

UNIQUENESS OF POSITIVE SOLUTIONS OF THE HEAT EQUATION

Dedicated to Professor Masanori Kishi on his 60th birthday

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1. Introduction

Let D be an unbounded domain in \mathbf{R}^2 and u a nonnegative solution of the heat equation on D . We consider the property (U) for D :

$$(U): \quad u = 0 \quad \text{on} \quad \partial_p D \Rightarrow u = 0 \quad \text{on} \quad D,$$

where $\partial_p D$ is the parabolic boundary of D . In the case of $D = \mathbf{R} \times (0, T)$ or $(0, \infty) \times (0, T)$, it is known that the property (U) holds (see [6]).

In this paper, by using a special form of the boundary Harnack principle, we shall show the following generalization.

Theorem. For $T \in (-\infty, \infty]$ and an upper semicontinuous function φ on \mathbf{R} , we set

$$D(\varphi, T) = \{(x, t); \varphi(x) < t < T\}.$$

If φ is bounded below, then the property (U) holds for $D(\varphi, T)$.

By Theorem we obtain the following

Corollary. Let φ , T and $D(\varphi, T)$ be as in Theorem, and assume that φ is bounded below. Let u, v be positive solutions of the heat equation on $D(\varphi, T)$. If $u - v$ vanishes continuously on $\partial_p D(\varphi, T)$, then $u = v$ on $D(\varphi, T)$.

On the other hand, for $D = \{(x, t); mt < x\}$, the property (U) does not hold (see Lemma 7, Proposition 2). By using the Appell transformation, we shall show that this is critical (see Proposition 1).

2. Preliminaries

For a domain D in \mathbf{R}^2 we denote by $\partial_p D$ the set of $(x, t) \in \partial D$ (the boundary of D) satisfying $V \cap D \cap (\mathbf{R} \times (t, \infty)) \neq \emptyset$ for every neighborhood V of (x, t) and call it the parabolic boundary of D . For $X = (x, t) \in D$, we denote by ω_D^X

the parabolic measure at X with respect to D . The parabolic measure ω_D^X is supported by $\partial_p D \cap \mathbf{R} \times (-\infty, t]$ and for any bounded continuous function f on $\partial_p D$, the function $\int f d\omega_D^X$ of X is the solution of the Dirichlet problem with boundary value f . A boundary point $Y \in \partial_p D$ is said to be regular if for every bounded continuous function f on $\partial_p D$,

$$\lim_{\substack{X \rightarrow Y \\ X \in D}} \int f d\omega_D^X = f(Y).$$

REMARK 1. We note that every boundary point in $\partial_p D(\varphi, T)$ is regular, where φ, T and $D(\varphi, T)$ are as in Theorem (see for example [1]).

We say that u is a parabolic function on a domain D if

$$(\partial/\partial t - \partial^2/\partial x^2) u = 0 \quad \text{on } D.$$

Recalling the Perron-Wiener-Brelot method, we have

Lemma 1. *Let D be a domain in \mathbf{R}^2 and u a nonnegative parabolic function on D . If u is continuous on $D \cup \partial_p D$, we have*

$$u(X) \geq \int u d\omega_D^X$$

on D .

Let W be the fundamental solution of the heat equation, defined by

$$\begin{aligned} W(x, t; y, s) &= (4\pi(t-s))^{-1/2} \exp\left(-\frac{(x-y)^2}{4(t-s)}\right) & \text{for } t > s \\ &= 0 & \text{for } t \leq s. \end{aligned}$$

We put

$$G(x, t; y) = W(x, t; y, 0) - W(x, t; -y, 0)$$

and for $m \in \mathbf{R}$,

$$K_m(x, t; s) = \frac{x - mt}{t - s} W(x, t; ms, s).$$

Then the function $G(x, t; y)$ of y is the density of the parabolic measure at (x, t) with respect to $(0, \infty) \times (0, T)$ ($T > 0$). By Lemma 1 we obtain the following

Lemma 2. *Let u be a nonnegative parabolic function on $D = (0, \infty) \times (0, T)$ for $T > 0$. If u is continuous on $\partial_p D \cup D$, then we have*

$$u(x, t) \geq \int_0^\infty G(x, t; y) u(y, 0) dy.$$

On the other hand, Kaufman and Wu showed in [2] that $K_m(x, t; s)$ is the

density at (ms, s) of the parabolic measure at (x, t) with respect to $\Omega_m = \{(x, t); mt < x\}$. More precisely, we have

Lemma 3. For m, T_0 and T_1 in \mathbf{R} , we set

$$\Omega = \{(x, t); T_0 < t < T_1, mt < x\} .$$

Then for $(x, t) \in \Omega$,

$$\omega_{\Omega}^{(x, t)} = K_m(x, t; s) ds$$

on $\{(ms, s); T_0 \leq s \leq T_1\}$.

3. The boundary Harnack principle

For $\alpha > 0$, we denote by τ_{α} the parabolic dilation defined by $\tau_{\alpha}(x, t) = (\alpha x, \alpha^2 t)$. The heat equation is stable for every parabolic dilation. This implies the following

REMARK 2. The parabolic measure $\omega_{\tau_{\alpha}^{-1}(D)}^{\tau_{\alpha}(X)}$ is equal to the τ_{α} -image of ω_D^X .

For $Y \in \mathbf{R}^2$ and $r > 0$, we put

$$V_r(Y) = \{Y + (x, t); -r^2 < t < r^2, -r + tr^{-1} < x < r\} ,$$

and

$$V'_r(Y) = V_r(Y) \setminus \{Y + (x, t); x \leq 0, t \leq 0\} .$$

Since Y is a regular boundary point of $V'_r(Y)$, we have the following lemma (use Remark 2).

Lemma 4. For any $\varepsilon > 0$, there exists $0 < \beta < 1$ independent of r such that

$$\omega_{V'_r(Y)}^X (\partial V'_r(Y) \setminus \{Y + (x, t); x \leq 0, t \leq 0\}) < \varepsilon$$

for all $X \in V'_{\beta r}(Y)$.

To prove our main theorem, the following lemma plays an important role, which is a special form of the boundary Harnack principle.

Lemma 5. Let φ be a decreasing upper semicontinuous function on $[0, \infty)$ satisfying $\varphi(0) = 2$ and $\varphi \geq 0$. Put

$$D = \{(x, t); 0 < x, \varphi(x) < t < 2\} .$$

Then there exists a constant $C > 0$ such that for any nonnegative parabolic function u on D vanishing continuously on $\partial_p D$

$$u(x, t) \leq Cu(x, 1)$$

for every (x, t) with $1 < x < \infty, \varphi(x-1) < 1/2$ and with $\varphi(x) < t < 1/2$.

Proof. For each $x_0 \in (1, \infty)$ with $\varphi(x_0 - 1) < 1/2$, we set

$$D_{x_0} = D \cap \{(x, t); 0 < x < x_0 + 2, \varphi(x_0 + 1) < t < 2\} .$$

First we shall show that for $Y_0 = (y_0, \varphi(x_0 + 1))$ with $x_0 + 1 < y_0 < x_0 + 3/2$, there exists a constant $C_0 > 0$ independent of x_0 and y_0 such that for $r \leq s \leq 1/4$ and any $x \in D_{x_0} \setminus V_s(Y_0)$,

$$(1) \quad \omega_{D_{x_0}}^x(V_r(Y_0) \cap \partial D_{x_0}) \leq C_0 \omega_{D_{x_0}}^{A_{2r}^s(Y_0)}(V_r(Y_0) \cap \partial D_{x_0}) ,$$

where $A_r(Y_0) = Y_0 + (r, 2r^2)$. Put $f(x) = \omega_{D_{x_0}}^x(V_r(Y_0) \cap \partial D_{x_0})$. It suffices to show that

$$(2) \quad f(X) \leq C_0 f(A_{2^k r}(Y_0)), X \in D_{x_0} \setminus V_{2^k r}(Y_0)$$

for every positive integer k with $2^k r < 1/4$.

By the maximum principle and the Harnack inequality, we have

$$\begin{aligned} f(X) &\leq 1 = C_1 \omega_{V_r'(Y_0)}^{A_{r/2}(Y_0)}(\partial V_r'(Y_0) \cap \{Y_0 + (x, t); t \leq 0, x \leq 0\}) \\ &\leq C_1 \omega_{V_r'(Y_0) \cap D_{x_0}}^{A_{r/2}(Y_0)}(\partial D_{x_0} \cap \partial V_r'(Y_0)) \\ &\leq C_1 f(A_{r/2}(Y_0)) \\ &\leq C_1 C_2 f(A_r(Y_0)) , \end{aligned}$$

for some $C_1, C_2 > 0$ independent of r , which shows (2) for $k=0$. By Remark 2, we have

$$f(x) \leq C_1 C_2^2 f(A_{2r}(Y_0)) .$$

By using Lemma 4 for $\varepsilon = C_2^{-1}$, there exists $0 < \beta < 1$ such that for any $Y \in \partial V_{2r}(Y_0) \cap \partial D_{x_0}$ and $X \in V_{\beta r}(Y) \cap D_{x_0}$,

$$\begin{aligned} f(X) &\leq C_1 C_2^2 f(A_{2r}(Y_0)) \omega_{V_{\beta r}(Y)}^x(\partial V_r'(Y_0) \cap \{Y + (x, t); t \leq 0, x \leq 0\}^c) \\ &\leq C_1 C_2 f(A_{2r}(Y_0)) , \end{aligned}$$

where $\{\cdot\}^c$ denotes the complement of a set $\{\cdot\}$. By the Harnack inequality, there exists $C_3 > 0$ such that

$$f(X) \leq C_3 f(A_{2r}(Y_0)), X \in \partial V_{2r}(Y_0) \cap D_{x_0} \cap \bigcap_{Y \in \partial V_{2r} \cap D_{x_0}} V_{\beta r}(Y)^c ,$$

so that by the maximum principle,

$$f(X) \leq C_0 f(A_{2r}(Y_0)), X \in D_{x_0} \setminus V_{2r}(Y_0)$$

with $C_0 = \max(C_1 C_2, C_3)$, which shows (2) for $k=1$. Inductively we have (2) for every k with $2^k r < 1/4$.

Similarly we see that for $Y_0 = (y_0, t_0) \in \partial D_{x_0}$ with $y_0 \geq x_0 + 2/3, t_0 \leq 1/2$ and for $r \leq s \leq 1/4$,

$$(3) \quad \omega_{D_{x_0}}^x(U_r(Y_0) \cap \partial D_{x_0}) \leq C' \omega_{D_{x_0}}^{B_s(Y_0)}(U_r(Y_0) \cap \partial D_{x_0}), X \in D_{x_0} \setminus U_s(Y_0)$$

with some constant $C' > 0$, where

$$U_r(Y) = \{Y+(x, t); -r^2 < t < r^2, -r < x < r-tr^{-1}\},$$

and

$$B_r(Y) = Y+(-r, 2r^2).$$

From (1), (3) and the Harnack inequality, it follows that with some constant $C > 0$,

$$\omega_{D_{x_0}}^{(x,t)}(V_r(Y_0) \cap \partial D_{x_0}) \leq C \omega_{D_{x_0}}^{(x_0,1)}(V_r(Y_0) \cap \partial D_{x_0}),$$

for every $Y_0=(y_0, t_0) \in \partial D_{x_0}$ with $y_0 \leq x_0+1, t_0 \leq 1/2$ and every $(x, t) \in D_{x_0}$ with $x \leq x_0, t \leq 1/2$. Since for $t < 1, \omega_{D_{x_0}}^{(x,t)}$ is absolutely continuous with respect to $\omega_{D_{x_0}}^{(x_0,1)}$, we have

$$\omega_{D_{x_0}}^{(x,t)} \leq C \omega_{D_{x_0}}^{(x_0,1)} \quad \text{on} \quad \partial D_{x_0} \cap [x_0+1, x_0+2] \times [\varphi(x_0+1), 1/2]$$

for every $(x, t) \in D_{x_0}$ with $x \leq x_0, t \leq 1/2$, which proves Lemma 5.

4. The uniqueness of positive parabolic functions

First we show the following plain

Lemma 6. *Let φ be a decreasing upper semicontinuous function bounded below on $[a, \infty)$ for some $a \in \mathbf{R}$, and put*

$$D = \{(x, t); a < x, \varphi(x) < t < \varphi(a)\}.$$

If a nonnegative parabolic function u on D vanishes continuously on $\partial_p D$, then $u=0$ on D .

Proof. We may assume that $a=0, \varphi(0)=3$ and $\lim_{x \rightarrow \infty} \varphi(x)=0$. It suffices to show that $u(x, 1)=0$ if $(x, 1) \in D$.

Let $(x_0, 1) \in D$ be fixed. We choose $y_0 > x_0+1$ with $\varphi(y_0-1) < 1/4$. For $y > y_0$, we put

$$J_y = \{(x, t); x < y, 0 < t < 2\}.$$

By the Harnack inequality and Lemma 5, there exists a constant $C > 0$ such that

$$u(y, t) \leq C u(y, 1)$$

for $y > y_0$ and $\varphi(y) < t < 1$. Hence we have

$$\begin{aligned} u(x_0, 1) &= \int u(y, t) d\omega_{D \cap J_y}^{(x_0,1)} \\ &\leq C u(y, 1) \omega_{D \cap J_y}^{(x_0,1)}(\{y\} \times [0, 1]) \end{aligned}$$

$$\begin{aligned} &\leq C u(y, 1) \omega_{y_0}^{(y, 1)}(\{y\} \times [0, 1]) \\ &= C u(y, 1) \int_0^1 K_0(y-x_0, 1; s) ds, \end{aligned}$$

for every $y > y_0$ (see Lemma 3). Since Lemma 2 gives

$$\int_{y_0}^\infty u(y, 1) \int_0^1 K_0(y-x_0, 1; s) ds dy < \infty,$$

we have

$$u(x_0, 1) \leq C \liminf_{y \rightarrow \infty} (u(y, 1) \int_0^1 K_0(y-x_0, 1; s) ds) = 0,$$

which shows Lemma 6.

Now we shall prove our main theorem.

Theorem. For $T \in (-\infty, \infty]$ and an upper semicontinuous function φ bounded below on \mathbf{R} , we set

$$D(\varphi, T) = \{(x, t); \varphi(x) < t < T\}.$$

Let u be continuous on $D(\varphi, T) \cup \partial_p D(\varphi, T)$ and nonnegative parabolic on $D(\varphi, T)$. If $u=0$ on $\partial_p D(\varphi, T)$, then $u=0$ on $D(\varphi, T)$.

Proof. Put

$$\psi(x) = \min(\sup_{y \leq x} \varphi(y), \sup_{y \geq x} \varphi(y)).$$

Then clearly $\varphi \leq \psi$. We can easily check that the upper semicontinuity of ψ follows from that of φ . Furthermore there exists $a \in [-\infty, \infty]$ such that ψ is increasing on $(-\infty, a)$ and decreasing on (a, ∞) . From the definition of ψ , it follows that for a nonnegative parabolic function u on $D(\varphi, T)$ vanishing continuously on $\partial_p D(\varphi, T)$, $u(x, t) = 0$ for every (x, t) with $\varphi(x) \leq t < \psi(x)$. Hence

$$u = 0 \quad \text{on} \quad \partial_p D(\psi, T).$$

Then Lemma 6 gives $u(x, t) = 0$ for $(x, t) \in D(\psi, T)$ with $t < \sup \psi$, which yields Theorem.

Proof of Corollary. Let u, v be positive parabolic functions on $D(\varphi, T)$ such that $u-v$ vanishes continuously on $\partial_p D(\varphi, T)$. Then for any $\varepsilon > 0$ and $T' < T$, we can find an upper semicontinuous function φ_ε such that $\partial_p D(\varphi_\varepsilon, T') \subset D(\varphi, T)$ and that

$$|u-v| < \varepsilon \quad \text{on} \quad D(\varphi, T') \setminus D(\varphi_\varepsilon, T').$$

Lemma 1 gives

$$u(X) - \int u d\omega_{D(\varphi_\varepsilon, T')}^X \geq 0 \quad \text{for } X \in D(\varphi_\varepsilon, T'),$$

so that by Theorem we have

$$u(X) = \int u d\omega_{D(\varphi_\varepsilon, T')}^X \quad \text{for } X \in D(\varphi_\varepsilon, T')$$

(see Remark 1). The similar equality holds for v . Therefore for $X \in D(\varphi_\varepsilon, T')$,

$$|u(X) - v(X)| \leq \int |u - v| d\omega_{D(\varphi_\varepsilon, T')}^X < \varepsilon,$$

which shows Corollary.

5. Application

In this section, by using the Appell transformation, we shall give some application of our theorem.

For $(x, t) \in \mathbf{R} \times (-\infty, 0)$, we define

$$\tau(x, t) = (-t^{-1}x, -t^{-1})$$

and

$$Au(x, t) = W(x, t; 0, 0)u(t^{-1}x, -t^{-1}),$$

for a function u on a domain D in $\mathbf{R} \times (-\infty, 0)$. The Appell transformation gives a one-to-one correspondence of the parabolic functions on D to the parabolic functions on $\tau(D)$.

By using the Appell transformation, we have

Lemma 7. For $a \in (-\infty, \infty]$ and an upper semicontinuous function ψ on $(-\infty, a)$, we set

$$\Omega(\psi) = \{(x, t); t < a, \psi(t) < x\}.$$

If $\limsup_{t \downarrow 0} t \psi(-t^{-1}) < \infty$, then the property (U) does not hold.

This and our main theorem imply the following

Proposition 1. Let $\psi_\alpha(t) = (-t)^\alpha$ and $\Omega(\psi_\alpha)$ be as in Lemma 7. If $\alpha > 1$, the property (U) holds for $\Omega(\psi_\alpha)$ but if $\alpha \leq 1$, then the property (U) does not hold.

EXAMPLE. Let $\psi(t) = -t \log \log \log(-t)$ on $(-\infty, -e^e)$. Then the property (U) holds for $\Omega(\psi)$.

Finally we consider an integral representation of positive parabolic functions on $\Omega_m = \{(x, t); mt < x\}$.

Proposition 2. For every positive parabolic function u on Ω_m , there exist

positive Borel measures μ, ν and a constant $C \geq 0$ such that

$$u(x, t) = \int_{-\infty}^{\infty} K_m(x, t; s) d\mu(s) + \int_{-m/2}^{\infty} k(x, t; \lambda) d\nu(\lambda) \\ + C(x - mt) \exp(m^2 t/4 - mx/2),$$

where $k(x, t; \lambda) = \exp(\lambda^2 t + \lambda x) - \exp((\lambda + m)^2 t - (\lambda + m)x)$. The measures μ, ν and the constant C are uniquely determined.

This is a modification of the following proposition by the Appell transformation. Kaufman and Wu [2] and Mair [3] obtained the integral representation for $m=0$.

Proposition 3 (see [6]). *For every positive parabolic function u on $(0, \infty) \times (0, \infty)$, there exist unique positive measures μ, ν on \mathbf{R} such that*

$$u(x, t) = \int_{-\infty}^{\infty} K_0(x, t; s) d\mu(s) + \int_0^{\infty} G(x, t; y) d\nu(y).$$

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References

- [1] L. Evans and R. Gariepy: *Wiener criterion for the heat equation*, Arch. Rational Mech. Anal., **78**(1982), 293–314.
- [2] R. Kaufman and J.-M. Wu: *Parabolic potential theory*, J. Differential Equation (2) **43** (1982), 204–234.
- [3] B.A. Mair: *Integral representations for positive solutions of the heat equation on some unbounded domains*, Illinois J. Math. (1) **30**(1986), 175–184.
- [4] J. Moser: *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17**(1964), 101–134.
- [5] N.A. Watson: *Green functions, potentials and the Dirichlet problem for the heat equation*, Proc. London Math. Soc. (3) **33**(1976), 251–298.
- [6] D.V. Widder: *The heat equation*, Academic Press, Orlando, Florida, 1975.

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