Mori, M.
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FREDHOLM DETERMINANT FOR PIECEWISE MONOTONIC TRANSFORNATIONS

MAKOTO MORI

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1. Introduction

Let F be a piecewise C^2 transformation from a finite union of bounded intervals I to itself. We assume

(A1) The lower Lyapunov number ξ is positive:

$$\xi = \liminf_{x\to\infty} \operatorname*{ess\,inf}_{n\in I} \frac{1}{n} \log|F^{n\prime}(x)| > 0 .$$

(A2) The mapping F is nondgenerate:

$$\operatorname*{ess\,inf}_{x\in I} |F'(x)| > 0 .$$

(A3) There exists a finite partition of I into subintervals, F is monotone on each of the subintervals, and the restrictions of F, F' and F'' to each of the subintervals can be extended continuously to its closure.

Here F^n stands for the n-th iterate of F:

$$F^n(x) = \begin{cases} F(F^{n-1}(x)) & n \geq 1 . \\ x & n = 0 , \end{cases}$$

In the present paper, we are concerned with the spectrum of the Perron-Frobenius operator P. The Perron-Frobenius operator P associated with F is originally a nonnegative contraction operator defined on L^1, the set of integrable functions, by

$$\int Pf(x)\, g(x)\, dx = \int f(x)\, g(F(x))\, dx ,$$

where g belongs to L^∞, the set of bounded measurable functions. The spectrum problem of P as an operator on L^1 is rather trivial: for instance, it is found in [14] that the unit disk is contained in the spectum of the Perron-Frobenius operator on L^1. Therefore, we will restrict P to BV, the set of functions with bounded variation. We consider that BV is a subspace of L^1 functions which admit versions with bounded variation. We define the norm on BV by

$$V(f) = \operatorname{var}(f) + \int |f|\, dx,$$

where

$$\operatorname{var}(f) = \inf\{\text{the total variation of } \hat{f} \text{ such that } \hat{f} = f \text{ a.e.}\}.$$

Then BV is an invariant subspace of P under the assumptions (A1)-(A3) (cf. [8]). We will study the spectum of the Perron-Frobenius operator P regarded as an operator acting on BV and denote the spectum by Spec(F).

As is discussed heuristically in Y. Oono and Y. Takahashi [13], the formal determinant of $I-zP$, called the Fredholm determinant, coincides essentially with the reciprocal of the zeta function (Ruelle-Artin-Mazur zeta function):

$$\zeta(z) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} z^n \sum_{x=F^n(x)} |F^{n\prime}(x)|^{-1}\right],$$

which is completely determined by the periodic orbits and the gradients at these points. In other words, Spec(F) is expected to be determined by the zeros of the Fredholm determinant. Our purpose is carried it out in a rigorous way.

Our rseult is as follows:

Theorem A. *Assume the conditions* (A1)-(A3) *above. The reciprocal* $1/\zeta(z)$ *of the zeta function has analytic extension to the complex domain* $\{z: |z| < e^{\xi}\}$, *and*

$$\operatorname{Spec}(F) \cap \{\lambda: |\lambda| > e^{-\xi}\} = \{z^{-1}: 1/\zeta(x) = 0,\ |z| < e^{\xi}\}.$$

Remark. (i) If the dynamical system is weakly mixing, then the topological entropy $h(F)$ equals the lower Lyapunov number ξ.

(ii) We only need to study the spectrum of P which is greater than $e^{-\xi}$ in modulus because it is proved in G. Keller [5] that

$$\{z: |z| < e^{-h(F)}\} \subset \operatorname{Spec}(F).$$

The importance of the study of the spectrum of P consists in the fact that the most of the ergodic properies of the dynamical system can be obtained from Spec(F).

(a) Under the conditions above, the eigenspace associated with the eigenvalue 1 determines the ergodic components of the dynamical system. In particular, the eigenspace is a subspace of the space BV and is isomorphic to the space of F-invariant density functions. This was first proved for F which is piecewise C^2 by A. Lasota and J.A. Yorke [8] and extended to the case $C^{1+\varepsilon}(\varepsilon>0)$ by G. Keller [6]. The condition that F is C^1 is not sufficient: there exists an example which has no finite absolutely continuous invariant probability measure (cf. [2]). The condition (A1) is also essential: there exists an example for which $F'(x)>1$ except only one point x_0 $(F'(x_0)=1)$ and it also has no finite absolutely continuous invariant probability measure (cf. [14], [18]).

(b) The eigenvalues on the unit circle determine the mixing properties of the dynamical system:

Suppose that 1 is the simple eigenvalue of P (this is always possible when we restrict F to one of the ergodic components). Then the dynamical system induced by the mapping F is mixing if and only if 1 is the only one eigenvalue on the unit circle (actually, it is Bernoulic cf. [1]).

(c) The eigenvalues in the unit disk determine the decay rate of correlations for good test functions, although as we stated before, the supremum of decay rates for functions belonging L^1 equals 1. If we restrict P to BV, we can determine the decay rate of correlation, because, as will seen soon below, $\int f(x) g(F^n(x))\, d\mu$ is described in terms of Fredholm determinant if $f \in BV$ and $g \in L^\infty$. Suppose that 1 is the unique eigenvalue on the unit circle. Recall eigenvalues are isolated in the domain $\{\lambda: |\lambda| > e^{-h(F)}\}$. Let η be the second largest eigenvalue in modulus, then η is the decay rate of correlation, that is, for any $f \in BV$

$$\int f(x) g(F^n(x))\, d\mu - \int f\, d\mu \int g\, d\mu = O(\eta^n),$$

where μ is the invariant probability measure and $g \in L^\infty$ ([3], [11]).

Now let us state the main ideas to prove Theorem A. We want to define the Fredholm determinant $\det(I - zP)$, but the Fredholm determinant can not be defined in the usual sense, because P is not a compact operator. This is the difficulty in establishing Theorem A. We need three ideas to overcome it.

The first idea is to use the renewal equation (§3), which will mark the structure of the dynamical system clear ([4]). Let us illustrate how to construct a renewal equation. For $f \in BV$ and $g \in L^\infty$, put

$$(f, g)(z) = \sum_{n=0}^{\infty} z^n \int f(x) g(F^n(x))\, dx.$$

Then, we get the formal expression

$$(f, g)(z) = \int (I - zP)^{-1} f(x) g(x)\, dx,$$

which suggests that the spectrum problem of P will be reduced to the problem of singularities of the complex function $(f, g)(z)$. Hence it is expected that $(f, g)(z)$ is asymptotically equal to $C/\det(I - zP)$ for some constant C. Notice that

$$(f, g)(z) = \int f(x) g(x)\, dx + z(Pf, g)(z).$$

Hence, if we can construct functions f_i and coefficients $\phi_{i,j}$ such that

$$zPf_i(x) = \sum_j \phi_{i,j}(z) f_j(x),$$

then we will obtain an equation

$$s = \chi + \Phi(z) s ,$$

where s, χ are the vectors with components $(f_i, g)(z)$, $\int f_i(x) g(x)\, dx$ and $\Phi(z)$ is the matrix with components $\phi_{i,j}(z)$, respectively. This is an analogue to what is called the renewal equation in the theory of Markov processes. Our goal is to prove the determinant $\det(I-\Phi(z))$ plays the role that the formal determinant $\det(I-zP)$ is expected to do. We will call $\det(I-\Phi(z))$ the Fredholm determinant.

The construction of $\Phi(z)$ is straightforward for Markov mappings (cf. [4]). This method can be applied to certain simple non-Makov mappings such as β-transformations and unimodal linear transformations. In the latter cases, the renewal equation was constructed in [9] on the usual symbolic dynamics and the Fredholm determinant is proved to be an analytic funciton in $|z|<e^{\xi}$. Thus we can see that the singularities of $(f, g)(z)$ are the zeros of the Fredholm determinant, and Spec(F) is characterized by the zeros of the Fredholm determinant. In [10], we extended the results to some piecewise linear transformations with different slopes. But this method cannot be applied to general cases even if they are piecewise linear.

The second idea is to introduce the signed symbolic dynamics (§2). In the above cases, the usual symbolic dynamics are determined essentially by the itinerary of the only one division point of the partition stated in (A3), but for general piecewise monotonic transformations, we have to trace the itinearies of all the division points even if they are piecewise linear. If we persist in using the usual symbolic cynamics the difficulty would arise because we have to trace the orbits of both endpoints of each subinterval at a time. On the signed symbolic dynamics, it suffices to trace each orbit separately to construct a renewal equation. Thus, for piecewise linear transformation, we succeed to define a finite dimensional matrix $\Phi(z)$ which we call the Fredholm matrix. Then Spec(F) is characterized, just as in [9] and [10], by the zeros of the Fredholm determinant $\det(I-\Phi(z))$.

As we showed in [11], these two ideas are sufficient to prove Theorem A when the mapping F is a piecewise linear transformation ([11]). The Fredholm matrix which, as is mentioned above, is determined by the orbits of the endpoints of the subintervals corresponding to the aplhabets, has sufficient information to describe the structure of the periodic orbits (cf. [4], [14], [15]). In particular, we can calculate the zeta function $\zeta(z)$ by the Fredholm determinant and get

$$\det(I-\Phi(z)) = 1/\zeta(z) .$$

Piecewise linear cases are also discussed in F. Hofbauer and G. Keller [3]. Some related topics can be found in [7], [12] and [16].

The third idea is necessary to approximate general piecewise monotonic transformations. It is the approximation of F by formal "piecewise linear transformations" F_N as stated in §3. These formal piecewise linear transformations are defined on the symbolic dynamics on which F is realized. Moreover, their Fredholm matrices $\Phi_N(z)$ are proved to be finite matrices as in [11]. Then the spectrum of the Perron-Frobenius operator P can be characterized by using $\det(I-\Phi_N(z))$ as follows:

Theorem B. *Let z_0 be a complex number such that $|z_0|<e^{\xi}$. Then z_0^{-1} belongs to* Spec(F) *if and only if there exists a sequence $\{z_N\}_{N=1}^{\infty}$ such that $\lim_{N\to\infty} z_N=z_0$ and $\det(I-\Phi_N(z_N))=0$.*

By Theorem B, we can calculate the spectrum of P and then we can prove Theorem A.

The proof of Theorem B will be given in §6. We define in §5 the zeta functions $\zeta_N(z)$ corresponding to F_N and show that they converge to $\zeta(z)$ in the unit disk $|z|<1$. Since $\det(I-\Phi_N(z))=1/\zeta_N(z)$, then the proof of Theorem A is reduced to the proof of the uniform boundedness of $\det(I-\Phi_N(z))$ in N for any fixed z with $|z|<e^{\xi}$ (Proposition 7.1 in §7).

The necessary properties of $\Phi_N(z)$ are summarized in §5, we will descuss the limit $\Phi(z)$ of $\Phi_N(z)$ in §4, and the notations which we use throught this article are listed in §2.

2. Notations

2.1. Alphabets, Words and Sentences

We will define several notations most of which are used in [11]. We denote by $\{\langle a\rangle\}_{a\in A}$ the partition of I into subintervals which satisfies the condition (A3) in the introduction. Thus on each subinterval $\langle a\rangle$ the mapping F is monotone and can extend to $\mathrm{cl}\langle a\rangle$ as C^2 function, where $\mathrm{cl}\,J$ stands for the closure of a set J. We also denote the interior and the boundary of a set J by $\mathrm{int}\,J$ and ∂J, respectively.

We call each element a of the index set A an apphabet. For an alphabet a, we define sgn a by

$$\operatorname{sgn} a = \operatorname{sgn}(F'|_{\mathrm{int}\langle a\rangle}) = \begin{cases} + & \text{if} \quad F'(x)>0 \quad \text{for} \quad x\in \mathrm{int}\langle a\rangle\,, \\ - & \text{if} \quad F'(x)<0 \quad \text{for} \quad x\in \mathrm{int}\langle a\rangle\,. \end{cases}$$

A finite sequence of alphabets will be called a word and for a word $w=a_1\cdots a_N(a_i\in A)$ we denote

$$|w| = N \quad \text{(the length of } w)\,,$$

$$w[K] = a_K, \quad (1 \leq K \leq N),$$
$$[w]_M = a_1 \cdots a_M \quad (1 \leq M \leq N),$$
$$\operatorname{sgn} w = \prod_{i=1}^{N} \operatorname{sgn} a_i,$$
$$\theta^K w = a_{K+1} \cdots a_N \quad (0 \leq K < N),$$
$$\langle w \rangle = \bigcap_{i=1}^{N} F^{-i+1}(\langle a_i \rangle) = \{x \in I: F^{i-1}(x) \in \langle a_i \rangle, 1 \leq i \leq N\}.$$

Thus $\langle w \rangle$ is the subinterval corresponding to a word w. We denote the empty word by ε and define $|\varepsilon| = 0$ and $\operatorname{sgn} \varepsilon = +$. Let W_N be the set of all words w with length N and $\langle w \rangle \neq \emptyset$ and set $W = \bigcup_{N=0}^{\infty} W_N$, where $W_0 = \{\varepsilon\}$.

We call an infinite sequence of alphabets $\alpha = a_1 a_2 \cdots$ a sentence and denote the N-th coordinate by

$$\alpha[N] = a_N,$$

the initial N-word by

$$[\alpha]_N = a_1 \cdots a_N,$$

and the shifted sequences by

$$\theta^K \alpha = a_{K+1} a_{K+2} \cdots \quad (K \geq 0).$$

For instance, the word $a_2 \cdots a_{N+1}$ is denoted by $[\theta\alpha]_N$. For words $u = a_1 \cdots a_N$, $v = b_1 \cdots b_M$, and a sentence $\alpha = c_1 c_2 \cdots$, we denote $u \cdot v = a_1 \cdots a_N b_1 \cdots b_M$ and $u \cdot \alpha = a_1 \cdots a_N c_1 c_2 \cdots$.

Let

$$\{\alpha\} = \bigcap_{N=1}^{\infty} \operatorname{cl} \langle [\alpha]_N \rangle.$$

We denote by S the set of all sentences which satisfy $\{\alpha\} \neq \emptyset$. By the assumption $\xi > 0$ the set $\{\alpha\}$ consists of exactly one point if $\alpha \in S$. We denote by S_0 the subset of those sentences $\alpha \in S$ for which $\bigcap_{N=1}^{\infty} \langle [\alpha]_N \rangle \neq \emptyset$.

We introduce orders in the following way. We write

$$x <_\sigma y = \begin{cases} x < y & \text{if} \quad \sigma = +, \\ x > y & \text{if} \quad \sigma = -. \end{cases}$$

(i) For alphabets $a_1, a_2 \in A$,

$$a_1 < a_2 \quad \text{if} \quad x_1 < x_2 \quad \text{for any} \quad x_i \in \langle a_i \rangle \quad (i = 1, 2).$$

(,i) For words $w_1, w_2 \in W_M$ and for $N \leq M$,

$$w_1 < w_2 \quad \text{if} \quad [w_1]_N = [w_2]_N \quad \text{and} \quad w_1[N+1] <_\sigma w_2[N+1] \quad \text{fot some } N,$$

where $\sigma = \operatorname{sgn} [w_1]_N$.

iii) For sentences α_1, α_2,

$$\alpha_1 < \alpha_2 \quad \text{if} \quad [\alpha_1]_N < [\alpha_2]_N \quad \text{for some } N.$$

The following is an immediate consequence of the definitions above.

Lemma 2.1. ([11]) (i) *For words $w_1, w_2 \in W(w_1, w_2 \neq \varepsilon)$, $w_1 < w_2$ if and only if $x_1 < x_2$ for any $x_i \in \langle w_i \rangle$.*

(ii) *For sentences $\alpha_1, \alpha_2 \in S$, $(\{\alpha_1\} \neq \{\alpha_2\})$, $\alpha_1 < \alpha_2$ if and only if $\{\alpha_1\} < \{\alpha_2\}$.*

We consider the topology on S induced by the order.

2.2. Plus and Minus Expansions

For each $x \in I$, we define a sentence $\alpha^x = a_1^x a_2^x \cdots \in S_0$, called the expansion of x, by the condition $F^{i-1}(x) \in \langle a_i^x \rangle$ for all i. Then, $x = \{\alpha^x\}$ since $\xi > 0$. On the other hand, with each $\beta \in S_0$ we can associate a unique $x \in I$ such that $\alpha^x = \beta$. Therefore, we can identify I with S_0. Moreover, by the assumption (A3), F, F' and F'' can naturally extend to the functions on S and F can be considered as the shift operator on S. Thus we mainly consider them on the symbolic dynamics hereafter.

As we saw in [11], the structure of the dynamics becomes much clearer if it is considered on the signed symbolic dynamics. For $x \in I$, let us denote

$$x^+ = \sup_{y < x} \alpha^y$$

and

$$x^- = \inf_{y > x} \alpha^y.$$

The sentences x^+ and x^- are called the plus expansion and the minus expansion of a point x, respectively. We denote $\tilde{S} = \{x^+, x^- : x \in I\}$.

Note that by the assumption (A3) we can define the values of F at x^+ and x^- by

$$F(x^+) = \lim_{y \uparrow \{x^+\}} F(y), \qquad F(x^-) = \lim_{y \downarrow \{x^-\}} F(y). \tag{2.1}$$

For a word w, we denote the expansions of endpoints of $\langle w \rangle$ by

$$w^+ = \sup_{y \in \langle w \rangle} \alpha^y = (\sup_{y \in \langle w \rangle} y)^+$$

and

$$w^- = \inf_{y \in \langle w \rangle} \alpha^y = (\inf_{y \in \langle w \rangle} y)^-.$$

Among the elements of $\tilde{S}$ they are of special importance. For a word w, we write $\tilde{w}$ to express one of w^{σ}, $\sigma\in\{+,-\}$. We distinguish the signed words $u^{\sigma}\in\tilde{W}_N$ and $v^{\tau}\in\tilde{W}_M$ if $N\neq M$ even when $\sigma=\tau$ and $\{u^{\sigma}\}=\{v^{\tau}\}$. We express $[\alpha]_N^{\sigma}$ for a sentence α instead of $([\alpha]_N)^{\sigma}$.

Let

$$\varepsilon(x^{\sigma})=\varepsilon(w^{\sigma})=\sigma \quad \sigma\in\{+,-\},$$

and we call $\varepsilon(x^{\sigma})$ and $\varepsilon(w^{\sigma})$ the sign of x^{σ} and w^{σ}. Since there will occur no confusion, we use the conventions that $\varepsilon(\theta^n w^{\sigma})=\sigma$ for $n\geq 1$ whenever such an expression appears in below. Now set

$$\begin{aligned}\tilde{A} &= \{a^+, a^-: a\in A\} \quad \text{(the signed alphabet set)},\\ \tilde{W}_N &= \{w^+, w^-: w\in W_N\},\\ \tilde{W} &= \bigcup_{N=1}^{\infty}\tilde{W}_N \quad \text{(the signed word set)},\end{aligned}$$

We define an order on $\tilde{S}$ by

(i) $\alpha^{\sigma}>\beta^{\tau}$ if $\alpha>\beta$ $(\alpha,\beta\in S,\ \sigma,\tau\in\{+,-\})$,

(ii) $\alpha^{+}<\alpha^{-}$ $(\alpha\in S)$.

Note that (ii) is natural by the definition of x^+ and x^-.

As a notation, we adopt for $\tilde{\alpha},\tilde{\beta}\in\tilde{S}$

$$(\tilde{\alpha},\tilde{\beta})=\begin{cases}(\{\tilde{\alpha}\},\{\tilde{\beta}\}) & \text{if } \{\tilde{\alpha}\}\leq\{\tilde{\beta}\},\\ (\{\tilde{\beta}\},\{\tilde{\alpha}\}) & \text{otherwise.}\end{cases} \tag{2.2}$$

2.3. Additional Notations

Denote a formal "derivative" F'_N (introduced in the next section) on the symbolic dynamics by

$$F'_N(\tilde{\alpha})=\frac{F([\tilde{\alpha}]_N^+)-F([\tilde{\alpha}]_N^-)}{\text{Lebes}\,\langle[\tilde{\alpha}]_N\rangle}, \tag{2.3}$$

where Lebes J is the Lebesgue measure of a measurable set J. We define for a sentence α

$$C(\alpha)=\begin{cases}F'(\alpha)^{-1} & \text{if } \alpha\in S,\\ F'_N(\alpha)^{-1} & \text{if } \langle[\alpha]_N\rangle\neq\emptyset \text{ and } \langle[\alpha]_{N+1}\rangle=\emptyset,\end{cases} \tag{2.4}$$

and for $N\geq 1$

$$C_N(\alpha)=\begin{cases}F'_N(\alpha)^{-1} & \text{if } \langle[\alpha]_N\rangle\neq\emptyset,\\ C(\alpha) & \text{otherwise.}\end{cases} \tag{2.5}$$

For convenience, we set $C_0(\alpha)=0$.

We also define

(2.6) $$C^n(\alpha) = \prod_{i=0}^{n-1} C(\theta^i \alpha),$$

(2.7) $$C_N^n(\alpha) = \prod_{i=0}^{n-1} C_N(\theta^i \alpha).$$

and

(2.8) $$\Delta_N(\alpha) = |C_N(\alpha)| - |C_{N-1}(\alpha)|.$$

For a statement L, we adopt

$$\delta[L] = \begin{cases} 1 & \text{if } L \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

We need the expression $\delta[L]=\frac{1}{2}$ frequently in the following. Thus we write

$$\begin{aligned} \sigma[L] &= \delta[L]-\tfrac{1}{2} \\ &= \begin{cases} +\frac{1}{2} & \text{if } L \text{ is true,} \\ -\frac{1}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

We also use the expression $\delta[x<y\colon \varepsilon]$, $\sigma[x<y\colon \varepsilon]$ instead of $\delta[x<_\varepsilon y]$, $\sigma[x<_\varepsilon y]$ ($\varepsilon \in \{+, -\}$) for typographical convenience, respectively.

3. Renewal equation

For an interval J, we define

$$s^J(z; x) = \sum_{u=0}^{\infty} z^n \sum_y |F^{n\prime}(y)|^{-1},$$

where the summation $\sum_y$ is taken over all $y \in J$ such that $F^n(y)=x$. For an interval J and $g \in L^\infty$, set

$$s_g^J(z) = (1_J, g)(z) = \sum_{n=0}^{\infty} z^n \int 1_J(x) g(F^n(x))\, dx$$

where 1_J is the indicator function of a set J. Then we get

$$s_g^J(z) = \sum_{n=0}^{\infty} z^n \int P^n\, 1_J(x) g(x)\, dx.$$

Therefore,

$$s_g^J(z) = \int s^J(z; x) g(x)\, dx.$$

For simplicity, if $J=\langle u \rangle$, we write $s^J(z; x)$, $s_g^J(z)$ and 1_J in place of $s^u(z; x)$, $s_g^u(z)$ and 1_u, respectively.

As in the previous paper ([11]), we will extend $s^u(z; x)$ ($u \in W$) to a function on the signed symbolic dynamics by dividing $s^u(z; x)$ into the sum of two quantities $s^{u^+}(z; x)$ and $s^{u^-}(z; x)$ given in (3.1) below. On the symbolic dynamics, the mapping F acts as a shift operator and $-\log|F'|$ plays the role of a weight function. In our case for a general piecewise monotonic transformation, this weight function may depend on all the coordinates, and the construction of the renewal equation becomes difficult. Therefore, as we stated in Introduction, we approximate F by formal "piecewise linear transfomration" F_N and F' by a formal "derivative" F'_N which depends only on first N coordinates.

As we defined in (2.3), we set

$$F'_N(\tilde{\alpha}) = \frac{F([\tilde{\alpha}]_N^+) - F([\tilde{\alpha}]_N^-)}{\text{Lebes}\,\langle[\tilde{\alpha}]_N\rangle}\,.$$

It is convenient to denote the shift operator by F_N when we consider the weight function $-\log|F'_N|$. Although F'_N is the approximation of the weight function F', we may say that F_N is a formal piecewise linear transformation which approximates the mapping F.

REMARK. The formal "piecewise linear transformation" F_N may differ from the piecewise linear transformation whose graph connects points $(u^-, F(u^-))$ and $(u^+, F(u^+))$ for each $u \in W_N$, and the symboloc dynamics of such a piecewise linear trasfoimation does not necessarily coincide with the symbolic dynamics of F.

We now proceed to construct the renewal equation which is our main tool. First of all, for $\tilde{\alpha} \in \tilde{S}$, $z \in \boldsymbol{C}$ and $x \in I$, define a formal power series $s^{\tilde{\alpha}}(z; x)$ by

$$\text{(3.1)} \qquad s^{\tilde{\alpha}}(z; x) = s^{\tilde{\alpha}}(z; x; F) \\ = \sum_{w \in \tilde{W}} z^{|w|} |C^{|w|}(w \cdot x)| \sigma[w \cdot x < \tilde{\alpha} : \varepsilon(\tilde{\alpha})] \, \delta[w[1] = \tilde{\alpha}[1], (\theta w) \cdot x \in S]\,.$$

Lemma 3.1. (i) *Let α^-, $\beta^+ \in \tilde{S}$, and suppose that $\alpha^-[1] = \beta^+[1]$ and $\alpha^- < \beta^+$. Then*

$$s^{\alpha^-}(z; x) + s^{\beta^+}(z; x) = s^{(\alpha^-, \beta^+)}(z; x) \quad \text{a.e.}x.$$

(ii) *Suppose that $\{\tilde{\alpha}\} = \{\tilde{\beta}\}$, $\tilde{\alpha}[1] = \tilde{\beta}[1]$ and $\varepsilon(\tilde{\alpha})\,\varepsilon(\tilde{\beta}) = -(\tilde{\alpha}, \tilde{\beta} \in \tilde{S})$. Then*

$$\text{(3.2)} \qquad s^{\tilde{\alpha}}(z; x) = -s^{\tilde{\beta}}(z; x) \quad \text{a.e.}x.$$

Proof. This lemma immediately follows from the definition (3.1) and the fact that

$$\sigma[w \cdot x < \alpha^-] + \sigma[w \cdot x > \beta^+] = \delta[w \cdot x \in [\alpha^-, \beta^+)]\,.$$

Now let us construct the renewal equation on the signed word set $\tilde{W}$ by using $s^{\tilde{\alpha}}(z; x)$ defined by (3.1). Let for $\tilde{\alpha} \in \tilde{S}$, $z \in \boldsymbol{C}$ and $x \in I$, we define formal power series $\chi(\tilde{\alpha}, x)$ and $\chi^{\tilde{\alpha}}(z; x)$ by

$$\chi(\tilde{\alpha}, x) = \sigma[x < \tilde{\alpha} : \varepsilon(\tilde{\alpha})]$$

and

$$\chi^{\tilde{\alpha}}(z;x)=\begin{cases}\chi(\tilde{\alpha},x) & \text{if}\quad \{\theta\tilde{\alpha}\}\in\bigcup_{w\in W}\partial\langle w\rangle\,,\\ \sum_{n=0}^{\infty} z^n\, C^n(\tilde{\alpha})\,\chi(\theta^n\,\tilde{\alpha},x) & \text{otherwise.}\end{cases}$$

Also we denote for $g\in L^\infty$

$$s_g^{\tilde{\alpha}}(z)=s_g^{\tilde{\alpha}}(z;F)=\int s^{\tilde{\alpha}}(z;x)\,g(x)\,dx$$

and

$$\chi_g^{\tilde{\alpha}}(z)=\chi_g^{\tilde{\alpha}}(z;F)=\int \chi^{\tilde{\alpha}}(z;x)\,g(x)\,dx\,.$$

Note that for $\tilde{\alpha}\in\tilde{W}$

$$\chi(\tilde{\alpha},x)=\mathcal{E}(\tilde{\alpha})\,\sigma[x,<\tilde{\alpha}]\quad a.e.x,$$

and that

$$\chi_g^{\alpha^-}(z)+\chi_g^{\beta^+}(z)=\int_{\{\alpha^-\}}^{\{\beta^+\}} g(x)\,dx \tag{3.3}$$

for α^-, $\beta^+\in\tilde{S}$ if $\alpha^-<\beta^+$ and $\{\theta\alpha^-\}$, $\{\theta\beta^+\}\in\bigcup_{w\in W}\partial\langle w\rangle$.

DEFINITION. We denote by $s_g(z)=s_g(z;F)$ and $\chi_g(z)=\chi_g(z;F)$ the infinite dimensional vectors whose components are the formal power series $s_g^{\tilde{u}}(z)$ and $\chi_g^{\tilde{u}}(z)$, $\tilde{u}\in\tilde{W}$, respectively.

Set for $\tilde{\alpha}\in\tilde{S}$ and $\tilde{v}\in\tilde{W}$

$$\begin{aligned}\phi(\tilde{\alpha},\tilde{v})&=\mathcal{E}(\tilde{\alpha})\,\{C_{|v|}-C_{|v|-1}\}\,(\tilde{\alpha}[1]\cdot\tilde{v})\,\sigma[\tilde{v}\le\theta\tilde{\alpha}]\\&=\Delta_{|v|}(\tilde{\alpha}[1]\cdot\tilde{v})\,\sigma[\tilde{v}\le\theta\tilde{\alpha}]\,,\end{aligned}\tag{3.4}$$

and define a matrix $\Phi(z)=\Phi(z;A)$ on $\tilde{W}$ with the alphabet set A by

$$\Phi(z)_{\tilde{u},\tilde{v}}=\begin{cases}z\,\phi(\tilde{u},\tilde{v}) & \text{if}\quad\{\theta\tilde{u}\}\in\bigcup_{w\in W}\partial\langle w\rangle\,,\\ \sum_{n=0}^{\infty} z^{n+1}\,C^n(\tilde{u})\,\phi(\theta^n\,\tilde{u},\tilde{v}) & \text{otherwise.}\end{cases}\tag{3.5}$$

This is the generalization of the Fredholm matrix $\Phi_*(z)$ given in the previous paper ([11]).

Lemma 3.2. *Let* $J_N=(\theta\tilde{\alpha},[\theta\tilde{\alpha}]_N^{-\varepsilon(\tilde{\alpha})})$. *Then*

$$\begin{aligned}s^{\tilde{\alpha}}(z;x)=\chi(\tilde{\alpha},x)+\sum_{N=1}^{\infty}\sum_{v\in W_N} z\Delta_N(\tilde{\alpha}[1]\cdot v)\,\{\sigma[v<[\theta\tilde{\alpha}]_N:\mathcal{E}(\tilde{\alpha})\operatorname{sgn}\tilde{\alpha}[1]]\\+\delta[v=[\theta\tilde{\alpha}]_N,\operatorname{sgn}\tilde{\alpha}[1]=-]\}\,s^v(z;x)+z\sum_{N=1}^{\infty}\Delta_N([\tilde{\alpha}]_{N+1})\operatorname{sgn}\tilde{\alpha}[1]\,s^{J_N}(z;x)\,.\end{aligned}$$

Proof. First of all let $\alpha\in\tilde{S}$ and write (3.1) as

$$\begin{aligned}s^{\tilde{\alpha}}(z;x)&=\chi(\tilde{\alpha},x)\\&+\sum_{u\in W}\delta[u\cdot x\in S]\,z^{|u|+1}\,|C^{|u|+1}(\tilde{\alpha}[1]\cdot u\cdot x)|\,\sigma[\tilde{\alpha}[1]\cdot u\cdot x<\tilde{\alpha}:\mathcal{E}(\tilde{\alpha})]\,.\end{aligned}\tag{3.6}$$

Now we approximate F by F_N $(N\geq 1)$ or C by C_N. Recall that

$$\Delta_N(\tilde{\alpha}) = \{|C_N|-|C_{N-1}|\}\,(\tilde{\alpha})$$

and

$$|C(\tilde{\alpha})| = \sum_{N=0}^{\infty} \Delta_N(\tilde{\alpha})\,,$$

Then for $v\in W_N$ and $u\cdot x\in\langle v\rangle$ we get

$$\begin{aligned}&\sigma[u\cdot x<\theta\tilde{\alpha}: \varepsilon(\tilde{\alpha})\ \mathrm{sgn}\ \tilde{\alpha}[1]]\\ &\quad= \sigma[v<[\theta\tilde{\alpha}]_N: \varepsilon(\tilde{\alpha})\ \mathrm{sgn}\ \tilde{\alpha}[1]]+\delta[v=[\theta\tilde{\alpha}]_N,\ \mathrm{sgn}\ \tilde{\alpha}[1]=-]\\ &\qquad+\mathrm{sgn}\ \tilde{\alpha}[1]\ \delta[u\cdot x\in J_N]\,.\end{aligned}$$

Therefore the sum $u\in W$ in the right hand side can be divided into the sums in $N\geq 1$, $v\in W_N$ and $u\cdot x\in\langle v\rangle$ as follows:

(3.7)

$$\begin{aligned}&\sum_{u\in W}\delta[u\cdot x\in S]\,z^{|u|+1}|C^{|u|+1}(\tilde{\alpha}[1]\cdot u\cdot x)|\,\sigma[\tilde{\alpha}[1]\cdot u\cdot x<\tilde{\alpha}: \varepsilon(\tilde{\alpha})]\\ &\quad=\sum_{u\in W} z|C(\tilde{\alpha}[1]\cdot u\cdot x)|\,\sigma[u\cdot x<\theta\tilde{\alpha}: \varepsilon(\tilde{\alpha})\ \mathrm{sgn}\ \tilde{\alpha}[1]]\ \delta[u\cdot x\in S]\,z^{|u|}|C^{|u|}u\cdot x)|\\ &\quad=\sum_{N=1}^{\infty}\sum_{v\in W_N}\sum_{u\in W,u\cdot x\in\langle v\rangle} z\Delta_N(\tilde{\alpha}[1]\cdot v)\ \delta[u\cdot x\in S]\,z^{|u|}|C^{|u|}(u\cdot x)|\\ &\qquad\times\sigma[u\cdot x<\theta\tilde{\alpha}: \varepsilon(\tilde{\alpha})\ \mathrm{sgn}\ \tilde{\alpha}[1]]\\ &\quad=\sum_{N=1}^{\infty}\sum_{v\in W_N} z\Delta_N(\alpha[1]\cdot v)\ \delta[u\cdot x\in S]\,z^{|u|}|C^{|u|}(u\cdot x)|\\ &\qquad\sum_{u\in W,u\cdot x\in\langle v\rangle}\{\sigma[v<[\theta\tilde{\alpha}]_N: \varepsilon(\tilde{\alpha})\ \mathrm{sgn}\ \tilde{\alpha}[1]]\\ &\qquad+\delta[v=[\theta\tilde{\alpha}]_N,\ \mathrm{sgn}\ \tilde{\alpha}[1]=-]+\mathrm{sgn}\ \tilde{\alpha}[1]\ \delta[u\cdot x\in J_N]\}\\ &\quad=\sum_{N=1}^{\infty}\sum_{v\in W_N} z\Delta_N(\tilde{\alpha}[1]\cdot v)\ s^v(z;x)\\ &\qquad\times\{\sigma[v<[\theta\tilde{\alpha}]_N: \varepsilon(\tilde{\alpha})\ \mathrm{sgn}\ \tilde{\alpha}[1]]+\delta[v=[\theta\tilde{\alpha}]_N,\ \mathrm{sgn}\ \tilde{\alpha}[1]=-]\}\\ &\qquad+\sum_{N=1}^{\infty} z\Delta_N([\tilde{\alpha}]_{N+1})\ \mathrm{sgn}\ \tilde{\alpha}[1]\ s^{J_N}(z;x)\,.\end{aligned}$$

This porves the Lemma.

Now our renewal equation is as follows:

Proposition 3.3. *As a formal power series, we get the equation:*

$$s_g(z) = \chi_g(z)+\Phi(z)\,s_g(z)\,. \tag{3.8}$$

Proof. By Lemma 3.1 and Lemma 3.2, the right hand side of (3.7) is equal to

$$\sum_{N=1}^{\infty}\sum_{\tilde{v}\in\tilde{W}_N} z\Delta_N(\tilde{\alpha}[1]\cdot\tilde{v})\ s^{\tilde{v}}(z;x) \tag{3.9}$$

$$\times \{\sigma[v<[\theta\tilde{\alpha}]_N : \mathcal{E}(\tilde{\alpha}) \operatorname{sgn} \tilde{\alpha}[1]]+\delta[v=[\theta\tilde{\alpha}]_N, \operatorname{sgn} \tilde{\alpha}[1] = -] \\ +\operatorname{sgn} \tilde{\alpha}[1]\, \delta[\tilde{v}=[\theta\tilde{\alpha}]_{|v|}^{-\varepsilon(\tilde{\alpha})}]\} \\ + \sum_{N=1}^{\infty} z\Delta_N([\tilde{\alpha}]_{N+1}) \operatorname{sgn} \tilde{\alpha}[1]\, s^{\theta\tilde{\alpha}}(z; x)\,.$$

Recall that $x^+<x^-$ and that $\mathcal{E}(\tilde{\alpha})=\mathcal{E}(\theta\tilde{\alpha})$, thus

$$\sigma[v<[\theta\tilde{\alpha}]_N : \mathcal{E}(\tilde{\alpha}) \operatorname{sgn} \tilde{\alpha}[1]]+\delta[v = [\theta\tilde{\alpha}]_N, \operatorname{sgn} \tilde{\alpha}[1] = -]] \\ +\operatorname{sgn} \tilde{\alpha}[1]\, \delta[\tilde{v} = [\theta\tilde{\alpha}]_N^{-\varepsilon(\tilde{\alpha})}]+\delta[\tilde{v}=\theta\tilde{\alpha} \quad \text{as an expansion}] \\ = \mathcal{E}(\tilde{\alpha}) \operatorname{sgn} \tilde{\alpha}[1]\, \sigma[\tilde{v}\leq\theta\tilde{\alpha}]\,. \tag{3.10}$$

Consequently we get (3.8) if $\{\theta\tilde{\alpha}\} \in \bigcup_{w\in W} \partial\langle w\rangle$, indeed (3.8) follows from substituting (3.10) for (3.9), Now let for $\tilde{\alpha}$ which satisfies $\{\theta\tilde{\alpha}\} \notin \bigcup_{w\in W} \partial\langle w\rangle$. Since

$$\sum_{N=1}^{\infty} z\Delta_N([\tilde{\alpha}]_{N+1}) = z|C(\tilde{\alpha})| = z \operatorname{sgn} \tilde{\alpha}[1]\, C(\tilde{\alpha})\,,$$

we get from (3.9)

$$s^{\tilde{\alpha}}(z; x) = \chi(\tilde{\alpha}, x)+ \sum_{N=1}^{\infty} \sum_{\tilde{v}\in\tilde{W}_N} z\Delta_N(\tilde{\alpha}[1]\cdot\tilde{v})\, \{\sigma[v<[\theta\tilde{\alpha}]_N : \mathcal{E}(\tilde{\alpha}) \operatorname{sgn} \tilde{\alpha}[1]] \\ +\delta[v = [\theta\tilde{\alpha}]_N, \operatorname{sgn} \tilde{\alpha}[1] = -]+\operatorname{sgn} \tilde{\alpha}[1]\, \delta[\tilde{v} = [\theta\tilde{\alpha}]_N^{-\varepsilon(\tilde{\alpha})}]\}\, s^{\tilde{v}}(z; x) \\ +zC(\tilde{\alpha})\, s^{\theta\tilde{\alpha}}(z; x)\,. \tag{3.11}$$

Then applying (3.11) to $s^{\theta\tilde{\alpha}}(z; x)$ $(n\geq 1)$ repeatedly, we get (3.8). The proof is completed.

Corollary 3.4. (i) *For* $\tilde{\alpha}\in\tilde{W}$,

$$s^{\tilde{\alpha}}(z; x) = \chi(\tilde{\alpha}, x)+ \lim_{N\to\infty} \{ \sum_{v\in W_N} z|C_N(\tilde{\alpha}[1]\cdot v)|\, \{\sigma[v<[\theta\tilde{\alpha}]_N : \mathcal{E}(\tilde{\alpha}) \operatorname{sgn} \tilde{\alpha}[1]] \\ +\delta[v = [\theta\tilde{\alpha}]_N, \operatorname{sgn} \tilde{\alpha}[1] = -]\}\, s^{v}(z; x)+z|C_N(\tilde{\alpha})| \operatorname{sgn} \tilde{\alpha}[1]\, s^{J_N}(z; x)\}\,.$$

Moreover, if $F=F_N$ for some N, then we get:

$$s^{\tilde{\alpha}}(z; x) = \chi(\tilde{\alpha}, x)+ \sum_{v\in W_N} z|C_N(\tilde{\alpha}[1]\cdot v)|\, \{\sigma[v<[\theta\tilde{\alpha}]_N : \mathcal{E}(\tilde{\alpha}) \operatorname{sgn} \tilde{\alpha}[1]] \\ +\delta[v = [\theta\tilde{\alpha}]_N, \operatorname{sgn} \alpha[1] = -]\}\, s^{v}(z; x)+z|C_N(\tilde{\alpha})| \operatorname{sgn} \tilde{\alpha}[1]\, s^{J_N}(z; x)\,,$$

where $J_N=(\theta\tilde{\alpha}, [\theta\tilde{\alpha}]_N^{-\varepsilon(\tilde{\alpha})})$, as before.

(ii) *Suppose that $F=F_N$ for some N. Then for α^-, $\beta^+\in\tilde{W}$ such that $\alpha^-<\beta^+$ and $[\alpha^-]_N=[\beta^+]_N$, we get*

$$s^{(\alpha^-,\beta^+)}(z; x) = \chi(\alpha^-, x)+\chi(\beta^+, x)+z|C(\alpha^-)|\, s^{J}(z; x)\,,$$

where $J=(\theta\alpha^-, \theta\beta^+)$.

Proof. The assertion (i) is a direct consequence of Lemma 3.2, and the assertion (ii) follows from (i).

4. Operator Φ

Now we will specify the domain and the renge of the operator $\Phi(z)$ so that $\Phi(z)$ be the bounded operator from a Banach space, denoted by $\mathscr{B}$ in below, to another Banach space $\mathscr{X}$. For a while, we fix a real number ρ so that $e^{-\xi}<\rho<1$, and we only consider z which satisfies $|z|<1/\rho$. By the definition of ξ we can take a constant $K_1=K_1(\rho)>1$ such that

$$\operatorname*{ess\,sup}_{x\in I} |C^n(x)|<K_1\rho^n\,, \tag{4.1}$$
$$\text{Lebes}\,\langle u\rangle<K_1\rho^{|u|} \quad (u\in W)\,.$$

Lemma 4.1. *Let* $1\leq N\leq\infty$.
(i) *The norm* $V((F'_N)^{-1}))$ *is uniformly bounded in* N.
(ii) *Denote for* $u\in W$ *and* $|u|\leq N\leq\infty$

$$h_{u,N}(x)=\begin{cases}|C_N(x)|-|C_{|u|}(x)| & \textit{if}\quad x\in\langle u\rangle\,,\\ 0 & \text{otherwise.}\end{cases}$$

Then we get

$$V(h_{u,N})\leq 2K_2\,\text{Lebers}\,\langle u\rangle\,, \tag{4.2}$$

where

$$K_2=\frac{\sup_{x\in I}|F''(x)|}{(\inf_{x\in I}|F'(x)|)^2}\,.$$

Proof. First note that

$$\left|\left(\frac{1}{F'(x)}\right)'\right|=\left|\frac{F''(x)}{F'(x)^2}\right|\leq K_2\,.$$

Therefore

$$\text{var}\,(h_{u,N})\leq K_2\,\text{Lebes}\,\langle u\rangle$$

and

$$\left|\int h_{u,N}\,dx\right|\leq\sup|f_{\langle u\rangle}|\,\text{Lebes}\,\langle u\rangle$$
$$\leq K_2\,\text{Lebes}\,\langle u\rangle\,.$$

This proves Lemma.

We denote by $\mathcal{S}$ the space of those vectors $(s^{\tilde{v}})_{\tilde{v}\in\tilde{W}}$ with $s^{\tilde{v}}\in \boldsymbol{C}$ whose components satisfy the relations

$$\mathcal{E}(\tilde{u})\,s^{\tilde{u}}=\mathcal{E}(\tilde{v})\,s^{\tilde{v}} \tag{4.3}$$

whenever $\{\tilde{u}\}=\{\tilde{v}\}$ and $\tilde{u}[1]=\tilde{v}[1]$, that is, whenever $\tilde{u}$ and $\tilde{v}$ express the same point.

Now let $\mathcal{B}=\mathcal{B}(z; F)$ be the set of $s=(s^{\tilde{u}})\in\mathcal{S}$ which satisfies the following three conditions:

(i) The following limit exists for $\tilde{\alpha}\in\tilde{S}$, and coincide with $s^{\tilde{u}}$ if $\tilde{\alpha}=\tilde{u}\in\tilde{W}$.

$$s^{\tilde{\alpha}} = \lim_{N\to\infty} \mathcal{E}(\tilde{\alpha})\, s^{[\tilde{\alpha}]_N^+} = -\lim_{N\to\infty} \mathcal{E}(\tilde{\alpha})\, s^{[\tilde{\alpha}]_N^-}, \tag{4.4}$$

(ii)
$$||s|| = ||s||_\infty + ||s||_v < \infty .$$

where

$$||s||_\infty = \sup_{\tilde{w}\in\tilde{W}} |s^{\tilde{w}}| ,$$

and

$$||s||_v = \sup_{V(f)=1} |\langle f, s\rangle|,$$

where

$$\langle f, s\rangle = \lim_{u\to\infty} \sup \sum_{u\in W_N} \frac{\int_{\langle u\rangle} f(x)\,dx}{\text{Lebes}\,\langle u\rangle} s^u \tag{4.5}$$

(iii)
$$\sup_{u\in W} \left| \frac{s^u - z\,|C_{|u|}(u)|\, s^{\theta u}}{\text{Lebes}\,\langle u\rangle} \right| < \infty .$$

REMARK. We will prove in Lemma 6.1 that for any $f\in BV$ the limit in (4.5) exists if $z^{-1}\notin \text{Spec}(F)$.

Lemma 4.2. (i) *If* $||s_g(z)||_v<\infty$ *for* $g\in L^\infty$, *then* $||s_g(z)||_\infty<\infty$.
(ii) *Let*

$$||s||_r = \sup_{N\geq 1} \sup_{\tilde{\alpha},\tilde{\beta}\in\tilde{S}} \{|s^{\tilde{\alpha}}+s^{\beta}|r^{-N} : [\tilde{\alpha}]_N = [\tilde{\beta}]_N \quad \text{and} \quad \mathcal{E}(\tilde{\alpha})\,\mathcal{E}(\tilde{\beta}) = -\} .$$

If $||s||_r<\infty$ *for some* $0<r<1$, *then* $||s||_v<\infty$.

Proof. (i) By Corollary 3.4 (i), we get

$$s^{\tilde{\alpha}}(z; x) = \chi(\tilde{\alpha}, x) + \lim_{N\to\infty} [\sum_{v\in W_N} z\,|C_N(\tilde{\alpha}[1]\cdot v)|\, \{\sigma[v<[\theta\tilde{\alpha}]_N : \mathcal{E}(\tilde{\alpha})\,\text{sgn}\,\tilde{\alpha}[1]]$$
$$+\delta[v = [\theta\tilde{\alpha}]_N, \text{sgn}\,\alpha[1] = -]\}\, s^v(z, x) + z\,|C(\tilde{\alpha})|\,\text{sgn}\,\tilde{\alpha}[1]\, s^{J_N}(z; x)] \tag{4.6}$$

By Lemma 4.1 (i), the third term in the right-hand side of (4.6) converges to zero as $N\to\infty$ and the sum in the second term is absolutely convergent uniformly in N. This proves (i).

Note that for $f\in BV$ the inequality

$$\sum_{u\in W} |f_u| \, r^{|u|} \leq \frac{V(f)}{1-r}$$

holds for a suitable choice of a version of f_u such that $f=\sum_{u\in W} f_u \, 1_u$. Hence

$$\begin{aligned} ||s||_v &= \sup_{V(f)=1} \sum_{u\in W} |f_u \, s^u| \\ &\leq \sup_{V(f)=1} \sum_{u\in W} |f_u| \, r^{|u|} \, ||s||_r \\ &\leq \frac{||s||_r}{1-r} . \end{aligned} \tag{4.7}$$

This proves (ii).

Now we define the range of $I-\Phi(z)$. Let

$$\mathcal{X} = \mathcal{X}(z) = \{\chi_g(z) : g\in L^\infty\} .$$

For a vector $s=\chi_g(z)\in\mathcal{X}$, we define norm by

$$||s|| = ||\chi_g(z)|| = ||g||_\infty = \operatorname*{ess\,sup}_{x\in I} |g(x)| .$$

and $s^{\tilde{\alpha}}$ is defined by (4.4) for $\tilde{\alpha}\in\tilde{S}\backslash\tilde{W}$.

The definition of $\chi_g(z)$ is stated in §3. The mapping $g\mapsto\chi_g(z)$ is clearly onto and by (3.3) it is one to one.

Lemma 4.3. *A vector $s\in S$ belongs to $\mathcal{X}$ if and only if it satisfies* (4.3) *and the following three conditions.*

(i) *The limit*

$$t^{\tilde{\alpha}} = \varepsilon(\tilde{\alpha}) \lim_{\tilde{v}\to\tilde{\alpha}} \varepsilon(\tilde{v}) \, s^{\tilde{v}}$$

exists for any $\tilde{\alpha}\in\tilde{S}$ where $\tilde{v}\in\tilde{W}$ and $\{\theta\tilde{v}\}\in\bigcup_{w\in W}\partial\langle w\rangle$.

(ii) *There holds*

$$s^{\tilde{u}} = \begin{cases} t^{\tilde{u}} & \text{for } \tilde{u}\in\tilde{W} \text{ and } \{\theta\tilde{v}\}\in\bigcup_{w\in W}\partial\langle w\rangle , \\ \sum_{n=0}^{\infty} z^n \, C_n(\tilde{u}) \, t^{\theta^n \tilde{u}} & \text{for } \tilde{u}\in\tilde{W} \text{ and } \{\theta\tilde{v}\}\notin\bigcup_{w\in W}\partial\langle w\rangle . \end{cases} \tag{4.8}$$

(iii) *For any $x\in I$, the limit*

$$\lim_{N\to\infty} \frac{s^{[\alpha^x]_N}}{\text{Lebes}\,\langle[\alpha^x]_N\rangle}$$

exists and bounded where $s^u=s^{u^+}+s^{u^-}$.

Proof. If $s=\chi_g(z)$ for some $g\in L^\infty$, then from the definition of $\chi_g(z)$ it satisfies the conditions (i)-(iii). Conversely, suppose that $s\in S$ satisfies (i)-(iii). Let

$$g(x)=\lim_{N\to\infty}\frac{s^{[\alpha^x]_N}}{\text{Lebes}\,\langle[\alpha^x]_N\rangle}\,.$$

Then $g\in L^\infty$ and $s=\chi_g(z)$. This proves Lemma.

Proposition 4.4. *The operator $I-\Phi(z)$ is bounded from $\mathcal{B}$ into $\mathcal{X}$.*

Proof. We will only prove the case when $\{\theta\tilde{u}\}\in\bigcup_{w\in W}\partial\langle w\rangle$; we can prove other cases in similar ways.

Recalling the condition (4.3), we get

$$\begin{aligned}(4.9)\qquad (\Phi(z)\,s)^{\tilde{u}} &= \sum_{\tilde{v}\in\tilde{V}}\Phi(z)_{\tilde{u},\tilde{v}}\,s^{\tilde{v}}\\ &=\lim_{N\to\infty}\sum_{v\in W_N}\sum_{\tilde{w}\,:\,\langle w\rangle\supset\langle v\rangle}\Phi(z)_{\tilde{u},\tilde{w}}\,s^{\tilde{v}}\\ &=\lim_{N\to\infty}\sum_{v\in W_N}\sum_{\tilde{w}\,:\,\langle w\rangle\supset\langle v\rangle}\varepsilon(\tilde{u})\,z\{C_{|w|}-C_{|w|-1}\}\,(u[1]\cdot\tilde{w})\\ &\quad\times\sigma[\tilde{w}\le\theta\tilde{u}]\,s^{\tilde{v}}\\ &=\lim_{N\to\infty}\sum_{v\in W_N}\varepsilon(\tilde{u})\,C_N(u[1]\cdot v)\,\sigma[v\le\theta\tilde{u}]\,s^{v}\end{aligned}$$

Hence, by Lemma 4.1, the right hand term of (4.9) is bounded. This shows $|(\Phi(z)\,s)^{\tilde{u}}|$ is bounded and $\Phi(z)s$ ($s\in\mathcal{B}$) is well-defined. Next note that

$$\begin{aligned}(4.10)\qquad (\Phi(z)\,s)^{u} &= (\Phi(z)\,s)^{u+}+(\Phi(z)\,s)^{u-}\\ &=\lim_{N\to\infty}\sum_{v\in W_N\,:\,\langle u[1]\cdot v\rangle\subset\langle u\rangle} z\,C_N(u[1]\cdot v)\,s^{v}\,.\end{aligned}$$

Therefore by (4.2) and assumption (iii) of $s\in\mathcal{B}$ that

$$\sup_{u\in W}\left|\frac{s^u-z\,|C_{|u|}(u)|\,s^{\theta u}}{\text{Lebes}\,\langle u\rangle}\right|<\infty\,,$$

we get

$$\sup_{u\in W}\left|\frac{\{(I-\Phi(z))\,s^u\}}{\text{Lebes}\,\langle u\rangle}\right|<\infty\,.$$

The proofs of latter parts are easy to be verified, hence we omit them.

5. Estimation of $\Phi_{M,N}(z)$

In this section, we will consider the mappings F_N and define their Fredholm matrices. Set $\tilde{W}(M)-\bigcup_{K=1}^{M}\tilde{W}_K$ and $\tilde{W}(\infty)=\tilde{W}$.

Let $\mathcal{S}(N)$ be the set of those vectors with index set $\tilde{W}(N)$ which satisfy the condition (4.3) with supremum norm denoted by $||\cdot||_N$. It is convenient to write $\mathcal{S}$ in place of $\mathcal{S}(\infty)$. We denote the restriction of $s\in\mathcal{S}(M)$ to $\mathcal{S}(N)$ by $s|_N$ for $N\le M\le\infty$.

DEFINITION. (i) We define a square matrix $\Phi_{N\,N}(z)$ indexed by $\tilde{W}(M)$

with $N\leq M\leq\infty$ by

$$\Phi_{M,N}(z)_{\tilde{u},\tilde{v}}=\begin{cases}\Phi(z)_{\tilde{u},\tilde{v}} & \text{if } \tilde{u}\in\tilde{W}(M),\ \tilde{v}\in\tilde{W}(N)\,,\\ 0 & \text{otherwise,}\end{cases}$$

and we denote $\Phi_{N,N}(z)$, simply, by $\Phi_N(z)$.

(ii) The lower Lyapunov number assoicated with F_N is defined by

$$\xi_N=\liminf_{n\to\infty}\ \operatorname*{ess\,inf}_{x\in I}\frac{1}{n}\log|F_N^{n\prime}(\alpha^x)|\,.$$

(iii) The zeta function $\zeta_N(z)$ corresponding to F_N is defined by

$$\zeta_N(z)=\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}z^n\sum_{\alpha=\theta^n\alpha}|F_N^{n\prime}(\alpha)|^{-1}\right].$$

REMARK. (i) Each component of the matrix $\Phi_{M,N}$ is analytic in the disk $|z|<e^{\xi_N}$.

(ii) Suppose that s is an eigenvector of $\Phi_{M,N}(z)$ associated with an eigenvalue λ ($\lambda\neq 0$) for some $M>N$, then $s|_N$ is also an eigenvecotr of $\Phi_N(z)$ with the same eigenvalue λ.

(iii) The zeta function $\zeta_N(z)$ converges to $\zeta(z)$ in the unit disk $|z|<1$. In §7, we will prove it also converges in the disk $|z|<e^{\xi}$.

Lemma 5.1. ([11]) *If* 0 *is an eigenvalue of* $I-\Phi_N(z)$ *restricted to* $\mathcal{S}(N)$, *then* 0 *is also an eigenvalue of* $I-\Phi_N(z)$.

This is Lemma 4.2 in [11] and we omit the proof. The key point of the proof is to show that the kernel of $I-\Phi_N(z)$ is contained in $\mathcal{X}+(I-\Phi_N(z))\,\mathcal{X}$ (Lemma 4.6 [11]).

REMARK. For a partition of I into subintervals which satisfies the assumption (A3), one can express the Fredholm determinant as

$$\det\{I-\Phi_N(z)\}=\{\exp R_N(z)\}/\zeta_N(z)$$

with a certain function $R_N(z)$ which depends only on the periodic orbits passing through the endpoints of $\langle a\rangle$, $a\in A$. Furthermore we can always take $R_N(z)=0$ if we make a suitable choices of subintervals which division points of the paritition belong to (cf. [11] for detail). Thus we get

$$\det\{I-\Phi_N(z)\}=1/\zeta_N(z)$$

for such a partition. Hereafter, we always assume the partition is suitable in this sense.

Lemma 5.2. (i) *The operator* $\Phi_{M,N}(z)$ *for* $N\leq M<\infty$ *is the bounded oper-*

ator from $\mathcal{S}(M)$ to itself.

(ii) *The restriction of $s_g(z; F_N)$ to $\tilde{W}(M)$ for $N \leq M \leq \infty$ satisfies the renewal equation*

$$s_g(z; F_N)|_M = \chi_g(z; F_N)|_M + \Phi_{M,N}(z)\, s_g(z; F_N)|_M .$$

Proof. The assertions follows directly from Proposition 3.3 and Proposition 4.4.

We will begin with the basic property of $\Phi_{M,N}(z)$.

Lemma 5.3. (i) *For $s \in \mathcal{S}(M)$,*

$$(5.1)\quad \{\Phi_{M,N}(z)\, s\}^{\tilde{u}} = \textstyle\sum_1 \varepsilon(\tilde{u})\, z\{|C_N(u[1]\cdot v^+)| - |C_N(u[1]\cdot v^{\uparrow -})|\}\, \sigma[v^+ \leq \theta\tilde{u}]\, s^{v^+} + \varepsilon(\tilde{u})\, z\, |C_N(\tilde{u})|\, s^{\theta\tilde{u}} + \sum_{\tilde{a}\,:\,\theta\tilde{a} \notin \cup_{w\in W} \partial\langle w\rangle} \varepsilon(\tilde{u})\, z\, |C_N(u[1]\cdot\tilde{a})|\, \sigma[\tilde{a} \leq \theta\tilde{u}]\, s^{\tilde{a}},$$

where the sum $\sum_1$ is taken over all $v^+ \in \tilde{W}_N$ such that $\{v^+\} \neq \{\theta\tilde{u}\}$ and $\{\theta v^+\} \in \cup_{w\in W} \partial\langle w\rangle$.

(ii) *There is a constant K_3 and for $N_1 > N_2$ we get,*

$$(5.2)\quad \|\Phi_{N_1,N_2}(z) - \Phi_{N_1}(z)\|_{N_1} < K_3\, \rho^{N_2} .$$

Proof. (i) First note that $s^{v^+} = -s^{v^{\uparrow -}}$, where $v^{\uparrow} = \min_{w\in W}\{w > v\}$. Then

$$\{\Phi_{M,N}(z)\, s\}^{\tilde{u}} = \sum_{\tilde{v}\in\tilde{W}_N} \varepsilon(\tilde{u})\, z\, |C_N(u[1]\cdot\tilde{v})|\, \sigma[\tilde{v} \leq \theta\tilde{u}]\, s^{\tilde{v}}$$
$$= \textstyle\sum_2 \varepsilon(\tilde{u})\, z\{|C_N(u[1]\cdot v^+)|\, \sigma[v^+ \leq \theta\tilde{u}] - |C_N(u[1]\cdot v^{\uparrow -})|\, \sigma[v^{\uparrow -} \leq \theta\tilde{v}]\} s^{\tilde{v}^+} + \sum_{\tilde{a}\,:\,\theta\tilde{a} \notin \cup_{w\in W} \partial\langle w\rangle} \varepsilon(\tilde{u})\, z\, |C_N(u[1]\cdot\tilde{a})|\, \sigma[\tilde{a} \leq \theta\tilde{u}]\, s^{\tilde{a}},$$

where the sum $\sum_2$ is taken over all $v^+ \in \tilde{W}_N$ such that $\{\theta v^+\} \in \cup_{w\in W} \partial\langle w\rangle$. Therefore, we get the proof of (i) by dividing the sum $\sum_2$ into the sum $\sum_1$ and $v^+ \in \tilde{W}_N$ such that $\{v^+\} = \{\theta\tilde{u}\}$.

(ii) Note that

$$|C_{N+1}(\alpha) - C_N(\alpha)| < \sup_{y_1 y_2 \in \langle[\alpha]_N\rangle} |F'(y_1)^{-1} - F'(y_2)^{-1}| < K_1 K_2\, \rho^N .$$

Then since

$$\{\Phi_{N_1,N_2}(z) - \Phi_{N_1}(z)\}_{\tilde{u},\tilde{v}} = \begin{cases} 0 & \text{if } \tilde{v} \in \tilde{W}(N_2), \\ \{\Phi_{N_1}(z)\}_{\tilde{u},\tilde{v}} & \text{otherwise,} \end{cases}$$

we get (ii).

The determinant $\det(I - \Phi_N(z))$ does not depend on the choice of alphabet set in the following sense:

Lemma 5.4. *Let $\Phi(z; W_K)$ is the Fredholm matrix induced by the alphabet set W_K. Then for $K \leq N \leq M < \infty$ and $|z| < e^{\xi_N}$, we get*

$$\det(I - \Phi_{M,N}(z; W_K)) = \det(I - \Phi_N(z)) . \tag{5.3}$$

In particular,

$$\Phi(z; W_1) = \Phi(z; A) = \Phi(z) .$$

Proof. We sweep out $\tilde{W}(K-1)$-elements of the matrix $I - \Phi_N(z)$. We continue the following procedure from $L=1$ to $K-1$:

For $\tilde{v} \in \tilde{W}_L$, let $\tilde{u} \in \tilde{W}_{L+1}$ be the signed word which satisfies $\{\tilde{v}\} = \{\tilde{u}\}$ and $\mathcal{E}(\tilde{v}) = \mathcal{E}(\tilde{u})$. Firstly, subtract the $\tilde{u}$-row from the $\tilde{v}$-row. Note that the components of the $\tilde{v}$-row equal the corresponding components of the $\tilde{v}$-row except diagonal components, thus the $(\tilde{v}, \tilde{v})$-component becomes 1, the $(\tilde{v}, \tilde{u})$-component becomes -1 and the rest of the components of the $\tilde{v}$-row become 0. Next, add the $\tilde{v}$-column to the $\tilde{u}$-column. Then all the components of the $\tilde{v}$-row become 0 except the $(\tilde{v}, \tilde{v})$-component which becomes 1. This shows $\det(I - \Phi(z; W_L)) = \det(I - \Phi(z; W_{L+1}))$. Repeating this, the equality $\det(I - \Phi_N(z)) = \det(I - \Phi_N(z; W_K))$ follows. We get (5.3) since for $M > N$ there holds $\det(I - \Phi_N(z; W_K)) = \det(I - \Phi_{M,N}(z; W_K))$.

Lemma 5.5. (i)

$$\lim_{N \to \infty} \xi_N = \xi . \tag{5.4}$$

(ii) *For sufficiently large N, an inequality*

$$\operatorname*{ess\,sup}_{x \in I} |C_N^n(\alpha^x)| < K_1 \rho^n$$

holds.

Proof. (i) For $\alpha \in S$ and $N \leq \infty$, set

$$C_N(z; \alpha) = \sum_{n=0}^{\infty} |C_N^n(\alpha)| z^n .$$

Hereafter we use the converntion $F_\infty = F$ and $\xi_\infty = \xi$. We denote the convergence radius of $F_N(z; \alpha)$ by $\xi_N(\alpha)$. Then $\xi_N = \operatorname{ess\,inf}_{x \in I} \xi_N(\alpha^x)$.

Recall that F'_N converges to F' uniformly. We get

$$\begin{aligned} \left| \frac{F'(\alpha)}{F'_N(\alpha)} \right| &= 1 + \left| \frac{F'(\alpha) - F'_N(\alpha)}{F'_N(\alpha)} \right| \\ &\leq (1 - \mathcal{E}(N))^{-1} , \end{aligned} \tag{5.5}$$

where we set

$$\varepsilon(N)=\frac{\sup_{\beta\in S}|F'_N(\beta)-F'(\beta)|}{\inf_{\beta\in S}|F'(\beta)|}.$$

Hence,

$$|C_N(z;\alpha)|\leq\sum_{n=0}^{\infty}\prod_{j=0}^{n-1}\left|\frac{C_N(\theta^j\alpha)}{C(\theta^j\alpha)}\right||C(\theta^j\alpha)||z^n|$$
$$\leq\sum_{n=0}^{\infty}(1-\varepsilon(N))^{-n}|C^n(\alpha)||z^n|.$$

This shows

$$\xi_N(\alpha)\geq(1-\varepsilon(N))\,\xi_\infty(\alpha).$$

By the similar estmiate as (5.5) from below we get

$$\xi_N(\alpha)\leq(1+\varepsilon(N))\,\xi_\infty(\alpha).$$

This completes the proof (i) and the proof of (ii) follows from (i).

6. Characterization of Spec (F) by Fredholm matrix

In Lemma 6.1 and Lemma 6.2 below, we characterize Spec (F) in terms of $\Phi(z)$ or, in other words, in terms of $s_g(z)$:

$$s_g^u(z)=\int(I-zP)^{-1}\,1_u(x)\,g(x)\,dx.$$

Then, in Lemma 6.4 and Lemma 6.5 below we study the relations between the spectrum of $\Phi(z)$ and the spectrum of $\Phi_N(z)$. Combining these results, we characterize Spec (F) in terms of $\Phi_N(z)$ in Theorem 6.3. This will lead us to the proof of Theorem B.

Let

$$\mathcal{D}=\{z:\,|z|<e^{\xi},\,z^{-1}\notin\mathrm{Spec}\,(F)\}.$$

Note that $s_g^u(z)$ is defined in the unit disk $|z|<1$ and has an analytic extension to the domain $\mathcal{D}$.

Lemma 6.1. *Let z satisfy $|z|<e^{\xi}$. Then, $z^{-1}\in\mathrm{Spec}(F)$ if and only if* $\sup_{||g||=1}||s_g(z)||=\infty$.

Proof. Recall the definitions in (4.5):

$$||s_g(z)||=||s_g(z)||_\infty+||s_g(z)||_v,$$
$$||s_g(z)||_v=\sup_{V(f)=1}|\langle f,s_g(z)\rangle|,$$

and

$$\langle f,s_g(z)\rangle=\lim_{N\to\infty}\sup\sum_{u\in W_N}f_u\,s_g^u(z),$$

where

$$f_u = \frac{\int_{\langle u\rangle} f(x)\, dx}{\text{Lebes}\,\langle u\rangle}.$$

We denote $f_{W_N}(x)=\sum_{u\in W_N} f_u\, 1_u(x)$. Then for any $f\in BV$, f_{W_N} satisfies the conditions:

(i) $\lim_{N\to\infty} f_{W_N}=f$ (in L^1),

(ii) $V(f_{W_N})\leq V(f)$ for $N=1, 2, \cdots,$.

Therefore, since for $g\in L^\infty$

$$\lim_{N\to\infty}\int (I-zP)^{-1} f_{W_N}(x)\, g(x)\, dx = \int (I-zP)^{-1} f(x)\, g(x)\, dx,$$

if $|z|<1$ and also if $z\in\mathcal{D}$ (i.e. $z^{-1}\notin \mathrm{Spec}(F)$), that is, the above lim sup can be replaced by lim:

$$\langle f, s_g(z)\rangle = \lim_{N\to\infty} \sum_{u\in W_N} f_u\, s_g^u(z).$$

Suppose first that $\sup_{\|g\|=1} \|s_g(z)\|=\infty$. Then

$$\sum_{u\in W_N} f_u\, s_g^u(z) = \int (I-zP)^{-1} f_{W_N}(x)\, g(x)\, dx$$

must be unbounded for some $f\in BV$ and $g\in L^\infty$. Therefore, there exists $f_N=\sum_{u\in W_N}(f_N)_u\, 1_u$ and $g_N\in L^\infty$ such that $V(f_N)\leq 1$, $\|g_N\|=1$ and

$$\lim_{N\to\infty} \Big|\sum_{u=W_N} (f_N)_u\, s_{g_N}^u(z)\Big| = \infty .$$

In this case, it follows that $z^{-1}\in \mathrm{Spec}(F)$, because

$$\sum_{u\in W_N} (f_N)_u\, s_{g_N}^u(z) = \int (I-zP)^{-1} f_N(x)\, g_N(x)\, dx .$$

Conversely, suppose that $\sup_{\|g\|=1} \|s_g(z)\|<\infty$. Then for any $f\in BV$ and $g\in L^\infty$, $\int (I-zP)^{-1} f_{W_N}(x)\, g(x)\, dx$ converges to $(f, g)(z)=\langle f, s_g(z)\rangle$ as $N\to\infty$. This shows $z^{-1}\notin \mathrm{Spec}(F)$.

Lemma 6.2. *Suppose that $z^{-1}\notin \mathrm{Spec}(F)$ and that there exists a vector $s\in\mathcal{B}$ satisfying $(I-\Phi(z))\, s=\chi_g(z)$. Then for any $u\in W$ the equality $s^u=s_g^u(z)$ holds, where we set $s^u=s^{u^+}+s^{u^-}$.*

REMARK. There may be a vector $s\neq s_g(z)$ which satify $(I-\Phi(z))\, s=\chi_g(z)$, but this lemma says, if $z^{-1}\notin \mathrm{Spec}(F)$, the vector $(s^u)_{u\in W}$ is unique. Hence $\langle f, s\rangle$ is also uniquely determined for any $f\in BV$ and $g\in L^\infty$.

Proof. Suppose that there exists s which satisfies $(I-\Phi(z_0))\, s=\chi_g(z_0)$ for z_0 such that $z_0^{-1}\notin \mathrm{Spec}(F)$. Note that

$$P\,1_u(x) = \sum_{F(y)=x} |C(y)|\,1_u(y)$$
$$(6.1) \qquad = \lim_{N\to\infty} \sum_{F(y)=x} |C_N(y)|\,1_u(y)$$
$$= \lim_{N\to\infty} \sum_{v\in W_N\,:\,\langle u[1]\cdot v\rangle \subset \langle u\rangle} |C_N(u[1]\cdot v)|\,1_v(y)\,.$$

Recall Corollary 3.4(i), then

$$s^u = \int_{\langle u\rangle} g(x)\,dx + \lim_{N\to\infty} \sum_{v\in W_N\,:\,\langle u[1]\cdot v\rangle \subset \langle u\rangle} z_0 |C_N(u[1]\cdot v)|\,s^v\,.$$

This shows that the equation

$$(6.2) \qquad s^u = \sum_{n=0}^{N-1} z_0^n \int P^k\,1_u(x)\,g(x)\,dx + z_0^N \langle P^N\,1_u, s\rangle\,.$$

holds for $N=1$, and for general cases we can show (6.2) using a similar formula for $P^N\,1_u$ to (6.1). Denote

$$s_N^u(z) = \sum_{n=0}^{N-1} z^n \int P^k\,1_u(x)\,g(x)\,dx + z^N \langle P^N\,1_u, s\rangle\,.$$

Then $s_N^u(z)$ coincides with s^u at $z=z_0$ by (6.1). On the other hand, it converges to $s_g^u(z)$ as $N\to\infty$ in $|z|<1$. Therefore, as we assumed $z_0^{-1}\notin \mathrm{Spec}(F)$, it follows $s^u=s_g^u(z_0)$. This proves Lemma.

For a piecewise linear transformation, as we have already seen in [11], the Fredholm matrix can be reduced to a finite dimensional matrix, and so we can naturally define its determinant $D(z)$ and show that a complex number z^{-1} with $|z|<e^{\xi}$ belongs to $\mathrm{Spec}(F)$ if and only if $D(z)=0$. In our case for a general piecewise monotonic transformation, we can characterize $\mathrm{Spec}(F)$ in terms of $I-\Phi_N(z)$, $N=1, 2, \cdots$, as follows:

Theorem 6.3. *For a complex number z in the disk $|z|<e^{\xi}$, z^{-1} belongs to* $\mathrm{Spec}(F)$ *if and only if $\|(I-\Phi_N(z))^{-1}\|_N$ is unbounded.*

We divide the proof of Theorem 6.3 in two parts.

Proof of Theorem 6.3: "only if" part. It suffices to prove the following lemma.

Lemma 6.4. *Assume $\|(I-\Phi_N(z))^{-1}\|_N$ is unbounded. Then there exists $s\in\mathcal{B}$ such that $\|s\|=1$ and $(I-\Phi(z))\,s=0$.*

In fact, if $\|(I-\Phi_N(z))^{-1}\|_N$ is unbounded, then there exists $s\in\mathcal{B}$ such that $\|s\|=1$ and $(I-\Phi(z))\ s=0$. Therefore, by Lemma 6.2, we get $z^{-1}\in\mathrm{Spec}(F)$. This completes the proof of only if part of Theorem 6.3.

Proof of Lemma 6.4. We choose $r<1$ so that $r>\rho$ and $r>|z|\rho$. Denote by λ_N the least eigenvalue in modules of $(I-\Phi_N(z))$ on $\mathcal{S}(N)$. Since we assume that $\|(I-\Phi_N(z))^{-1}\|_N$ is unbounded, λ_N decreases to zero as N tends to infinity. By Remark (ii) in §5, λ_N is also the least eigenvalue in modulus of $(I-\Phi_{M,N}(z))$ restricted to $S(M)$. In particular, there exists the corresponding normalized eigenvector $s_N\in\mathcal{S}(\infty)\|s_{N|_\Lambda}\|_N=1$ such that the restriction of s_N to $\mathcal{S}(M)$ satisfies the condition

$$(I-\Phi_{M,N}(z))\, s_{N|_M}=\lambda_N\, s_{N|_M}\,. \tag{6.3}$$

If $\|s_N\|$ is bounded, then we can construct $s\neq 0$ and $s\in\mathcal{B}$ by diagonal method from $\tilde{s}_N=s_N/\|s_N\|$. By (6.3), it is easy to see that $(I-\Phi(z))\, s=0$.

Hence we will prove the boundedness of $\|s_N\|$. Let $\varepsilon>0$ satisfy $r(1-\varepsilon)^{-1}<1$. Then there exists an integer N_1 such that for $N\geq N_1|\lambda_N|<\varepsilon$. Set

$$K_4=\max\left\{\frac{K_1(K_1K_2+2\# A+1)}{\{1-r(1-|\varepsilon|)^{-1}\}},\ \max_{\tilde{u}\in\tilde{W}(N_1)}|s_N^{\tilde{u}}|\right\}.$$

Note that

$$|C_N(u[1]\cdot v^+)-C_N(u[1]\cdot v^{\uparrow -})|\leq K_1K_2|C_N(u[1]\cdot v^+)|\,\mathrm{Lebes}\langle u[1]\cdot v\rangle\,,$$

where $v^{\uparrow}=\min_{w\in W}\{w>v\}$. Since $\|s_{N|_N}\|_N=1$, we only need to consider $\tilde{W}_{N+K}$ $(K>0)$. Let $\tilde{u}\in\tilde{W}_{N+K}$ for which there exists no $\tilde{v}\in\tilde{W}_N$ such that $\{\tilde{u}\}=\{\tilde{v}\}$. Since

$$\{(I-\Phi_{N+K,N}(z))\, s_{N|_{N+K}}\}^{\tilde{u}}=\lambda_N\, s_N^{\tilde{u}}\,,$$

we get by Lemma 5.3 (i)

$$\begin{aligned}(1-|\lambda_N|)|s_N^{\tilde{u}}|\leq{}&|z|\,|C_N(\tilde{u})|\,|s_N^{\theta\tilde{u}}|\\&+\sum_{v\in W_N}|z|\,|C_N(u[1]\cdot v)|\,K_1K_2\,\mathrm{Lebes}\langle u[1]\cdot v\rangle\\&+\sum_{\tilde{a}}|z|\,|C_N(u[1]\cdot\tilde{a})|\,|s_N^{\tilde{a}}|.\end{aligned} \tag{6.4}$$

Hence, we get by the induction on K

$$\begin{aligned}|s_N^{\tilde{u}}|\leq K_1K_2\,&\{\sum_{j=1}^{K}(1-|\lambda_N|)^{-j}|z|^j[\sum_{v\in W_N}|C_N^j(u[1,j]\cdot v)|+\sum_{\tilde{a}\in\tilde{A}}|C_N^j(u[1,j]\cdot\tilde{a})|]\}\\&+(1-|\lambda_N|)^{-K}|z|^K|C_N^K(\tilde{u})|\,|s_N^{\theta^K\tilde{u}}|\end{aligned}$$

Consequently,

$$|s_N^{\tilde{u}}|\leq K_1(K_1K_2+2\# A+1)\{1-r(1-|\lambda_N|)^{-1}\}^{-1}\,, \tag{6.5}$$

for sufficiently large N whenever $r(1-|\lambda_N|)^{-1}<1$. Then the inequality (6.5) shows that $\|s_N\|_\infty\leq K_4$.

Next we will estimate $\|s_N\|_r$. It can be done in a similar way. Let $u=(\alpha^-,\beta^+)$, α, $\beta\in\tilde{W}$ and $[\alpha^-]_M=[\beta^+]_M$. Then we get as (6.4)

$$(1-|\lambda_N|)|s_N^u| \leq |z||C_N(u)||s_N^{\theta u}| + \sum_{\substack{\langle v\rangle\subset\langle\theta u\rangle \\ v\in W_N}} |z||C_N(u[1]\cdot v)|K_1 K_2 \,\mathrm{Lebes}\,\langle u[1]\cdot v\rangle .$$

Then we also get by (6.5)

$$\begin{aligned}|s_N^u| &\leq (1-|\lambda_N|)^{-M}|z|^M|C_N^M(u)||s_N^{\theta^M u}| \\ &\quad + K_1 K_2 \sum_{j=1}^{M}\sum_{\substack{\langle v\rangle\subset\langle\theta^j u\rangle \\ v\in W_N}} (1-|\lambda_N|)^{-j}|z|^j|C_N^j(u[1]\cdot v)|\,\mathrm{Lebes}\,\langle u[j]\cdot v\rangle \\ &\leq (1-|\lambda_N|)^{-M}K_1 r^M K_4 + (K_1)^2 K_2 \sum_{j=0}^{M}(1-|\lambda_N|)^{-j} r^j \rho^{M-j} \\ &\leq \{K_1 K_4 + (K_1)^3 K_2[1-\rho r^{-1}(1-|\lambda_N|)^{-1}]\}\, r^M .\end{aligned}$$

Hence setting

$$K_5 = K_4 + \{2K_4 + 2\,(K_1 K_4 + (K_1)^3 K_2[1-\rho r^{-1}(1-\varepsilon)^{-1}])\}/(1-r)\,,$$

we get

$$\begin{aligned}||s_N|| &= ||s_N||_\infty + ||s_N||_v \\ &\leq ||s_N||_\infty + \frac{||s_N||_r}{1-r} \\ &\leq K_5\,,\end{aligned}$$

and so $s_N \in \mathscr{B}$. The proof is completed.

Proof of Theorem 6.3: "if" part. It suffices to prove the following lemma.

Lemma 6.5. *Let* $|z|<e^{\xi}$. *Assume* $||(I-\Phi_N(z))^{-1}||_N$ *is bounded. Then there exists the* limit $s=\lim_{N\to\infty}(I-\Phi_N(z))^{-1}\chi_g(z)\in\mathscr{B}$ *uniformly on* $\{g\in L^\infty: ||g||=1\}$ *and it satisfies* $((I-\Phi(z))\,s=\chi_g(z)$.

Let $B=\{z\colon |z|<e^{\xi}$ and $||(I-\Phi_N(z))^{-1}||$ is unbounded$\}$. Then by Lemma 6.4, the set B has no accumulation point since so does Spec(F). Thus, if $||(I-\Phi_N(z))^{-1}||$ is bounded, then by Lemma 6.5 we get $\lim_{N\to\infty} s_g(z; F_N)= s_g(z)\in\mathscr{B}$. Hence $z^{-1}\notin$ Spec(F). This proves the if part of Theorem 6.3.

Proof of Lemma 6.5 We choose $r<1$ so that $r>\rho$ and $r>|z|\rho$. Fix $g\in L^\infty$ with $||g||_\infty=1$. For $\tilde u\in\tilde W_M$, $L\geq M\geq N$, we get by Lemma 5.2 (ii),

$$s_g^{\tilde u}(z; F_N) = \{(I-\Phi_{L,N})^{-1}(\chi_{g|L})\}^{\tilde u}\,.$$

We proved in Lemma 5.3 (ii) that for $N_1\geq N_2$

$$||\Phi_{N_1,N_2}(z)-\Phi_{N_1}(z)||_{N_1}<K_3\,\rho^{N_2}\,.$$

Then we get

$$\begin{aligned}
&\|\{s_g(z; F_{N_1})-s_g(z; F_{N_2})\}|_{N_2}\|_{N_2} \\
&\quad = \|\{(I-\Phi_{N_1,N_2})^{-1}(\Phi_{N_1}-\Phi_{N_1,N_2})(I-\Phi_{N_1})^{-1}(\chi_{g|N_1})\}|_{N_2}\|_{N_2} \\
&\quad \leq \|(I-\Phi_{N_2})^{-1}\|_{N_2}\|\Phi_{N_1}-\Phi_{N_1,N_2}\|_{N_1}\|(I-\Phi_{N_1})^{-1}\|_{N1}\|g\|_\infty .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\{s_g(z; F_{N_1})-s_g(z; F_{N_2})\}|_{N_2}\|_{N_2} \\
\leq K_6\,\rho^{N_2}\,,
\end{aligned} \tag{6.6}$$

for some constant K_6, this shows the existence of $s=\lim_{N\to g} s_g(z; F_N)$.

Now we will show s belongs to $\mathcal{B}$. It is clear that s satisfies (4.3), (4.4) and $\|s\|_\infty<\infty$. It remains to prove the condition (ii) of the definition of $\mathcal{B}$. By Lemma 4.2, it is sufficient to show $\|s_g\|_r<\infty$. It is sufficient to show that $t(M, N)\,r^{-M}$ is uniformly bounded in M and N, where

$$t(M, N) = \sup\,\{|s_g^{(\tilde{\alpha},\tilde{\beta})}(z; F_N)| : \varepsilon(\tilde{\alpha})\,\varepsilon(\tilde{\beta}) = -,\ [\tilde{\alpha}]_M = [\tilde{\beta}]_M,\ \tilde{\alpha},\tilde{\beta}\in\tilde{S}\}\,.$$

It is sufficient to show the boundedness of $t(N, N)\,r^{-N}$. Indeed, we get by Corollary 3.4 (ii),

$$\begin{aligned}
s_g^{(\tilde{\alpha},\tilde{\beta})}(z; F_N) &= s_g^{\tilde{\alpha}}(z; F_N)+s_g^{\tilde{\beta}}(z; F_N) \\
&= \sum_{n=0}^{M-N-1} z^n\, C_N^n(\tilde{\alpha})\,\chi_g^{J(n)}+z^{M-N}\, C_N^{M-N}(\tilde{\alpha})\, s_g^{J(M-N)}(z; F_N)\,,
\end{aligned}$$

where $J(n)=(\theta^n\tilde{\alpha}, \theta^n\tilde{\beta})$. Note first

$$\begin{aligned}
|\chi_g^{J(n)}| &\leq \|g\|\,\mathrm{Lebes}\,J(n) \\
&\leq \|g\|\,K_1\,\rho^{M-n}\,.
\end{aligned}$$

Since $e^{-\xi}<\rho<1$, there exists an integer N_0 auch that $|C_N^L(\tilde{\alpha})|<\rho^L$ fro $N\geq N_0$. Therefore for $M-N\geq N_0$

$$\begin{aligned}
t(M, N) &\leq \sum_{n=0}^{M-N-1}(K_1)^2(|z|\rho)^n\,\rho^{(M-n)}+(|z|\rho)^{M-N}\,t(N, N) \\
&\leq K_7(|z|\rho)^M+(|z|\rho)^{M-N}\,t(N, N)
\end{aligned} \tag{6.7}$$

for some constant K_7.

On the other hand, we get by (6.6) for $N=N_1$ and $N-N_0=N_2$

$$t(N, N)\leq t(N, N-N_0)+2K_6\,\rho^{(N-N_0)}\,,$$

then by (6.7)

$$t(N, N)\leq(|z|\rho)^{N_0}\,t(N-N_0, N-N_0)+\{K_7|z|^N+2K_6\,\rho^{-N_0}\}\,\rho^N\,.$$

Hence we inductively get

$$t(N, N)\,r^{-N}\leq N\,\{K_7+2K_6\,\rho^{-N_0}\}\left[\max\left\{\frac{|z|\rho}{r}, \frac{\rho}{r}\right\}\right]^{N-N_0}\max_{1\leq L\leq N_0} t(L, L)\,.$$

Therefore by (6.7), we get

$$t(M, N)\, r^{-M} \leq K_7 \left(\frac{|z|\rho}{r}\right)^M + \left(\frac{|z|\rho}{r}\right)^{M-N} t(N, N)\, r^{-N}$$
$$\leq K_7 \left(\frac{|z|\rho}{r}\right)^M$$
$$+ \left(\frac{|z|\rho}{r}\right)^{M-N} N\{K_7 + 2K_6\, \rho^{-N_0}\} \left[\max\left\{\frac{|z|\rho}{r}, \frac{\rho}{r}\right\}\right]^{N-N_0} \max_{1\leq L\leq N_0} t(L, L)$$

This shows $||s||_r < \infty$. Hence $||s||_v < \infty$ by Lemma 4.2 (ii). This proves $s \in \mathcal{B}$ and

$$(I - \Phi(z))\, s = \lim_{N\to\infty} (I - \Phi_{\infty, N}(z))\, s_g(z; F_N)$$
$$= \chi_g(z)\,.$$

This proves Lemma.

REMARK. The above proof also shows the following: for $0 < r < 1$, let

$$\mathcal{D}_r = \{z^{-1} \colon |z| < re^{\xi}\}\,,$$
$$\mathcal{F}_r = \{f \in L^1 \colon f = \sum_{u\in W} f_u\, 1_u \quad \text{with} \quad \sum_{u\in W} |f_u|\, r^{|u|} < \infty\}\,,$$

and $\mathrm{Spec}_r(F)$ be the spectrum of the Perron-Frobenius operator restricted to $\mathcal{F}_r$, Then

$$\mathrm{Spec}(F) = \bigcup_{0<r<1} \{\mathrm{Spec}_r(F) \cap \mathcal{D}_r\}\,,$$

Now we can prove Theorem B, which states $\mathrm{Spec}(F)$ can be approximated by the reciprocals of z_N such that $I - \Phi_N(z_N)$ has eigenvalue 0. It can be done by a standard way.

Proof of Theorem B. Assume that $z_0^{-1} \in \mathrm{Spec}(F)$. Let $f = \sum_{u\in W} f_u\, 1_u \in BV$ be an eigenfunction of P corresponding to an eigenvalue z_0^{-1}. Set

$$a_N(z) = \{\sum_{u\in W} f_u\, s_g^u(z; F_N)\,(z)\}^{-1}\,,$$
$$a(z) = (f, g)\,(z)^{-1}\,.$$

Since eigenvalues are isolated, we can find $g \in L^\infty$ and a nieghborhood U of z_0 such that $a(z)$ is analytic in U, $a(z_0) = 0$ and $|a(z)| > 0$ in U except at z_0. Then $||(I - \Phi(z))^{-1}||_N$ is bounded in N for each $z \in \partial U$ and $a_N(z)$ converges to $a(z)$ on ∂U. Hence,

$$|a_N(z) - a(z)| < |a(z)|$$

holds on ∂U for sufficiently large N. Hence by Rouché's theorem there exists unique zero z_N of $a_N(z)$ in U for sufficiently large N. This means that 0 is an eigenvalue of $I-\Phi_N(z_N)$ restricted to $\mathcal{S}(N)$. As we noticed in Lemma 5.1, 0 is also an eigenvalue of $I-\Phi_N(z_N)$. Hence

$$\det(I-\Phi_N(z_N)) = 0 .$$

On the contrary, if $z_0 \notin \mathrm{Spec}(F)$, there exists a neighborhood U of z_0 such that $U \cap \mathrm{Spec}(F) = \emptyset$. Note that $\langle f, s_g(z; F_N)(z)\rangle$ converges to $\langle f, s_g(z; F)\rangle$ uniformly in $g \in L^\infty$ with $||g||=1$ and in z on any compact subset of $\mathcal{D}$ if $f \in BV$. Therefore $(f, g)\ (z; F_N) = \langle f, s_g(z; F_N)\rangle$ is bounded in U for sufficiently large N. This shows $\det(I-\Phi_N(z)) \neq 0$ in U for sufficiently large N. This completes the proof.

7. Zeta function

In this section, we will prove the following.

Proposition 7.1. *Let* $|z| < e^{\xi}$. *Then for any fixed* z, $\det(I-\Phi_N(z))$ *is bounded.*

From Proposition 7.1, we can prove Theorem A.

Proof of Theorem A. Recall

$$\zeta(z) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} z^n \sum_{x=F^n(x)} |F^{n'}(x)|^{-1}\right].$$

In §4, we have seen the following:

(i) $\det(I-\Phi_N(z))$ is equal to $1/\zeta_N(z)$ and is analytic in the disk $|z| < e^{\xi_N}$ (cf. [11]).

(ii) ξ_N converges to ξ (cf. Lemma 5.5).

(iii) $\zeta_N(z)$ converges to $\zeta(z)$ in the unit disk $|z| < 1$.

It follows from Proposition 7.1. that $1/\zeta_N(z) = \det(I-\Phi_N(z))$ converges to $1/\zeta(z)$ and it is analytic in the disk $|z| < e^{\xi}$. Suppose $1/\zeta(z_0) = 0$, and let us show $z_0^{-1} \in \mathrm{Spec}(F)$. By using Rouche's theorem, one can find a sequence $\{z_N\}$ such that $\lim_{N\to\infty} z_N = z_0$ and $1/\zeta_N(z_N) = \det(I-\Phi_N(z_N)) = 0$. Therefore, by Theorem B, $z_0^{-1} \in \mathrm{Spec}(F)$.

On the other hand, Theorem B also says that if $z_0^{-1} \in \mathrm{Spec}(F)$, then there eixts a sequence $\{z_N\}$ such that $z_N \to z_0$ and $\det(I-\Phi_N(z_N)) = 1/\zeta_N(z_N) = 0$. This shows $1/\zeta(z_0) = 0$. Hence, Thoerem A is proved.

Proof of Proposition 7.1. So far we have considered the signed word sets $\tilde{W}_K$ and $\tilde{W}_L$ $(K \neq L)$ are different sets. Now we want to identify the signed word set $\tilde{W}_K$ with a subset of $\tilde{W}_L$ for $K < L$. For this sake we need some additional notations. Let

$$\hat{\tilde{W}}_1 = \{\tilde{u} \in \tilde{W}_N \colon \text{there exists } \tilde{v} \in \tilde{W}_1 \text{ such that } \{\tilde{u}\} = \{\tilde{v}\} \text{ and } \mathcal{E}(\tilde{u}) = \mathcal{E}(\tilde{v})\} ,$$

and for $K \geq 2$

$$\hat{W}_K = \{\tilde{u} \in \tilde{W}_N \colon \text{there exists } \tilde{v} \in \tilde{W}_K \text{ wuch that } \{\tilde{u}\} = \{\tilde{v}\} \text{ and } \mathcal{E}(\tilde{u}) = \mathcal{E}(\tilde{v})\} .$$

Then, $\tilde{W}_N$ is the disjoint union of $\hat{W}_K$:

$$\hat{W}_K \cap \hat{W}_L = \emptyset \quad (K \neq L)$$

and

$$\tilde{W}_N = \bigcup_{K=1}^{N} \hat{W}_K .$$

In the following, we omit the suffix N and denote F_N by F.

Note that by Lemma 5.4,

$$\det(I - \Phi_N(z)) = \det(I - \Phi_N(z; W_N)) .$$

Since the dimension of the matrix $I - \Phi_N(z; W_N)$ is smaller than that of $I - \Phi_N(z)$, we will estimate $\det(I - \Phi_N(z; W_N))$.

First step. Let us sweep out the matrix $\Phi(z; W_N)$, the Fredholm matrix with the alphabet set W_N, which is of the form

$$\Phi(z; W_N)_{\tilde{u},\tilde{v}} = \begin{cases} \mathcal{E}(\tilde{u})\, zC(u[1]\cdot\tilde{v})\, \sigma[\tilde{v} \leq \theta\tilde{u}] & \tilde{u} \notin \hat{W}_1 \\ \mathcal{E}(\tilde{u}) \sum_{n=1}^{\infty} z^n\, C^n(\tilde{u})\, \sigma[\tilde{v} \leq \theta^n \tilde{u}] & \tilde{u} \in \hat{W}_1 . \end{cases}$$

We use the following matrix U indexed by $\tilde{W}_N$:

$$U_{\tilde{u},\tilde{v}} = \begin{cases} 1 & \text{if } \tilde{u} = \tilde{v} , \\ -1 & \text{if } \tilde{v} = v^+, \tilde{u} = v^{\uparrow -} , \quad \text{and} \quad \tilde{u}, \tilde{v} \notin \hat{W}_1 , \\ 0 & \text{otherwise,} \end{cases}$$

where $v^\uparrow = \min_{w \in W} \{w > v\}$ as before. The multiplication of U from right results in subtracting $v^{\uparrow -}$-column from v^+-column, and the multiplication of U^{-1} from left results in adding v^+-row to $v^{\uparrow -}$-row $(v^+ \notin \hat{W}_1)$.

Consequently, we can compute the components of $U^{-1}\Phi(z; W_N)\, U$ as follows:

Case 1. (i-a) $\tilde{u} \in \hat{W}_1$ and $\tilde{v} \in \hat{W}_1$,
(i-b) $\tilde{u} \in \hat{W}_1$, $\tilde{v} \notin \hat{W}_1$ and $\mathcal{E}(\tilde{v}) = -$,
(ii-a) $\tilde{u} \notin \hat{W}_1$, $\mathcal{E}(\tilde{u}) = +$ and $\tilde{v} \in \hat{W}_1$
(ii-b) $\tilde{u} \notin \hat{W}_1$, $\mathcal{E}(\tilde{u}) = +$, $\tilde{v} \notin \hat{W}_1$ and $\mathcal{E}(\tilde{v}) = -$.

In these cases, the multiplication of U leaves the $\tilde{u}$-row unchanged and also that of U^{-1} leaves the $\tilde{v}$-column unchanged, thus

$$\{U^{-1}\Phi(z; W_N)\, U\}_{\tilde{u},\tilde{v}} = \Phi(z; W_N)_{\tilde{u},\tilde{v}} .$$

Case 2. $\tilde{u} \notin \hat{W}_1$ and $\mathcal{E}(\tilde{u}) = -$. Recalling the fact that the $\tilde{u}$-row equals the $u^{\downarrow +}$-row with opposite sign, we get

$$\{U^{-1}\Phi(z; W_N)\, U\}_{\tilde{u},\tilde{v}} = 0\,.$$

Case 3. In the rest cases, the multiplication of U leaves the $\tilde{u}$-row unchangd and the resultant $\tilde{v}$-column is the original $\tilde{v}$-column subtracted from the $v^{\uparrow -}$-column.

(i-a) If $\tilde{u}\notin \hat{W}_1$, $\mathcal{E}(\tilde{u})=+$, $\tilde{v}\notin \hat{W}_1$, $\mathcal{E}(\tilde{v})=+$ and $\theta\tilde{u}=\tilde{v}$, then

$$\{U^{-1}\Phi(z; W_N)\, U\}_{\tilde{u},\tilde{v}} = z\{C(u[1]\cdot v)+C(u[1]\cdot v^{\uparrow})\}\ \sigma[\tilde{v}\leq\theta\tilde{u}]\,.$$

(i-b) If $\tilde{u}\notin \hat{W}_1$, $\mathcal{E}(\tilde{u})=+$, $\tilde{v}\notin \hat{W}_1$ $\mathcal{E}(\tilde{v})=+$ but $\theta\tilde{u}\neq\tilde{v}$, then

$$\{U^{-1}\Phi(z; W_N)\, U\}_{\tilde{u},\tilde{v}} = z\{C(u[1]\cdot v)-C(u[1]\cdot v^{\uparrow})\}\ \sigma[\tilde{v}\leq\theta\tilde{u}]\,.$$

(ii) If $\tilde{u}\in \hat{W}_1$, $\tilde{v}\notin \hat{W}_1$, and $\mathcal{E}(\tilde{v})=+$, then

$$\{U^{-1}\Phi(z; W_N)\, U\}_{\tilde{u},\tilde{v}} = \mathcal{E}(\tilde{u})\sum_{n=1}^{\infty} z^n\, C^{n-1}(\tilde{u})$$
$$\{C(u[n]\cdot v)-C(u[n]\cdot v^{\uparrow})\}\ \sigma[\tilde{v}\leq\theta^n\,\tilde{u}]\,.$$

Thus all the components of the $\tilde{u}$-rows equal 0 if $u^-\notin \hat{W}_1$ and all the components of the u^+-columns also equal 0 if $u^+\in \hat{W}_N$. Therefore, we may ignore them from the matrix $U^{-1}(I-\Phi(z; W_N))\, U$ and get a smaller square matrix $I-\Psi(z)$ indexed by AW^+, where $AW^+=\hat{W}_1\cup\{u^+\in\cup_{K=2}^{N-1}\hat{W}_K\}$.

On the set AW^+ we introduce an order for $\tilde{u}\in\hat{W}_K$ and $\tilde{v}\in\hat{W}_K$ in a similar way as before: define $\tilde{u}>\tilde{v}$ if

(1) $K>L$

or

(2) $K=L$ and $\tilde{u}>\tilde{v}$ as signed words.

We denote by

$$W(\tilde{u}) = \{\tilde{v}\in\tilde{W}_N\colon \tilde{v}>\tilde{u}\}\,.$$

Second setp. We can find the following Lemma in p.54 of [17].

Lemma 7.2. *Let $M=(M_{i,j})_{i,j=1,\cdots,n}$ be a square matrix and set for $1\leq j\leq n$*

$$M(j) = \sum_{l=1}^{\infty}\sum_{i_1,\cdots,i_l<j}\prod_{p=1}^{l} M_{i_p,i_{p+1}}\,,$$

where $i_1=i_{l+1}=j$. Then,

$$\det(I-M) = \prod_{j=1}^{n}(1-M(j))\,.$$

Remark. If we consider the graph with (weighted) incidence matrix M then $M(j)$ is a sum over primitive cycles which start from j and pass only elments less than j and return to j.

Now set

$$\psi_{\tilde{u}}(z)=\sum_{n=1}^{\infty}\sum_{\tilde{u}_2,\cdots,\tilde{u}_n\in W(\tilde{u})}\prod_{i=1}^{n}\Psi(z)_{\tilde{u}_i,\tilde{u}_{i+1}} \quad \text{where} \quad \tilde{u}_1=\tilde{u}_{n+1}=\tilde{u}\,,$$

Then by Lemma 7.2, we get

$$\det(I-\Psi(z))=\prod_{\tilde{u}\in AW^+}(1-\psi_{\tilde{u}}(z))\,, \tag{7.1}$$

Therefore, it only suffices to show the uniform boundedness in N of the sum $\sum_{\tilde{u}\in AW^+}|\psi_{\tilde{u}}(z)|$.

We now fix $\varepsilon>0$ so that $r\,e^{\varepsilon}<1$. As we have seen in Lemma 5.4,

$$1/\zeta_N(z)=\det(I-\Phi_N(z))=\det(I-\Phi_{M,N}(z;\,W_K))$$

for any $K\leq N\leq M$. Recall that the length of subintervals $\langle w\rangle$ ($w\in W_K$) decreases uniformly to 0 as K tends to infinity. Therefore, if necessary, taking W_K for sufficinently large K as an alphabet set A, we may assume the subintervals $\{\langle a\rangle\}_{a\in A}$ so small that

$$\sup_{a\in A}\sup_{x,y\in\langle a\rangle}|F'(x)/F'(y)|<e^{\varepsilon}\,. \tag{7.2}$$

Let $\tilde{u}_1,\tilde{u}_2,\cdots\in\hat{W}_N$ satisfy $[\theta\tilde{u}_i]_{N-1}=[\tilde{u}_{i+1}]_{N-1}$ $(i\geq 1)$ and $\tilde{u}_1[1]\cdot\tilde{u}_2[1]\cdots\in S$. Then we get by (7.2)

$$\xi-\varepsilon<\liminf_{n\to\infty}\frac{1}{n}\log\prod_{i=1}^{n}|F'(\tilde{u}_i)|<\xi+\varepsilon \tag{7.3}$$

Third step

Lemma 7.3. *Let $u^+, v_{1,i}^+, v_{2,i}^+\in\hat{W}_1$ such that $v_{1,i}^+[1]=v_{2,i}^+[1]$ $(1\leq i\leq n)$. Then there exists a constant K_8 such that*

$$\prod_{i=1}\sum_{v_i^+\,:\,v_{1,i}^+\geq v_i^+\geq v_{2,i}^+}|\Psi(z)_{v_{i-1}^+,v_i^+}|\leq K_8\{re^{\varepsilon}\}^n\,,$$

where $v_0^+=u^+$.

Proof. By the definition of $\Psi(z)$, we set

$$\sum_{v^+\,:\,v_1^+\geq v^+\geq v_2^+}\Psi(z)_{u^+,v^+}$$
$$=\begin{cases}0 & \text{if } \theta v^+\notin\langle v_1^+[1]\rangle\,,\\ \dfrac{z}{2}\{C(u[1]\cdot v_1^{\uparrow})+C(u[1]\cdot v_1)\} & \text{if } v_1^+\geq\theta u^+>v_2^+\,,\\ \dfrac{z}{2}\{C(u[1]\cdot v_1^{\rightarrow})-C(u[1]\cdot v_1)\} & \text{if } \theta u^+>v_1^+ \text{ and } \theta u^+\in\langle v_1^+[1]\rangle,\\ \dfrac{z}{2}\{-C(u[1]\cdot v_1^{\uparrow})+C(u[1]\cdot v_1)\} & \text{if } \theta u^+\leq v_2^+ \text{ and } \theta u^+\in\langle v_1^+[1]\rangle\,.\end{cases}$$

Consequently Lemma 7.3 follows from the estimate (6.3).

Fourth step. Set

$$\psi_{u^+,n}(z)=\sum_{u_2^+,\cdots,u_n^+\in W(u^+)}^{n}\prod_{i=1}\Psi(z)_{u_i^+,u_{i+1}^+}$$

with $u_1^+=u_{n+1}^+=u^+$. Then we write $\psi_{u^+}(z)$ as

$$\psi_{u^+}(z)=\sum_{n=1}^{\infty}\psi_{u^+,n}(z)\,.$$

Note here, since $\theta u^+\notin W(u^+)$, there exists K_9 such that $|\Psi(z)_{u^+,u_2^+}|\leq K_9$ Lebes $\langle u_2\rangle$. Therefore by Lemma 7.3, we get

$$\begin{aligned}\sum_{u^+\notin\hat{W}_1}|\psi_{u^+,n}(z)| &=\sum_{u_2^+\cdots,u_n^+\in W(u^+)}\prod_{i=2}^{n}|\Psi(z)_{u_i^+,u_{i+1}^+}|\sum_{u^+\notin\hat{W}_1}|\Psi(z)_{u^+,u_2^+}|\\ &\leq\sum_{u_2^+\notin\hat{W}_1}K_8\{re^{\varepsilon}\}^{n-1}K_9\,\text{Lebes}\,\langle u_2\rangle\,.\end{aligned}$$

Hence,

$$\sum_{u^+\notin\hat{W}_1}|\psi_{u^+}(z)|\leq K_8K_9\{1-re^{\varepsilon}\}^{-1}\,.$$

We can estimate the summation over W_1 in a similar manner but taking care of the infinite sum $\sum_{N=0}^{\infty}z^n|C^n|$.

Hence follows the uniform boundedness of $\sum_{u^+\in AW^+}|\psi_{u^+}(z)|$. Thus det $(I-\Phi_N(z))$ is uniformly bounded in N in the disk $|z|<e^{\xi}$. Consequently $1/\zeta(z)$ has analytic extension to the disk $|z|<e^{\xi}$. This proves the Proposition 7.1.

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Department of Mathmatics
National Defense Academy