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ADDITIVE FUNCTIONALS, NOWHERE RADON AND KATO CLASS SMOOTH MEASURES ASSOCIATED WITH DIRICHLET FORMS

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0. Introduction

The theory of regular Dirichlet forms $(E,\mathcal{F})$ associated with a locally compact separable metric space $\mathcal{X}$ and a positive Radon measure m s.t. $\text{supp}[m]=\mathcal{X}$ is a well developed subject, both from the potential analytic and the probabilistic point of view. It has its origins in work by Beurling-Deny and was particularly pursued by Fukushima and Silverstein see e.g. [19], [27] and references therein. It presents, at least in the symmetric case, a natural extension of the continuous functions framework of classical and axiomatic potential theory in the functional analytical (L^2-functions) direction, covering in particular a stochastic calculus for generators with coefficients which are not restricted to be functions (the associated processes need not be semimartingales). This theory has turned out, in the last 15 years, to be particularly suited for applications in quantum theory, see e.g. [4], [5], [20], [1], [8]. In this field, but also in other contexts, see e.g. [2], there is the necessity of studying certain generalized functionals of the processes (of Feynman-Kac type), corresponding to singular perturbations of a given Dirichlet form (e.g. the one associated with the Laplacian over $\boldsymbol{R}^d$). This is discussed e.g. in [2], [10], [28], [29], [30], [16], [3], [1], [15], [11], [12], [23], [22] and references therein. Many of the discussions have been concerned with functionals associated with measures in the so called Kato class (cfr. [9], [26]). They are particular cases of smooth measures (in the sense of [19]) for the Dirichlet form associated with the Laplacian. It is natural to ask oneselves what happens if one tries to carry through similar constructions using an *arbitrary* smooth measure associated with a general (regular) Dirichlet form. In the present paper we initiate such a study. We give results on the structure

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of the space S of all smooth measures associated with a given Dirichlet form, as well as on their associated Feynman-Kac functionals and generalised resolvents. By so doing we prepare our way for a systematic study of singular perturbations of Dirichlet forms, to be carried through in a successive publication [6]. Let us now describe in more details the setting.

We consider a regular Dirichlet form $(E, \mathcal{F})$ on $L^2(\mathcal{X}, m)$ where $\mathcal{X}$ is a locally compact separable metric space and m is a positive Radon measure on $\mathcal{X}$ with supp $[m]=\mathcal{X}$. Let $M=(\Omega, X_t, \zeta, P_x)$ be a Hunt process on $\mathcal{X}$ which is m-symmetric and associated with $(E, \mathcal{F})$.

Following M. Fukushima, a function $A:[0, \infty)\times\Omega\to[-\infty, \infty]$ is said to be an AF (*additive functional*) if

(i) $A_t(\cdot)$ is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t$ is the smallest completed σ-algebra which contains $\sigma\{X_s: s\leq t\}$;

(ii) there exist a defining set $\Lambda\in\mathcal{F}_\infty$ and an exceptional set $N\subset\mathcal{X}$ with Cap(N) $=0$ such that $P_x(\Lambda)=1$ for all $x\in\mathcal{X}-N$, $\theta_t\Lambda\subset\Lambda$ for all $t>0$ (θ_t denotes the shift operator on Ω) and for each $\omega\in\Lambda$, $A_0(\omega)=0$, $|A_t(\omega)|<\infty$ for $t<\zeta(\omega)$, $A.(\omega)$ is right continuous and has left limit, $A_{t+s}(\omega)=A_t(\omega)+A_s(\theta_t\omega)$ for $s, t\geq 0$.

An AF A is called a PCAF (*positive continuous* AF) if A is an AF and $A.(\omega)$ is non-negative and continuous function for each ω in its defining set Λ.

Given a PCAF A, there exists a unique Borel measure μ on $\mathcal{X}$, which is called the *Revuz measure of A*, such that

$$\lim_{t\downarrow 0}\frac{1}{t}E_{h\cdot m}\left[\int_0^t f(X_s)dA_s\right]=\langle f\cdot\mu, h\rangle:=\int_{\mathcal{X}} h(x)(f\cdot\mu)(dx) \tag{0.1}$$

for all γ-excessive functions h and $f\in\mathcal{B}^+$ ($\mathcal{B}^+$ denotes all non-negative Borel functions on $\mathcal{X}$, $\gamma\geq 0$ is a constant).

As pointed out by M. Fukushima in his recent paper [18] the above definition of PCAF is a generalization to the ordinary concept of PCAF (which we shall refer to as the classical definition of PCAF) with the goal of relaxing the finiteness requirement on PCAF's in the classical definition and to get a broader but simpler class of associated Revuz measures.

Denote by S the totality of the associated Revuz measures of PCAF's. The elements in S are called *smooth measures*. A simple analytical description of S has been given as follows [19]:

A Borel measure μ on $\mathcal{X}$ is in S if and only if μ charges no set of zero capacity and there exists an increasing sequence of compact sets $\{F_n\}_{n\geq 1}$ such that $\mu(F_n)<\infty$ for each n, $\mu(\mathcal{X}-\cup F_n)=0$ and Cap$(K-F_n)\to 0$ for any compact set K.

From the above description it is easy to see that S contains all positive Radon measures charging no set of zero capacity. It is also known that any measure in S can be approximated by (Radon) measures of finite energy integral.

In this paper we shall point out two extremely contrasting properties of S. On the one hand, we shall show in Section 1 that there are many smooth measures μ which are nowhere Radon in the sense that $\mu(G)=\infty$ for all non-empty open sets $G\subset\mathscr{X}$. Thus the class S is much bigger than it has been realized up to date. Nevertheless on the other hand we shall show in Section 2 that the class S is so nice that each measure in S can be approximated by measures in Kato class (see Definition 2.1). Recall that in the classical case of regular Dirichlet forms, corresponding to the Laplace operator, the measures of Kato class play an important role in connecting the Schrödinger semigroups and Feynman-Kac formula, cf. e.g. [9], [26], [12], [6,7], [23], [24], [13], [14].

As an application of the approximation of smooth measures by the measures in Kato class, in Section 3 we shall prove some properties of the generalized resolvents which are very useful in the study of perturbations of Dirichlet forms. In the last Section we provide some alternative descriptions of the smooth measures in Kato class.

For the applications of the above results in quantum mechanics and the study of perturbations of Dirichlet forms the reader is refer to our sequential paper [6].

Some of the results in the present paper and in [6] have been described in [7].

1. Smooth measures which are "nowhere Radon"

In this section we shall prove that there exist smooth measures μ which are "nowhere Radon" in the sense that $\mu(G)=\infty$ for all non-empty open sets G of $\mathscr{X}$.

Theorem 1.1. *Let B be a subset of zero capacity and ν be a smooth measure with* supp $[\nu]\supset B$. *Then there exists at least one smooth measure μ which is equivalent to ν and such that $\mu(G)=\infty$ for all open sets G with the property that $G\cap B\neq\emptyset$.*

Proof. Set $B_1=\{x\in B: \nu \text{ is finite on a neighborhood of } x\}$. Without loss of generality we may assume that B_1 is non-empty. Otherwise ν itself is a desired smooth measure. Since $\mathscr{X}$ is separable, there exists a countable subset B_2 of B_1, say $B_2=\{x_j, j\geq 1\}$ such that $\bar{B}_2\supset B_1$. For each j we choose a decreasing sequence of small balls $\{G_{j,k}\}_{k\geq 1}$ having x_j as their common center such that

$$\bigcap_{k\geq 1} G_{j,k}=x_j \tag{1.1}$$

and

$$\text{Cap}\,(G_{j,k})+\nu(G_{j,k})\leq 2^{-k}. \tag{1.2}$$

Such a sequence exists because we have $\text{Cap}(\{x_j\})=0$ which implies $\nu(\{x_j\})=0$, and ν is finite on a neighborhood of x_j. Notice that $\nu(G_{j,k})$ is also strictly positive since supp $[\nu]\supset\bar{B}$.

Define for each j

$$f_j(x)=\begin{cases}\dfrac{k}{\nu(G_{j,k})} & \text{when } x\in G_{j,k}-G_{j,k+1},\ k=1,2,\cdots\\ 1 & \text{otherwise.}\end{cases} \tag{1.3}$$

Because f_j is bounded outside $G_{j,k}$ for each k, we can choose a positive number c_j such that

$$c_j \sup_{x\in\mathcal{X}-G_{j,j}} f_j(x)\leq 2^{-j}. \tag{1.4}$$

Now we define

$$f(x)=\sum_{j\geq 1} c_j f_j(x)$$

and

$$\mu(dx)=f(x)\nu(dx)\,.$$

We are going to show that μ is a smooth measure with the required properties. Since $0<f<\infty$, ν-almost everywhere, it is evident that $\mu=f\cdot\nu$ is equivalent to ν, and accordingly μ charges no set of zero capacity. Let $\{E_n\}_{n\geq 1}$ be an increasing sequence of compact sets such that $\nu(E_n)<\infty$, $\nu(\mathcal{X}-\cup_{n\geq 1}E_n)=0$ and $\lim_{n\to\infty}$ Cap $(K-E_n)=0$ for any compact set K. We set for each n:

$$G_n=(\cup_{1\leq j\leq n}G_{j,2n})\cup(\cup_{j>n}G_{j,j}) \tag{1.5}$$

and

$$F_n=E_n-G_n\,.$$

Then $\mu(F_n)<\infty$, $\mu(\mathcal{X}-\bigcup_{n\geq 1}F_n)=0$ and $\lim_{n\to\infty}$ Cap $(K-F_n)=0$ for any compact set K. Thus μ is a smooth measure. Let G be an open set such that $G\cap B\neq\emptyset$. If $G\cap(B-B_1)\neq\emptyset$ then $\mu(G)\geq\nu(G)=\infty$ by the definition of B_1. Suppose $G\cap(B-B_1)=\emptyset$, then $G\cap B_1\neq\emptyset$, which implies $G\cap B_2\neq\emptyset$. Let $x_j\in G\cap B_2$. Then for k large enough we have $G_{j,k}\subset G$ and consequently

$$\mu(G)\geq c_j\int_{G_{j,k}}\frac{k}{\nu(G_{j,k})}\nu(dx)\geq c_jk\,.$$

Letting $k\to\infty$ we obtain $\mu(G)=\infty$. Thus μ is as desired. ■

Corollary 1.2. *Suppose that each single-point set of $\mathcal{X}$ is a set of zero capacity. Then for any countable dense subset B of $\mathcal{X}$ we can construct a nowhere*

Radon smooth measure μ_B *as in Theorem* 1.1, *with* $\nu=m$. μ_B *is "nowhere Radon" in the sense that* $\mu_B(G)=\infty$ *for all non-empty open sets* G *of* $\mathscr{X}$.

For example: if $\mathscr{X}=\boldsymbol{R}^d$, $d\geq 2$ and $(E,\mathscr{F})$ is the classical Dirichlet form associated with the Brownian motion, then each single-set point is a set of zero capacity. Corollary 1.2 asserts that to any countable dense subset $B\subset\boldsymbol{R}^d$ there exists a smooth measure μ_B on $\boldsymbol{R}^d$ which is "nowhere Radon". In fact in this case we can exhibit explicitly examples of nowhere Radon smooth measures μ_B, as shown in the following proposition.

Proposition 1.3. *Let* $\{x_j\}_{j\geq 1}$ *be a dense subset of* $\boldsymbol{R}^d (d\geq 2)$ *and* $\{\alpha_j\}_{j\geq 1}$ *be an arbitary sequence of real numbers. Then there exists a sequence* $\{c_j\}_{j\geq 1}$ *of strictly positive real numbers such that the measure* μ *definited by* $\mu(dx)=f(x)dx$ *with*

$$f(x)=\sum_{j\geq 1} c_j|x-x_j|^{\alpha_j} \tag{1.6}$$

is a smooth measure on $\boldsymbol{R}^d$ (*with respect to the classical Dirichlet form associated with the Laplacian*).

Proof. For each j, we can choose a decreasing sequence of small balls $\{G_{j,k}\}_{k\geq 1}$ with x_j as their common center such that (1.1) and (1.2) hold (with respect to the classical capacity associated with the Laplacian and the Lebesgue measure ν on $\boldsymbol{R}^d$). Set $E_j=\{x\colon |x|\leq j\}$ and choose positive numbers c_j such that

$$c_j \sup_{x\in E_j-G_{j,j}} |x-x_j|^{\alpha_j}\leq 2^{-j}.$$

Let $\mu(dx)=f(x)dx$ with f defined by (1.6). Then similarly as in the proof of Theorem 1.1 we can show that μ is a smooth measure. Hence $\{c_j\}_{j\geq 1}$ is a sequence with the required properties. ■

REMARK 1.4.

(i) Suppose that there exists a natural number j_0 such that $\alpha_j\leq -d$ for all $j\geq j_0$, then $\int_G f(x)dx=\infty$ for all non-empty open sets G, i.e., μ is "nowhere Radon". Suppose that $\alpha_j=-j$ for all j, then the function f defined by (1.6) satisfies the property:

$$\int_G |f(x)|^p\,dx=\infty \quad \text{for any non-empty open set } G \text{ and any } p>0 \tag{1.7}$$

(ii) In [29] Stollmann and Voigt constructed a regular potential V satisying (1.7). Our construction of f by (1.6) is in fact similar to the construction of V in [29], but with a different choice of $\{c_j\}_{j\geq 1}$. Nevertheless, since we can prove that $W_2^1(\boldsymbol{R}^d)\cap L^2(\boldsymbol{R}^d,\mu)$ is dense in $L^2(\boldsymbol{R}^d,\nu)$ for any smooth measure μ (cfr. [6]), the function f defined by (1.6) (with the choice of $\{c_j\}_{j\geq 1}$ in the above Proposition) is still regular in the sense of [29].

By Theorem 1.1 we can also construct smooth measures which are concentrated on some subsets of $\mathscr{X}$, or even singular with respect to the reference measure m, but such that they are "nowhere Radon" on their support. We give here several examples.

EXAMPLE 1.5. Consider the case of $\boldsymbol{R}^d (d\geq 2)$ with the classical Dirichlet form associated with the Laplacian. Let F be a closed $d-1$-dimensional manifold and ν be the $d-1$-dimensional Lebesgue measure on F. Then ν is a smooth measure. By Theorem 1.1 we can construct a smooth measure μ which is singular with respect to the Lebesgue measure m on $\boldsymbol{R}^d$ (since μ is equivalent to ν), and $\mu(G)=\infty$ for all non-empty relatively open subsets G of F.

EXAMPLE 1.6. Let $\mathscr{X}=D\cup\partial D$ where D is a bounded domain of $\boldsymbol{R}^d (d\geq 2)$ with C^3 boundary ∂D. Let m be the Lebesgue measure on $\mathscr{X}$ and $(E, \mathscr{F})$ be the maximal Markovian extension of the form

$$E(u, u) = \frac{1}{2}\int |\nabla u|^2 m(dx), \qquad u\in C_0^\infty(D).$$

Then $(E, \mathscr{F})$ is a regular Dirichlet form on $L^2(\mathscr{X}, m)$ corresponding to the Laplacian operator with Neumann boundary condition on ∂D. Denote by ν the area measure of ∂D. Obviously ν is singular with respect to m. But ν is a smooth measure. In fact ν is the Revuz measure of the boundary local time of the reflecting Brownian motion on $\mathscr{X}$ (c.f. [23]). We can also prove that each single point of ∂D is of zero capacity. Thus by Theorem 1.1 there exists a smooth measure μ concentrated on ∂D (hence singular with respect to m) such that $\mu(G)=\infty$ for all non-empty relatively open subsets G of ∂D.

EXAMPLE 1.7. Let $\mathscr{X}=\boldsymbol{R}^{3N}$ and let us write $x\in\boldsymbol{R}^{3N}$ as $\{x_1, \cdots, x_N\}$ with $x_i\in\boldsymbol{R}^3$. Set

$$\Phi(x) = \frac{1}{4\pi}\sum_{i<j}^{N} |x_i-x_j|^{-1}\exp(-\lambda|x_i-x_j|)$$

for some $\lambda\geq 0$. Let $m(dx)=\Phi^2(x)dx$ and define

$$E(u, v) = \int_{\mathscr{X}} \nabla u\cdot\nabla v\, m(dx)$$

for u and v in $C_0^1(\boldsymbol{R}^{3N})$. Then E is positive and closable and it produces a regular Dirichlet form $(E, \mathscr{F})$ on $L^2(\mathscr{X}; m)$ ([5]). The energy operator H (associated with E) is a realization of a Hamiltonian of N particles interacting by δ-interactions. Notice that in this case each single-point set is of zero capacity. Let us set

$$D = \{x = \{x_1, \cdots, x_N\}:\ x_i = x_j \text{ for some } 1\leq i<j\leq N\}.$$

Applying Theorem 1.1 we can construct a smooth measure μ such that $\mu(G)=\infty$ for all open sets G such that $G\cap D\neq\emptyset$.

Now consider the positive quadratic form E^{μ}:

$$E^{\mu}(u,v)=E(u,v)+\int_{\mathcal{X}}uv\mu(dx)\,,\quad u,v\in\mathcal{F}\cap L^2(\mu)\,.$$

It can be shown that E^{μ} is a Dirichlet form (c.f [6]). We then obtain a self-adjoint operator $H^{\mu}=H+\mu$ which describes a Hamiltonian of N particles interacting by δ-interactions plus Coulomb-like interactions.

2. Smooth measures in Kato class

Let us denote by $B(\mathcal{X})$ the family of Borel functions on $\mathcal{X}$. For $f\in B(\mathcal{X})$ set

$$\|f\|_q=\inf_{\mathrm{Cap}(N)=0}\ \sup_{x\in\mathcal{X}-N}|f(x)|$$

(where the index q reminds us of "quasi everywhere").

For a given smooth measure μ, we shall denote by A^{μ} the unique (up to equivalence) PCAF such that μ is the Revuz measure of A^{μ} (we say that two PCAF A_t^1 and A_t^2 are equivalent if $P^x(A_t^1=A_t^2)=1$ for quasi everywhere $x\in X$, in the sense of [19]).

DEFINITION 2.1. A smooth measure μ is said to be in *Kato class* ($\mu\in S_K$ in notation), if

$$\lim_{t\downarrow 0}\|E.A_t^{\mu}\|_q=0\,.$$

REMARK 2.2. In the classical case of M being Brownian motion on $\boldsymbol{R}^d$, S_K coincides with the generalized Kato class GK_d introduced in [12], and a function f is in Kato class of functions if and only if $f\cdot dx\in S_K$ (c.f. [9]).

Some analytic descriptions of S_K will be given in Section 4.

The importance of the class S_K is its connection with the Feynman-Kac semigroups by the inequality contained in the following proposition.

Proposition 3.3. *Let $\mu\in S_K$. Then there exist constants c and β such that*

$$\|E.e^{A_t^{\mu}}\|_q\leq ce^{\beta t},\quad \forall t>0\,.$$

Proof. This is easily proven by applying Khaminskii's inequality and using the semigroup property of the Feynman-Kac functionals. (c.f. [9]) ■

We denote by S_0 the positive Radon measures of finite energy integral and introduce the family S_{K_0} as follows:

$$S_{K_0}=\{\mu\in S_K\cap S_0:\mu(\mathcal{X})<\infty\}\,.$$

Theorem 2.4. *A positive Borel measure μ on $\mathscr{X}$ is smooth if and only if there exists an increasing sequence $\{F_n\}_{n\geq 1}$ of compact sets satisfying the following properties*

$$(2.2)\qquad \text{(i)}\quad I_{F_n}\cdot\mu\in S_{K_0}\,,\quad \forall n\geq 1$$

$$(2.3)\qquad \text{(ii)}\quad \mu(\mathscr{X}-\cup F_n)=0\,,$$

$$(2.4)\qquad \text{(iii)}\quad \lim_{n\to\infty}\mathrm{Cap}\,(K-F_n)=0\quad \textit{for any compact set } K.$$

Proof. It suffices to prove the "only if" part. First we assume $\mu\in S_0$. Let us set

$$\mathcal{U}(x)=E_x\int_0^\infty e^{-s}dA_s^\mu$$

and

$$\mathcal{U}_t(x)=E_x\int_0^t e^{-s}dA_s^\mu=\mathcal{U}(x)-e^{-t}E_x\mathcal{U}(X_t)\,.$$

Then we have $\mathcal{U}\in\mathscr{F}$, which implies $\mathcal{U}_t\in\mathscr{F}$ and $\mathcal{U}_t\to 0$ (as $t\to 0$) in E_1-norm. (Recall that $\|f\|_{E_1}^2:=E(f,f)+\int_{\mathscr{X}}f^2m(dx)$). Moreover each $\mathcal{U}_t$ is quasicontinous. Thus similarly as in the proof of [19] Th. 3.1.4 we may find a decreasing sequence of open sets $\{G_n\}_{n\geq 1}$ and a subsequence of $\{t_k\}_{k\geq 1}$ such that $\mathrm{Cap}\,(G_n)\to 0$, $t_k\downarrow 0$ and $\mathcal{U}_{t_k}\to 0$ uniformly on each $\mathscr{X}-G_n$. Notice also that $\mathcal{U}_t$ is decreasing pointwise on $\mathscr{X}-N$ as $t\downarrow 0$ (N denotes the exceptional set of A^μ). We have actually proved that

$$(2.5)\qquad \lim_{t\downarrow 0}\ \sup_{x\in\mathscr{X}-(G_n\cup N)}|\mathcal{U}_t(x)|=0\qquad \text{for each } n.$$

Let $\{E_n\}_{n\geq 1}$ be an increasing sequence of compact sets such that $\mu(E_n)<\infty$ and $\lim_{n\to\infty}\mathrm{Cap}\,(K-E_n)=0$ for any compact K. We define $F_n=E_n-G_n$. We claim that $I_{F_n}\cdot\mu\in S_{K_0}$. It suffices to show that $I_{F_n}\mu\in S_K$. Let us set

$$\bar{\mathcal{U}}_t(x)=E_x\int_0^t e^{-s}I_{F_n}(X_s)dA_s^\mu$$

for fixed n. Then we have

$$(2.6)\qquad \lim_{t\downarrow 0}\ \sup_{x\in F_n-N}|\bar{\mathcal{U}}_t(x)|\leq\lim_{t\downarrow 0}\ \sup_{x\in F_n-N}|\mathcal{U}_t(x)|=0\,.$$

Define

$$\tau=\mathrm{lif}\,\{t>0\colon X_t\in F_n\}$$

and

$$g(t-s,x)=E_x\int_0^{t-s}e^{-r}I_{E_n}(X_r)dA_r^\mu\,,\quad t-s\geq 0\,.$$

By the strong Markovian property we have for q.e. $x\in\mathscr{X}-F_n$,

$$(2.7)\qquad \mathcal{U}_t(x) = E_x\left[I_{\{t>\tau\}}\int_\tau^t e^{-r} I_{F_n}(X_r)dA_r^\mu\right] = E_x[I_{\{t>\tau\}}e^{-\tau}g(t-\tau, X_\tau)] \leq \sup_{x\in F_n-N} |\mathcal{U}_t(x)|\,.$$

Comparing (2.6) and (2.7) we obtain

$$\lim_{t\downarrow 0} \|E.\int_0^t e^{-s} I_{F_n}(X_s)dA_s^\mu\|_q = 0\,,$$

which is equivalent to $I_{F_n}\cdot\mu\in S_K$. It is easy to check that $\{F_n\}_{n\geq 1}$ satisfies (2.3) and (2.4). Thus the theorem is proved for $\mu\in S_0$.

For a general measure $\mu\in S$, we may take an increasing sequence of compact sets $\{E_j\}_{j\geq 1}$ such that $I_{E_j}\cdot\mu\in S_0$, $\mu(\mathcal{X}-\cup E_j)=0$ and Cap $(K-E_j)\to 0$ for any compact set K. For each j, we can find a sequence of compact sets $\{F_{jn}\}_{n\geq 1}$ such that $\{F_{jn}\}_{n\geq 1}$ satisfies (2.1)—(2.4) for the measure $I_{E_j}\cdot\mu$. Now we define

$$F_n = \cup_{j=1}^n (F_{jn}\cap E_j)\,.$$

Then $\{F_n\}_{n\geq 1}$ is an increasing sequence of compact sets such that $I_{F_n}\cdot\mu\in S_{K_0}$ for each n. It is also easy to check (2.3) and (2.4) for $\{F_n\}_{n\geq 1}$ by the observation that for any compact set K,

$$K-F_n\subset(K-E_j)\cup(K-F_{jn})\,,\quad j\leq n\,.$$

The proof is thus completed. ■

Corollary 2.5. *Let B be a Borel set of $\mathcal{X}$. Then* Cap $(B)=0$ *if and only if $\mu(B)=0$ for all $\mu\in S_{K_0}$.*

Proof. This follows from Theorem 2.4 and [19] Theorem 3.3.2. ■

3. Generalized resolvents

We first introduce some notations.

Let $\mu=\mu^+-\mu^-$ be a signed Borel measure on $\mathcal{X}$. If $|\mu|=\mu^++\mu^-$ is a smooth measure, then μ is called a *signed smooth measure*, and we shall write $\mu\in S-S$. It is evident that $\mu\in S-S$ if and only if μ^+ and μ^- are both smooth measures. For $\mu\in S-S$ we shall set $A^\mu:=A^{\mu^+}-A^{\mu^-}$. We shall call μ the *Revuz measure* of A^μ.

Let $\mu\in S_K$. By Proposition 2.3 we know there exist constants c and β such that $\|E.e^{A_t^\mu}\|_q\leq ce^{\beta t}$. Let us introduce the notation

$$(3.1)\qquad \beta(\mu) := \inf\{\beta\geq 0 : \|E.e^{A_t^\mu}\|_q\leq ce^{\beta t} \text{ for some } c>0\}\,.$$

For f and g in $\mathcal{B}(\mathcal{X})$ and a Borel measure μ on $\mathcal{X}$, we shall sometimes use the notation $\langle f, g\cdot\mu\rangle$ or $\langle f, g\rangle_\mu$ to denote the integral $\int_{\mathcal{X}} fg\mu(dx)$. $\langle f, g\rangle_m$ will be simply denoted by (f, g).

For $\alpha\geq 0$ a constant, μ and ν in $S-S$ and $f\in\mathcal{B}(\mathcal{X})$, let us introduce the notation

$$(3.2)\qquad \mathcal{U}_\nu^{\alpha+\mu}f(x)=E_x\left[\int_0^\infty e^{-\alpha t-A_t^\mu}f(X_t)dA_t^\nu\right]$$

provided the right hand side makes sense. In particular, if $\nu=m$, we shall simply write $\mathcal{U}^{\alpha+\mu}f$ instead of $\mathcal{U}_\nu^{\alpha+\mu}f$.

Theorem 3.1. *Let* $\alpha\geq 0$, $\mu\in S-S$, $\{\mu_1,\mu_2,\nu\}\subset S$ *and* $f\in\mathcal{B}^+(\mathcal{X})$. *If* $\mathcal{U}_\nu^{\alpha+\mu}f(x)<\infty$, *then*

$$(3.3)\quad \mathcal{U}_\nu^{\alpha+\mu+\mu_1-\mu_2}f(x)+\mathcal{U}_{\mu_1}^{\alpha+\mu+\mu_1-\mu_2}\mathcal{U}_\nu^{\alpha+\mu}f(x)=\mathcal{U}_\nu^{\alpha+\mu}f(x)+\mathcal{U}_{\mu_2}^{\alpha+\mu+\mu_1-\mu_2}\mathcal{U}_\nu^{\alpha+\mu}f(x)\,.$$

If $\mathcal{U}_\nu^{\alpha+\mu+\mu_1-\mu_2}f(x)<\infty$, *then*

$$(3.4)\qquad \mathcal{U}_\nu^{\alpha+\mu+\mu_1-\mu_2}f(x)+\mathcal{U}_{\mu_1}^{\alpha+\mu}\mathcal{U}_\nu^{\alpha+\mu+\mu_1-\mu_2}f(x)$$
$$=\mathcal{U}_\nu^{\alpha+\mu}f(x)+\mathcal{U}_{\mu_2}^{\alpha+\mu}\mathcal{U}_\nu^{\alpha+\mu+\mu_1-\mu_2}f(x)\,.$$

Proof. It suffices to prove (3.3). Assume that $\mathcal{U}_\nu^{\alpha+\mu}f(x)<\infty$. By the Markovian property we have

$$E_x\left[\int_t^\infty e^{-\alpha s-A_s^\mu}f(X_s)dA_s^\nu\,|\,\mathcal{F}_t\right]=e^{-\alpha t-A_t^\mu}\mathcal{U}_\nu^{\alpha+\mu}f(X_t)\quad P_x\text{ a.s.}$$

Applying the above equality and Fubini's theorem we obtain

$$(3.5)\quad E_x\left[\int_0^\infty {}^{-\alpha t-A_t^\mu}\left(\int_0^t e^{-A_s^{\mu_1}+A_s^{\mu_2}}\,dA_s^{\mu_2}\right)f(X_t)\,dA_t^\nu\right]=\mathcal{U}_{\mu_2}^{\alpha+\mu+\mu_1-\mu_2}\mathcal{U}_\nu^{\alpha+\mu}f(x)\,.$$

Similarly

$$(3.6)\quad E_x\left[\int_0^\infty e^{-\alpha t-A_t^\mu}\left(\int_0^t e^{-A_s^{\mu_1}+A_s^{\mu_2}}\,dA_s^{\mu_1}\right)f(X_t)dA_t^\nu\right]=\mathcal{U}_{\mu_1}^{\alpha+\mu+\mu_1-\mu_2}\mathcal{U}_\nu^{\alpha+\mu}f(x)\,.$$

Now (3.3) follows from (3.5), (3.6) and the following identity:

$$e^{-A_t^{\mu_1}+A_t^{\mu_2}}+\int_0^t e^{-A_s^{\mu_1}+A_s^{\mu_2}}\,dA_s^{\mu_1}=1+\int_0^t e^{-A_s^{\mu_1}+A_s^{\mu_2}}\,dA_s^{\mu_2}\,.$$ ∎

REMARK. (3.3) and (3.4) are in fact a generalization of the resolvent properties. For this reason we call (3.3) and (3.4) the *generalized resolvent formula* and accordingly $\mathcal{U}_\nu^{\alpha+\mu}f$ (provided it makes sense) is called *a generalized resolvent.*

In what follows α will always stand for a non-negative constant without explicitely mentioning it.

Theorem 3.2.

i) *Let* $\mu\in S_{K_0}$ *and* $\alpha>\beta(\mu)$. *Then*

(3.7) $\mathcal{U}^{\alpha-\mu} f \in \mathcal{F} \cap L^2(\mathcal{X}; \mu)$ *for all* $f \in L^2(\mathcal{X}; m)$.

(ii) *Let* $\mu = \mu^+ - \mu^-$ *such that* $\mu^+ \in S$ *and* $\mu^- \in S_{K_0}$, *and* $f \in L^2(\mathcal{X}; m)$ *be non-negative. If* $\mathcal{U}^{\alpha+\mu} f \in L^2(\mathcal{X}; m)$, *then* $\mathcal{U}^{\alpha+\mu} f \in \mathcal{F} \cap L^2(\mathcal{X}; |\mu|)$. (3.8)

(iii) *Let* $\mu \in S - S$ *and* $f \in L^2(\mathcal{X}; m)$ *be non-negative. If* $\mathcal{U}^{\alpha+\mu} f \in L^2(\mathcal{X}; m + \mu^-)$, *then* $\mathcal{U}^{\alpha+\mu} f \in \mathcal{F} \cap L^2(\mathcal{X}; |\mu|)$. (3.9)

(iv) *Let* $\mu \in S - S$, *then for all* ν_1 *and* ν_2 *in* S *and non-negative Borel functions* f_1 *and* f_2,

$$\langle \mathcal{U}^{\mu}_{\nu_1} f_1, f_2 \cdot \nu_2 \rangle = \langle \mathcal{U}^{\mu}_{\nu_2} f_2, f_1 \cdot \nu_1 \rangle \tag{3.10}$$

and

$$E_{f_2 \cdot \nu_2}\left[\int_0^t e^{-A_s^\mu} f_1(X_s) dA_s^{\nu_1}\right] = E_{f_1 \cdot \nu_1}\left[\int_0^t e^{-A_s^\mu} f_2(X_s) dA_s^{\nu_2}\right], \quad t \geq 0. \tag{3.11}$$

Proof. The proof is splitted into several lemmas.

Lemma 3.3. *Let* $\mu \in S_K$ *and* $\alpha > \beta(\mu)$. *Then*

$$||\mathcal{U}^{\alpha-\mu} 1||_q + ||\mathcal{U}^{\alpha-\mu}_{\mu} 1||_q < \infty .$$

Here 1 *stands for the function on* $\mathcal{X}$ *identically equal to* 1.

Proof. It is obvious that $||\mathcal{U}^{\alpha-\mu} 1||_q < \infty$ for $\alpha > \beta(\mu)$. That $||\mathcal{U}^{\alpha-\mu}_{\mu} 1||_q < \infty$ holds can be seen from the formula

$$e^{-\alpha t + A_t^\mu} + \alpha \int_0^t e^{-\alpha s + A_s^\mu} ds = 1 + \int_0^t e^{-\alpha s + A_s^\mu} dA_s^\mu. \tag{3.12}$$ ∎

Lemma 3.4. *Suppose that* $\mu \in S_{K_0}$, $\alpha > \beta(\mu)$ *and* $f \in L^2(\mathcal{X}; m) \cap \mathcal{B}_{qb}(\mathcal{X})$, *then* $\mathcal{U}^{\alpha-\mu} f \in \mathcal{F}$ (*where* $\mathcal{B}_{qb}(\mathcal{X}) := \{f \in \mathcal{B}(\mathcal{X}) : ||f||_q < \infty\}$).

Proof. Without loss of generality we may assume that $f \geq 0$. Since $f \in \mathcal{B}_{qb}(\mathcal{X})$, applying the above lemma we can see that $||\mathcal{U}^{\alpha-\mu} f||_q < \infty$. Thus $(\mathcal{U}^{\alpha-\mu} f) \cdot \mu \in S_0$ because $\mu \in S_{K_0} \subset S_0$. Consequently $\mathcal{U}^{\alpha}_{\mu} \mathcal{U}^{\alpha-\mu} f = \mathcal{U}^{\alpha}_{(\mathcal{U}^{\alpha-\mu} f) \cdot \mu} 1 \in \mathcal{F}$. But by Theorem 3.1 we have $\mathcal{U}^{\alpha-\mu} f = \mathcal{U}^{\alpha} f + \mathcal{U}^{\alpha}_{\mu} \mathcal{U}^{\alpha-\mu} f$, hence $\mathcal{U}^{\alpha-\mu} f \in \mathcal{F}$. ∎

Lemma 3.5. *Let* $\mu, \nu \in S_{K_0}$, $\alpha > \beta(\mu)$. *Then* $\mathcal{U}_\nu{}^{\alpha-\mu} 1 \in \mathcal{F} \cap L^2(\mathcal{X}; \mu)$.

Proof. Take a large number $\beta > \beta(\mu + \nu)$. Then

$$||\mathcal{U}_\nu{}^{\beta-\mu} 1||_q \leq ||\mathcal{U}^{\beta-(\mu+\nu)}_{\mu+\nu} 1||_q < \infty .$$

Consequently

$$\mathcal{U}_\mu{}^\beta \mathcal{U}_\nu{}^{\beta-\mu} 1 = \mathcal{U}^{\beta}_{(\mathcal{U}_\nu{}^{\beta-\mu} 1 \cdot \mu)} 1 \in \mathcal{F} \cap \mathcal{B}_{qb}(\mathcal{X}) .$$

By Theorem 3.1 we obtain

$$\mathcal{U}_\nu^{\beta-\mu}1 = \mathcal{U}_\nu^\beta 1 + \mathcal{U}_\mu^\beta \mathcal{U}_\nu^{\beta-\mu}1 \in \mathcal{F} \cap \mathcal{B}_{qb}(\mathcal{X}).$$

Again by Theorem 3.1 and Lemma 3.3 we obtain

$$\mathcal{U}_\nu^{\alpha-\mu}1 = \mathcal{U}_\nu^{\beta-\mu}1 + (\beta-\alpha)\mathcal{U}^{\alpha-\mu}\mathcal{U}_\nu^{\beta-\mu}1 \in \mathcal{F} \cap \mathcal{B}_{qb}(\mathcal{X}).$$

Now we can write

$$\mathcal{U}_\nu^{\alpha-\mu}1 = \mathcal{U}_\nu^\alpha 1 + \mathcal{U}_\nu^\alpha \mathcal{U}_\nu^{\alpha-\mu}1$$

and hence

$$E_\alpha(\mathcal{U}_\nu^{\alpha-\mu}1, \mathcal{U}_\nu^{\alpha-\mu}1) = E_\alpha(\mathcal{U}_\nu^\alpha 1, \mathcal{U}_\nu^{\alpha-\mu}1) + \langle \mathcal{U}_\nu^{\alpha-\mu}1, (\mathcal{U}_\nu^{\alpha-\mu}1)\cdot\mu\rangle < \infty .$$

Thus $\mathcal{U}_\nu^{\alpha-\mu}1 \in L^2(\mathcal{X}; \mu)$ ■

Lemma 3.6. *If $\mu \in S_{K_0}$, $\alpha > \beta(\mu)$, then*

$$\langle \mathcal{U}_{\nu_1}^{\alpha-\mu} f_1, f_2 \cdot \nu_2 \rangle = \langle \mathcal{U}_{\nu_2}^{\alpha-\mu} f_2, f_1 \cdot \nu_1 \rangle$$

for all ν_1, $\nu_2 \in S$ and $f_1, f_2 \in \mathcal{B}^+(\mathcal{X})$.

Proof. Let ν_1, $\nu_2 \in S_{K_0}$. We have

$$\mathcal{U}_{\nu_1}^{\alpha-\mu}1 = \mathcal{U}_{\nu_1}^\alpha 1 + \mathcal{U}_\mu^\alpha \mathcal{U}_{\nu_1}^{\alpha-\mu}1 .$$

Consequently

$$E_\alpha(\mathcal{U}_{\nu_1}^{\alpha-\mu}1, \mathcal{U}_{\nu_2}^{\alpha-\mu}1) = \langle \mathcal{U}_{\nu_2}^{\alpha-\mu}1, \nu_1 \rangle + \langle \mathcal{U}_{\nu_2}^{\alpha-\mu}1, \mathcal{U}_{\nu_1}^{\alpha-\mu}1 \rangle_\mu .$$

Similarly

$$E_\alpha(\mathcal{U}_{\nu_1}^{\alpha-\mu}1, \mathcal{U}_{\nu_2}^{\alpha-\mu}1) = \langle \mathcal{U}_{\nu_1}^{\alpha-\mu}1, \nu_2 \rangle + \langle \mathcal{U}_{\nu_1}^{\alpha-\mu}1, \mathcal{U}_{\nu_2}^{\alpha-\mu}1 \rangle_\mu .$$

Comparing the above two identities we obtain

$$\langle \mathcal{U}_{\nu_1}^{\alpha-\mu}1, \nu_2 \rangle = \langle \mathcal{U}_{\nu_2}^{\alpha-\mu}1, \nu_1 \rangle .$$

Noticing that $\nu \in S_{K_0}$ and $0 \le f \le \mathcal{B}_{qb}(\mathcal{X})$ imply $f\cdot\nu \in S_{K_0}$, the proof can be completed by applying the monotone convergence theorem. ■

Lemma 3.7. *If $\mu \in S_{K_0}$, $g \in L^2(\mathcal{X}; \mu)$, $\alpha > 0$. then*

$$\mathcal{U}_\mu^\alpha g \in \mathcal{F} \cap L^2(\mathcal{X}; \mu) .$$

Proof. Let us set $\|\mathcal{U}_\mu^\alpha 1\|_q = c$. We have

$$|\mathcal{U}_\mu^\alpha g|^2 \le \left(E.\int_0^\infty e^{-\alpha t}\, dA_t^\mu\right)\left(E.\int_0^\infty e^{-\alpha t}|g(X_t)|^2\, dA_t^\mu\right) \le c\,\mathcal{U}_\mu^\alpha |g|^2.$$

Applying the previous lemma we obtain

$$\int_{\mathcal{X}} |\mathcal{U}_\mu^\alpha g|^2 \mu(dx) \le c\langle \mathcal{U}_\mu^\alpha |g|^2, \mu\rangle = c\langle \mathcal{U}_\mu^\alpha 1, |g|^2\mu\rangle \le c^2\|g\|^2_{L^2(\mathcal{X};\,\mu)} < \infty.$$

Similarly we can prove

$$\int_{\mathcal{X}} |\mathcal{U}_\mu{}^\alpha g|^2 m(dx) \leq c\|\mathcal{U}^\alpha 1\|_q \|g\|^2_{L^2(\mathcal{X};\,\mu)} < \infty .$$

Now applying the relation (0.1), we have

$$\frac{1}{t}(\mathcal{U}_\mu{}^\alpha g - e^{-\alpha t} P_t \mathcal{U}_\mu{}^\alpha g,\ \mathcal{U}_\mu{}^\alpha g) = \frac{1}{t} E_{\mathcal{U}_\mu{}^\alpha g}\int_0^t e^{-\alpha t} g(X_t) dA_t^\mu \rightarrow \langle \mathcal{U}_\mu{}^\alpha g, g\rangle_\mu < \infty .$$

By [19] Lemma 1.3.4 we get $\mathcal{U}_\mu{}^\alpha g \in \mathcal{F}$. ■

3.8 Proof of Theorem 3.2 (i)

Let $\mu \in S_{K_0}$, $\alpha > \beta(\mu)$ and $f \in L^2(\mathcal{X}; m)$.

Let us set $c = \|\mathcal{U}^{\alpha-\mu} 1\|_q + \|\mathcal{U}_\mu{}^{\alpha-\mu} 1\|_q$. By Lemma 3.3 we know $c < \infty$. Similarly as in the proof of the above lemma, we have

$$\int_{\mathcal{X}} |\mathcal{U}^{\alpha-\mu} f|^2 \mu(dx) \leq c\langle \mathcal{U}^{\alpha-\mu}|f|^2, \mu\rangle = c(\mathcal{U}_\mu{}^{\alpha-\mu} 1, |f|^2) \leq c^2 \|f\|^2_{L^2(\mathcal{X};\,m)} \cdot$$

Thus $\mathcal{U}^{\alpha-\mu} f \in L^2(\mathcal{X}; \mu)$. By Theorem 3.1 we have $\mathcal{U}^{\alpha-\mu} f = \mathcal{U}^\alpha f + \mathcal{U}_\mu{}^\alpha \mathcal{U}^{\alpha-\mu} f$. Applying Lemma 3.7 we get $\mathcal{U}^{\alpha-\mu} f \in \mathcal{F}$, which completes the proof of Theorem 3.2 (i).

To prove Theorem 3.2.(ii) we still need a Lemma.

Lemma 3.9. *Let $\mu = \mu^+ - \mu^-$ with $\mu^+ \in S$ and $\mu^- \in S_{K_0}$. Then*

$$\mathcal{U}^{\alpha+\mu} f \in \mathcal{F} \cap L^2(\mathcal{X}; |\mu|)$$

for all $\alpha > \beta(\mu^-)$ and $f \in L^2(\mathcal{X}; m)$.

Proof. Without loss of generality we may assume that $f \geq 0$. By Theorem 3.2. (i) we have

$$\mathcal{U}^{\alpha+\mu} f \leq \mathcal{U}^{\alpha-\mu^-} f \in L^2(\mathcal{X}; \mu^-) \cap L^2(\mathcal{X}; m) .$$

By Theorem 3.1 we can write

$$\mathcal{U}^{\alpha+\mu} f + \mathcal{U}^\alpha_{\mu^+} \mathcal{U}^{\alpha+\mu} f = \mathcal{U}^\alpha f + \mathcal{U}^\alpha_{\mu^-} \mathcal{U}^{\alpha+\mu} f \tag{3.13}$$

Hence

$$\mathcal{U}^\alpha_{\mu^+} \mathcal{U}^{\alpha+\mu} f \leq \mathcal{U}^\alpha f + \mathcal{U}^\alpha_{\mu^-} \mathcal{U}^{\alpha+\mu} f . \tag{3.14}$$

By Lemma 3.7 we have $\mathcal{U}^\alpha_{\mu^-} \mathcal{U}^{\alpha+\mu} f \in \mathcal{F}$. Thus by [19] Lemma 3.3.2 $\mathcal{U}^\alpha_{\mu^+} \mathcal{U}^{\alpha+\mu} f \in \mathcal{F}$ because both sides of (3.14) are α-excessive functions. Now (3.13) enables us to write

$$0 \leq E_\alpha(\mathcal{U}^{\alpha+\mu} f, \mathcal{U}^{\alpha+\mu} f) = (\mathcal{U}^{\alpha+\mu} f, f) + \langle \mathcal{U}^{\alpha+\mu} f, {}^\alpha\mathcal{U}^{\alpha+\mu} f\rangle_{\mu^-} - \langle \mathcal{U}^{\alpha+\mu} f, \mathcal{U}^{\alpha+\mu} f\rangle_{\mu^+}$$

which shows $\mathcal{U}^{\alpha+\mu} f \in L^2(\mathcal{X}; \mu^+)$. The proof is completed. ■

3.10. Proof of Theorem 3.2 (ii)

Let $\mu=\mu^+-\mu^-$ with $\mu^+\in S$ and $\mu^-\in S_{K_0}$, $0\leq f\in L^2(\mathcal{X}; m)$ and $\mathcal{U}^{\alpha+\mu}f\in L^2(\mathcal{X}; m)$. We first take a large number $\beta>\beta(\mu^-)$. Then by the above lemma we have

$$\mathcal{U}^{\beta+\mu}f\in\mathcal{F}\cap L^2(\mathcal{X}; |\mu|),\ \mathcal{U}^{\beta+\mu}\mathcal{U}^{\alpha+\mu}f\in\mathcal{F}\cap L^2(\mathcal{X}; |\mu|).$$

But using Theorem 3.1 we can write

$$\mathcal{U}^{\alpha+\mu}f=\mathcal{U}^{\beta+\mu}f+(\beta-\alpha)\mathcal{U}^{\beta+\mu}\mathcal{U}^{\alpha+\mu}f.$$

Consequently $\mathcal{U}^{\alpha+\mu}f\in\mathcal{F}\cap L^2(\mathcal{X}; |\mu|)$, which proves Theorem 3.2. (ii). ■

We now proceed to

3.11. Proof of Theorem 3.2 (iii)

Let $\mu=\mu^+-\mu^-$ with $\mu^+\in S$ and $\mu^-\in S$. Suppose that $0\leq f\in L^2(\mathcal{X}; m)$ and $\mathcal{U}^{\alpha+\mu}f\in L^2(\mathcal{X}; m+\mu^-)$. By Theorem 2.4 we can take an increasing sequence of compact sets $\{F_n\}_{n\geq 1}$ such that $I_{F_n}\cdot\mu^-\in S_{K_0}$, $\mu^-(\mathcal{X}-\cup F_n)=0$ and Cap $(K-F_n)\to 0$ for any compact set K. Let $\mu_n=\mu^+-I_{F_n}\cdot\mu^-$, then

$$0\leq\mathcal{U}^{\alpha+\mu_n}f\uparrow\mathcal{U}^{\alpha+\mu}f\qquad \text{q.e.}.$$

Applying Theorem 3.2 (ii) we get

$$\mathcal{U}^{\alpha+\mu_n}f\in\mathcal{F}\cap L^2(\mathcal{X}; |\mu_n|),\quad n\geq 1.$$

Employing (3.13) we can write

$$\begin{aligned}(3.15)\quad 0&\leq E_\alpha(\mathcal{U}^{\alpha+\mu_n}f, \mathcal{U}^{\alpha+\mu_m}f)\\ &=(\mathcal{U}^{\alpha+\mu_n}f, f)+\langle\mathcal{U}^{\alpha+\mu_n}f, \mathcal{U}^{\alpha+\mu_m}f\rangle_{I_{F_m}\cdot\mu^-}-\langle\mathcal{U}^{\alpha+\mu_n}f, \mathcal{U}^{\alpha+\mu_m}f\rangle_{\mu^+}.\end{aligned}$$

In particular

$$\langle\mathcal{U}^{\alpha+\mu_n}f, \mathcal{U}^{\alpha+\mu_n}f\rangle_{\mu^+}\leq(\mathcal{U}^{\alpha+\mu_n}f, f)+\langle\mathcal{U}^{\alpha+\mu_n}f, \mathcal{U}^{\alpha+\mu_n}f\rangle_{I_{F_n}\cdot\mu^-}.$$

Since $\mathcal{U}^{\alpha+\mu}f\in L^2(\mathcal{X}; m+\mu^-)$, by the monotone convergence theorem we obtain $\mathcal{U}^{\alpha+\mu}f\in L^2(\mathcal{X}; \mu^+)$. Now by virtue of the fact that $\mathcal{U}^{\alpha+\mu}f\in L^2(\mathcal{X}; m+|\mu|)$, (3.15) enables us to conclude that $\{\mathcal{U}^{\alpha+\mu_n}f\}_{n\geq 1}$ is an E_α-cauchy sequence. Thus $\lim_{n\to\infty}\mathcal{U}^{\alpha+\mu_n}f=\mathcal{U}^{\alpha+\mu}f\in\mathcal{F}$. The proof of Theorem 3.2 (iii) is thus completed.

To prove Theorem 3.2.(iv) we first prove the following

Lemma 3.12. *Let $\mu=\mu^+-\mu^-$ with $\mu^+\in S$ and $\mu^-\in S_{K_0}$. If $\mathcal{U}^{\alpha+\mu}f\in L^2(\mathcal{X}; m)$ for any $f\in L^2(\mathcal{X}; m)$, then $\mathcal{U}_\nu^{\alpha+\mu}1\in\mathcal{F}\cap L^2(\mathcal{X}; |\mu|)$ for any $\nu\in S_{K_0}$.*

Proof. Take a large number $\beta>\beta(\mu^-)$. By Lemma 3.5 we have

$$0\leq\mathcal{U}_\nu^{\beta+\mu}1\leq\mathcal{U}_\nu^{\beta-\mu^-}1\in L^2(\mathcal{X}; m+\mu^-)$$

for any $\nu\in S_{K_0}$. By Theorem 3.1 we can write

$$\mathcal{U}_\nu^{\beta+\mu}1+\mathcal{U}_{\mu+}^{\beta+\mu}\mathcal{U}_\nu^{\beta+\mu}=\mathcal{U}_\nu^{\beta}1+\mathcal{U}_{\mu-}^{\beta}\mathcal{U}_\nu^{\beta+\mu}1\,.$$

Thus similarly as in the argument used for proving Lemma 3.9 we may conclude

$$\mathcal{U}_\nu^{\beta+\mu}1\in\mathcal{F}\cap L^2(\mathcal{X};\,|\mu|)\,,\quad \text{for all } \nu\in S_{K_0}\,.$$

Applying again Theorem 3.1 we have

$$\mathcal{U}_\nu^{\alpha+\mu}1=\mathcal{U}_\nu^{\beta+\mu}1+(\beta-\alpha)\mathcal{U}^{\alpha+\mu}\mathcal{U}_\nu^{\beta+\mu}1\,.$$

By Theorem 3.2 (ii) we then obtain $\mathcal{U}_\nu^{\alpha+\mu}1\in\mathcal{F}\cap L^2(\mathcal{X};\,|\mu|)$ for all $\nu\in S_{K_0}$, which concludes the proof of Lemma 3.12.

3.13. Proof of Theorem 3.2 (iv)

Let $\{\mu_n\}_{n\geq 1}$ be specified as in the proof of Theorem 3.2 (iii). Let $n\geq 1$ be fixed, by Lemma 3.12 we have for α large enough,

$$\mathcal{U}_\nu^{\alpha+\mu_n}1\in\mathcal{F}\cap L^2(\mathcal{X};\,|\mu_n|)\qquad \text{for all } \nu\in S_{K_0}.$$

By Theorem 3.1 we can write for $\alpha>\beta(I_{F_n}\cdot\mu^-)$

$$\mathcal{U}_{\mu^+}^{\alpha}\mathcal{U}_\nu^{\alpha+\mu_n}1=\mathcal{U}_\nu^{\alpha}1+\mathcal{U}_{I_{F_n}\cdot\mu^-}^{\alpha}\mathcal{U}_\nu^{\alpha+\mu_n}1-\mathcal{U}_\nu^{\alpha+\mu_n}1\,.$$

Thus similarly as in the proof of Lemma 3.6 we obtain

$$\langle\mathcal{U}_{\nu_1}^{\alpha+\mu_n}1,\,\nu_2\rangle=\langle\mathcal{U}_{\nu_2}^{\alpha+\mu_n}1,\,\nu_1\rangle\,. \tag{3.16}$$

For fixed ν_1, $\nu_2\in S_{K_0}$, the above equality holds in fact for all $\alpha+\beta$ instead of α with $\beta\geq 0$. Thus by the uniqueness of the Laplace transform we get for ν_1 and ν_2 in S_{K_0}:

$$E_{\nu_2}\left[\int_0^t e^{-\alpha s-A_s^{\mu_n}}\,dA_s^{\nu_1}\right]=E_{\nu_1}\left[\int_0^t e^{-\alpha s-A_s^{\mu_n}}\,dA_s^{\nu_2}\right],\quad t\geq 0\,. \tag{3.17}$$

The above implies

$$E_{\nu_2}\left[\int_0^t f(s)e^{-\alpha s-A_s^{\mu_n}}\,dA_s^{\nu_1}\right]=E_{\nu_1}\left[\int_0^t f(s)e^{-\alpha s-A_s^{\mu_n}}\,dA_s^{\nu_2}\right],\quad t\geq 0\,, \tag{3.18}$$

for all non-negative Borel functions $f\in\mathcal{B}((0,\infty))$. In particular taking $f(s)=e^{\alpha s}$, we get

$$E_{\nu_2}\left[\int_0^t e^{-A_s^{\mu_n}}\,dA_s^{\nu_1}\right]=E_{\nu_1}\left[\int_0^t e^{-A_s^{\mu_n}}\,dA_s^{\nu_2}\right],\quad \forall t\geq 0\,. \tag{3.19}$$

Now (3.11) and (3.10) follow easily from (**3**.19) by the monotone convergence theorem. ∎

REMARK 3.14. Some special cases of (3.10) were first observed by M. Fukushima and Y. Oshima. Using a different method, they proved in [21] Lemma 3.1 that for any $\mu\in S$ and $f,g\in C_0(\mathcal{X})$, one has $(\mathcal{U}^{\alpha+\mu}g,f)=(\mathcal{U}^{\alpha+\mu}f,g)$. In [25] Th. 4.2.1 and Th. 4.2.3 Oshima proved that

$$\langle \mathcal{U}_{\mu}^{p+q\mu}f, g\cdot\mu\rangle=\langle \mathcal{U}_{\mu}^{p+q\mu}g, f\cdot\mu\rangle$$

and

$$(\mathcal{U}_{\mu}^{p+q\mu}f, g) = \langle \mathcal{U}^{p+q\mu}g, f\cdot\mu\rangle$$

for all $p, q\geq 0$ real numbers, $\mu\in S$ and $f,g\in\mathcal{B}^+(\mathcal{X})$. His results are available for the non-symmetric regular Dirichlet space. For applications of our results we refer to [6].

4. An analytic description of Kato's class

Recall that $||\cdot||_q$ is defined at the beginning of Section 2.

Lemma 4.1. *Let $\mu\in S$. Then*

$$||\mathcal{U}_{\mu}{}^{\alpha}1||_{q}| = ||\mathcal{U}_{\mu}{}^{\alpha}1||_{L^\infty(m)} \tag{4.1}$$

$$||E.A_t^{\mu}||_q = ||E.A_t^{\mu}||_{L^\infty(m)}\,. \tag{4.2}$$

Proof We prove only (ii), (i) being easier. Obviously we have

$$||\mathcal{U}_{\mu}{}^{\alpha}1||_{L^\infty(m)}\leq||\mathcal{U}_{\mu}{}^{\alpha}1||_q\,.$$

To prove the inverse inequality, suppose that $||\mathcal{U}_{\mu}{}^{\alpha}1||_{L^\infty(m)}=C<\infty$. Take an increasing sequence of compact sets $\{F_n\}_{n\geq 1}$ such that $\mu_n:=I_{F_n}\cdot\mu\in S_0$, $\mu(\mathcal{X}-\cup F_n)=0$ and $\lim_{n\to\infty}$ Cap $(K-F_n)=0$ for any compact set K. Then $\mathcal{U}_{\mu_n}^{\alpha}1$ is quasi-continuous. By [19] Lemma 3.1.4 there exists a set N_n of zero capacity such that $\mathcal{U}_{\mu_n}^{\alpha}1(x)\leq C$ on $\mathcal{X}-N_n$ Set $N_0=\mathcal{X}-\cup_{n=1}^{\infty}F_n$ and $N=\cup_{n=0}^{\infty}N_n$. Then Cap $(N)=0$ and $\mathcal{U}_{\mu}{}^{\alpha}1(x)\leq C$ for $x\in\mathcal{X}-N$ by the monotone convergence theorem. That is: $||\mathcal{U}_{\mu}{}^{\alpha}1||_q\leq C=||\mathcal{U}_{\mu}{}^{\alpha}1||_{L^\infty(m)}$. ■

In the sequel let $(p_t)_{t\geq 0}$ be the Markovian transition function of M and μ be a given Borel measure on $\mathcal{X}$. We make use of the following notations (whenever the expressions involved make sense).

$$\mu T_t f := \langle\int_0^t p_s f ds,\ \mu\rangle := E_{\mu}\Big[\int_0^t f(X_s)ds\Big]$$

$$\mu\mathcal{U}^{\alpha}f := \langle\int_0^{\infty} e^{-\alpha s}p_s f ds,\ \mu\rangle := E_{\mu}\Big[\int_0^{\infty} e^{-\alpha s}f(X_s)ds\Big]\,.$$

Theorem 4.1. *Let μ be a smooth measure. Then the following assertions are equivalent to each other:*

(i) $\mu\in S_K$;

(ii) $\lim_{\alpha\uparrow\infty}||\mathcal{U}_{\mu}{}^{\alpha}1||_q=0$;

(iii) μT_t *is a bounded functional on* $L^1(\mathcal{X}; m)$ *and* $\lim_{t\downarrow 0} ||\mu T_t||=0$*;*

(iv) $\mu\mathcal{U}^\alpha$ *is a bounded functional on* $L^1(\mathcal{X}; m)$ *and* $\lim_{\alpha\uparrow\infty} ||\mu\mathcal{U}^\alpha||=0$.

In (iii) *and* (iv) $||\cdot||$ *denotes the operator norm of a functional on* $L^1(\mathcal{X}; m)$.

Proof By the additive property of A_t^μ we have

$$\mathcal{U}_\mu^\alpha 1 = E.\int_0^\infty e^{-\alpha s}\,dA_s^\mu = E.\left[\sum_{n=0}^\infty e^{-\alpha nt} E_{X_{nt}} \int_0^t e^{-\alpha s}\,dA_s^\mu\right].$$

Consequently

$$||\mathcal{U}_\mu^\alpha 1||_q \leq (1-e^{-\alpha t})^{-1} ||E.A_t^\mu||_q\,,$$

from which the assertion (i)$\Rightarrow$(ii) follows. On the other hand obviously we have

$$||E.A_t^\mu||_q \leq e^{\alpha t} ||\mathcal{U}_\mu^\alpha 1||_q$$

which implies the assertion (ii)$\Rightarrow$(i). Noticing that $L^\infty(\mathcal{X}; m)$ is the dual space of $L^1(\mathcal{X}; m)$, the assertion (i)$\Leftrightarrow$(iii) follows from (3.11) and (4.2). Similarly the assertion (ii)$\Leftrightarrow$(iv) follows from (3.10) and (4.1).

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