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LARGE DEVIATION FOR A CLASS OF CURRENT-VALUED PROCESSES

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1. Introduction

In this paper, we are concerned with the large deviation problem for two typical current-valued processes among those that are induced by random curves: one is induced by Brownian motion, the other is by geodesic flow. For both processes, the law of large numbers, the central limit theoremes have been studied and there are some studies discussed relations between asymptotic behaviours of Brownian motion and geodesic flow (see e.g., Ledrappier [4]). These results suggest that the deviation funccions for two current-valued processes may coincide or at least have some connections, but since at present this remains unclear, we content ourselves to determine the deviation functions for those two currect-valued processes. Let M be a compact Riemannian manifold. We denote by $\Lambda^1(M)$ and $\Lambda^1(M)'$ be the smooth 1-forms on M and the currents, respectively. We denote by $(\mathcal{D}_1)_p$ the completion of $\Lambda^1(M)/\mathrm{Ker}\|\cdot\|_p$ with respect to the norm $\|\ \|_p$ (see, e.g., [5])., The dual spece of $(\mathcal{D}_1)_p$ is denoted by $(\mathcal{D}_1)'_p$. For a $\Lambda^1(M)'$-valued process $Y=(Y_t)_{t\in T}$, where $T=[0,\infty)$ or $\boldsymbol{R}$, we define the following quantities: Given a family of probability measures $\{m_x\}_{x\in M}$,

(1.1a) $$\bar{\lambda}(\Gamma)=\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in M} m_x\left[\frac{1}{t}Y(t)\in\Gamma\right],$$

(1.1b) $$\underline{\lambda}(\Gamma)=\liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in M} m_x\left[\frac{1}{t}Y(t)\in\Gamma\right],$$

for any Borel set Γ in $(\mathcal{D}_1)'_p$ and

(1.2) $$\Lambda[\alpha]=\lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in M} E^{m_x}[e^{\langle Y(t),\alpha\rangle}].$$

We call a function k an upper [resp. a lower] deviation function if

(1.3a) $$\underline{\lambda}(\Gamma)\geq-\inf\{k(\xi);\xi\in(\mathring{\Gamma})\}$$

(1.3b) $$[\text{resp. } \bar{\lambda}(\Gamma)\leq-\inf\{k(\xi);\xi\in(\bar{\Gamma})\}].$$

In particular, we call simply k a deviation function if it is lower semi-continuous and is both an upper and a lower deviation function. The objective of this paper is to determine the deviation function for the above-mentioned current-valued processes.

First let $X=(X_t, P_x, x\in M)$ be the Brownian motion on M, i.e., the diffusion generated by $\frac{1}{2}\Delta$. We consider the following stochastic line integral:

$$\langle Y(t), \alpha\rangle = Y_t(\alpha) = \int_{X[0,t]} \alpha \tag{1.1}$$

for smooth 1-form α on M. The random process $\{Y_t\}$ can be regarded as a current-valued process (Ochi [5]). In this case, $m_x=P_x$, $x\in M$. Our result is the following.

Theorem 1.1. *There exists a deviation function I for which*

$$I[\xi] = \sup_{\alpha\in\mathcal{D}_1} \{\langle\xi, \alpha\rangle - \Lambda[\alpha]\} \tag{1.5}$$

holds. The deviation function I can be written explicitely as

$$I[\xi] = \sup_{\|\alpha\|_p=1} \inf_{f\in C} \frac{1}{2}\left[\frac{\langle\xi, \alpha\rangle^2}{\sigma^2(\alpha, f)} + \int_M \|df\|^2 \, dv_M\right], \tag{1.6}$$

where we set

$$C = \{f\in C^2(M); f>0, \int f^2 \, dv_M = 1\}$$

and

$$\sigma^2(\alpha, f) = \inf_{\beta\sim\alpha} \int_M \|\beta\|^2 f^2(x) \, dv_M(x) . \tag{1.7}$$

($\beta\sim\alpha$ means that $\beta-\alpha$ is homologous to zero).

Avellaneda [1] treated the large deviation problem for the random process $(Y_t(\alpha^1), \cdots, Y_t(\alpha^k))$, where $\alpha^1, \cdots, \alpha^k (k=\dim H^1(M))$ is a basis of $H^1(M)$. Our result can be considered as a generalization of the result of [1]. Although, in our formulation, the state space of the process $\{Y_t\}$ is infinite dimensional, the method used in [1] is also applicable to our situation with some adaptations.

For the second one, we consider the process induced by geodesic flow $G=\{G^t\}$ over a compact Riemannian manifold M. Let SM be the bundle of unit tangent vectors. We denote the normalized measure on SM by $dm=c.dM(x) d\sigma_x(v)$, where dM denotes the Riemannian volume of M, $d\sigma_x(v)$ is the uniform measure on the unit sphere and c is the normalizing constant. This measure is an invariant measure of the geodesic flow. We take $m_x=m$, $x\in M$. In this case, the integral corresponding to (1.1) becomes an ordinary line integral:

$$\langle Y(t), \alpha\rangle = Y_t(\alpha) = \int_{G[0,t]} \alpha . \tag{1.8}$$

The following theorem concerns with the case of geodesic flow.

Theorem 1.2. *There exists a deviation function* I *for which* (1.7) *holds. The deviation function* $I(\xi)$ *is given by*

$$I(\xi) = \inf_{\Phi(\mu)=\xi} q(\mu)\,,$$

where Φ *is the mapping defined by* (4.1). *In the above,* q *is the deviation function of geodesic flow and can be written explicitely as*

$$q(\mu) = h_\mu - \chi^+(\mu)\,,$$

where $\chi^+(\mu)$ *is the positive Lyapunov exponent.*

As is seen above, the large deviation of the first level for the current-valued process induced by geodesic flow reduces to that of the second level for the geodesic flow. The latter has been studied by Takahashi [8], [9] for much wider class of dynamical systems.

2. Lemmas

First we introduce a notation. For a smooth vector field b on M, we set $L^b = \frac{1}{2}\Delta_M + b$. We denote by $\{Q_x^b\}_{x\in M}$ the diffusion measure generated by the operator L^b. For any Borel set Γ in $(\mathcal{D}_1)'_p$, we set

$$\mathcal{P}(t, \Gamma) = \inf_{x\in M} P_x[\frac{1}{t} Y(t)\in\Gamma]\,. \tag{2.1}$$

Lemma 2.1. *There exist positive constants* C *and* K *such that for any* $\rho>0$ *and* $s>0$,

$$\sup_{x\in M} Q_x^b[\frac{1}{t}\,\|Y(s)\|'_p>\rho] \le \frac{K}{\rho^2 s}\, e^{-\rho^2 t^2/2Cs}\,.$$

Proof. It is sufficient to prove the case $b\equiv 0$. We may assume that $\delta\alpha^n = 0$, $n=1, 2, \cdots$, since the drift term does not cause any difficulties. First note that $Y_n(t)\,(=Y_t(\alpha^n))$ is a martingale with $\frac{1}{t}\langle Y_n\rangle(t)\to(\alpha^n, \alpha^n)$, as $t\to\infty$. Using the representation theorem for continuous martingales, there exists a 1-dimensional Brownian motion B such that $Y_n(t)=B(\langle Y_n\rangle(t))$. Thus we have

$$\begin{aligned} P_x(|Y_n(s)|>\frac{t\rho}{b_n}) &= P_x(|B(\langle Y_n\rangle(s))|>\frac{t\rho}{b_n}) \\ &\le P_x(\sup_{0\le\sigma\le\|\alpha^n\|_q^2 s}|B(\sigma)|>\frac{t\rho}{b_n}) \le 2\exp[-\frac{\rho^2 t^2}{2b_n^2\|\alpha^n\|_q^2 s}] \end{aligned}$$

Now choosing a sequence b_n as $b_n^{-1}=C^{-1}n^a\,\|\alpha^n\|_q^2$ with $C=\sum_{n=1}^{\infty}n^a\,\|\alpha^n\|_q^2(<\infty)$, we have

$$P_x[\frac{1}{t}\|Y(s)\|_p'>\rho]\leq P_x[\frac{1}{t^2}\sum_{n=1}^{\infty}Y_n(s)^2>\rho^2]\leq\sum_{n=1}^{\infty}P_x[|Y_n(s)|^2>\frac{t^2\rho^2}{b_n^2}]$$
$$\leq 2\sum_{n=1}^{\infty}\exp[\frac{-\rho^2t^2}{2b_n^2\|\alpha^n\|_q^2 s}]=2\sum_{n=1}^{\infty}\exp[-\frac{\rho^2t^2}{2Cs}n^a]$$
$$\leq K\frac{1}{\rho^2 s}\exp[-\frac{\rho^2t^2}{2Cs}],$$

which proves the lemma.

Lemma 2.2. *We set* $\mu_x^Q(t,\Gamma)=Q_x^b(\frac{1}{t}Y(t)\in\Gamma)$. *There exists a positive constant* A_2 *such that for any* $\delta>0$, $\Gamma\in\mathcal{B}((\mathcal{D}_1)_p')$, $t>2$, $(x,y)\in M\times M$,

$$\mu_x^Q(t,\Gamma)\leq A_2\,\mu_y^Q(t,\Gamma^\delta)+A_2\,\delta^{-2}\,e^{-t^2\delta^2/2C},$$

where $\Gamma^\delta=\{\xi\in(\mathcal{D}_1)_p';\ \mathrm{dist}(\xi,\Gamma)<\delta\}$.

Proof. In the proof, we omit the superscript b of Q^b.

$$\mu_x^Q(t,\Gamma)=Q_x(\frac{1}{t}Y(t)\in\Gamma)$$
$$\leq Q_x(\frac{1}{t}(Y(t)-Y(1))\in\Gamma^{\delta/2})+Q_x(\frac{1}{t}\|Y(1)\|_p'>\delta/2).$$

By Lemma 2.1, we have

$$Q_x(\frac{1}{t}\|Y(1)\|_p'\geq\delta/2)\leq K\delta^{-2}\,e^{-\delta^2t^2/2C}.$$

The first term can be estimated as follows.

$$Q_x[\frac{1}{t}(Y(t)-Y(1))\in\Gamma^{\delta/2}]=E^{Q_x}[Q_x[\frac{1}{t}(Y(t)-Y(1))\in\Gamma^{\delta/2}/\mathcal{F}_1]]$$
$$=E^{Q_x}[Q_{X(1)}[\frac{1}{t}\,Y(t-1)\in\Gamma^{\delta/2}]/\mathcal{F}_1]$$
$$\leq C_3\,E^{Q_y}[Q_{X(1)}[\frac{1}{t}\,Y(t-1)\in\Gamma^{\delta/2}]]$$
$$=C_3\,E^{Q_y}[Q_y[\frac{1}{t}(Y(t)-Y(1))\in\Gamma^{\delta/2}/\mathcal{F}_1]]$$
$$=C_3\,Q_y[\frac{1}{t}(Y(t)-Y(1))\in\Gamma^{\delta/2}]$$
$$=C_3\,Q_y[\frac{1}{t}(Y(t)-Y(1))\in\Gamma^{\delta/2};\ \frac{1}{t}\|Y(1)\|<\delta/2]$$
$$+C_3\,Q_y[\frac{1}{t}(Y(t)-Y(1))\in\Gamma^{\delta/2};\ \frac{1}{t}\|Y(1)\|\geq\delta/2]$$

$$\leq C_3 Q_y[\frac{1}{t} Y(1) \in \Gamma^\delta] + C_3 Q_y[\frac{1}{t} \| Y(1) \| \geq \delta/2]$$

$$\leq C_3 Q_y[\frac{1}{t} Y(1) \in \Gamma^\delta] + C_3 K \delta^{-2} e^{-\delta^2 t^2/2C},$$

which proves the lemma.

Next we show that the law of large numbers for the current-valued process Y. Let α_ξ be the unique element of $(\mathcal{D}_1)_p$ such that $\langle \xi, \alpha \rangle = (\alpha_\xi, \alpha)_{L^2}$ for any $\alpha \in (\mathcal{D}_1)_p$. We introduce the vector field $b[\xi]$ by $b = b^i \frac{\partial}{\partial x^i}$, $b^i = g^{ki} \alpha_{\xi,k}$. We write $Q_x^\xi = Q_x^{b[\xi]}$. Then we have

Proposition 2.1. *For any* $\xi \in (\mathcal{D}_1)'_p$, *we have*

$$Q_x^\xi(\lim_{t\to\infty} \frac{1}{t} Y(t) = \xi) = 1.$$

Proof. We write $\alpha_\xi = \sum \theta_i \alpha^i$. We set

$$Y_t(\alpha) = M_t(\alpha) + N_t(\alpha), \quad M_t(\alpha) = \int_0^t \alpha_i(r_s)\, dB_s^i, \quad N_t(\alpha) = \int_0^t \alpha(b[\xi])(X_s)\, ds.$$

By the definition of $b[\xi]$, we have $\alpha(b[\xi])(x) = \langle \alpha, \alpha_\xi \rangle$. Thus,

$$\sup_{\|\alpha\|_p=1} |\frac{1}{t} Y_t(\alpha) - \langle \xi, \alpha \rangle| \leq \sup_{\|\alpha\|'_p=1} |\frac{1}{t} M_t(\alpha)| + \sup_{\|\alpha\|_p=1} |\frac{1}{t} N_t(\alpha) - \langle \xi, \alpha \rangle|.$$

For the second term, we write $\alpha = a_i \alpha^i$, $\langle \xi, \alpha^i \rangle = \xi^i$. Then,

$$|\frac{1}{t} N_t(\alpha) - \langle \xi, \alpha \rangle|^2 = |\sum_i a_i^2 [\frac{1}{t} \int_0^t \langle \alpha^i, \alpha_\xi \rangle (X_s) ds - \xi^i]|^2$$

$$\leq (\sum_i a_i^2) \sum_i |\frac{1}{t} \int_0^t \langle \alpha^i, \alpha_\xi \rangle (X_s)\, ds - \xi^i|^2.$$

We thus have

$$\sup_{\|\alpha\|_p=1} |\frac{1}{t} N_t(\alpha) - \langle \xi, \alpha \rangle|^2 \leq \sum_i |\frac{1}{t} \int_0^t \langle \alpha^i, \alpha_\xi \rangle (X_s)\, ds - \xi^i|^2.$$

Noting that

$$|\frac{1}{t} \int_0^t \langle \alpha^i, \alpha_\xi \rangle (X_s)\, ds - \xi^i|^2 \leq C \|\alpha^i\|_q^2$$

and

$$|\frac{1}{t} \int_0^t \langle \alpha^i, \alpha_\xi \rangle (X_s)\, ds - \xi^i|^2 \to |\frac{1}{t} \int_M \langle \alpha^i, \alpha_\xi \rangle\, dm_M - \xi^i|^2 = 0,$$

we have

$$\sup_{||\alpha||_p=1} |\frac{1}{t} N_t(\alpha)-\langle\xi, \alpha\rangle| \to 0, t \to \infty .$$

For the first term, note that $M^*(t)=\sup_{||\alpha||_p=1}|M_t(\alpha)|$ is a submartingale and $M^*(t)^2 \leq \sum_{n=1}^\infty M_t(\alpha^n)^2$. We have

$$E_x^Q [M^*(t)^2] \leq \int_0^t E_x^Q [\sum_{n=1}^\infty |\alpha^n|^2(X_s)]\, ds$$

$$\leq \int_0^t \sum_{n=1}^\infty ||\alpha^n||_q^2 ds \leq (\sum_{n=1}^\infty ||\alpha^n||_q^2)\, t .$$

By Doob's inequality, we have

$$P_x(\sup_{A\leq t\leq B} \frac{1}{t} M^*(t)>\varepsilon) \leq \frac{1}{\varepsilon^2 A^2} E_x^Q[M^*(B)^2] \leq \frac{KB}{\varepsilon^2 A^2}$$

which implies

$$Q_x(\lim_{t\to\infty} \frac{1}{t} M^*(t) = 0) = 1 .$$

The following lemma plays an important role in the proof of our theorem.

Lemma 2.3. (i) *For any* $\xi\in(\mathcal{D}_1)'_p$, *and* $r>0$, *there exists a* $T_0=T_0(\xi, r)>0$ *such that for any* $t>T_0$.

$$\inf_{x\in M} P_x [\frac{1}{t} Y(t)\in B_r(\xi)] \geq \frac{1}{2} \exp [-tr-t(2||\alpha_\xi||_q)^2] .$$

(ii) *There exist positive constants K and C such that for any* $R>0$,

$$\sup_{x\in M} P_x [\frac{1}{t} ||Y(t)||'_p>R] \leq \frac{K}{R^2 t} \exp [-\frac{R^2 t}{2C}] .$$

Proof. Since the second part is a special case of Lemma 2.1, i.e., $s=t$ and $\rho=R$, we need only to show (i). Fix an orthonormal basis α^i of $(\mathcal{D}_1)_p$ and write $\alpha_\xi=\sum_{i=1}^\infty \theta_i \alpha^i$ and $\xi^i=\langle\xi, \alpha^i\rangle$. Then we have $\xi^j=\sum_{i=1}^\infty \theta_i(\alpha^i, \alpha^j)_{L^2}$. If we choose p sufficiently large, we have $\sum_i \theta_j \xi^j=(\alpha_\xi, \alpha_\xi)_{L^2}$. Recall that Q_x^ξ is the diffusion measure generated by the operator $\frac{1}{2}\Delta_M+b[\xi]$. By the Cameron-Martin formula,

$$P_x [\frac{1}{t} Y(t)\in B_r(\xi)] = \int_{\{\frac{1}{t} Y(t)\in B_r(\xi)\}} e^{-Y_t(\alpha_\xi)-\frac{1}{2}\int_0^t |\alpha_\xi|^2(X_s)ds}\, dQ_x^\xi .$$

Noting that

$$\int_0^t |\alpha_\xi|^2(X_s)\, ds \leq ||\alpha_\xi||_q^2\, t ,$$

$$|Y_t(\alpha_\xi)| \leq tr+t\, ||\alpha_\xi||_{L^2}^2 \quad \text{on the set} \quad \{\frac{1}{t} Y(t)\in B_r(\xi)\}$$

and Lemma 2.2, we have

$$\inf_{x\in M} P_x[\frac{1}{t} Y(t)\in B_r(\xi)]$$
$$\geq e^{-t[r+2\|\alpha_\xi\|_q^2]}[Q^\xi_{x_0}(\frac{1}{t}\ Y(t)\in B_{r/2}(\xi))-A_2\, r^{-2}\, e^{-t^2r^2/4C}]\,.$$

Now by virtue of Proposition 2.1, the proof of (i) is completed if we choose $T_0=T_0(\xi, r)>0$ such that for any $t>T_0$, $A_1\, r^{-2}\, e^{-t^2r^2/4C}<\frac{1}{4}$ and $Q^\xi_x(\frac{1}{t}Y(t)\in B_r(\xi))>\frac{3}{4}$, which proves the lemma.

The following lemma is easily shown from the definitions of $\mathcal{P}$.

Lemma 2.4. (i) *For any* $\xi\in(\mathcal{D}_1)'_p$, $r>0$, *we have*

$$\mathcal{P}(t_1+t_2, B_r(\xi))\geq\mathcal{P}(t_1, B_r(\xi))\,\mathcal{P}(t_2, B_r(\xi))\,.$$

(ii) *For any* $\xi_1, \xi_2\in(\mathcal{D}_1)'_p$, $r_1>0$, $r_2>0$, *and* $\lambda\in(0, 1)$,

$$\mathcal{P}(t, \lambda B_{r_1}(\xi_1)+(1-\lambda)\, B_{r_2}(\xi_2))\geq\mathcal{P}(\lambda t_1, B_{r_1}(\xi_1))\,\mathcal{P}((1-\lambda)\, t, B_{r_2}(\xi_2))\,.$$

3. Proof of Theorem 1.1

First we show that the existence of the deviation function. Set

$$l(\xi, r) = \sup_{t>T_0(\xi,r)} [-\frac{1}{t}\log\mathcal{P}(t, B_r(\xi))]\,, \tag{3.2}$$

and

$$I[\xi] = \sup_{r>0} l(\xi, r) = \lim_{r\to 0} l(\xi, r)\,. \tag{3.3}$$

Since by virtue of Lemma 2.3 and Lemma 2.4, the function $S(t)=-\log\mathcal{P}(t, B_r(\xi))$ is subadditive and satisfies $\sup_{t>0}\frac{S(t)}{t}<\infty$, we have

$$\sup_{t>0}\frac{S(t)}{t} = \lim_{t\to\infty}\frac{S(t)}{t}\,,$$

that is,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathcal{P}(t, B_r(\xi)) = -l(\xi, r)\,.$$

By Lemma 2.4, $I[\xi]$ is then convex and lower semicontinuous. Now we show that I is in fact the deviation function. We first show the lower bound (1.3a). For any Borel set Γ and $\xi\in\mathring{\Gamma}$, there exists a positive constant r_0 such that for any $r\in(0, r_0)$, $\overline{B_r(\xi)}\subset\Gamma$. We then have

$$\lim_{t\to\infty} \frac{1}{t} \log \inf_{x\in M} P_x[\frac{1}{t} Y(t)\in\Gamma]$$
$$\geq \liminf_{t\to\infty} \frac{1}{t} \log \inf_{x\in M} P_x[\frac{1}{t} Y(t)\in B_{r_0}(\xi)] = -l(\xi, r_0) \geq -I[\xi] .$$

To show the upper bound (1.3b), we divide two cases. Assume that $\bar{\Gamma}$ is compact. Since $I[\xi]$ is lower semicontinuous, for any ε, and $\xi\in(\mathcal{D}_1)'_p$, there exists a positive constant $r=r(\xi)$ such that

$$I[\xi]\leq l(\xi, r)+\varepsilon . \tag{3.4}$$

By the compactness of $\bar{\Gamma}$, we can choose $\xi_1, \cdots, \xi_N$ such that $\bar{\Gamma}\subset \bigcup_{i=1}^N B_{r(\xi_i)/2}(\xi_i)$. Using this and Lemma 2.2, we have

$$P_x[\frac{1}{t} Y(t)\in\Gamma]\leq P_x[\bigcup_{i=1}^N B_{r(\xi_i)/2}(\xi_i)]\leq A_2\, \mathcal{P}(t, \bigcup_{i=1}^N B_{r(\xi_i)/2}(\xi_i))+A_2\, \delta^{-2}\, e^{-t^2\delta^2/2C}$$

for some $\delta>0$. Thus, we have

$$\limsup \frac{1}{t} \log \sup_{x\in M} P_x[\frac{1}{t} Y(t)\in\Gamma]$$
$$\leq \limsup \frac{1}{t} \log[A_2\, \mathcal{P}(t, \bigcup_{i=1}^N B_{r(\xi_i)/2}(\xi_i))+A_2\, \delta^{-2}\, e^{-t^2\delta^2/2C}]$$
$$\leq \max[-l(\xi_1, r_1), \cdots, -l(\xi_N, r_N)]\leq -\min l(\xi_i, r_i)$$
$$\leq -\inf_{\xi\in\bar{\Gamma}} I[\xi]+\varepsilon .$$

Next let $\bar{\Gamma}$ be not compact. By Lemma 2.3 it holds

$$\frac{||\xi||'_p}{2C}\leq I[\xi]\leq 2\, ||\alpha_\xi||_q^2 .$$

Together with the fact that I is lower semicontinuous, it follows that

$$\inf_{\xi\in\bar{\Gamma}} I[\xi] = \inf_{\xi\in\bar{\Gamma}\cap \overline{B_R(0)}} I[\xi]$$

Thus we have

$$\limsup_{t\to\infty} \frac{1}{t} \log P_x[\frac{1}{t} Y(t)\in\bar{\Gamma}]$$
$$\leq \limsup_{t\to\infty} \frac{1}{t} \log \{P_x[\frac{1}{t} Y(t)\in\bar{\Gamma}\cap\overline{B_R(0)}]+P_x[\frac{1}{t} ||Y(t)||'_p>R]\}$$
$$\leq \max \{\limsup_{t\to\infty} \frac{1}{t} P_x[\frac{1}{t} Y(t)\in\bar{\Gamma}\cap\overline{B_R(0)}], -\frac{R^2}{2C}\}$$
$$\leq \max\{-\inf_{\xi\in\bar{\Gamma}\cap\overline{B_R(0)}} I[\xi], -\frac{R^2}{2C}\}$$
$$= -\inf_{\xi\in\bar{\Gamma}} I[\xi] ,$$

where we used the result of case 1.

To prove (1.5), we use the result described in Stroock [7]. Let $E=(\mathcal{D}_1)'_p$, $E^*=(\mathcal{D}_1)_p$. For $\alpha\in E^*$, set

$$\mu_x^\alpha(t, A)=P_x[\langle\frac{1}{t}\,Y(t),\alpha\rangle\in A],\ A\in\mathcal{B}(\boldsymbol{R}^1)\,,$$

$$l^\alpha(A)=\lim_{t\to\infty}\frac{1}{t}\log\inf_{x\in M}\mu_x^\alpha(t, A),\ A \text{ is convex, open in } \boldsymbol{R}^1\,,$$

and

$$\begin{aligned}I^\alpha(\theta)&=-\inf\{l^\alpha(A)\,|\,A \text{ is convex, open and } \theta\in A\}\,,\\&=-\lim_{\delta\downarrow 0} l^\alpha(B_\delta(\theta)),\ \theta\in\boldsymbol{R}^1\,.\end{aligned}$$

Then by Lemma 3.18 in Stroock [7], we have

$$I[\xi]=\sup_{\alpha\in(\mathcal{D}_1)_p} I^\alpha(\langle\xi,\alpha\rangle)\,,\ \ \xi\in(\mathcal{D}_1)'_p\,.$$

If we set

$$\Lambda^\alpha(\theta)=\lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in M}E^{P_x}[e^{\theta Y_t(\alpha)}],\ \alpha\in(\mathcal{D}_1)_p\,,$$

then we have

$$\Lambda^\alpha(\theta)=\Lambda[\theta\alpha],\ \theta\in\boldsymbol{R}^1,\ \alpha\in\Lambda_p^1\,.$$

By a result of Avellaneda [1], it holds that

$$I^\alpha(\tau)=\sup_{\theta\in\boldsymbol{R}^1}[\theta\tau-\Lambda^\alpha(\theta)],\ \tau\in\boldsymbol{R}^1\,.$$

We thus obtain

$$\begin{aligned}I[\xi]&=\sup_{\alpha\in(\mathcal{D}_1)_p} I^\alpha(\langle\xi,\alpha\rangle)=\sup_{\alpha\in(\mathcal{D}_1)_p}\ \sup_{\theta\in\boldsymbol{R}^1}\{\langle\xi,\theta\alpha\rangle-\Lambda[\theta\alpha]\}\\&=\sup_{\alpha\in(\mathcal{D}_1)_p}[\langle\xi,\alpha\rangle-\Lambda[\alpha]]\,,\end{aligned}$$

which completes the proof of (1.5). For the proof of (1.6), we first quote here the following Proposition due to Avellaneda [1, Lemma 3.5].

Lemma 3.1. *For $\alpha\in\Lambda^1(M)$, set*

$$\Lambda_\alpha^*(\tau)=\sup_{\theta\in\boldsymbol{R}^1}[\tau\theta-\Lambda^\alpha(\theta)]\,.$$

Then it holds

$$\Lambda_\alpha^*(\theta)=\inf_{f\in C^2(M),\int f^2dv_M=1,f>0}\frac{1}{2}\Big[\frac{\theta^2}{\sigma^2(\alpha,f)}+\int_M\|df\|^2\,dv_M(x)\Big]\,.$$

Now the proof of (1.6) proceeds as follows:

$$\begin{aligned} I[\xi] &= \sup_{\alpha\in(\mathcal{D}_1)_p} [\langle\xi,\alpha\rangle-\Lambda[\alpha]] \\ &= \sup_{\|\alpha\|_p=1}\sup_{\theta>0}[\theta\langle\xi,\alpha\rangle-\Lambda[\theta\alpha]] \\ &= \sup_{\|\alpha\|_p=1}\sup_{\theta>0}[\theta\langle\xi,\alpha\rangle-\Lambda^{\alpha}(\theta)]\,. \end{aligned}$$

Since $\Lambda^{\alpha}(\theta)=\Lambda[\theta\alpha]$, we have $\Lambda^{\alpha}(-\theta)=\Lambda[-\theta\alpha]=\Lambda^{-\alpha}(\theta)$. Noting the fact $\Lambda^{\alpha}(0)=0$, we thus have

$$\begin{aligned} &\sup_{\|\alpha\|_p=1}\sup_{\theta\in\mathbf{R}^1}[\theta\langle\xi,\alpha\rangle-\Lambda^{\alpha}(\theta)] \\ &= \sup_{\|\alpha\|_p=1}\Lambda^*_{\alpha}(\langle\xi,\alpha\rangle) \\ &= \sup_{\|\alpha\|_p=1}\inf_{f\in C}\frac{1}{2}\Big[\frac{\langle\xi,\alpha\rangle^2}{\sigma^2(\alpha,f)}+\int_M \|df\|^2\,dv_M(x)\Big]\,, \end{aligned}$$

which completes the proof of (1.6).

4. Proof of Theorem 1.2

In this section, we consider the current-valued process induced by geodesic flow. Let $\boldsymbol{G}=\{G^t\}$ be the geodesic flow on SM. For $\alpha\in\Lambda(M)$ and $\omega\in M$, the integral (1.8) is of the form:

$$Y_t(\alpha)=Y_t(\alpha,\omega)=\int_{G([0,t],\omega)}\alpha=\int_0^t f[\alpha]\,(G^s\omega)\,ds\,.$$

We fix the following notation. $\mathcal{M}=\mathcal{M}(SM)$ is the space of probability measures on SM. $\mathcal{M}(\boldsymbol{G})=\mathcal{M}(SM,\boldsymbol{G})$ denotes the set of the elements of $\mathcal{M}(SM)$ which are invariant under $\boldsymbol{G}$. Let us consider the following mapping.

$$\Phi\colon\ \mathcal{M}\to\Lambda_1'(M)$$

which is defined by

$$\langle\Phi[\mu],\alpha\rangle=\int_{SM} f[\alpha]\,(\omega)\,d\mu(\omega)=\langle\mu,f[\alpha]\rangle\,. \tag{4.1}$$

Proposition 4.1. *Φ is a continuous mapping in the weak topology for $\mathcal{M}$ and the topology defined by the metric $\|\cdot\|_p'$ for $\Lambda_1'(M)$.*

Proof. Assume that μ_n converges weakly to μ. Take an orthonormal base $\{\alpha^n\}$ in $(\mathcal{D}_1)_p$. Then we have

$$\begin{aligned} \|\Phi[\mu_n]-\Phi[\mu]\|_p' &= \sup_{\|\alpha\|=1}|\langle\Phi[\mu_n],\alpha\rangle-\langle\Phi[\mu],\alpha\rangle| \\ &\le \sum_m\Big(\int f[\alpha^m]\,d\mu_n-\int f[\alpha^m]\,d\mu\Big)^2\,. \end{aligned}$$

For any $\varepsilon>0$, there exists an m_0 such that if $m\geq m_0$ then $\sum_{k=m_0}^{\infty}||\alpha^k||_q<\varepsilon$. We thus have

$$\limsup_{n\to\infty}||\Phi[\mu_n]-\Phi[\mu]||_p'\leq\limsup_{n\to\infty}\sum_{m=1}^{m_0}\left(\int f[\alpha^m]\,d\mu_n-\int f[\alpha^m]\,d\mu\right)^2+2C\varepsilon$$
$$=2C\varepsilon\,,$$

which completes the proof.

Following Takahashi [8], [9], we define the quantities $Q(G)$, $q(\mu)$ as follows.

$$Q(G)=\limsup_{t\to\infty}\frac{1}{t}\log m(\omega\in M;\,\varepsilon_{t,\omega}\in G)\,,\qquad\text{for}\quad G\in\mathcal{B}(\mathcal{M})\,,\tag{4.2}$$

where $\varepsilon_{t,\omega}\in\mathcal{M}$ is defined by

$$\varepsilon_{t,\omega}(\cdot)=\frac{1}{t}\int_0^t\delta_{G^s\omega}(\cdot)\,ds\,,\tag{4.3}$$

and

$$q(\mu)=\inf\{Q(G);\,\mu\in G,\,G\text{: open in }\mathcal{M}\}\,.\tag{4.4}$$

Corresponding to these quantities, we define the following:

$$Q^*(G^*)=\limsup_{t\to\infty}\frac{1}{t}\log\left[m(\omega\in SM\,|\,\Phi[\varepsilon_{t,\omega}]\in G^*)\right]\,,\quad G^*\in\mathcal{B}(\Lambda_1'(M))\,,\tag{4.2a}$$

and

$$I(Y)=\inf\{Q^*(G^*);\;Y\in G^*,\,G^*\text{ open in }\Lambda_1'(M)\}\,.\tag{4.4a}$$

By the very definition of $\varepsilon_{t,\omega}$ and Φ, we see

$$\Phi[\varepsilon_{t,\omega}]=\frac{1}{t}\,Y(t,\omega)\,.\tag{4.5}$$

To characterize the deviation function, we define L^t: $L^1(m)\to L^1(m)$ by

$$\int v(x)\,L^t u(x)\,m(dx)=\int v(G^t x)\,u(x)\,m(dx)$$

for $u\in L^1(m)$, $v\in C(SM)$. Let A be an infinitesimal generator of $\{L^t\}$ with domain $D(A)$. Next we define the relative entropy. Let α be a finite partition of SM. We denote $\alpha_n(t)=\bigvee_{i=0}^{n-1}G^{-it}\alpha$. We set

$$H(\mu\,|\,m;\alpha)=-\sum_{E\in\alpha}\mu(E)\log\frac{\mu(E)}{m(E)}\,.$$

For μ is $\{G^t\}$-invariant, we set

$$h_t(\mu\,|\,m;\alpha)=\lim_{n\to\infty}\frac{1}{n}H(\mu\,|\,m;\alpha_n(t))$$

and

$$h(\mu|m)=h(\mu|m, SM, \{G^t\})=\sup_\alpha t^{-1} h_t(\mu|m;\alpha)$$

The following theorem is due to Takahashi and actually holds for more general flows.

Theorem 4.1. (Takahashi [9, Theorem 2, Theorem 3])
(1)

$$\begin{aligned} q(\mu) &= \frac{1}{t}\inf\{\int \log\left(\frac{L^t u}{u}\right) d\mu;\ u\in C^+(SM)\} \\ &= \inf\{\int \frac{Au}{u}\, d\mu;\ u\in D(A)\cap C^+(SM)\}\ , \end{aligned}$$

where $C^+(SM)=\{u\in C(SM); u(x)>0, \forall x\in SM\}$. *Furthermore,* $q(\mu)=-\infty$ *if* μ *is not* $\{G^t\}$*-invariant.*
(2) $q(\mu)=h(\mu|m)$.
(3) $q(\mu)=\underline{q}(\mu)$, *where* $\underline{q}(\mu)$ *is defined by*

$$\underline{Q}(G)=\liminf_{t\to\infty}\frac{1}{t}\log m\{x\in SM;\ \varepsilon_{t,x}\in G\}$$

$$\underline{q}(\mu)=\inf\{\underline{Q}(G);\ G \text{ open},\ \mu\in\mathcal{M}\}\ .$$

REMARK. The assertion (3) in the above theorem implies $q(\mu)$ is in fact the deviation function (see Orey [6, Proposition 1.1. (vii)]).

Proof of Theorem 1.2. The first assertion is a consequence of contraction principle. To show the second assertion, it is sufficient to show that $h(\mu|m)=h_\mu-\chi^+(\mu)$ in view of the above theorem. Although it may be known this equality for the geodesic flow, we present the proof for the completeness. The inequality $q(\mu)\geq h_\mu-\chi^+(\mu)$ can be shown by a modification of the proof of Takahashi [9]. We show $q(\mu)\leq h_\mu-\chi^+(\mu)$. Take the function ϕ for which $\int\phi d\mu=-\chi^+(\mu)$ (See e.g., Bowen-Ruelle [2]). By a result of E. Franco [3, Prop. (2.11)],

$$m(B)\leq C_\varepsilon \exp\left[S_{nt}\,\phi(z)-P(\phi)\,nt\right],\ z\in B\ ,$$

where P denotes the pressure and we set $S_t\,\phi(z)=\int_0^t \phi(G^s x)\,ds$. Note that the pressure $P(\phi)=0$. Taking the logarithm of both sides, multiplying $\mu(B)$ for both sides and adding $-\int_B S_{nt}\,\phi(y)\,d\mu(y)$, we have

$$\mu(B)\log m(B)-nt\int\phi d\mu$$

$$\leq \mu(B) \log m(B) - \int_B S_{nt}\,\phi(y)\, d\mu(y)$$
$$\leq \mu(B) \log C_\varepsilon + \mu(B)\, S_{nt}\,\phi(z) - \int_B S_{nt}\,\phi(y)\, d\mu(y)$$
$$\leq \mu(B) \log C_\varepsilon + \mu(B)\, S_{nt}\,\phi(z) - \int_B \{K - S_{nt}\,\phi(z)\}\, d\mu(y)$$
$$= \mu(B)\,[\log C_\varepsilon + K]\,.$$

Therefore

$$\sum_{E \in \alpha_n(t)} \mu(E) \log m(E) \leq nt \int \phi d\mu + \log C_\varepsilon + K\,.$$

Thus we have

$$h_t(\mu \,|\, m; \alpha) \leq t h_\mu + t \int \phi d\mu\,,$$

this implies

$$h(\mu \,|\, m) \leq h_\mu + \int \phi d\mu\,.$$

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