability $\hat{P}$.

We state two examples of $(\hat{\Omega},\hat{\Sigma},\hat{P})$ that will be treated in §4.

(3.13) *Poisson noise.* We recall that $N(\cdot,\omega)$ is a Poisson random measure with intensity measure $\nu\mu(\nu>0)$ on the probability space (Ω,Σ,P). We define $\pi:\Omega\to\hat{\Omega}$ as

$$\pi(\omega)(dx)=\kappa N(dx,\omega)\quad(\kappa>0).$$

Then the Poisson noise $(\hat{\Omega},\hat{\Sigma},\hat{P})$ is defined as the image measure of P by π.

(3.14) *Poisson integral potential.* Let $\phi(x,y)$ be a non-negative function on $E^{\langle\infty\rangle}\times E^{\langle\infty\rangle}$, which satisfies following conditions: for any $i\in I^0$,

$$\phi(x,y)=\phi(\Phi_i(x),\Phi_i(y))\quad \text{for } x,y\in E,$$
$$\phi(x,y)=0\quad \text{if } x\in E_i,\ y\in E_i^c \text{ or } x\in E_i^c,\ y\in E_i$$
$$0<\int_{E\times E}\phi(x,y)\mu(dx)\mu(dy)<\infty.$$

Define $\pi: \Omega \to \hat{\Omega}$ by

$$\pi(\omega)(dx) = \int_{E^{\langle\infty\rangle}} \phi(x, y) N(dy, \omega) \mu(dx) .$$

Then the Poisson integral potential $(\hat{\Omega}, \hat{\Sigma}, \hat{P})$ is defined as the image measure of P by π.

3-3. Proof of Lemma (3.3)(1) and (3.10)(1)

For a positive Radon measure η on E, we define a Dirichlet form $(\mathcal{F}_\sharp, \mathcal{E}_\eta)$ on $L^2(E; \mu)$ as

$$\mathcal{E}_\eta(f, g) = \mathcal{E}(f, g) + \int_E f(x) g(x) \eta(dx) \quad \text{for } f, g \in \mathcal{F}_\sharp .$$

The self-adjoint operator on $L^2(E; \mu)$ associated with $(\mathcal{F}_\sharp, \mathcal{E}_\eta)$ is denoted by $H_{\sharp, \eta}$, and other notions are distinguished by the subscript η.

Proposition (3.15). ([5]) *For any positive Radon measure η on E,*
(1) *$\mathcal{F}$ is a Hilbert space with the inner product $\mathcal{E}_{\eta+\mu}$,*
(2) *$\mathcal{F}_0$ is a closed subspace of $\mathcal{F}$ and* $\dim (\mathcal{F}_0^{\perp, \mathcal{E}_{\eta+\mu}})^\dagger = M$.

We need the next version of the min-max principle.

Proposition (5.16). *For $\varphi_1, \cdots, \varphi_{n-1} \in \mathcal{F}_\sharp$, let*

$$\begin{aligned} &\tilde{\lambda}_\sharp(\varphi_1, \cdots, \varphi_{n-1}) \\ &\quad = \inf \left\{ \frac{\mathcal{E}_{\eta+\mu}(u, u)}{(u, u)_{L^2(E;\mu)}} ; u \in \mathcal{F}_\sharp, \mathcal{E}_{\eta+\mu}(u, \varphi_i) = 0 \quad \text{for } i = 1, 2, \cdots, n-1 \right\} . \end{aligned}$$

We then have

$$\lambda_{\sharp, \eta+\mu, n} = \sup_{\substack{\varphi_i \in \mathcal{F}_\sharp \\ 1 \le i \le n-1}} \tilde{\lambda}_\sharp(\varphi_1, \cdots, \varphi_{n-1}) . \tag{3.17}$$

Proof. Let the right hand side of (3.17) be $\tilde{\lambda}_n$. It suffices to prove the following:
(1) if $\dim[\mathrm{Ran}(\mathrm{Pr}_{(-\infty, a]})] \ge n$, then $\tilde{\lambda}_n \le a$,
(2) if $\dim[\mathrm{Ran}(\mathrm{Pr}_{(-\infty, a]})] < n$, then $\tilde{\lambda}_n \ge a$,
where we simply write $\mathrm{Pr}_{(-\infty, a]}$ for $\mathrm{Pr}_{\sharp, \eta+\mu, (-\infty, a]}$.

If $\dim[\mathrm{Ran}(\mathrm{Pr}_{(-\infty, a]})] \ge n$, then there exists an n-dimensional subspace V of $\mathcal{F}_\sharp$ such that for any $\psi \in V$

$$\mathcal{E}_{\eta+\mu}(\psi, \psi) \le a(\psi, \psi) .$$

Because $(\mathcal{F}_\sharp, \mathcal{E}_{\eta+\mu})$ is a Hilbert space, for any $\varphi_1, \cdots, \varphi_{n-1} \in \mathcal{F}_\sharp$, there exists

† We denote by $\mathcal{L}^{\perp, \mathcal{E}_{\eta+\mu}}$ the orthogonal complement of a subset $\mathcal{L}$ of the Hilbert space $(\mathcal{F}_\sharp, \mathcal{E}_{\eta+\mu})$

$\psi\in\mathcal{F}_{\sharp}$ such that $\psi\neq 0$ and $\psi\in V\cap[\varphi_1, \cdots, \varphi_{n-1}]^{\perp,\mathcal{E}_{\eta+\mu}}$.† Therefore we have $\tilde{\lambda}_{\sharp}(\varphi_1, \cdots, \varphi_{n-1})\leq a$ for any $\varphi_1, \cdots, \varphi_{n-1}\in\mathcal{F}_{\sharp}$, that is, $\tilde{\lambda}_n\leq a$.

If $\dim[\mathrm{Ran}(\mathrm{Pr}_{(-\infty,a]})]\leq n-1$, there exist $\varphi_1^{(0)}, \cdots, \varphi_{n-1}^{(0)}\in\mathcal{F}_{\sharp}$ such that $[\varphi_1^{(0)}, \cdots, \varphi_{n-1}^{(0)}]=\mathrm{Ran}(\mathrm{Pr}_{(-\infty,a]})$. We can see that $\psi\in\mathrm{Ran}(\mathrm{Pr}_{(a,\infty)})$ if and only if $\varphi\in[\varphi_1^{(0)}, \cdots, \varphi_{n-1}^{(0)}]^{\perp,\mathcal{E}_{\eta+\mu}}$. Indeed, let $\varphi_i^{(0)}$ be an eigenfunction belonging to the eigenvalue $\lambda_i\in[1, a)$ $(i=1, \cdots, n-1)$ of $H_{\sharp,\eta+\mu}$. Then,

$$\begin{aligned}\psi\in\mathrm{Ran}(\mathrm{Pr}_{(a,\infty)})&\Leftrightarrow(\psi, \varphi_i^{(0)}) = 0 \quad \text{for } i = 1, \cdots, n-1,\\ &\Leftrightarrow\mathcal{E}_{\eta+\mu}(\psi, \varphi_i^{(0)}) = \lambda_i(\psi, \varphi_i^{(0)}) = 0 \quad \text{for } i = 1, \cdots, n-1.\end{aligned}$$

Therefore $\tilde{\lambda}_{\sharp}(\varphi_1^{(0)}, \cdots, \varphi_{n-1}^{(0)})\geq a$, that is, $\tilde{\lambda}_n\geq a$. ■

Now we prove Lemma (3.3)(1) and Lemma (3.10)(1) but it is enough to treat the case that $m=0$.

Lemma (3.18). *For any $\omega\in\Omega$ and $\hat{\omega}\in\hat{\Omega}$,*

(1) $$0\leq k_{\omega}(\lambda)-k_{0,\omega}(\lambda)\leq M.$$

(2) $$0\leq k_{\hat{\omega}}(\lambda)-k_{0,\hat{\omega}}(\lambda)\leq M.$$

Proof. We first show the inequalities (2). The first inequaliyt is derived from the min-max principle and the inclusion $\mathcal{F}_0\subset\mathcal{F}$. As for the second inequality, there exists by Proposition (3.15) a complete orthonormal system $\{\varphi_i^{(0)}\}$ of $(\mathcal{F}, \mathcal{E}_{\hat{\omega}+\mu})$ such that

$$[\varphi_1^{(0)}, \cdots, \varphi_M^{(0)}] = \mathcal{F}_0^{\perp,\mathcal{E}_{\hat{\omega}+\mu}}.$$

For any $\varphi_1, \cdots, \varphi_{n-1}\in\mathcal{F}_0$, we then have

$$\tilde{\lambda}(\varphi_1, \cdots, \varphi_{n-1}, \varphi_1^{(0)}, \cdots, \varphi_M^{(0)}) = \tilde{\lambda}_0(\varphi_1, \cdots, \varphi_{n-1})\,.$$

Therefore $\lambda_{\hat{\omega}+\mu,M+n}\geq\lambda_{0,\hat{\omega}+\mu,n}$. Because $\lambda_{\sharp,\hat{\omega}+\mu,n}=\lambda_{\sharp,\hat{\omega},n}+1$, we get $k_{0,\hat{\omega}}(\lambda)\geq k_{\hat{\omega}}(\lambda)-M$. The inequalities in (1) are derived in the same manner by noting that $\mathcal{F}_{0,\omega}$ is a closed subspace of the Hilbert space $(\mathcal{F}_{\omega}, \mathcal{E}_{\mu})$ and $\dim(\mathcal{F}_{0,\omega}^{\perp,\mathcal{E}_{\mu}})\leq M$. ■

4. Lifschitz tails for the integrated density of states

In this section we investigate the behavior of $k(\lambda)$ and $\hat{k}(\lambda)$ when λ tends to 0, and observe the Lifschitz singularity.

4-1. The Laplacian with Poisson obstacles

Theorem (4.1). *Let C be the constant in Lemma* (2.10). *Then for $\lambda\in[0, C)$,*

† $[\varphi_1, \cdots, \varphi_{n-1}]$ means the subspace generated by $\varphi_1, \cdots, \varphi_{n-1}$.

$$\frac{1}{N}\left(\frac{\lambda}{\lambda_{0,1}}\right)^{d_s/2}\exp\left\{-\nu N\left(\frac{\lambda}{\lambda_{0,1}}\right)^{-d_s/2}\right\}\leq k(\lambda)\leq N\left(\frac{\lambda}{C}\right)^{d_s/2}\exp\left\{-\frac{\nu}{N}\left(\frac{\lambda}{C}\right)^{-d_s/2}\right\}.$$

Proof of the lower estimate. If $N(E^{\langle m\rangle},\omega)=0$, then $k_{0,\omega}^{\langle m\rangle}(\lambda)=k_0^{\langle m\rangle}(\lambda)$. Thus we have

$$\begin{aligned}\boldsymbol{E}[k_{0,\omega}^{\langle m\rangle}(\lambda)]&\geq\boldsymbol{E}[k_{0,\omega}^{\langle m\rangle}(\lambda);N(E^{\langle m\rangle},\omega)=0]\\&=k_0^{\langle m\rangle}(\lambda)P(N(E^{\langle m\rangle},\omega)=0)=k_0^{\langle m\rangle}(\lambda)e^{-\nu N^m},\quad\text{for any } m.\end{aligned}$$

Let d_s be the spectral dimension of the nested fractal (Ψ,E): $d_s=\dfrac{2\log N}{\log N-\log(1-c)}$ (see [2]). For any $\lambda\in(0,\lambda_{0,1})$, by Corollary (2.9), we choose n such that $\lambda_{0,1}^{\langle n\rangle}\leq\lambda<\lambda_{0,1}^{\langle n-1\rangle}$ so $k_0^{\langle n\rangle}(\lambda)\geq1$, and we also have $\dfrac{1}{N^n}\geq\dfrac{1}{N}\left(\dfrac{\lambda}{\lambda_{0,1}}\right)^{d_s/2}$. Therefore it follows from (3.5) that

$$\begin{aligned}k(\lambda)&\geq\frac{1}{N^n}e^{-\nu N^n}\\&\geq\frac{1}{N}\left(\frac{\lambda}{\lambda_{0,1}}\right)^{d_s/2}\exp\left\{-\nu N\left(\frac{\lambda}{\lambda_{0,1}}\right)^{-d_s/2}\right\},\quad\lambda\in(0,\lambda_{0,1}).\end{aligned}\tag{4.2}$$

Notice that $C\leq\lambda_{0,1}$ by the next lemma.

Proof of the upper estimate. We use the following lemma:

Lemma (4.3). *Let C be the constant in Lemma* (2.10) *and c the constant in* (2.6).

(1) $$C\leq\lambda_{0,1}\wedge\lambda_2.$$

(2) $$\left(\frac{1-c}{N}\right)^nC\leq\lambda_{\omega,1}^{\langle n\rangle}\quad\textit{if}\quad N(E^{\langle n\rangle},\omega)\neq0.$$

Proof. From Lemma (2.10) it follows that for any $f\in\mathcal{F}_0$

$$C(f,f)_{L^2(E;\mu)}\leq C\max_{x\in E}f^2(x)=C\max_{\substack{x\in E\\y\in\partial E}}\{|f(x)-f(y)|^2\}\leq\mathcal{E}(f,f).$$

Hence $C\leq\lambda_{0,1}$. (2) is then a consequence of (2.9). Note that $\lambda_1=0$ and that the eigenfunction of λ_1 is a constant function on E. Thus, if $\Pr_{(-\infty,0]}f=0$, then $\int_E f\,d\mu=0$ and

$$\begin{aligned}\int_E f^2(x)\mu(dx)&=\int_E\left\{\int_E(f(x)-f(y))\mu(dy)\right\}^2\mu(dx)\\&\leq\max_{x,y\in E}\{|f(x)-f(y)|^2\}.\end{aligned}$$

Therefore we get $C\leq\lambda_2$ by Lemma (2.10). ■

For any $\lambda\in(0, C)$, we choose n such that

$$\left(\frac{1-c}{N}\right)^{n+1}C\leq\lambda<\left(\frac{1-c}{N}\right)^{n}C.$$

If $k_{\omega}^{\langle n\rangle}(\lambda)>0$, then $\lambda\geq\lambda_{\omega,1}^{\langle n\rangle}$. From this and (3.5), it follows that

$$\begin{aligned}k(\lambda)&\leq\frac{1}{N^{n}}\boldsymbol{E}[k_{\omega}^{\langle n\rangle}(\lambda)]\\&=\frac{1}{N^{n}}\boldsymbol{E}[k_{\omega}^{\langle n\rangle}(\lambda);\ \lambda_{\omega,1}^{\langle n\rangle}<\left(\frac{1-c}{N}\right)^{n}C].\end{aligned}$$

By virtue of Lemma (4.3), we get

$$\begin{aligned}k(\lambda)&\leq\frac{1}{N^{n}}\boldsymbol{E}[k_{\omega}^{\langle n\rangle}(\lambda);\ N(E^{\langle n\rangle},\omega)=0]\\&=\frac{k^{\langle n\rangle}(\lambda)}{N^{n}}\exp(-\nu N^{n})=\frac{1}{N^{n}}\exp(-\nu N^{n}).\end{aligned}$$

Since $\frac{1}{N^{n}}\leq N\left(\frac{\lambda}{C}\right)^{d_s/2}$, we obtain

$$k(\lambda)\leq N\left(\frac{\lambda}{C}\right)^{d_s/2}\exp\left\{-\frac{\nu}{N}\left(\frac{\lambda}{C}\right)^{-d_s/2}\right\},\quad\lambda\in(0, C).\tag{4.4}$$

REMARK. On account of the fact that $C\leq\lambda_{0,1}$, $N\geq2$, the upper bound and the lower bound of (4.1) do not coincide.

4-2. Random Schrödinger operators

We first make assumptions on the distribution of $\text{\textcircled{a}}(E)$ and $\text{\textcircled{a}}(\overset{o}{E})$.

(4.5) ASSUMPTION.
(1) There exist positive constants ε_0, γ and δ such that $\hat{P}(\text{\textcircled{a}}(E)<\varepsilon)\geq\gamma\varepsilon^{\delta}$ for any $\varepsilon\in(0, \varepsilon_0)$.
(2) $\hat{P}(\text{\textcircled{a}}(\overset{o}{E})\in dx)$ does not reduce to a δ-distribution concentrated at the origin.

Theorem (4.6). *Under the above assumptions,*

$$\lim_{\lambda\downarrow0}\frac{\log[-\log\hat{k}(\lambda)]}{\log\lambda}=-\frac{d_s}{2}.$$

Proof of the lower estimate. Let φ be a normalized eigenfunction belonging to the eigenvalue $\lambda_{0,1}$. Then $\varphi^{\langle m\rangle}=\frac{1}{N^{m/2}}\sigma_m^{-1}\varphi$ is a normalized eigenfunction belonging to the eigenvalue $\lambda_{0,1}^{\langle m\rangle}$. If $\text{\textcircled{a}}(E^{\langle m\rangle})\leq\varepsilon$, we then have

$$\begin{aligned}\lambda_{0,\hat{\omega},1}^{\langle m\rangle} &\leq \mathcal{E}^{\langle m\rangle}(\varphi^{\langle m\rangle}, \varphi^{\langle m\rangle}) + \int_{E^{\langle m\rangle}} (\varphi^{\langle m\rangle}(x))^2 \hat{\omega}(dx) \\ &\leq \lambda_{0,1}^{\langle m\rangle} + \sup_{x\in E^{\langle m\rangle}} (\varphi^{\langle m\rangle}(x))^2 \hat{\omega}(E^{\langle m\rangle}) \\ &\leq \lambda_{0,1}^{\langle m\rangle} + \frac{\varepsilon}{N^m} \sup_{x\in E} \varphi^2(x)\,.\end{aligned}$$

For $\varepsilon\in(0, \varepsilon_0)$, let $\varepsilon_m=(1-c)^m\varepsilon$ and $L=\sup_{x\in E}\varphi^2(x)$. By the above inequality and Corollary (2.9), if $\hat{\omega}(E^{\langle m\rangle})\leq\varepsilon_m$ then $\lambda_{0,\hat{\omega},1}^{\langle m\rangle}\leq(\frac{1-c}{N})^m(\lambda_{0,1}+\varepsilon L)$. For any $\lambda\in(0, \lambda_{0,1}+\varepsilon L)$, let m be the integer such that $(\frac{1-c}{N})^m(\lambda_{0,1}+\varepsilon L)\leq\lambda<(\frac{1-c}{N})^{m-1}(\lambda_{0,1}+\varepsilon L)$. Then we obtain

$$\begin{aligned}\hat{k}(\lambda) &\geq \frac{1}{N^m}\hat{E}[k_{0,\hat{\omega}}^{\langle m\rangle}(\lambda)] \geq \frac{1}{N^m}\hat{P}(\lambda\geq\lambda_{0,\hat{\omega},1}^{\langle m\rangle}) \\ &\geq \frac{1}{N^m}\hat{P}((\frac{1-c}{N})^m(\lambda_{0,1}+\varepsilon L)\geq\lambda_{0,\hat{\omega},1}^{\langle m\rangle}) \\ &\geq \frac{1}{N^m}\hat{P}(\hat{\omega}(E^{\langle m\rangle})\leq\varepsilon_m) \\ &\geq \frac{1}{N^m}\hat{P}(\cap_{i\in I_{m,0}^0}\{\hat{\omega}(E_i)<(\frac{1-c}{N})^m\varepsilon\}) \\ &= \frac{1}{N^m}\hat{P}(\hat{\omega}(E)<(\frac{1-c}{N})^m\varepsilon)^{N^m} \geq \frac{1}{N^m}\{\gamma(\frac{1-c}{N})^{m\delta}\varepsilon^\delta\}^{N^m}.\end{aligned}$$

Therefore

$$\hat{k}(\lambda)\geq\frac{1}{N}(\frac{\lambda}{\lambda_{0,1}+\varepsilon L})^{d_s/2}\left\{\gamma\varepsilon^\delta(\frac{1-c}{N}\,\frac{\lambda}{\lambda_{0,1}+\varepsilon L})^\delta\right\}^{N(\lambda/(\lambda_{0,1}+\varepsilon L))^{-d_s/2}} \quad \lambda\in(0, \lambda_{0,1}+\varepsilon L). \tag{4.7}$$

Proof of the upper estimate. We use the following lemma:

Lemma (4.8). *For $L>0$ and m, suppose $\varepsilon\geq\frac{C(1-c)^m}{L}$. Then*

$$\lambda_{\hat{\omega},1}^{\langle m\rangle}\geq(\frac{1-c}{N})^m\frac{C}{1+\varepsilon} \quad \textit{if } \hat{\omega}(E^{\langle m\rangle})\geq L.$$

Proof. For any $f\in\mathcal{F}^{\langle m\rangle}$ and $\varepsilon>0$, we have

$$f^2(x)\hat{\omega}(E^{\langle m\rangle})\leq(1+\varepsilon)\int_{E^{\langle m\rangle}}(f(x)-f(y))^2\hat{\omega}(dy)+(1+\frac{1}{\varepsilon})\int_{E^{\langle m\rangle}}f^2(y)\hat{\omega}(dy)\,,$$

and by Corollary (2.11)

$$\hat{\omega}(E^{\langle m\rangle})N^{-m}(f,f)_{L^2(E^{\langle m\rangle};\mu)}\leq\hat{\omega}(E^{\langle m\rangle})\sup_{x\in E^{\langle m\rangle}}f^2(x)$$
$$\leq(1+\varepsilon)(1-c)^{-m}C^{-1}\hat{\omega}(E^{\langle m\rangle})\mathcal{E}^{\langle m\rangle}(f,f)+(1+\frac{1}{\varepsilon})\int_{E^{\langle m\rangle}}f^2(y)\hat{\omega}(dy)\,.$$

Under the assumptions in the lemma, it holds that

$$N^{-m}(f,f)_{L^2(E^{\langle m\rangle};\mu)}$$
$$\leq(1+\varepsilon)(1-c)^{-m}C^{-1}\{\mathcal{E}^{\langle m\rangle}(f,f)+\frac{1}{\varepsilon L}(1-c)^m C\int_{E^{\langle m\rangle}}f^2(y)\hat{\omega}(dy)\}$$
$$\leq(1+\varepsilon)(1-c)^{-m}C^{-1}\mathcal{E}^{\langle m\rangle}_{\hat{\omega}}(f,f)\quad\text{for any } f\in\mathcal{F}^{\langle m\rangle}\,.$$

■

By Assumption (4.5), there exists a positive constant $\varepsilon>0$ such that $h_0=\hat{P}(\hat{\omega}(\overset{o}{E})\geq\varepsilon)>0$. For a fixed n, let $\varepsilon_n=\dfrac{2C(1-c)^n}{\varepsilon h_0 N^n}$ and $L_m=\dfrac{\varepsilon h_0}{2}N^m$ for $m\geq n$. For any $\lambda\in(0,(\dfrac{1-c}{N})^n\dfrac{C}{1+\varepsilon_n})$, we choose $m\geq n$ such that

$$(\frac{1-c}{N})^{m+1}\frac{C}{1+\varepsilon_n}\leqslant\lambda<(\frac{1-c}{N})^m\frac{C}{1+\varepsilon_n}\,.$$

Note that $\lambda<\lambda^{\langle m\rangle}_{\hat{\omega},2}$ by Lemma (4.3)(1). Then, it follows from Lemma (4.8) that

$$\hat{k}(\lambda)\leq\frac{1}{N^m}\hat{E}[k^{\langle m\rangle}_{\hat{\omega}}(\lambda)]=\frac{1}{N^m}\hat{P}(\lambda^{\langle m\rangle}_{\hat{\omega},1}\leq\lambda)$$
$$\leq\frac{1}{N^m}\hat{P}(\lambda^{\langle m\rangle}_{\hat{\omega},1}<(\frac{1-c}{N})^m\frac{C}{1+\varepsilon_n})$$
$$\leq\frac{1}{N^m}\hat{P}(\hat{\omega}(E^{\langle m\rangle})<L_m)\,.$$

For $\boldsymbol{i}\in I^0$, let $h_{\boldsymbol{i}}(\hat{\omega})$ be the random variables defined by

$$h_{\boldsymbol{i}}(\hat{\omega})=\begin{cases}1, & \text{for } \hat{\omega}(\overset{o}{E}_{\boldsymbol{i}})\geq\varepsilon\\ 0, & \text{for } \hat{\omega}(\overset{o}{E}_{\boldsymbol{i}})<\varepsilon\,,\end{cases}$$

and let $h(\hat{\omega})=h_{(\dots,0,0)}(\hat{\omega})$. We then have the following.

$$\sum_{\boldsymbol{i}\in I^0_{m,0}}h_{\boldsymbol{i}}(\hat{\omega})=\#\{\boldsymbol{i}\in I^0_{m,0};\,\hat{\omega}(\overset{o}{E}_{\boldsymbol{i}})\geq\varepsilon\}\,.$$
$$\sum_{\boldsymbol{i}\in I^0_{m,0}}h_{\boldsymbol{i}}(\hat{\omega})\geq\frac{h_0}{2}N^m\Rightarrow\hat{\omega}(E^{\langle m\rangle})\geq L_m\,.$$

Thus we obtain

$$\hat{k}(\lambda)\leq\frac{1}{N^m}\hat{P}(\sum_{i\in I^0_{m,0}}(\frac{h_0}{2}-h_i(\hat{\omega}))>0)$$
$$\leq\frac{1}{N^m}\hat{E}[\exp[y\sum_{i\in I^0_{m,0}}\{\frac{h_0}{2}-h_i(\hat{\omega})\}]]$$
$$=\frac{1}{N^m}e^{-(1/2)yh_0N^m}\exp[N^mF(y)],$$

where $F(y)=\log\hat{E}[\exp\{-y(h(\hat{\omega})-h_0)\}]$. Note that $\hat{E}[h(\hat{\omega})]=h_0$ and that $|h(\hat{\omega})-\frac{1}{2}|\leq\frac{1}{2}$. Hence $F(0)=F'(0)=0$ and $F''(y)\leq\frac{1}{4}$. Therefore $F(y)\leq\frac{1}{8}y^2$. From this, it follows that

$$\hat{k}(\lambda)\leq\frac{1}{N^m}e^{-(1/2)yh_0N^m+(1/8)y^2N^m}.$$

By setting $y=2h_0$ we conclude that

$$\hat{k}(\lambda)\leq N(\frac{1+\varepsilon_n}{C}\lambda)^{d_s/2}\exp\{-\frac{h_0^2}{2N}(\frac{1+\varepsilon_n}{C}\lambda)^{-d_s/2}\} \tag{4.9}$$
$$\lambda\in(0,(\frac{1-c}{N})^n\frac{C}{1+\varepsilon_n}).$$

In the rest of this section, we assume the concrete structures (3.13), (3.14) for the probability space $(\hat{\Omega},\hat{\Sigma},\hat{P})$ and obtain more precise information on $\hat{k}(\lambda)$. We identify $\hat{\Omega}$ with $\pi(\Omega)$ and write $\hat{\omega}$ for $\pi(\omega)$, $\omega\in\Omega$.

4-3. Poisson noise

Theorem (4.10).

$$-\nu N\lambda_{0,1}^{d_s/2}\leq\varliminf_{\lambda\downarrow 0}\lambda^{d_s/2}\log\hat{k}(\lambda)\leq\varlimsup_{\lambda\downarrow 0}\lambda^{d_s/2}\log\hat{k}(\lambda)\leq-\frac{\nu}{N}C^{d_s/2}.$$

Proof of the lower estimate. Using the Dirichlet-Neumann bracketing (3.12), the following lower estimate is obtained in the same manner as in §4-1:

$$\hat{k}(\lambda)\geq\frac{1}{N}(\frac{\lambda}{\lambda_{0,1}})^{d_s/2}\exp\{-\nu N(\frac{\lambda}{\lambda_{0,1}})^{-d_s/2}\}\quad\lambda\in(0,\lambda_{0,1}). \tag{4.11}$$

Proof of the upper estimate. We use the next lemma:

Lemma (4.12). *For any m, let $\varepsilon\geq\frac{(1-c)^mC}{\kappa}$. Then*

$$\lambda_{\omega,1}^{\langle m\rangle}\geq(\frac{1-c}{N})^m\frac{C}{1+\varepsilon}\quad \textit{if } \hat{\omega}(E^{\langle m\rangle})\neq 0.$$

Proof. Note that $\hat{\omega}(E^{\langle m\rangle})\neq 0$ implies $\hat{\omega}(E^{\langle m\rangle})\geq\kappa$. Then the lemma easily follows from Lemma (4.8). ■

For any $m\geq 0$, let $\varepsilon_m=\frac{(1-c)^m C}{\kappa}$. On account of Lemma (4.12), the upper estimate is obtained in the same way as in §4-1 by replacing C with $(\frac{1-c}{N})^m\frac{C}{1+\varepsilon_m}$ in (4.4): for any $m\geq 0$

$$\hat{k}(\lambda)\leq N(\frac{1+\varepsilon_m}{C}\lambda)^{d_s/2}\exp\{-\frac{\nu}{N}(\frac{1+\varepsilon_m}{C}\lambda)^{-d_s/2}\} \tag{4.13}$$
$$\lambda\in(0,(\frac{1-c}{N})^m\frac{C}{1+\varepsilon_m}).$$

4-4. Poisson integral potential

We first recall that the Poisson integral potential was defined in (3.14). Since $\int_{E\times E}\phi(x,y)\mu(dx)\mu(dy)$ is finite and strictly positive,

$$\mu(\{y;\int_E\phi(x,y)\mu(dx)>0\})=\tau>0. \tag{4.14}$$

Theorem (4.15). *For $\phi(x,y)$ satisfying the assumption* (3.14),

$$-\nu N\lambda_{0,1}^{d_s/2}\leq\varliminf_{\lambda\downarrow 0}\lambda^{d_s/2}\log\hat{k}(\lambda)\leq\varlimsup_{\lambda\downarrow 0}\lambda^{d_s/2}\log\hat{k}(\lambda)\leq-\frac{\nu\tau}{N}C^{d_s/2}.$$

Proof of the lower estimate. Noting that

$$\int_{E^{\langle m\rangle}}f^2(x)\hat{\omega}(dx)=\int_{E^{\langle m\rangle}}\left\{\int_{E^{\langle m\rangle}}\phi(x,y)f^2(x)\mu(dx)\right\}N(dy,\omega),$$

we have $k_{0,\hat{\omega}}^{\langle m\rangle}(\lambda)=k_0^{\langle m\rangle}(\lambda)$ if $N(E^{\langle m\rangle},\omega)=0$. Thus we can get the following lower estimate in the same manner as in §4-1:

$$\hat{k}(\lambda)\geq\frac{1}{N}(\frac{\lambda}{\lambda_{0,1}})^{d_s/2}\exp\{-\nu N(\frac{\lambda}{\lambda_{0,1}})^{-d_s/2}\}\quad\lambda\in(0,\lambda_{0,1}). \tag{4.16}$$

Proof of the upper estimate. For fixed $n,l\geq 0$, let $\varepsilon_n=C(1-c)^n$, and for $m\geq l+n$, $L_{m,n}=(1-c)^{m-n}$. For any $\lambda\in(0,(\frac{1-c}{N})^{n+l}\frac{C}{1+\varepsilon_n})$, let $m\geq l+n$ be the integer such that

$$(\frac{1-c}{N})^{m+1}\frac{C}{1+\varepsilon_n}\leq\lambda<(\frac{1-c}{N})^m\frac{C}{1+\varepsilon_n}.$$

Note that $k_{\hat{\omega}}^{\langle m\rangle}(\lambda)\leq 1$ because $\lambda<\lambda_2^{\langle m\rangle}\leq\lambda_{\hat{\omega},2}^{\langle m\rangle}$. From this and (3.12), it follows that

$$\hat{k}(\lambda) \leq \frac{1}{N^m} \hat{E}[k_{\hat{\omega}}^{\langle m \rangle}(\lambda)] = \frac{1}{N^m} \hat{P}(\lambda_{\hat{\omega},1}^{\langle m \rangle} \leq \lambda)$$
$$\leq \frac{1}{N^m} \hat{P}(\lambda_{\hat{\omega},1}^{\langle m \rangle} < (\frac{1-c}{N})^m \frac{C}{1+\varepsilon_n}) .$$

Let $A(\xi)$ and $A^{\langle m \rangle}(\xi)$ be the sets defined by

$$A(\xi) = \{y; \int_E \phi(x, y) \mu(dx) > \xi\} ,$$
$$A^{\langle m \rangle}(\xi) = \{y; \int_{E^{\langle m \rangle}} \phi(x, y) \mu(dx) > \xi\} ,$$

and let $\mu_\xi = \mu(A(\xi))$. It is easy to see that $\mu(A^{\langle m \rangle}(\xi)) = N^m \mu_\xi$, and that $\mu_\xi \to \tau$ as $\xi \downarrow 0$. On account of Lemma (4.8), it holds that

$$\hat{k}(\lambda) \leq \frac{1}{N^m} \hat{P}(\hat{\omega}(E^{\langle m \rangle}) < L_{m,n})$$
$$\leq \frac{1}{N^m} P(\int_{A^{\langle m \rangle}(\xi)} N(dy, \omega) \int_{E^{\langle m \rangle}} \mu(dx) \phi(x, y) < L_{m,n})$$
$$\leq \frac{1}{N^m} P(N(A^{\langle m \rangle}(\xi), \omega) < \frac{L_{m,n}}{\xi}) .$$

By setting $\xi = \xi_l = (1-c)^l \geq L_{m,n}$, we conclude that

$$\hat{k}(\lambda) \leq \frac{1}{N^m} \exp\{-\nu \mu(A^{\langle m \rangle}(\xi_l))\} = \frac{1}{N^m} \exp(-\nu N^m \mu_{\xi_l})$$

$$\leq N(\frac{1+\varepsilon_n}{C}\lambda)^{d_s/2} \exp\{-\frac{\nu \mu_{\xi_l}}{N}(\frac{1+\varepsilon_n}{C}\lambda)^{-d_s/2}\} \tag{4.17}$$
$$\lambda \in (0, (\frac{1-c}{N})^{n+l} \frac{C}{1+\varepsilon_n}) .$$

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