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INVARIANT DIFFERENTIAL OPERATORS ON THE GRASSMANN MANIFOLD $SG_{2,n-1}(R)$

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0. Introduction. The Grassmann manifold $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})=SO(n+1)/SO(n-1)\times SO(2)$ with its canonical Riemannian metric is known to be a Riemannian symmetric space of rank 2. Hence the algebra $\boldsymbol{D}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ of $SO(n+1)$-invariant (linear) differential operators on $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ is generated by two differential operators.

It is the aim of our paper to exhibit simultaneous eigenspace decomposition of appropriate generators $\Delta_0^\wedge$ and $\Delta_1^\wedge$ of the algebra $\boldsymbol{D}(\boldsymbol{SG}_{2n-1}(\boldsymbol{R}))$. We have obtained in [7] the followings:

(1) the eigenspace decomposition of Δ_0 restricted to $\boldsymbol{K}^*(S^n, g_0)$ is given, where g_0 is the canonical metric on S^n and Δ_0 is the Lichnerowicz operator acting on the graded algebra $\boldsymbol{S}^*(S^n, g_0)$ of symmetric tensor fields on the standard sphere (S^n, g_0) and $\boldsymbol{K}^*(S^n, g_0)$ is the graded subalgebra of $\boldsymbol{S}^*(S^n, g_0)$ generated by Killing vector fields,

(2) Radon transform $\wedge$:

$$\boldsymbol{S}^*(S^n, g_0)\to C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$$

intertwines Δ_0 with the Laplace Beltrami operator $\Delta_0^\wedge$ on $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$, i.e.,

$$(\Delta_0\xi)^\wedge = \Delta_0^\wedge\xi^\wedge$$

for $\xi\in\boldsymbol{S}^*(S^n, g_0)$,

(3) the eigenspace decomposition obtained in (1) is transferred to that of $\Delta_0^\wedge$, since the kernel of the Radon transform restricted to $\boldsymbol{K}^*(S^n, g_0)$ is the principal ideal generated by $g_0/2-1$ and the image of $\boldsymbol{K}^*(S^n, g_0)$ is uniformly dense in $C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$.

In the present paper a new differential operator Δ_1 which acts on $\boldsymbol{S}^*(S^n, g_0)$ with analogous properties as (1), (2), (3) above is constructed.

Especially $\Delta_0^\wedge$ together with the differential operator $\Delta_1^\wedge$ corresponding to Δ_1 by the Radon transform are found to be a set of generators of the algebra $\boldsymbol{D}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$.

In **1** and **2**, we recall the results obtained in [7] with some improvements.

where $\langle\, ,\rangle$ and $d\sigma$ are the inner product at each point of M and the volume element induced from the Riemannian metric, respectively. Now define five fundamental linear operators on the graded algebra $\boldsymbol{S}^*(M)$.

DEFINITION 1.1. (1) Denote by T^* the linear map of degree 2:

$$\boldsymbol{S}^p(M)\in\xi\mapsto\frac{1}{2}g\circ\xi\in\boldsymbol{S}^{p+2}(M)\,.$$

(2) Denote by T the adjoint operator of T^*:

$$(T^*\xi,\eta)=(\xi,T\eta)\,.$$

Evidently, $\boldsymbol{S}^p(M)\ni\xi\mapsto T\xi\in\boldsymbol{S}^{p-2}(M)$, i.e., T is of degree -2.

(3) Denote by δ^* the linear map:

$$\boldsymbol{S}^p(M)\ni\xi\mapsto\delta^*\xi:=\frac{1}{2!}[g,\xi]\in\boldsymbol{S}^{p+1}(M)\,,$$

(4) Denote by δ the adjoint operator of $\delta^*\colon\boldsymbol{S}^p(M)\to\boldsymbol{S}^{p-1}(M)$ defined as

$$(\delta\xi,\eta)=(\xi,\delta^*\eta)\quad \xi\in S^p(M),\ \eta\in S^{p-1}(M).$$

(5) As the fifth operator let us define the degree operator d by

$$\boldsymbol{S}^p(M)\ni\xi\mapsto d\xi:=p\xi\in\boldsymbol{S}^p(M)\,.$$

Then δ^* and d are derivations on $\boldsymbol{S}^*(M)$, i.e.,

$$\delta^*(\xi\circ\eta)=(\delta^*\xi)\circ\eta+\xi\circ\delta^*\eta\,,$$

and

$$d(\xi\circ\eta)=(d\xi)\circ\eta+\xi\circ d(\eta)\,,$$

The proofs of these two assertions are direct and easy.

Lemma 1.1. ([7] pp. 54–55)

$$[T,\delta]=0,\quad [T^*,\delta^*]=0,\quad [\delta^*,T]=\delta,\quad [T^*,\delta]=\delta^*,$$

$$[T,T^*]=\frac{n+2d}{2},\quad [T^m,T^*]=\frac{mn+2dm+2m^2-2m}{2}T^{m-1}\,.$$

DEFINITION 1.2. $\Delta_0:=-2\sum_{a,b=0}^{m}g^{ab}\nabla_a\nabla_b-[\delta,\delta^*]$ acting on $\boldsymbol{S}^*(M)$ is called the *Lichnerowicz operator* on (M,g). The restriction of Δ_0 to $C^\infty(M)$ coincides with the ordinary *Laplace Beltrami operator*, which we denote by the same notation Δ_0.

Lemma 1.1. ([7] p. 55)

$$[\Delta_0, T] = 0\,, \quad [\Delta_0, T^*] = 0\,.$$

Lemma 1.3. ([6] Lemma 1.5) *Let (M, g) be a locally symmetric Riemannian manifold. Then*

$$[\Delta_0, \delta^*] = 0\,, \quad [\Delta_0, \delta] = 0\,.$$

$$\mathbf{Ker}\ \delta^*(M, g) := \sum_{p\geqq 0} (Ker\ \delta^*) \cap \boldsymbol{S}^p(M, g) \quad (direct\ sum)$$

is a graded subalgebra of $\boldsymbol{S}^*(M)$. Each element of **Ker** $\delta^*(M, g)$ is called *a Killing tensor field.* The graded subalgebra of **Ker** $\delta^*(M, g)$ generated by **Ker** $\delta^*(M, g) \cap \boldsymbol{S}^1(M)$, is denoted as

$$\boldsymbol{K}^*(M, g) = \sum_{p\geqq 0} \boldsymbol{K}^p(M, g)\ (\subseteq \mathbf{Ker}\ \delta^*(M, g)) \quad (direct\ sum).$$

Theorem 1.1. ([7] p. 62) (1) **Ker** $\delta^*(S^n, g_0)$ *coincides with* $\boldsymbol{K}^*(S^n, g_0)$.

(2) *For any $\xi \in \boldsymbol{K}^*(S^n, g_0)$, there exists a differential operator D_ξ with ξ as its symbol tensor field such that*

$$[D_\xi, \Delta_0] = 0\,.$$

2. Differential operators acting on $\boldsymbol{S}^*(S^n)$.

Lemma 2.1. ([6]) *Let M_i $(i=1, 2)$ be differentialbe manifolds. There are subalgebras $\tilde{\boldsymbol{E}}(M_i)$ $(i=1, 2)$ of $\boldsymbol{E}(M_1 \times M_2)$, being canonically isomorphic to $\boldsymbol{E}(M_i)$ $(i=1, 2)$ respectively, each one of which is the centralizer of the other in $\boldsymbol{E}(M_1 \times M_2)$.*

Let ι: $S^n \to \boldsymbol{R}^{n+1}$ be the canonical imbedding of S^n onto the unit sphere in $\boldsymbol{R}^{n+1}$. It induces a trivialization $\tilde{\iota}$: $S^n \times \boldsymbol{R} \to \boldsymbol{R}^{n+1} - \{0\}$ of the real line bundle $\boldsymbol{R}^{n+1} - \{0\}$ defined by $(x, t) \mapsto \iota(x, t) = e^t x$. By Lemma 2.1 a vector field ξ on S^n is uniquely identified with the vector field $\tilde{\xi}$ on $S^n \times \boldsymbol{R}$ such that

$$(2.\ 1) \qquad [\tilde{\xi}, t] = 0 \quad \text{and} \quad [\tilde{\xi}, \partial/\partial t] = 0\,.$$

The condition (2. 1) for $\tilde{\xi} = \sum_{a=0}^n \tilde{\xi}^a \dfrac{\partial}{\partial x^a} \in \boldsymbol{E}^1(\boldsymbol{R}^{n+1} - \{0\})$ is written as

$$(2.\ 1)' \qquad \sum_{a=6}^n \tilde{\xi}^a x^a = 0 \quad \text{and} \quad \sum_{b=0}^n x^b \frac{\partial \tilde{\xi}^a}{\partial x^b} = \tilde{\xi}^a \ (0 \leqq a \leqq n)\,,$$

since $r\tilde{\iota}(x, t) = e^t$ and $\tilde{\iota}_*(\partial/\partial t) = \sum_{a=0}^n x^a \dfrac{\partial}{\partial x^a}$, where $r^2 = \sum_{a=0}^n (x^a)^2$.

More generally, we can identify $\boldsymbol{E}^*(S^n)$ with the subalgebra

$$(2.\ 2) \qquad \boldsymbol{E}^*(S^n) := \{D \in \boldsymbol{E}^*(\boldsymbol{R}^{n+1} - \{0\}) \mid [D, r^2] = 0 \ \text{and}\ [D, \sum_{a=0}^n x^a \partial/\partial x^a] = 0\}$$

of $\boldsymbol{E}^*(\boldsymbol{R}^{n+1} - \{0\})$ in virtue of Lemma 2. 1. Each coefficient $\tilde{\xi}^{a_1 \cdots a_k}$ of $D \in \boldsymbol{E}^p(S^n)$

($p\geqq k\geqq 0$) is a homogeneous function of degree k with respect to the variables $x^0, \cdots, x^n$. This identification is transferred to the identification of the two algebras $\boldsymbol{S}^*(S^n)$ and $\tilde{\boldsymbol{S}}^*(S^n):=\tilde{\sigma}^*(\tilde{\boldsymbol{E}}^*(S^n))$, where $\tilde{\sigma}$ is the symbol operator of $\boldsymbol{E}^*(\boldsymbol{R}^{n+1}-\{0\})$. Let us identify $\boldsymbol{S}^*(S^n)$ with $\tilde{\boldsymbol{S}}^*(S^n)$ via the symbol operators σ^* of $\boldsymbol{E}^*(S^n)$ and $\tilde{\sigma}^*$ of $\tilde{\boldsymbol{E}}^*(S^n)$.

Namely,

$$(1/p!)\Xi^{a_1\cdots a_p}(\partial/\partial x^{a_1})\circ\cdots\circ(\partial/\partial x^{a_p})\in\boldsymbol{S}^p(\boldsymbol{R}^{n+1}-\{0\})$$

is in $\tilde{\boldsymbol{S}}^p(S^n)$ if and only if

$$\sum_{a=0}^{n}\Xi^{a_1\cdots a_{p-1}a}x^a=0 \quad\text{and}\quad \sum_{a=0}^{n}\frac{\partial\Xi^{a_1\cdots a_p}}{\partial x^a}x^a=p\Xi^{a_1\cdots a_p}. \tag{2.3}$$

From now on, the componentwise expression of $\xi\in\boldsymbol{S}^p(\boldsymbol{R}^{n+1}-\{0\})$ will be expressed as

$$\xi=\frac{1}{p!}\sum_{a_1,\cdots,a_p=0}^{n}\xi^{a_1\cdots a_p}y_{a_1}\cdots y_{a_p},$$

where $\xi^{a_1\cdots a_p}\in C^\infty(\boldsymbol{R}^{n+1}-\{0\})$. Here y_i ($0\leqq i\leqq n$) are regarded as current coordinates of $T^*(\boldsymbol{R}^{n+1}-\{0\})_x=\{\sum y_i dx^i_{|x}\}$ at $x=(x_0, \cdots, x_n)$.

That is, we regard a contravariant symmetric tensor field of degree p as a homogeneous polynomial of order p with respect to y_i's.

Denote by $\breve{\boldsymbol{E}}(\boldsymbol{R}^{n+1}\text{-}\{0\})$ the set of all differential operators of $2(n+1)$ variables $x^0, \cdots, x^n, y_0, \cdots, y_n$ the coefficients of which being C^∞ with respect to the variables x^i's on $\boldsymbol{R}^{n+1}\text{-}\{0\}$ and polynomials with respect to the variables y_j's. Elements of $\breve{\boldsymbol{E}}(\boldsymbol{R}^{n+1}\text{-}\{0\})$ are differential operators acting on symmetric tensor fields on $\boldsymbol{R}^{n+1}-\{0\}$.

Lemma 2.2. (1) *A symmetric tensor field* $\xi\in\boldsymbol{S}^p(\boldsymbol{R}^{n+1}-\{0\})$ *belongs to* $\tilde{\boldsymbol{S}}^p(S^n)$ *if and only if*

$$\sum_{a=0}^{n}x^a\partial\xi/\partial y_a=0 \quad \textit{and} \quad \sum_{a=0}^{n}x^a\partial\xi/\partial x^a=p\xi.$$

(2) *If* $\xi\in\boldsymbol{S}^p(\boldsymbol{R}^{n+1}-\{0\})$, *then* $\sum_{a=0}^{n}y_a\partial\xi/\partial y_a=p\xi$.

Proof. (1) is another expression of (2.3) in terms of differential operators belonging to $\breve{\boldsymbol{E}}(\boldsymbol{R}^{n+1}-\{0\})$. (2) is evident. Q.E.D.

DEFINITION 2.1. (1) Denote by $\breve{I}$ the left ideal in $\breve{\boldsymbol{E}}(\boldsymbol{R}^{n+1}-\{0\})$ generated by $\sum_{a=0}^{n}x^a\partial/\partial x^a-\sum_{a=0}^{n}y_a\partial/\partial y_a$ and $(1/r^2)\sum_{a=0}^{n}x^a\partial/\partial y_a$.

(2) Put

$$\widetilde{\boldsymbol{EO}}(S^n) :=$$

$$\{D \in \breve{\boldsymbol{E}}(\boldsymbol{R}^{n+1}-\{0\}) \mid [D, \sum_{a=0}^{n} x^a \partial/\partial x^a - \sum_{a=0}^{n} y_a \partial/\partial y_a] \subseteq \breve{I}$$

$$\text{and } [D, (1/r^2) \sum_{a=0}^{n} x^a \partial/\partial y_a] \subseteq \breve{I}\} .$$

Lemma 2.3. *$D \in \breve{\boldsymbol{E}}(\boldsymbol{R}^{n+1}-\{0\})$ preserves $\boldsymbol{S}^*(S^n)$ if and only if $D \in \widetilde{\boldsymbol{EO}}(S^n)$.*

Proof. This assertion is an immediate consequence of Lemma 2.2. Q.E.D.
Put

$$\breve{I}_0 := \widetilde{\boldsymbol{EO}}(S^n) \cap \breve{I} .$$

$\breve{I}_0$ is easily proved to be a two-sided ideal of $\widetilde{\boldsymbol{EO}}(S^n)$. So

$$\boldsymbol{EO}(S^n) := \widetilde{\boldsymbol{EO}}(S^n)/\breve{I}_0$$

can be regarded as an algebra of differential operators acting on symmetric tensor fields on S^n. Now we can regard the fundamental operators T^*, T, δ^*, δ, and d, as elements of $\boldsymbol{EO}(S^n)$. In the following, a representative in $\widetilde{\boldsymbol{EO}}(S^n)$ for each of these operators will be given explicitly.

Lemma 2.4. *The following operators $\tilde{T}$, $\tilde{T}^*$, $\tilde{\delta}$, $\tilde{\delta}^*$ and $\tilde{d}$ acting on $\boldsymbol{S}^*(S^n)$, give representatives for the fundamental operators:*

(1) $$\tilde{T}^* = (1/2) \sum_{a,b=0}^{n} (r^2\delta^{ab} - x^a x^b) y_a y_b \in \widetilde{\boldsymbol{EO}}(S^n) \cap \boldsymbol{S}^2(S^n) .$$

(2) $$\tilde{T} = (1/2r^2) \sum_{a=0}^{n} \partial^2/\partial y_a \partial y_a \in \widetilde{\boldsymbol{EO}}(S^n) .$$

(3) $$\tilde{\delta}^* = r^2 \sum_{a=0}^{n} y_a \partial/\partial x^a \in \boldsymbol{EO}(S^n) .$$

(4) $$\tilde{\delta} = -\sum_{a=0}^{n} (\partial^2/\partial x^a \partial y_a + r^{-2} \langle\!\langle x, y \rangle\!\rangle \partial^2/\partial y_a \partial y_a) \in \widetilde{\boldsymbol{EO}}(S^n) ,$$

where $\langle\!\langle x, y \rangle\!\rangle := \sum x^a y_a$.

Proof. The operators $\tilde{T}$, $\tilde{T}^*$, $\tilde{\delta}$, and $\tilde{\delta}^*$ in $\boldsymbol{E}^*(S^n)$ are introduced in [7] and proved to correspond to T, T^*, δ, and δ^*. They are expressed as above as elements of $\widetilde{\boldsymbol{EO}}(S^n)$, respectively. That a representative $\tilde{d}$ of the degree operator d is given by the Euler operator, can be observed immediately from the second equation of Lemma 2.3. Q.E.D.

DEFINITION 2.2. Define

(1) $$\kappa_{a,b} := x^a\partial/\partial x^b - x^b\partial/\partial x^a \in \boldsymbol{E}^1(S^n)$$

for $0 \leqq a,\ b \leqq n$ and $a \neq b$, and

(2) $$\check{\kappa}_{a,b} := x^a\partial/\partial x^b - x^b\partial/\partial x^a + y_a\partial/\partial y_b - y_b\partial/\partial y_a \in \widetilde{\boldsymbol{EO}}(S^n)$$

for $0 \leqq a,\ b \leqq n$ and $a \neq b$.

Lemma 2.5. *Between $\kappa_{a,b}$ and $\check{\kappa}_{a,b}$ we have the following relation:*

$$[\kappa_{a,b}, \xi] = \check{\kappa}_{a,b}(\xi)$$

for arbitrary $\xi \in \boldsymbol{S}^*(S^n, g_0)$, where the bracket product in the left-hand side is the one in $\boldsymbol{S}^*(S^n, g_0)$ defined in (1. 1).

Proof. This can be easily verified. Q.E.D.

Lemma 2.6.

$$(1)\ [\check{\kappa}_{a,b}, T] = 0\,, \qquad (2)\ [\check{\kappa}_{a,b}, T^*] = 0\,,$$
$$(3)\ [\check{\kappa}_{a,b}, \delta^*] = 0\,, \qquad (4)\ [\check{\kappa}_{a,b}, \delta] = 0\,.$$

Proof. In virtue of the Lemma 2.5 these can be easily verified. Q.E.D.

Denote by $\kappa^*_{a,b}$ the adjoint operators of $\kappa_{a,b}$ as elements of $\boldsymbol{E}^1(S^n)$, and $\check{\kappa}^*_{a,b}$ the adjoint operators of $\check{\kappa}_{a,b}$ with respect to the canonical linner product defined on $\boldsymbol{S}^*(S^n, g_0)$.

We can see easily

$$\kappa^*_{a,b} = -\kappa_{a,b} \quad \text{and} \quad \check{\kappa}^*_{a,b} = -\check{\kappa}_{a,b}\,.$$

Lemma 2.7. (1) *The Laplace Beltrami operator Δ_0 on (S^n, g_0) coincides with*

$$\sum_{a<b} \kappa^*_{a,b}\kappa_{a,b}$$

as a differential operator of order 2 *acting on $C^\infty(\boldsymbol{R}^{n+1} - \{0\})$.*

(2) *The Lichnerowicz operator on (S^n, g_0) coincides with*

$$\sum_{a<b} \check{\kappa}^*_{a,b}\check{\kappa}_{a,b} \in \widetilde{\boldsymbol{EO}}(S^n)\,.$$

Proof. (1) $\sum_{a<b}\kappa^*_{a,b}\kappa_{a,b}$ can be expanded as follows:

$$-r^2\sum_{a=0}^{n}\frac{\partial^2}{\partial x^a\partial x^a} + \sum_{a,b=0}^{n} x^a x^b\frac{\partial^2}{\partial x^a\partial x^b} + n\sum_{a=0}^{n} x^a\frac{\partial}{\partial x^a}\,.$$

This operator satisfies the following three conditions: (i) its symbol tensor field coincides with the (contravariant) metric tensor g_0; (ii) it is a self-adjoint linear differential operator; (iii) it annihilates the constant function 1. Such an operator must coincide with the Laplace Beltrami operator.

(2) Δ_0 on $\boldsymbol{S}^*(S^n, g_0)$ is known to be (cf. [7]):

$$[\delta, \delta^*]+2d(n+d-2)-8T^*T = -\sum_{a,b=0}^{n}(r^2\delta_{ab}-x^a x^b)\frac{\partial^2}{\partial x^a \partial x^b}$$

$$-2\langle\langle x, y\rangle\rangle \sum_{a=0}^{n}\frac{\partial^2}{\partial x^a \partial y_a}-2\langle\langle x, y\rangle\rangle^2 T+d(2n+d-3)-4T^*T\,,$$

where the operator d is as in Lemma 1.1 and the notation $\langle\langle\ ,\ \rangle\rangle$ is as in Lemma 2.4. On the other hand, $\sum_{a<b}\check{\kappa}^*_{a,b}\check{\kappa}_{a,b}$ is equal to

$$-1/2\sum_{a,b=0}^{n}(x^a\frac{\partial}{\partial x^b}-x^b\frac{\partial}{\partial x^a}+y_a\frac{\partial}{\partial y_b}-y_b\frac{\partial}{\partial y_a})^2 =$$

$$-\sum_{a,b=0}^{n}(r^2\delta^{ab}-x^a x^b)\frac{\partial^2}{\partial x^a \partial x^b}+(n-1)d-2\langle\langle x, y\rangle\rangle\sum_{a=0}^{n}\frac{\partial^2}{\partial x^a \partial y_a}$$

$$-2\langle\langle x, y\rangle\rangle^2 T+(n+d-2)d-4T^*T\,.$$

This coincides with the Lichnerowicz operator reviewed above. Q.E.D.

DEFINITION 2.3. Define an endomorphism S of degree -2 on the graded algebra $\boldsymbol{S}^*(S^n, g_0)$:

$$\boldsymbol{S}^p(S^n, g_0)\ni\xi\mapsto S\xi\in\boldsymbol{S}^{p-2}(S^n, g_0)\,,$$

by

(1) $$S := \Delta_0 T-\lambda_{p,1}T+(16/3)T^*T^2+(1/3)[\delta^*, T\delta] \text{ on } \boldsymbol{S}^p(S^n, g_0)\,,$$

where

$$\lambda_{p,k} := 2(p-k)n+2p^2-4(k+1)p+4k^2+6k\,.$$
$$(\text{Eventually } \lambda_{p,1} = 2(p-1)n+2p^2-8p+10.)$$

Moreover we define

(2) $$B^*_j := 2j^2T^*+(\delta^*)^2 \quad (j\geqq 1),$$

(3) $$A^*_k := (\prod_{i=1}^{k} B^*_{2i})T^k \ (k\geqq 1),\ A^*_0 = 1\,.$$

DEFINITION 2.4. (1) Denote the restriction of T^* to $\boldsymbol{K}^*(S^n, g_0)$ by T^*_0.

(2) Denote the image of T^*_0 by $Im\ T^*_0(\subseteq\boldsymbol{K}^*(S^n, g_0))$ and denote the orthogonal complement of $Im\ T^*_0$ in $\boldsymbol{K}^*(S^n, g_0)$ by

$$\boldsymbol{P}^*(S^n, g_0)\,.$$

We have

$$\boldsymbol{K}^*(S^n, g_0) = Im\ T_0^* \oplus \boldsymbol{P}^*(S^n, g_0)$$

$$\boldsymbol{P}^*(S^n, g_0) = \sum_{p=0}^{\infty} \boldsymbol{P}^p(S^n, g_0) \quad (\textit{direct sum})$$

with $\boldsymbol{P}^p(S^n, g_0) = \boldsymbol{P}^*(S^n, g_0) \cap \boldsymbol{K}^p(S^n, g_0)$.

Lemma 2.8. (1) *As an endomorphism of degree* -2 *on the graded algebra* $\boldsymbol{S}^*(S^n)$, *S preserves* $\boldsymbol{K}^*(S^n, g_0)$ *invariant.*

(2) A_k^* *also preserves* $\boldsymbol{K}^*(S^n, g_0)$ *invariant.*

For the proof c.f. [7] Lemma 4.3.

Denote the orthogonal projection:

$$\boldsymbol{K}^*(S^n, g_0) \rightarrow \boldsymbol{P}^*(S^n, g_0)$$

by Π_0. Π_0 can be proved to be commutative with Δ_0 (cf. [7]). Let

$$C_k^* := \Pi_0 A_k^*.$$

C_k^*'s satisfy

$$\Delta_0 C_k^* - \lambda_{p,k} C_k^* + \frac{1}{(k+1)(2k+1)} C_{k+1}^* = 0 \text{ on } \boldsymbol{K}^p(S^n, g_0).$$

(cf. [7] p. 69, Lemma 4.3 and (4. 10).)

Define

$$P_{p,k} := \frac{n+2p-4k-3}{k!(n+2p-2k-3)!!} \sum_{i=k}^{[p/2]} \left(\frac{(-1)^{i-k}(n+2p-2k-2i-5)!!}{(2i)!\,(i-k)!} \right) C_i^*,$$

where $p \geqq 2k \geqq 0$. Denote the image of $P_{p,k}$: $\boldsymbol{K}^p(S^n, g_0) \rightarrow \boldsymbol{P}^p(S^n, g_0)$ by $E_{p,k}$.

Theorem 2.1. (1) *For* $p \geqq 2k \geqq 0$ *we have*

$$\Delta_0 P_{p,k} = \lambda_{p,k} P_{p,k} \text{ on } \boldsymbol{K}^p(S^n, g_0),$$

where $\lambda_{p,k}$ *is as in Definition* 2.3 (1).

(2) *We have the two direct sums:*

$$\boldsymbol{K}^p(S^n, g_0) = \sum_{k=0}^{[p/2]} (T^*)^k (\boldsymbol{P}^{p-2k}(S^n, g_0)),$$

$$\boldsymbol{P}^p(S^n, g_0) = \sum_{k=0}^{[p/2]} E_{p,k},$$

which thus together with (1) *give the eigenspace decomposition of* Δ_0 *on* $\boldsymbol{K}^*(S^n, g_0)$.

(3) *Every* $E_{p,k}$ *is nonzero for* $n \geqq 3$.

For the proof of (1) refer to [7] lemma 4.4. For the proof of (2) and (3) cf. [7] p. 69 and pp. 75–76.

In the remainder of this section we assume $n+1\geqq 4$.

DEFINITION 2.5. Define

(1) $$D_{abcd} := \frac{1}{2^3} \sum_{e,f,g,h=0}^{n} \delta^{efgh}_{abcd} \breve{\kappa}_{ef} \breve{\kappa}_{gh} \quad \in \widetilde{\boldsymbol{EO}}(S^n),$$

(2) $$\Delta_1 := \frac{1}{4!} \sum_{a,b,c,d=0}^{n} D^*_{abcd} D_{abcd} \quad \in \widetilde{\boldsymbol{EO}}(S^n).$$

Notice that D_{abcd} is a self-adjoint operator.

Theorem 2.2.

(1) $$[\breve{\kappa}_{ab}, \Delta_0] = 0 .$$

(2) $$[\breve{\kappa}_{ab}, \Delta_1] = 0 .$$

Proof. These are obtained by direct calculations. Q.E.D.

Theorem 2.3.

(1) $$[T^*, \Delta_1] = 0 .$$

(2) $$[T, \Delta_1] = 0 .$$

(3) $$[\delta^*, \Delta_1] = 0 .$$

(4) $$[\delta, \Delta_1] = 0 .$$

Proof. From Lemma 2.7 we obtain easily. Q.E.D.

Note that thus Δ_1 preserves $\boldsymbol{P}^p(S^n, g_0)$ invariant.

3. The eigenspace decomposition of $\boldsymbol{K}^*(S^n, g_0)$.

In this section we assume $n+1\geqq 4$.

Theorem 3.1. *As a differential operator acting on* $\boldsymbol{S}^*(S^n)$

$$\Delta_1 = -4T^*T\Delta_0+d(n+d-3)\Delta_0-16(T^*)^2T^2-2T^*\delta^2-2(\delta^*)^2T-(n+2d-4)\delta^*\delta+4(2d-3)n+2d^2-10d+11)T^*T-d(d-1)(n+d-2)(n+d-3) .$$

Proof. From the definition of Δ_1 in **2** and Lemma 2.4 we can obtain the result by direct calculations. Q.E.D.

Lemma 3.1.

$$\Delta_1 = (d+1)(n+d-2)\{\Delta_0-d(n+d-1)\}-2\delta^*T\delta^*+ (n+d-2)\delta\delta^*-6T^*S \text{ on } \boldsymbol{S}^*(S^n, g_0) .$$

Proof. Apply T^* to the operator S in its definition reviewed in **2**. Then we can express T^*S in terms of fundamental operators, from which we can eliminate $-4T^*\Delta_0 T$ in virtue of Theorem 3.1. The resulting relation is the required one. Q.E.D.

Theorem 3.2. *We have*

$$\Delta_1 = \sum_{k=0}^{[p/2]} \mu_{p,k} P_{p,k} \text{ on } \boldsymbol{P}^p(S^n, g_0)\,,$$

where

$$\mu_{p,k} = (p-2k)(p+1)(n+p-2)(n+p-3-2k)\,.$$

Thus $\boldsymbol{P}^p(S^n, g_0) = \sum_{k=0}^{[p/2]} E_{p,k}$ *gives the eigenspace decomposition of* Δ_1 *restricted to* $\boldsymbol{P}^p(S^n, g_0)$.

Proof. Restricting Δ_1 on $E_{p,k}$ we have from Lemma 2.8 (1) and Lemma 3.1,

$$\begin{aligned}\Delta_1 P_{p,k} &= (p+1)(n+p-2)\{\Delta_0 P_{p,k} - p(n+p-1)P_{p,k}\} \\ &= (p+1)(n+p-2)\{2(p-k)n + 2p^2 - 4(k+1)p \\ &\quad + 4k^2 + 6k - p(n+p-1)\} P_{p,k}\end{aligned}$$

which coincides with the desired eigenvalue. Q.E.D.

Lemma 3.2.

(1) $$Ker\ T^k \cap \boldsymbol{P}^p(S^n, g_0) \subset \sum_{l=0}^{k-1} E_{p,l}\,,$$

where $p \geqq 2k \geqq 0$.

(2) *Let* $\xi \in \boldsymbol{P}^p(S^n, g_0)$ *be an eigen tensor field of* Δ_0. *Then* $\xi \in E_{p,k}$ *if and only if* $T^k\xi \neq 0$ *and* $T^{k+1}\xi = 0$.

Proof. (1) From the definition of the projection operator $P_{p,k}$ in **2** the assertion follows immediately. (2) follows from (1) directly. Q.E.D.

Theorem 3.3. *Let* $\xi \in E_{p,k}$ *and let* ξ *be a simultaneous eigen tensor field of* Δ_i $(i=0, 1)$. *Then* ξ *has the eigenvalues* $\lambda_{p,k}$ *and* $\mu_{p,k}$ *for* Δ_0 *and* Δ_1, *respectively.*

Proof. From the commutativity of T with Δ_0 and from that $T^k\xi \neq 0$, $T^k\xi$ is proved to be an eigen tensor field of Δ_0 with eigenvalue $\lambda_{p,k}$. On the other hand, as $(k+1)\delta T^k\xi = [\delta^*, T^{k+1}]\xi = \delta^* T^{k+1}\xi = 0$, we obtain

$$\begin{aligned}\Delta_1(T^k\xi) &= (p-2k)(n+p-2k-3)\Delta_0(T^k\xi) - \\ &(p-2k)(n+p-2k-3)(p-2k-1)(n+p-2k-2)T^k\xi\,.\end{aligned}$$

Thus $T^k\xi$ is a simultaneous eigen tensor field of Δ_0 and Δ_1. $\mu_{p,k}$ is regained as an eigenvalue if we substitute the $\lambda_{p,k}$ in place of Δ_0 in the right-hand side of the expression of $\Delta_1(T^k\xi)$. The commutativity of Δ_1 with T obtained in Theorem 2.3 implies that $\Delta_1\xi=\mu_{p,k}\xi$. Q.E.D.

4. Main theorems.

Denote by $\boldsymbol{W}_2(\boldsymbol{R}^{n+1})$ the manifold of all 2-frames in the Euclidean space $(\boldsymbol{R}^{n+1}, g_0)$ of dimension $n+1$ and call it a *Stiefel manifold*. The submanifold of $\boldsymbol{W}_2(\boldsymbol{R}^{n+1})$ defined as the set of orthonormal 2-frames is denoted by $\boldsymbol{V}_2(\boldsymbol{R}^{n+1})$ and we call it an *orthogonal Stiefel manifold*. $\boldsymbol{V}_2(\boldsymbol{R}^{n+1})$ is identified with the homogeneous space

$$SO(n+1)/SO(n-1)\,.$$

Denote by $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ the Grassmann manifold of all oriented 2-planes in $\boldsymbol{R}^{n+1}$ passing through the origin. As is well known $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ is identified with the homogeneous space

$$SO(n+1)/SO(n-1)\times SO(2)\,.$$

The orthogonal Stiefel manifold $\boldsymbol{V}_2(\boldsymbol{R}^{n+1})$ can be regarded as a principal bundle with the base space $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$ and the structural group $SO(2)$, where the projection π_v is defined canonically.

Theorem 4.1. (cf. [7]) *Let $P(M,G)$ be a principal bundle with a Lie group G as its fibre. Let $\boldsymbol{F}^G(P)$ be the subalgebra of $\boldsymbol{E}(P)$ which consists of G-invariant differential operators on P. Then there is an isomorphism: $\boldsymbol{E}^G(P)/\boldsymbol{J}\cong\boldsymbol{E}(M)$, where $\boldsymbol{J}$ is the two-sided ideal of $\boldsymbol{E}^G(P)$ generated over $\boldsymbol{R}$ by G-invariant vertical vector fields.*

Applying Theorem 4.1 to the principal bundle

$$\boldsymbol{V}_2(\boldsymbol{R}^{n+1}) \rightarrow \boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$$

with $SO(2)$ as fibre, we obtain

$$\boldsymbol{E}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))\cong\boldsymbol{E}^{SO(2)}(\boldsymbol{V}_2(\boldsymbol{R}^{n+1}))/\boldsymbol{J}\,, \tag{4. 1}$$

where $\boldsymbol{J}$ is the two-sided ideal in $\boldsymbol{E}^{SO(2)}(\boldsymbol{V}_2(\boldsymbol{R}^{n+1}))$ generated by $SO(2)$-invariant vertical vector fields. On the other hand, there is a polar decomposition of the Stiefel manifold $\boldsymbol{W}_2(\boldsymbol{R}^{n+1})$:

$$\boldsymbol{W}_2(\boldsymbol{R}^{n+1})\cong P_2\times \boldsymbol{V}_2(\boldsymbol{R}^{n+1})\,, \tag{4. 2}$$

where P_2 is the space of real positive definite 2×2 symmetric matrices. (cf. ([7]) Applying Lemma 2.1, the polar decomposition assures the existence of two subalgebras each one of which is the centralizer of the other in $\boldsymbol{E}(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$ and the second one is canonically isomorphic to $\boldsymbol{E}(\boldsymbol{V}_2(\boldsymbol{R}^{n+1}))$. Thus a differential operator $D\in\boldsymbol{E}(\boldsymbol{V}_2(\boldsymbol{R}^{n+1}))$ can be identified with a differential operator $D^\dagger\in\boldsymbol{E}(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$ satisfying

$$[D^\dagger, \rho^2_{\alpha\beta}]=0\,,$$

$$[D^\dagger, \frac{\partial}{\partial\rho^2_{\alpha\beta}}]=0\,,$$

where each of $\rho^2_{\alpha\beta}(1\leqq\alpha, \beta\leqq 2)$ denotes the (α, β)-component of the P_2-part ρ^2 in the polar decomposition (4. 2). The totality of such operators is designated as $\boldsymbol{E}^\dagger(\boldsymbol{V}_2(\boldsymbol{R}^{n+1}))$.

Connecting the isomorphism (4. 1) with the identification "†" we obtain representatives in $\boldsymbol{E}(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$ of elements in $\boldsymbol{E}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$.

Let $\mathbf{Geod}(S^n, g_0)$ be the space of oriented geodesics on (S^n, g_0) with respect to the canonical metric. We have a natural identification

$$\iota\colon \boldsymbol{SG}_{2,n-1}(\boldsymbol{R})\to\mathbf{Geod}(S^n, g_0)\,.$$

Let $\xi\in\boldsymbol{S}^p(S^n)$. Define $\xi^\wedge\in C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ by

$$\xi^\wedge(\Gamma)=\frac{1}{2\pi p!}\int_{\gamma=\iota(\Gamma)}\langle\xi, \dot\gamma^p\rangle\, ds\,,$$

where $\dot\gamma^p$ is the p-th symmetric power in $\boldsymbol{S}^*(\gamma)$ of the unit tangent vector field $\dot\gamma$ along $\gamma=\iota(\Gamma)$. The mapping defined by

$$\boldsymbol{S}^*(S^n, g_0)\ni\xi\mapsto\xi^\wedge\in C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$$

is called the *Radon transform.*

DEFINITION 4.1. (1) Denote by (P^{ab}) the system of normalized Plücker coordinates P^{ab} of the Grassmann manifold $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$, where a system of Plücker coordinates P^{ab} is said to be *normalized* if and only if

$$\sum_{a<b}(P^{ab})^2=1\,.$$

(2) Denoted by

$$\boldsymbol{R}(P^{ab}\colon n\geqq b>a\geqq 0)$$

the subalgebra of $C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ generated by the normalized Plücker coordinates.

Theorem 4.2. (cf. [7])

(1) $$(\kappa_{a,b})^\wedge=P^{ab},$$

where (P^{ab}) *are the system of normalized Plücker coordinates of the Grassmann manifold* $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$).

(2) *The image of the Radon transform restricted to* $\boldsymbol{K}^*(S^n, g_0)$ *is the uniformly*

dense subalgebra $\boldsymbol{R}(P^{ab}: n \geqq b > a \geqq 0)$ *of* $C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$.

(3) *The kernel of the Radon transform retricted to* $\boldsymbol{K}^*(S^n, g_0)$ *is the ideal generated by* $g_0/2-1$.

Corollary.

$$(\xi \circ \eta)^\wedge = \xi^\wedge \eta^\wedge$$

for $\xi \in \boldsymbol{K}^*(S^n, g_0)$ *and* $\eta \in \boldsymbol{S}^*(S^n)$.

Proof. The assertion follows from the fact that $\langle \xi, \dot{\gamma}^p \rangle$ is constant along $\gamma = \iota(\Gamma)$. Q.E.D.

From now on we often confuse an element of $\boldsymbol{E}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ with its representative in $\boldsymbol{E}(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$ as well as an element of $C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ with its representative in $C^\infty(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$. For an element $q=(q_1, q_2)$ of $\boldsymbol{W}_2(\boldsymbol{R}^{n+1})$ the components of q_1, q_2 will be denoted by q_1^a, $q_2^a (0 \leqq a \leqq n)$, respectively.

DEFINITION 4.2. Define

$$\hat{\kappa}_{a,b} := q_1^a \partial/\partial q_1^b - q_1^b \partial/\partial q_1^a + q_2^a \partial/\partial q_2^b - q_2^b \partial/\partial q_2^a \,,$$

where $0 \leqq a < b \leqq n$, and

$$\hat{\Delta_0} := \sum_{a<b} \hat{\kappa}^*_{a,b} \hat{\kappa}_{a,b} \,,$$

where $\hat{\kappa}^*_{a,b}$ is a representative via the connecting isomorphism of (4.1) with $\dagger$, of the adjoint operator of Killing vector field on $(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}), g_1)$ corresponding to $\hat{\kappa}_{a,b}$, where g_1 is the canonically normalized $SO(n+1)$-invariant Riemannian metric on $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$. Moreover we define

$$\hat{D}_{abcd} := \frac{2^3}{1} \sum_{e,f,g,h=0}^{n} \delta^{efgh}_{abcd} \hat{\kappa}_{e,f} \hat{\kappa}_{g,h} \,,$$

$$\hat{\Delta_1} := \frac{1}{4!} \sum_{a,b,c,d=0}^{n} (\hat{D}_{abcd})^* \hat{D}_{abcd} \,.$$

Notice that $\hat{\Delta_0}$ is a representative of the Laplace Beltrami operator on $(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}), g_1)$ and expressed explicitly as

$$-(\delta^{ab} - q^a_\alpha q^b_\beta (\rho^2)^{\alpha\beta}) \rho^2_{\gamma\delta} \partial^2/\partial q^a_\gamma \partial q^b_\delta + (n-1) q^a_\alpha \partial/\partial q^a_\alpha \,,$$

where the convention of dummy indices is adopted. (cf. [7] p. 64.)

Theorem 4.3. *The natural* $SO(n+1)$*-action on* $\boldsymbol{S}^*(S^n)$ *and* $C^\infty(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ *commutes with the Radon transform.*

Proof. This follows from the definition of the Radon transform. Q.E.D.

Corollary. *Let* $\eta^\wedge \in C^\infty(\boldsymbol{W}_2(\boldsymbol{R}^{n+1}))$ *be the image of* $\eta \in \boldsymbol{S}^*(S^n)$ *by the Radon transform. Then*

$$\hat{\kappa}_{a,b}\eta^\wedge = (\kappa_{a,b}\eta)^\wedge .$$

Proof. The asseition is an infinitesimal version of Theorem 4.3. Q.E.D.

Theorem 4.4. *Let* $\eta \in \boldsymbol{S}^*(S^n)$ *and let* $\eta^\wedge$ *be its Radon transform.*

(1) $$(\Delta_0\eta)^\wedge = \Delta_0^\wedge \eta^\wedge,$$

(2) $$(\Delta_1\eta)^\wedge = \Delta_1^\wedge \eta^\wedge .$$

Proof. By the definitions of Δ_0, Δ_1, $(\Delta_0)^\wedge$, and $(\Delta_1)^\wedge$, the assertion follows from the previous corollary. Q.E.D.

DEFINITION 4.3. Denote by $E_{p,k}^\wedge$ the image of $E_{p,k}$ by the Radon transform.

Theorem 4.5. *Assume* $n+1>4$.

(1) $\Delta_0^\wedge$ *and* $\Delta_1^\wedge$ *are generators of the algebra* $\boldsymbol{D}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ *of* $SO(n+1)$-*invariant differential operators on* $\boldsymbol{SG}_{2,n-1}(\boldsymbol{R})$.

(2) $$\Delta_{0\,|E_{p,k}^\wedge}^\wedge = \lambda_{p,k}\,\mathbf{1}_{p,k} \quad \text{and} \quad \Delta_{1\,|E_{p,k}^\wedge}^\wedge = \mu_{p,k}\,\mathbf{1}_{p,k}$$

where $\mathbf{1}_{p,k}$ *is the identity operator of* $E_{p,k}^\wedge$. *The totality of*

$$E_{p,k}^\wedge,\ p \geqq 2k \geqq 0\,,$$

gives all of the simultaneous eigenspaces of $\Delta_0^\wedge$ *and* $\Delta_1^\wedge$.

(3) *Each* $E_{p,k}^\wedge$ *is an* $SO(n+1)$-*irreducible subspace.*

Proof. Notice that $\boldsymbol{SG}_{2,2}(\boldsymbol{R})$ is known to be globally homothetic to $S^2\times S^2$ with the canonical metric. So we omit to detail of $\boldsymbol{D}(\boldsymbol{SG}_{2,2}(\boldsymbol{R}))$ as the reduced case.

(1) That $\Delta_0^\wedge$ and $\Delta_1^\wedge$ are invariant differential operators is a direct consequence of Theorem 4.4. It is known that the algebra $\boldsymbol{D}(\boldsymbol{SG}_{2,n-1}(\boldsymbol{R}))$ is generated by two operators of order 2 and 4, respectively. (cf. [3] Ch.II.) It remains to show that $\Delta_0^\wedge$ and $\Delta_1^\wedge$ are algebraically independent. Suppose that they are not so, then we can write

$$\Delta_1^\wedge = a(\Delta_0^\wedge)^2 + b\Delta_0^\wedge + c$$

for some constants a, b and c ($(a, b, c) \neq (0, 0, 0)$). On the other hand, in virtue of Theorem 3.2 we can eassily verify that $\Delta_1^\wedge$ acts trivially on $\sum_{k=0}^\infty E_{2k,k}^\wedge$ (*direct sum*). If we restict the action of $\Delta_1^\wedge$ to each $E_{2k,k}^\wedge$, we would obtain an polynomial equation of one variable of order at most four with an infinite number of solutions $k=1, 2, \cdots$, from which we can conclude $a=b=c=0$. This is a contradiction. Thus our assertion (1) is proved.

(2) follows from Theorem 4.4 and Theorem 4.2 (2). In order to prove (3) we need

Lemma 4.2. *If* $\lambda_{p,k}=\lambda_{p',k'}$ *and* $\mu_{p,k}=\mu_{p',k'}$ *then* $p=p'$ *and* $k=k'$.

Proof. We can easily verify

(1) $$\lambda_{p,k}>\lambda_{p,k+1}\,,$$

(2) $$\lambda_{p,k}<\lambda_{p+1,k}\,,$$

(3) $$\lambda_{p,k}<\lambda_{p+1,k+1}\,,$$

On the other hand, we can see

(4) $$\mu_{p,k}>\mu_{p,k+1}\,,$$

(5) $$\mu_{p,k}<\mu_{p+1,k}\,,$$

but

(6) $$\mu_{p,k}>\mu_{p+1,k+1}$$

in contrast with (3). From these the required property follows immediately.
Q.E.D.

Proof of Theorem 4.5. (3) In virtue of Lemma 4.2, $E^{\wedge}_{p,k}$ is known to be maximal in $\boldsymbol{R}(P^{ab}\colon n\geqq b>a\geqq 0)$ as the subspace of simultaneous eigenfunctions of the eigenvalues $\lambda_{p,k}$ and $\mu_{p,k}$ for $\Delta^{\wedge}_0$ and $\Delta^{\wedge}_1$, respectively. For the irreducibility of $E^{\wedge}_{p,k}$, we refer to [2] p. 401, Corollary 3.3. Q.E.D.

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