Suetake, C.
Osaka J. Math.
29 (1992), 433–434

A REMARK ON FINITE POINT TRANSITIVE AFFINE PLANES WITH TWO ORBITS ON l_∞

CHIHIRO SUETAKE

(Received October 4, 1991)

In this note for the most part we shall use the notations of [1]. Let $\mathcal{P}=\pi\cup l_\infty$ be the projective extension of an affine plane and G a collineation group of $\mathcal{P}$. If p is a point of $\mathcal{P}$ and l is a line of $\mathcal{P}$, then $G(p, l)$ is the subgroup G consisting of all perspectivities in G with center p and axis l. If m is a line of $\mathcal{P}$, then $G(m, m)$ is the subgroup of all elations in G with axis m.

In [3] the author proved the following theorem on Kallaher's conjecture (See [1]).

Theorem 1. *Let π be a finite affine plane of order n with a collineation group G which is transitive on the affine points of π. Suppose that G has two orbits of length 2 and n-1 on l_∞. Then π is a translaiton plane and the group G contains the group of translations of π, except in the following case*:

(*) $|G(l_\infty, l_\infty)|=n=2^m$ *for some* $m\geq 1$, $G(p_1, l_\infty)=G(p_2, l_\infty)=1$ *and* $|G(p, l_\infty)|=2$ *for all* $p\in l_\infty-\{p_1, p_2\}$, *where* $\{p_1, p_2\}$ *is a G-orbie of length* 2 *on* l_∞.

The case (*) actually occurs when π is a desarguesian plane of order 2. Maharjan [2] studied the planes with property (*) under the condition that $n\leq 4$. The purpose of this note is to prove the following.

Proposition 2. *Assume* (*). *Then $n=2$ and G is a cyclic group of order* 4.

Maharjan [2] proved the proposition under the condition that $n\leq 4$.
The proposition, together with Theorem 1, gives the following.

Theorem 3. *Let π be a finite affine plane of order n with a collineation group G which is transitive on the affine points of π. If G has two orbits of length 2 and $n-1$ on l_∞, then one of the following statements holds*:

(i) *The plane π is a tanslation plane and the group G contains the group of translations of π.*

(ii) $n=2$ *and G is a cyclic group of order* 4.

In the rest of the note, we prove Proposition 2. Set $T=G(l_\infty, l_\infty)$ and $L=G_{P_1,P_2}$. Then $|G:L|=2$. Let O be an affine point of π. Set $l=P_1O$. Suppose that G_l is transitive on the points of l. Since $2|n$, there exists an involution σ in the center of a Sylow 2-subgroup of G_l. By Corollary 3.6.1 of [1], σ is a perspecitivity. Theorfore σ is an elation with the center P_1. On the other hand $G_l \cap T=1$. Thus $G(P_1, l)\neq 1$. This yields $P_2^{G_l}\neq P_2$, a contradiction. Hence we have the following.

(1) G_l is not transitive on the points on l.

As G leaves $\{P_1, P_2\}$ invariant, $G_{P_1}=L$. Hence, by Theorem 4.3 of [1], we have

(2) L is transitive on the lines through P_1.

Since $|G: L|=2$, (1) and (2) imply that L has exactly two orbits Ω_1 and Ω_1 on the points of π such that $|\Omega_1|=|\Omega_2|$. Hence,

(3) $|\Omega_1|=|\Omega_2|=n^2/2$.

Therefore $L_l(=G_l)$ has exactly two orbits Γ_1 and Γ_2 on the points of l. It follows from (2) and (3) that $\Gamma_1=\Omega_1\cap l$, $\Gamma_2=\Omega_2\cap l$ and $|\Gamma_1|=|\Gamma_2|=n/2$. Also we get $L_0\leq L_l$, $|G: L_0|=|G: L|\times|L: L_l|\times|L_l: L_0|=2\cdot n\cdot n/2=n^2$.

Suppose that $n-1\neq 1$. Let p be a prime such that $p|(n-1)$ and A a p-Sylow subgroup of L_0. Then since $n-1=|l_\infty-\{P_1, P_2\}|\,|\,|G|$ and $|G: L_0|=n^2=2^{2m}$, A also is a Sylow p-subgroup of G and $A\neq 1$. Since $(n-1, n/2)=1$, $|Fix(A)\cap l|\geq 2$. Assume that $N_G(A)\leq L$. Then since $L\trianglelefteq G$ and A is a p-Sylow subgroup of L, $LN_G(A)=G$ and so $L=G$, a contradiction. Therefore $N_G(A)\nleq L$. Let $\tau\in N_G(A)$-L. Then l^τ is through P_2. Hence there exists a point Q on l such that A fixes Q. Since A fixes P_1, O, Q, P_2, O^τ and Q^τ, A is a planar collineation group. In particular, $Fix(A)\cap(l_\infty-\Delta)\neq\phi$. This yields that G is not transitive on $l_\infty-\Delta$ by Theorem 3.6 of [1], a contradiction. Thus $n-1=1$ and so $n=2$.

We may assume that $\pi=PG(2,2)^{l_\infty}$, where $l_\infty=\langle(1,0,0)\rangle\langle(0,1,0)\rangle$. Let $P_1=\langle(1,0,0)\rangle$ and $P_2=\langle(0,1,0)\rangle$. Then, by direct computation, $G=\left\langle\begin{pmatrix}0&1&0\\1&0&0\\1&0&1\end{pmatrix}\right\rangle$. Clearly $|G|=4$. Thus Proposition 2 follows.

References

[1] M.J. Kallaher: Affine planes with transtive colleneation groups, North Holland, New York-Amesterdam-Oxford, 1982.

[2] H.B. Maharjan: Personal communication.

[3] C. Suetake: *On finite point transitive affine planes with two orbits on* l_∞, Osaka J. Math. **27** (1990), 271–276.

Amagasakiminami High School
Amagasaki, Hyogo 660
Japan