

## A PROBABILISTIC CONSTRUCTION OF THE HEAT KERNEL FOR THE $\bar{\partial}$ -NEUMANN PROBLEM ON A STRONGLY PSEUDOCONVEX SIEGEL DOMAIN

NAOMASA UEKI\*

(Received September 11, 1990)

### 1. Introduction

In this paper, we consider the heat equation for the  $\bar{\partial}$ -Neumann problem. This is an initial boundary value problem whose boundary condition includes *an imaginary directional differentiation*. In [7], Mallivain constructed the solution of the heat equation on a domain by using a method of singular perturbations and pointed out that a method related to the Fourier transform can be applied to this problem. On the other hand, a strongly pseudoconvex Siegel domain is well known as one of the most fundamental complex manifolds with boundary. This domain  $D$  can be regarded as the product of a Heisenberg group  $H_n$  and  $\mathbf{R}^+$ :  $D=H_n \times \mathbf{R}^+$ . In [4], Gaveau constructed explicitly the heat kernel for Kohn's Laplacian on  $H_n$ , by combining a probabilistic method and the Fourier transform. In this paper, by referring their works, we construct the heat kernel for the  $\bar{\partial}$ -Neumann problem on the Siegel domain  $D$  explicitly in terms of the theory of generalized Wiener functionals by Watanabe [13]. For the heat kernel on this domain, Stanton gave an explicit formula in the  $(0, q)$ -form case ( $q > 0$ ), by using methods of the partial differential equations [10], [11]. We here consider the general  $(p, q)$ -form case. The main part of our discussion is the proof of well-definedness of the heat kernel. In [12], our main results (Theorems 2.1 and 2.2 below) were announced.

We briefly explain our methods. The equation we consider is the following:

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} F(t, X) = -\square F(t, X), & t > 0, X \in D, \\ \lim_{t \rightarrow 0} F(t, X) = f(X) \in \Lambda^{p, q}(D), & \text{uniformly on } \bar{D}, \\ PF(t, X) = Q \left( \frac{\partial}{\partial r} - i \frac{\partial}{\partial u} \right) F(t, X) = 0 & \text{on } bD, \end{cases}$$

---

\*This work was supported by Yukawa Foundation and Grant-in-Aid for Scientific Research, The Ministry of Education, Science and Culture.

where

$$\square = -\frac{1}{4} \sum_{j=1}^{2n} X_j^2 - \frac{1}{2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial r^2} \right) + i(n-2C) \frac{\partial}{\partial u} - \sum_{j=1}^{2n} \frac{A_j}{\sqrt{2}} X_j - B.$$

For the notations see Section 2 below. Except for the term  $i\partial/\partial u$  in the boundary condition this equation has the same form as the  $d$ -Neumann boundary condition case that Airault [1], Ikeda and Watanabe [5] (see also [9]) considered. Hence, as in the  $d$ -Neumann case, we may expect the heat kernel of (1.1) can be expressed as

$$(1.2) \quad h_t(X, X') = E[m(t) M(t) K(t) \delta_{X'}(X(t), U(t) - i\phi(t), R(t))]$$

where  $(X(t), U(t), R(t))$  is the diffusion process generated by  $1/4 \sum_{j=1}^{2n} X_j^2 + 1/2 (\partial^2/\partial u^2 + \partial^2/\partial r^2)$ ,  $\phi(t)$  is the local time of the diffusion  $\{R(t), t \geq 0\}$  at 0 and  $\delta_{X'}$  is the Dirac  $\delta$  function on  $D$ .  $m(t)$  and  $M(t)$  are operator valued functionals for  $-i(n-2C) \partial/\partial u$  and  $\sum_{j=1}^{2n} A_j/\sqrt{2} X_j + B$ , respectively (see (2.12) and (3.4)).  $K(t) = I - PI_{\{\min_{0 \leq s \leq t} R(s) = 0\}}$  is the functional for the boundary condition. However the process  $(X(t), U(t) - i\phi(t), R(t))$  does not lie on  $D$ . Hence we do not know how to give a mathematical sense to the right hand side of (1.2).

Now we note that the metric, the differential structure and the complex structure of our case are *constant in the direction that we must consider the imaginary directional differentiation*. By using these facts, we can use the independence of the above processes and Fourier transform effectively. Then we can formally rewrite the expression (1.2) as

$$(1.3) \quad \begin{aligned} & h_t((x, u, r), (x', u', r')) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{i\lambda(u'-u)} E[e^{-i\lambda u(t)} m(t)] \\ & \quad \times \theta_t^\lambda(x, x') E[e^{-\lambda\phi(t)} K(t) | R(t) = r'] r_t(r, r') \end{aligned}$$

where

$$\begin{aligned} u(t) &= U(t) - u - S_x(t), \\ \theta_t^\lambda(x, x') &= E[e^{-i\lambda S_x(t)} M(t) \delta_{x'}(X(t))], \\ S_x(t) &= \sum_{j=1}^n 2 \int_0^t (X^{n+j}(s) \circ dX^j(s) - X^j(s) \circ dX^{n+j}(s)) \end{aligned}$$

and  $r_t(r, r')$  is the transition probability function of  $R(t)$ . We prove that this is the heat kernel for (1.1). In  $(0, q)$ -form case,  $M(t) = I$  and  $\theta_t^\lambda(x, x')$  is rewritten more explicitly ((2.18) below). However, in  $(p, q)$ -form case, we can not rewrite  $\theta_t^\lambda(x, x')$  explicitly as in  $(0, q)$ -form case. Then it is difficult to show the exponential decay of  $\theta_t^\lambda(x, x')$  in  $\lambda$ , which is necessary to show the well-definedness of the right hand side of (1.3). Now we note that  $\theta_t^\lambda(x, x')$  is, as a function of  $(t, x, x')$ , the heat kernel for the operator  $\square_t^\lambda$  defined by (5.26) below.

By using the semigroup property of this heat kernel, we can prove the exponential decay of  $\theta_i^\lambda(x, x')$  which is sufficient to prove the convergence of the right hand side of (1.3). This decay of  $\theta_i^\lambda(x, x')$  is an interesting property of the oscillatory integral over the Wiener space. Furthermore, by using the Malliavin calculus, we can prove the smoothness of (1.3). We note that we need not the partial Malliavin calculus used for the boundary value problem (e.g., [9]), because of the independence of the processes.

The organization of this paper is as follows. In Section 2, we first formulate the problem by following Stanton [10] and then state our main theorem. In Section 3, we give a probabilistic explanation of our formula (2.15) of the heat kernel; we give a few lemmas by which we expect the formula (2.15). In Section 4, we prove Theorem 2.2 below. This theorem plays a crucial role in this paper. In Section 5, we prove our main theorem, Theorem 2.1. In Section 6, we consider the short time asymptotic behavior of the heat kernel on the diagonal. Finally, in Section 7, we generalize the above results to certain domains with nondegenerate indefinite Levi forms.

## 2. Preliminaries and Main theorems

In this section, first of all, following Stanton [10], we review the formulation of the heat equation on a strongly pseudoconvex Siegel domain. The strongly pseudoconvex Siegel domain is defined by

$$D = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}, \operatorname{Im} w > |z|^2\}.$$

We consider the Hermitian metric for which  $\{Z_1, Z_2, \dots, Z_{n+1}\}$  is an orthonormal basis of  $T^{(1,0)}(D)$ , where

$$(2.1) \quad Z_j = \frac{\partial}{\partial z^j} + 2i\bar{z}^j \frac{\partial}{\partial w}, \quad j = 1, 2, \dots, n, \quad Z_{n+1} = i\sqrt{2} \frac{\partial}{\partial w}.$$

We note that the volume form for the metric is the restriction of  $2^n$  times the standard Euclidean volume element on  $\mathbf{C}^{n+1}$ . The differential forms given by

$$(2.2) \quad \omega^j = dz^j, \quad \omega^{n+1} = \frac{1}{i\sqrt{2}} (dw - 2i \sum \bar{z}^j dz^j)$$

form the dual basis for  $T_{(1,0)}^*(D)$ .

We now introduce several spaces of differential forms and several operators. Let  $\Lambda_0^{p,q}(D)$  be the space of  $C^\infty(p, q)$  forms with compact support in  $D$ . Let  $S^{p,q}(\bar{D})$  be the space of  $(p, q)$  forms on  $D$  whose coefficients relative to  $\omega^I \wedge \bar{\omega}^J$  can be extended to rapidly decreasing functions on  $\mathbf{C}^{n+1}$  and  $L_2^{p,q}(D)$  that of square integrable  $(p, q)$  forms on  $D$ . Then  $\bar{\partial}$  maps  $S^{p,q}(\bar{D})$  to  $S^{p,q+1}(\bar{D})$  and has a smallest closed extension to  $L_2^{p,q}(D)$ . For simplicity we also denote by  $\bar{\partial}$  its closed extension. Let  $\bar{\partial}^*$  be the adjoint of  $\bar{\partial}$  on  $L_2^{(p,q)}(D)$ . We define the  $\bar{\partial}$ -Laplacian

□ by

$$(2.3) \quad \begin{aligned} \square &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}, \\ \text{Dom } \square &= \{f \in L^{p,q}(D); f \in \text{Dom } \bar{\partial} \cap \text{Dom } \bar{\partial}^*, \\ &\quad \bar{\partial}f \in \text{Dom } \bar{\partial}^*, \bar{\partial}^*f \in \text{Dom } \bar{\partial}\}. \end{aligned}$$

Then the operator □ is a positive selfadjoint operator on  $L^{p,q}(D)$  (Folland-Kohn [3] Proposition 1.3.8).

To express the above operators explicitly, we prepare several notations. For  $\omega \in T^*D \otimes \mathcal{C}$ , let  $\text{ext}(\omega)$  be the exterior multiplication, i.e.,

$$\text{ext}(\omega) \eta = \omega \wedge \eta \quad \text{for } \eta \in \Lambda(T^*D \otimes \mathcal{C}),$$

and  $\text{int}(\bar{\omega})$  be interior multiplication, i.e., the dual operator of  $\text{ext}(\omega)$ . This  $\text{int}(\omega)$  is complex linear in  $\omega$ . We note that in some literature, e.g., Folland-Kohn [3],  $\text{int}(\omega)$  is conjugate linear in  $\omega$ . The condition for  $f \in S^{p,q}(\bar{D})$  to be in  $\text{Dom } \bar{\partial}^*$  is

$$\text{int}(\omega^{n+1})f \upharpoonright_{bD} = 0,$$

where  $bD$  is the boundary of the domain  $D$ . Thus the condition for  $f \in S^{p,q}(\bar{D})$  to be in  $\text{Dom } \square$  are

$$(2.4) \quad \text{int}(\omega^{n+1})f \upharpoonright_{bD} = 0, \quad \text{int}(\omega^{n+1})\bar{\partial}f \upharpoonright_{bD} = 0.$$

These mean the  $\bar{\partial}$ -Neumann boundary conditions. For a differential operator  $A$ , we define  $\underline{A}$  to be the operator acting only on the coefficients of  $\omega^I \wedge \bar{\omega}^J$ , i.e.,

$$\underline{A}(\sum f_{I\bar{J}} \omega^I \wedge \bar{\omega}^J) = \sum (Af_{I\bar{J}}) \omega^I \wedge \bar{\omega}^J.$$

Since  $\bar{\partial}\omega^j = \bar{\partial}\bar{\omega}^j = \bar{\partial}\omega^{n+1} = 0$  for  $j=1, 2, \dots, n$ , and

$$\bar{\partial}\omega^{n+1} = \sqrt{2} \sum_{j=1}^n \omega^j \wedge \bar{\omega}^j,$$

in terms of the above notations,  $\bar{\partial}$  and  $\bar{\partial}^*$  can be represented as follows:

$$(2.5) \quad \begin{aligned} \bar{\partial} &= \sum_{j=1}^{n+1} \text{ext}(\bar{\omega}^j) \bar{Z}_j + \sqrt{2} \sum_{j=1}^n \text{ext}(\omega^j) \text{ext}(\bar{\omega}^j) \text{int}(\bar{\omega}^{n+1}), \\ \bar{\partial}^* &= - \sum_{j=1}^{n+1} \text{int}(\omega^j) Z_j + \sqrt{2} \sum_{j=1}^n \text{ext}(\omega^{n+1}) \text{int}(\omega^j) \text{int}(\bar{\omega}^j), \end{aligned}$$

on  $S^{p,q}(\bar{D})$ .

We use as coordinates on  $D$   $(z, u, r)$  where  $u = \text{Re } w$  and  $r = \text{Im } w - |z|^2$  for  $w \in \mathcal{C}$ . Then we can regard  $D$  as the product of the boundary  $bD$  and  $\mathbf{R}^+$ . We identify  $bD$  with the Heisenberg group  $H_n$ . The group law on  $H_n$  is

$$(2.6) \quad \begin{aligned} (z', u') (z, u) &= (z' + z, u' + u + 2 \sum_{j=1}^n (x'^{n+j} x^j - x'^j x^{n+j})) \\ &\quad \text{for } (z', u'), (z, u) \in H_n, \end{aligned}$$

where  $z^j = x^j + ix^{n+j}$ . The metric we gave is the product metric of an invariant metric on  $H_n$  and the standard metric on  $\mathbb{R}^+$ . In terms of these coordinates,

$$(2.7) \quad Z_j = \frac{\partial}{\partial z^j} + i\bar{z}^j \frac{\partial}{\partial u}, \quad j = 1, 2, \dots, n, \quad Z_{n+1} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial r} + i \frac{\partial}{\partial u} \right).$$

Then we have

$$Z_j = 1/2 (X_j - iX_{n+j})$$

where

$$(2.8) \quad X_j = \frac{\partial}{\partial x^j} + 2x^{n+j} \frac{\partial}{\partial u}, \quad X_{n+j} = \frac{\partial}{\partial x^{n+j}} - 2x^j \frac{\partial}{\partial u}.$$

We also define the projections  $P$  and  $Q$  by

$$P = \text{ext}(\bar{\omega}^{n+1}) \text{int}(\omega^{n+1}), \quad Q = \text{int}(\omega^{n+1}) \text{ext}(\bar{\omega}^{n+1}).$$

Then the  $\bar{\partial}$ -Neumann boundary condition (2.6) is rewritten as follows:

$$(2.9) \quad Pf \upharpoonright_{\partial D} = Q \left( \frac{\partial}{\partial r} - i \frac{\partial}{\partial u} \right) f \upharpoonright_{\partial D} = 0$$

for  $f \in \mathcal{S}^{p,q}(\bar{D})$ . Now we see that this condition is similar to the absolute boundary condition except the term  $i\partial/\partial u$  (cf. Ikeda-Watanabe [5]).

The operator  $\square$  is expressed on  $\mathcal{S}^{p,q}(\bar{D})$  as follows:

$$(2.10) \quad \square = -\frac{1}{4} \sum_{j=1}^{2n} X_j^2 - \frac{1}{2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial r^2} \right) + i(n-2C) \frac{\partial}{\partial u} - \sum_{j=1}^{2n} \frac{A_j}{\sqrt{2}} X_j - B$$

where

$$\begin{aligned} C &= qQ + (q-1)P, \\ A_j &= \text{ext}(\omega^{n+1}) \text{int}(\bar{\omega}^j) - \text{ext}(\omega^j) \text{int}(\bar{\omega}^{n+1}), \quad j = 1, 2, \dots, n, \\ A_{n+j} &= i(\text{ext}(\omega^{n+1}) \text{int}(\bar{\omega}^j) + \text{ext}(\omega^j) \text{int}(\bar{\omega}^{n+1})), \quad j = 1, 2, \dots, n, \\ B &= -2 \sum_{j,k=1}^n \text{ext}(\omega^j) \text{ext}(\bar{\omega}^j) \text{int}(\omega^k) \text{int}(\bar{\omega}^k) \\ &\quad - 2 \sum_{j=1}^n (\text{int}(\omega^j) \text{ext}(\bar{\omega}^j) - \text{ext}(\omega^j) \text{int}(\bar{\omega}^j)) \text{ext}(\omega^{n+1}) \text{int}(\bar{\omega}^{n+1}). \end{aligned}$$

REMARK. 2.1. If  $p=0$ ,  $A_j = A_{n+j} = B = 0$  and  $\square$  acts diagonally on  $(0, q)$  forms. If  $p \neq 0$ ,  $\square$  does not act on diagonally but  $A_j$ ,  $A_{n+j}$  and  $B$  commutes with  $P$  and  $Q$ . Thus  $\square$  preserves the orthogonal decomposition  $\Lambda^{p,q}(T^*D) = \text{Ran } P \oplus \text{Ran } Q$ .

A fundamental solution of the heat equation is a one parameter family of

bounded operators  $H_t, t > 0$ , on  $L_2^{p,q}(D)$  such that for  $f \in \Lambda_0^{p,q}(D)$ ,

- (i) for  $t \in [0, T]$ ,  $\|H_t f\| \leq C$  where  $C$  is independent of  $t$  but may depend on  $T$  and  $f$ ;
- (ii)  $H_t f$  is differentiable in  $t$ ;
- (2.11) (iii)  $H_t f \in \text{Dom } \square$ ;
- (iv)  $(\partial/\partial t + \square) H_t f = 0$ ;
- (v)  $H_t f \rightarrow f$  in  $L_2^{p,q}(D)$  as  $t \rightarrow 0$ ;
- (iv)  $\square H_t f = H_t \square f$ .

Then we obtain the following results (Stanton [10]):

**Proposition 2.1.** *There is a unique fundamental solution  $H_t$ . Furthermore,  $H_t$  is the semigroup generated by  $-\square$ .*

Stanton proved this proposition in  $(0, q)$ -form case. Her proof is applicable to general  $(p, q)$ -form case.

Before we state our main theorem, we prepare a few notations. Let  $(W_0^{2n}, P)$  be a  $2n$ -dimensional Wiener space, i.e.,

$$W_0^{2n} = \{(x^1(\cdot), x^2(\cdot), \dots, x^{2n}(\cdot)): [0, \infty) \rightarrow \mathbf{R}^{2n}: \text{continuous, } x(0) = 0\}$$

and  $P$  is the Wiener measure. For any  $\varepsilon > 0$  and  $m \in \mathbf{N}$ , we define the End  $(\Lambda^{p,q}(T^* D))$ -valued process  $M_m^\varepsilon(t)$  by the solution of the SDE

$$(2.12) \quad \begin{cases} dM_m^\varepsilon(t) = M_m^\varepsilon(t) d\Xi_m^\varepsilon(t), \\ M_m^\varepsilon(0) = I \end{cases}$$

where

$$\Xi_m^\varepsilon(t) = \sum_{j=1}^{2n} \frac{\varepsilon}{\sqrt{m}} A_j x^j(t) + \frac{\varepsilon^2}{m} Bt.$$

When  $m=1$ , we omit the subscript  $m$ :  $\Xi_1^\varepsilon(t) =: \Xi^\varepsilon(t), M_1^\varepsilon(t) =: M^\varepsilon(t)$ . When  $m=\varepsilon=1$ , we also omit the superscript  $\varepsilon$ :  $\Xi_1^1(t) =: \Xi(t), M_1^1(t) =: M(t)$ . For any  $\alpha \in \mathbf{R}, \lambda \in \mathbf{R}$  and  $(x, u) \in H_n$ ,

$$(2.13) \quad f_i^\alpha(x, u, \lambda) = \frac{1}{2\pi} \exp(i\lambda u - t\lambda\alpha) E \left[ e^{-i\lambda S_0(t)} M(t) \delta_x \left( \frac{x(t)}{\sqrt{2}} \right) \right]$$

where

$$(2.14) \quad S_0(t) = \sum_{j=1}^n \int_0^t (x^{n+j}(s) \circ dx^j(s) - x^j(s) \circ dx^{n+j}(s)).$$

Here,  $\circ dx^j(t)$  means the Stratonovich differential. The right hand side of (2.13)

is well defined as a generalized expectation of a generalized Wiener functional [13]. Then our main theorem is stated as follows:

**Theorem 2.1.** *The unique fundamental solution  $H_t$  of the heat equation for the  $\bar{\partial}$ -Neumann problem on  $(p, q)$ -forms on  $D$  has the following smooth kernel:*

$$\begin{aligned}
 (2.15) \quad & h_t((x, u, r), (x', u', r')) \\
 &= P_t^{n-2q+2}((x, u)^{-1}(x', u')) e_t^-(r, r') P \\
 &\quad + P_t^{p-2q}((x, u)^{-1}(x', u')) e_t^+(r, r') Q \\
 &\quad + Q_t^{n-2q}((x, u)^{-1}(x', u'), r+r') Q
 \end{aligned}$$

where

$$\begin{aligned}
 (2.16) \quad & P_t^\alpha(x, u) = \int_{-\infty}^\infty d\lambda f_t^\alpha(x, u, \lambda) \exp\left(-\frac{t}{2} \lambda^2\right), \\
 & Q_t^\lambda((x, u), r) = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty d\lambda \cdot \lambda f_t^\alpha(x, u, \lambda) e^{r\lambda} \int_{\sqrt{i/2}(r/t+\lambda)}^\infty d\mu e^{-\mu^2}
 \end{aligned}$$

and

$$e_t^\pm(r, r') = \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left(-\frac{(r-r')^2}{2t}\right) \pm \exp\left(-\frac{(r+r')^2}{2t}\right) \right\}.$$

REMARK 2.2. In the case of  $p=0$ ,  $M(t)=I$  and so

$$\begin{aligned}
 (2.17) \quad & f_t^\alpha(x, u, \lambda) = \frac{1}{2\pi} \exp(i\lambda u - t\lambda\alpha) E \left[ e^{-i\lambda S_0(t)} \delta_x \left( \frac{x(t)}{\sqrt{2}} \right) \right] I \\
 &= f_t^{0,\alpha}(x, u, \lambda) I
 \end{aligned}$$

where

$$(2.18) \quad f_t^{0,\alpha}(x, u, \lambda) = \frac{1}{2\pi^{n+1}} \left( \frac{\lambda}{\sinh(t\lambda)} \right)^n \exp(i\lambda u - t\lambda\alpha - |x|^2 \lambda \coth(t))$$

([5]). Now we see that the  $h_t$  in (2.15) coincides with the heat kernel obtained by Stanton [10].

In our proof of Theorem 2.1, the most essential part is to show the convergence of the  $\lambda$ -integration in the right hand side of (2.16). For this purpose, we show Theorem 2.2 below. Before we state the theorem, we prepare a few notations. We use the multi-index notation:  $\alpha=(\alpha_1, \alpha_2, \dots, \alpha_{2n}) \in \mathbf{Z}_+^{2n}$ ,  $|\alpha|=\alpha_1+\alpha_2+\dots+\alpha_{2n}$ ,  $a^\alpha = a_1^{\alpha_1} a_2^{\alpha_2} \dots a_{2n}^{\alpha_{2n}}$  for  $a=(a_1, a_2, \dots, a_{2n}) \in \mathbf{R}^{2n}$  and  $\partial_x^\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x^2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x^{2n}}\right)^{\alpha_{2n}}$ . For  $A = \sum_{I, I' \in \mathcal{I}(p), J, J' \in \mathcal{I}(q)} A_{II'JJ'} \text{ext}(\omega^I \wedge \bar{\omega}^J) \text{int}(\bar{\omega}^I \wedge \omega^{J'}) \in \text{End}(\Lambda^{p,q}(T^*D))$ , we set

$$\|A\| := \left( \sum_{I, I' \in \mathcal{I}(p), J, J' \in \mathcal{I}(q)} |A_{II'JJ'}|^2 \right)^{1/2}$$

where  $\mathcal{I}(p) = \{I = (i_1, i_2, \dots, i_p); 1 \leq i_1 < i_2 < \dots < i_p \leq 2(n+1)\}$ , i.e.,  $\|A\|$  denotes the Hilbert-Schmidt norm. Then we obtain the following:

**Theorem 2.2.** (i) For any  $\varepsilon_0 > 0, \alpha, \beta \in \mathbf{Z}_+^{2n}$  and  $k \in \mathbf{Z}_+$ ,

$$(2.19) \quad \overline{\lim}_{\lambda \rightarrow \pm\infty} \frac{1}{|\lambda|} \log \sup_{\substack{x \in \mathbf{R}^{2n} \\ 0 < \varepsilon \leq \varepsilon_0}} \|x^\beta \partial_x^\alpha E [S_0(1)^k e^{-i\lambda S_0(1)} M^\varepsilon(1) \delta_x \left(\frac{x(1)}{\sqrt{2}}\right)]\| \leq -n.$$

(ii) For any  $\varepsilon_0 > 0, \alpha, \beta \in \mathbf{Z}_+^{2n}$  and  $k \in \mathbf{Z}_+$ ,

$$(2.20) \quad \overline{\lim}_{\lambda \rightarrow \pm\infty} \frac{1}{|\lambda|} \log \sup_{\substack{x \in \mathbf{R}^{2n} \\ 0 < \varepsilon \leq \varepsilon_0}} \|x^\beta \partial_x^\alpha E [S_0(1)^k e^{-i\lambda S_0(1)} (M^\varepsilon(1) - I) \delta_x \left(\frac{x(1)}{\sqrt{2}}\right)]\| / \varepsilon \leq -n.$$

By using scaling property of Brownian motions, we can easily prove the following:

**Corollary.** For any  $\alpha, \beta \in \mathbf{Z}_+^{2n}$  and  $k \in \mathbf{Z}_+$ ,

$$(2.21) \quad \overline{\lim}_{\lambda \rightarrow \pm\infty} \sup_{t_0 \leq t \leq t_1} \frac{1}{t|\lambda|} \log \sup_{x \in \mathbf{R}^{2n}} \|x^\beta \partial_x^\alpha E [S_0(t)^k e^{-i\lambda S_0(t)} M(t) \delta_x \left(\frac{x(t)}{\sqrt{2}}\right)]\| \leq -n.$$

### 3. A formal construction of the heat kernel

The purpose of this section is to show that  $h_t$  in (2.15) is a candidate for the integral kernel of  $H_t$ . We first consider the initial value problem (1.1). Let  $(W_0^{2(n+1)}, P)$  be a  $2(n+1)$ -dimensional Wiener space, i.e.,

$$W_0^{2(n+1)} = \{(x^1(\cdot), x^2(\cdot), \dots, x^{2n}(\cdot), u(\cdot), r(\cdot)): [0, \infty) \rightarrow \mathbf{R}^{2(n+1)}: \text{continuous}, (x(0), u(0), r(0)) = 0\}$$

and  $P$  is the Wiener measure. Then the diffusion process  $Z(t) = (X^1(t), X^2(t), \dots, X^{2n}(t), U'(t), R'(t))$  defined by

$$(3.1) \quad \begin{aligned} X^j(t) &= x^j + \frac{x^j(t)}{\sqrt{2}}, \quad j = 1, 2, \dots, 2n, \\ U'(t) &= u + u(t) + S_x(t), \\ R'(t) &= r + r(t), \\ S_x(t) &= \sum_{j=1}^n \int_0^t \sqrt{2} (X^{n+j}(s) \circ dx^j(s) - X^j(s) \circ dx^{n+j}(s)) \end{aligned}$$

is a solution of the following SDE on  $H_n \times \mathbf{R}$ :



$$(3.2) \quad \begin{cases} dZ(t) = \frac{1}{\sqrt{2}} \sum_{j=1}^{2n} X_j(Z(t)) \circ dx^j(t) + \frac{\partial}{\partial u} (Z(t)) \circ du(t) + \frac{\partial}{\partial r} (Z(t)) \circ dr(t), \\ Z(0) = (x^1, x^2, \dots, x^{2n}, u, r). \end{cases}$$

$R(t) := |r+r(t)|$  has the following decomposition by Skolohod:

$$(3.3) \quad R(t) = r + B(t) + \phi(t)$$

where

$$B(t) = \int_0^t \text{sgn}(r+r(s)) dx(s)$$

and

$$\phi(t) = \lim_{\eta \searrow 0} \frac{1}{2\eta} \int_0^t I_{(-\eta, \eta)}(r+r(s)) ds$$

is the local time of  $R(t)$  at 0. We define the  $\text{End}(\Lambda^{p,q}(T^*D))$ -valued process  $m(t)$  by the solution of the SDE

$$(3.4) \quad \begin{cases} dm(t) = m(t) \{-i(n-2C)\} du(t), \\ m(0) = I \end{cases}$$

and define the  $\text{End}(\Lambda^{p,q}(T^*D))$ -valued process  $K(t)$  by

$$(3.5) \quad K(t) = I_{\{\sigma \leq t\}} Q + I_{\{\sigma > t\}}$$

where  $\sigma = \inf \{s: R(s) = 0\}$ . Then we have the following lemma:

**Lemma 3.1.** *Let  $G(t, X)$  be a smooth form on  $[0, \infty) \times \bar{D}$  such that  $G(t, \cdot) \in S^{p,q}(\bar{D})$  for each  $t \geq 0$ , uniformly with respect to  $t \in [0, T]$  for any  $T > 0$ , i.e., any seminorms of  $G(t, \cdot)$  are bounded in  $t \in [0, T]$  for any  $T > 0$ . Then it holds that*

$$(3.6) \quad \begin{aligned} & m(t) M(t) K(t) e^{-i\lambda U(t)} \int_{-\infty}^{\infty} e^{i\lambda \zeta} G(s, X(t), \zeta, R(t)) d\zeta \\ & - K(0) e^{-i\lambda u} \int_{-\infty}^{\infty} e^{i\lambda \zeta} G(0, x, \zeta, r) d\zeta \\ & = \text{martingale} + \int_0^t m(s) M(s) K(s) e^{-i\lambda U(s)} \\ & \quad \times \int_{-\infty}^{\infty} e^{i\lambda \zeta} \left\{ \frac{\partial G}{\partial s} - \square G \right\} (s, X(s), \zeta, R(s)) d\zeta ds \\ & \quad + \int_0^t m(s) M(s) K(s) e^{-i\lambda U(s)} \\ & \quad \times \int_{-\infty}^{\infty} e^{i\lambda \zeta} \left\{ \frac{\partial G}{\partial r} - i \frac{\partial G}{\partial u} \right\} (s, X(s), \zeta, R(s)) d\zeta d\phi(s), \end{aligned}$$

where  $X(t) = (X^1(t), X^2(t), \dots, X^{2n}(t))$  and  $U(t) = U'(t) - i\phi(t)$ .

Proof. By using Itô's formula and noting that  $m(t)$ ,  $M(t)$  and  $K(t)$  are commutative, we have

$$\begin{aligned}
 & m(t) M(t) K(t) e^{-i\lambda U(t)} e^{i\lambda \zeta} G(t, X(t), \zeta, R(t)) \\
 & - m(0) M(0) K(0) e^{-i\lambda U(0)} e^{i\lambda \zeta} G(0, X(0), \zeta, R(0)) \\
 & = \int_0^t m(s) M(s) K(s) e^{-i\lambda U(s)} e^{i\lambda \zeta} \\
 & \quad \times \{ -i(n-2c+\lambda) G(s, X(s), \zeta, R(s)) du(s) \\
 & \quad + \sum_{j=1}^{2n} \left( A_j G + \frac{1}{\sqrt{2}} \frac{\partial G}{\partial x^j} \right) (s, X(s), \zeta, R(s)) dx^j(s) \\
 & \quad - i\lambda G(s, X(s), \zeta, R(s)) dS_x(s) + \frac{\partial G}{\partial r} (s, X(s), \zeta, R(s)) dB(s) \} \\
 (3.7) \quad & + \int_0^t m(s) M(s) K(s) e^{-i\lambda U(s)} e^{i\lambda \zeta} \left\{ \frac{\partial G}{\partial t} + \frac{1}{4} \sum_{j=1}^n \left( \frac{\partial}{\partial x^j} - 2i\lambda X^{n+j}(s) \right)^2 G \right. \\
 & \quad + \frac{1}{4} \sum_{j=1}^n \left( \frac{\partial}{\partial x^{n+j}} + 2i\lambda X^j(s) \right)^2 G - \frac{\lambda^2}{2} G + \frac{1}{2} \frac{\partial^2 G}{\partial r^2} - \lambda(n-2C) G \\
 & \quad + \sum_{j=1}^n \frac{A_j}{\sqrt{2}} \left( \frac{\partial}{\partial x^j} - 2i\lambda X^{n+j}(s) \right) G + \sum_{j=1}^n \frac{A_{n+j}}{\sqrt{2}} \left( \frac{\partial}{\partial x^{n+j}} + 2i\lambda X^j(s) \right) G \\
 & \quad \left. + BG \right\} (s, X(s), \zeta, R(s)) ds + \int_0^t m(s) M(s) K(s) e^{-i\lambda U(s)} e^{i\lambda \zeta} \\
 & \quad \times \left\{ \frac{\partial G}{\partial r} - \lambda G \right\} (s, X(s), \zeta, R(s)) d\phi(s).
 \end{aligned}$$

Since  $G$  is assumed to be smooth and rapidly decreasing in  $X$ , by integrating in  $\zeta$  variables, exchanging the order of integration and using the integration by parts, we obtain (3.6) from (3.7).  $\square$

If the solution  $F(t, X)$  of (1.1) is smooth in  $(t, X)$  and rapidly decreasing in  $X$  uniformly with respect to  $t \in [\delta, T]$  for any  $0 < \delta < T$ , by Lemma 3.1, we have

$$\begin{aligned}
 & e^{-i\lambda u} \int_{-\infty}^{\infty} d\zeta e^{i\lambda \zeta} F(T, x, \zeta, r) \\
 & = E \left[ m(T) M(T) K(T) e^{-i\lambda U(T)} \int_{-\infty}^{\infty} d\zeta e^{i\lambda \zeta} f(X(T), \zeta, R(T)) \right]
 \end{aligned}$$

and so

$$\begin{aligned}
 & F(t, x, u, r) \\
 (3.8) \quad & = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda E [m(t) M(t) K(t) e^{-i\lambda U(t)} \int_{-\infty}^{\infty} d\zeta e^{i\lambda \zeta} f(X(t), \zeta, R(t))].
 \end{aligned}$$

Now we set

$$\begin{aligned}
 & H_{t,\lambda}((x, u, r), (x', u', r')) \\
 (3.9) \quad & := \frac{1}{2\pi} E \left[ m(t) M(t) K(t) e^{-i\lambda U(t)} \int_{-\infty}^{\infty} d\zeta e^{i\lambda \zeta} \delta_{(x', u', r')} (X(t), \zeta, R(t)) \right]
 \end{aligned}$$

$$= \frac{1}{2\pi} e^{i\lambda(u'-u)} E[e^{-i\lambda u(t)} m(t)] E[e^{-\lambda S_x(t)} M(t) \delta_{x'}(X(t))] \\ \times E[e^{-\lambda\phi(t)} K(t) | R(t) = r'] r_t(r, r'),$$

where  $r_t(r, r')$  is the probability density function of  $R(t)$ . We can rewrite this  $H_{t,\lambda}((x, u, r), (x', u', r'))$  as follows:

**Lemma 3.2.**  $H_{t,\lambda}((x, u, r), (x', u', r'))$  is expressed as follows:

$$(3.10) \quad \begin{aligned} & H_{t,\lambda}((x, u, r), (x', u', r')) \\ &= f_t^{n-2q+2}((x, u)^{-1}(x', u'), \lambda) \exp\left(-\frac{t}{2} \lambda^2\right) e_t^-(r, r') P \\ &+ f_t^{n-2q}((x, u)^{-1}(x', u'), \lambda) \exp\left(-\frac{t}{2} \lambda^2\right) e_t^+(r, r') Q \\ &- \frac{2}{\sqrt{\pi}} \lambda f_t^{n-2q}((x, u)^{-1}(x', u'), \lambda) e^{r\lambda} \int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu \exp(-\mu^2) Q \end{aligned}$$

where  $e_t^\pm(r, r')$  and  $f_t^\alpha(x, u, \lambda)$  are defined in (2.13) and (2.16), respectively.

Proof. Since

$$(3.11) \quad \begin{cases} m(t)Q = \exp\left(-i(n-2q)u(t) + (n-2q)^2 \frac{t}{2}\right) Q, \\ m(t)P = \exp\left(-i(n-2q+2)u(t) + (n-2q+2)^2 \frac{t}{2}\right) P, \end{cases}$$

we have

$$(3.12) \quad \begin{cases} E[e^{-i\lambda u(t)} m(t)] Q = \exp\left(-\frac{t^2}{2} \lambda^2 - t\lambda(n-2q)\right) Q, \\ E[e^{-i\lambda u(t)} m(t)] P = \exp\left(-\frac{t^2}{2} \lambda^2 - t\lambda(n-2q+2)\right) P. \end{cases}$$

We also have

$$(3.13) \quad \begin{aligned} & E[e^{-i\lambda S_x(t)} M(t) \delta_{x'}(X(t))] \\ &= \exp(-i\lambda \sum_{j=1}^n 2(x^{n+j} x'^j - x^j x'^{n+j})) E\left[e^{-i\lambda S_0(t)} M(t) \delta_{x'-x} \left(\frac{x(t)}{\sqrt{2}}\right)\right]. \end{aligned}$$

By using Lemma 3.3 below, we obtain

$$(3.14) \quad \begin{aligned} & E[e^{-\lambda\phi(t)} K(t) | R(t) = r'] r_t(r, r') \\ &= \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left(-\frac{(r-r')^2}{2t}\right) - \exp\left(-\frac{(r+r')^2}{2t}\right) \right\} \\ &+ \int_0^\infty db e^{-\lambda b} \frac{2(r+r'+b)}{\sqrt{2\pi^3}} \exp\left(-\frac{(r+r'+b)^2}{2t}\right) Q \\ &= \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left(-\frac{(r-r')^2}{2t}\right) - \exp\left(-\frac{(r+r')^2}{2t}\right) \right\} P \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left(-\frac{(r-r')^2}{2t}\right) + \exp\left(-\frac{(r+r')^2}{2t}\right) \right\} Q \\
 & - \int_0^\infty db \lambda e^{-\lambda b} \frac{2}{\sqrt{2\pi t}} \exp\left(-\frac{(r+r'+b)^2}{2t}\right) Q.
 \end{aligned}$$

By (3.12), (3.13) and (3.14), we conclude (3.10).  $\square$

**Lemma 3.3.** (cf. Itô-McKean [6] p. 45) *Let  $r(\cdot)$  be a 1-dimensional Brownian motion. For  $r \geq 0$ , we set*

$$\begin{aligned}
 R(t, r) &= |r+r(t)|, \\
 \phi(t, r) &= \lim_{\eta \searrow 0} \frac{1}{2\eta} \int_0^t I_{(-\eta, \eta)}(r+r(s)) ds, \\
 \sigma(r) &= \inf \{t \geq 0, R(t, r) = 0\}.
 \end{aligned}$$

Then the joint distribution of  $(R(t, r), \phi(t, r))$  is

$$\begin{aligned}
 & P(R(t, r) \in da, \phi(t, r) \in db, \sigma(r) \leq t) \\
 (3.15) \quad &= \frac{2(r+a+b)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(r+a+b)^2}{2t}\right) dadb, \\
 & P(R(t, r) \in da, \phi(t, r) \in db, \sigma(r) > t) \\
 &= \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left(-\frac{(r-a)^2}{2t}\right) - \exp\left(-\frac{(r+a)^2}{2t}\right) \right\} \delta_0(b) dadb,
 \end{aligned}$$

$a, b \geq 0$ .

By (3.8) and Lemma 3.2 we can expect that (2.15) holds.

#### 4. Proof of Theorem 2.2

In this section, we prove Theorem 2.2. First we prepare a few lemmas. The main idea of the proof of Theorem 2.2 is to use semigroup property of a kernel as follows:

**Lemma 4.1.** For any  $m \in \mathbb{N}$  and  $\varepsilon > 0$ ,

$$\begin{aligned}
 & E \left[ S_0(1)^k e^{i\lambda S_0(1)} M_m^\varepsilon(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right) \right] \\
 &= \sum_{k_1 + \dots + k_m = k} \frac{k!}{k_1! \dots k_m!} \\
 (4.1) \quad & \times \int_{\mathbb{R}^{2n}} dx_1 E \left[ \left( \frac{S_0(1)}{m} \right)^{k_1} e^{i\lambda S_0(1)} M_m^\varepsilon(1) \delta_{x_1} \left( \frac{x(1)}{\sqrt{2m}} \right) \right] \\
 & \times \int_{\mathbb{R}^{2n}} dx_2 E \left[ \left( \frac{S_{x_1}(1)}{m} \right)^{k_2} e^{i\lambda S_{x_1}(1)} M_m^\varepsilon(1) \delta_{x_2} \left( x_1 + \frac{x(1)}{\sqrt{2m}} \right) \right] \\
 & \times \dots \times \int_{\mathbb{R}^{2n}} dx_{m-1} E \left[ \left( \frac{S_{x_{m-2}}(1)}{m} \right)^{k_{m-1}} e^{i\lambda S_{x_{m-2}}(1)} M_m^\varepsilon(1) \right]
 \end{aligned}$$

$$\begin{aligned} & \times \delta_{x_{m-1}} \left( x_{m-2} + \frac{x(1)}{\sqrt{2m}} \right) \\ & \times E \left[ \left( \frac{S_{x_{m-1}}(1)}{m} \right)^{l_m} e^{i\lambda S_{x_{m-1}}(1)} M_m^e(1) \delta_x \left( x_{m-1} + \frac{x(1)}{\sqrt{2m}} \right) \right]. \end{aligned}$$

Proof. By using scaling property and Markovian property of the Brownian motion, for any  $\zeta \in \mathbf{R}$ , we have

$$\begin{aligned} & \sum_{l=0}^{\infty} \frac{(i\zeta)^l}{l!} E \left[ S_0(1)^l e^{im\lambda S_0(1)} M^e(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right) \right] \\ & = E \left[ e^{i(m\lambda + \zeta)S_0(1)} M^e(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right) \right] \\ & = E \left[ e^{i(\lambda + \zeta/m)S_0(m)} M_m^e(m) \delta_x \left( \frac{x(m)}{\sqrt{2m}} \right) \right] \\ (4.2) \quad & = \int_{\mathbf{R}^{2n}} dx_1 E \left[ e^{i(\lambda + \zeta/m)S_0(1)} M_m^e(1) \delta_{x_1} \left( \frac{x(1)}{\sqrt{2m}} \right) \right] \\ & \times \int_{\mathbf{R}^{2n}} dx_2 E \left[ e^{i(\lambda + \zeta/m)S_{x_1}(1)} M_m^e(1) \delta_{x_2} \left( x_1 + \frac{x(1)}{\sqrt{2m}} \right) \right] \\ & \times \cdots \times \int_{\mathbf{R}^{2n}} dx_{m-1} E \left[ e^{i(\lambda + \zeta/m)S_{x_{m-2}}(1)} M_m^e(1) \delta_{x_{m-1}} \left( x_{m-2} + \frac{x(1)}{\sqrt{2m}} \right) \right] \\ & \times E \left[ e^{i(\lambda + \zeta/m)S_{x_{m-1}}(1)} M_m^e(1) \delta_x \left( x_{m-1} + \frac{x(1)}{\sqrt{2m}} \right) \right] \\ & = \sum_{l_1, \dots, l_m=0}^{\infty} \frac{1}{l_1! \cdots l_m!} (i\zeta)^{l_1 + \dots + l_m} \\ & \times \int_{\mathbf{R}^{2n}} dx_1 E \left[ \left( \frac{S_0(1)}{m} \right)^{l_1} e^{i\lambda S_0(1)} M_m^e(1) \delta_{x_1} \left( \frac{x(1)}{\sqrt{2m}} \right) \right] \\ & \times \int_{\mathbf{R}^{2n}} dx_2 E \left[ \left( \frac{S_{x_1}(1)}{m} \right)^{l_2} e^{i\lambda S_0(1)} M_m^e(1) \delta_{x_2} \left( x_1 + \frac{x(1)}{\sqrt{2m}} \right) \right] \\ & \times \cdots \times \int_{\mathbf{R}^{2n}} dx_{m-1} E \left[ \left( \frac{S_{x_{m-2}}(1)}{m} \right)^{l_{m-1}} e^{i\lambda S_{x_{m-2}}(1)} M_m^e(1) \right. \\ & \quad \times \delta_{x_{m-1}} \left( x_{m-2} + \frac{x(1)}{\sqrt{2m}} \right) \left. \right] \\ & \times E \left[ \left( \frac{S_{x_{m-1}}(1)}{m} \right)^{l_m} e^{i\lambda S_{x_{m-1}}(1)} M_m^e(1) \delta_x \left( x_{m-1} + \frac{x(1)}{\sqrt{2m}} \right) \right]. \end{aligned}$$

Since  $\zeta$  is arbitrary, we obtain (4.1).  $\square$

In the followings, we write  $C(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$  a constant which depends only on the parameter  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$  but may vary at each hand of the equations. We suppress the dependence on  $n$ . The next two lemmas give the necessary estimates in Malliavin calculus:

**Lemma 4.2.** For any  $p > 1$  and  $s \in \mathbf{Z}_+$ ,

$$(4.3) \quad \|M_m^\varepsilon(1)\|_{p,s} \leq C(p, s) \exp \left\{ \left( \frac{\varepsilon}{\sqrt{m}} \right)^p C(p, s) \right\}$$

and

$$(4.4) \quad \|M_m^\varepsilon(1) - I\|_{p,s} \leq C(p, s) \frac{\varepsilon}{\sqrt{m}} \exp \left\{ \left( \frac{\varepsilon}{\sqrt{m}} \right)^p C(p, s) \right\}$$

where  $\|\cdot\|_{p,s}$  is the Sobolev norm on the Wiener space (see e. g. [13]).

Proof. For any continuous adapted Hilbert space valued process  $N(\cdot)$  and  $0 \leq t \leq 1$ , by using the inequality of Burkholder (Ikeda-Watanabe [5] Theorem III-3.1), we have

$$(4.5) \quad \begin{aligned} & E \left[ \left\| \int_0^t N(s) d\Xi_m^\varepsilon(s) \right\|^p \right] \\ & \leq \sum_{j=1}^{2n} E \left[ \left\| \int_0^t N(s) A_j \frac{\varepsilon}{\sqrt{m}} dx^j(s) \right\|^p \right] (2n+1)^{p-1} \\ & \quad + E \left[ \left\| \int_0^t N(s) B \frac{\varepsilon^2}{m} ds \right\|^p \right] (2n+1)^{p-1} \\ & \leq C(p) \left\{ \sum_{j=1}^{2n} E \left[ \left( \int_0^t \|N(s)\|^2 \|A_j\|^2 \frac{\varepsilon^2}{m} ds \right)^{p/2} \right] \right. \\ & \quad \left. + E \left[ \int_0^t \|N(s)\|^p \|B\|^p \frac{\varepsilon^{2p}}{m^p} ds \right] t^{p-1} \right\} \\ & \leq C(p) \left( \frac{\varepsilon}{\sqrt{m}} \right)^p \int_0^t E[\|N(s)\|^p] ds. \end{aligned}$$

On the other hand, let  $H$  be the  $2n$ -dimensional Cameron-Martin space ([13]). It is easily seen that

$$\begin{aligned} D\Xi_m^\varepsilon(s)[h] &= \sum_{j=1}^{2n} \frac{\varepsilon}{\sqrt{m}} A_j h^j(s) \quad \text{for } h \in H, \\ D^r \Xi_m^\varepsilon(s) &= 0 \quad \text{for } r \geq 2. \end{aligned}$$

Then we see that

$$(4.6) \quad E \left[ \left\| \int_0^t N(s) d(D\Xi_m^\varepsilon(s)) \right\|^p \right] \leq C(p) \left( \frac{\varepsilon}{\sqrt{m}} \right)^p \int_0^t E[\|N(s)\|^p] ds.$$

Since, by (3.2),

$$D^r M_m^\varepsilon(t) = \delta_{r0} I + \int_0^t D^r M_m^\varepsilon(s) d\Xi_m^\varepsilon(s) + r \int_0^t D^{r-1} M_m^\varepsilon(s) d(D\Xi_m^\varepsilon(s)),$$

we have

$$(4.7) \quad \sum_{r=0}^{\infty} E[\|D^r M_m^\varepsilon(t)\|^p] \leq C(p, s) \left( 1 + \sum_{r=0}^{\infty} \left( \frac{\varepsilon}{\sqrt{m}} \right)^p \int_0^t E[\|D^r M_m^\varepsilon(s)\|^p] ds \right).$$

By using the equivalence of the Sobolev norms ([5] Theorem V-8.4), (4.7) is rewritten as follows:

$$(4.8) \quad \|M_m^\varepsilon(t)\|_{p,s}^p \leq C(p, s) \left(1 + \left(\frac{\varepsilon}{\sqrt{m}}\right)^p \int_0^t \|M_m(s)\|_{p,s}^p ds\right).$$

By using the lemma of Gronwall, we obtain (4.3) easily. Similarly we obtain (4.4).  $\square$

**Lemma 4.3.** *For any  $k \in \mathbb{N}$ ,  $\alpha \in \mathbb{Z}_+^{2n}$  and  $q > 1$ , there exists  $s \in \mathbb{N}$  such that*

$$(4.9) \quad \sup_{x \in \mathbb{R}^{2n}} (1 + |x|)^k \|\partial^\alpha \delta_x(x(1))\|_{q,-s} \leq \infty.$$

*Proof.* For any  $\kappa > 0$ , there exists  $s \in \mathbb{N}$  such that

$$\|(1 + |\cdot| - \Delta)^{-s} \phi(\cdot)\|_\infty \leq C(s, \kappa) \|(1 + |\cdot| - \Delta)^{-\kappa} \phi(\cdot)\|_2 \quad \text{for } \phi \in \mathcal{S}(\mathbb{R}^{2n})$$

where  $\|\cdot\|_\infty$  and  $\|\cdot\|_2$  are the  $L_\infty$ -norm and  $L_2$ -norm on  $\mathbb{R}^{2n}$ , respectively. By combining this with Theorem 2.1 of [13], we have

$$\|\partial^\alpha \delta_x(x(1))\|_{q,-s} \leq C(q, s, k) \|(1 + |\cdot| - \Delta)^{-\kappa} \partial^\alpha \delta_x(\cdot)\|_2.$$

Furthermore the  $L_2$ -norm is estimated as follows:

$$\begin{aligned} & (1 + |x|^2)^k \|(1 + |\cdot| - \Delta)^{-\kappa} \partial^\alpha \delta_x(\cdot)\|_2 \\ &= (1 + |x|^2)^k \sup_{\substack{g \in \mathcal{S}(\mathbb{R}^{2n}) \\ \|g\|_2 \leq 1}} \int_{\mathbb{R}^{2n}} (1 + |\xi|^2 - \Delta_\xi)^{-\kappa} \partial_\xi^\alpha \delta_x(\xi) g(\xi) d\xi \\ &= \sup_g |(1 + |x|^2)^k \partial_x^\alpha (1 + |x|^2 - \Delta_x)^{-\kappa} g(x)| \\ &= \sup_g |(1 + |x|^2)^k \int_{\mathbb{R}^{2n}} \frac{e^{i\xi x}}{(2\pi)^n} \xi^\alpha (1 + |\xi|^2 - \Delta_\xi)^{-\kappa} \hat{g}(\xi) d\xi| \\ &\leq C \sup_g \left( \int_{\mathbb{R}^{2n}} |(1 + |\xi|^2)^\kappa (1 - \Delta_\xi^k) \xi^\alpha (1 + |\xi|^2 - \Delta_\xi)^{-\kappa} \hat{g}(\xi)|^2 d\xi \right)^{1/2} \\ &\leq C \sup_g \left( \int_{\mathbb{R}^{2n}} |(1 + |\xi|^2 - \Delta_\xi)^{\kappa(k, \alpha, n) - \kappa} \hat{g}(\xi)|^2 d\xi \right)^{1/2} \end{aligned}$$

for some  $\kappa(k, \alpha, n) \geq 0$  determined by  $k$ ,  $\alpha$  and  $n$ . Thus, if  $\kappa$  is large enough, we have

$$(4.10) \quad (1 + |x|^2)^k \|\partial^\alpha \delta_x(x(1))\|_{q,-s} \leq C(q, s, k, \alpha).$$

Hence we conclude (4.9).  $\square$

Now we prove Theorem 2.2.

*Proof of Theorem 2.2.* We prove the theorem as  $\lambda \rightarrow -\infty$  only. The case as  $\lambda \rightarrow +\infty$  can be proved similarly.

We consider (i). We take  $m \in \mathbb{N}$  arbitrary. We exchange the order of differentiations and integrations in (4.1) formally:

$$\begin{aligned}
 (4.11) \quad & x^\beta \partial^\alpha E \left[ S_0(1)^k e^{i m \lambda S_0(1)} M_m^e(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right) \right] \\
 &= \int_{(\mathbb{R}^{2n})^{m-1}} dx_1 \cdots dx_{m-1} \Phi_{k, \alpha, \beta}^e(m, \lambda, x_1, \dots, x_{m-1}, x)
 \end{aligned}$$

where

$$\begin{aligned}
 (4.12) \quad & \Phi_{k, \alpha, \beta}^e(m, \lambda, x_1, \dots, x_{m-1}, x) \\
 &= \sum_{k_1 + \dots + k_m = k} \frac{k!}{k_1! \cdots k_m!} E \left[ \left( \frac{S_0(1)}{m} \right)^{k_1} e^{i \lambda S_0(1)} M_m^e(1) \delta_{x_1} \left( \frac{x(1)}{\sqrt{2m}} \right) \right] \\
 &\times E \left[ \left( \frac{S_{x_1}(1)}{m} \right)^{k_2} e^{i \lambda S_{x_1}(1)} M_m^e(1) \delta_{x_2} \left( x_1 + \frac{x(1)}{\sqrt{2m}} \right) \right] \\
 &\times \cdots \times E \left[ \left( \frac{S_{x_{m-2}}(1)}{m} \right)^{k_{m-1}} e^{i \lambda S_{x_{m-2}}(1)} M_m^e(1) \delta_{x_{m-1}} \left( x_{m-2} + \frac{x(1)}{\sqrt{2m}} \right) \right] \\
 &\times x^\beta \partial^\alpha E \left[ \left( \frac{S_{x_{m-1}}(1)}{m} \right)^{k_m} e^{i \lambda S_{x_{m-1}}(1)} M_m^e(1) \delta_x \left( x_{m-1} + \frac{x(1)}{\sqrt{2m}} \right) \right].
 \end{aligned}$$

We will estimate the right hand side of (4.11). Let  $\|\cdot\|$  be the Hilbert-Schmidt norm defined in Section 2. By using a property of  $\delta$ -functions and making the change of variables  $\sqrt{2m} x_j \rightarrow x_j$ ,

$$\begin{aligned}
 (4.13) \quad & \int_{(\mathbb{R}^{2n})^{m-1}} dx_1 \cdots dx_{m-1} \|\Phi_{k, \alpha, \beta}^e(m, \lambda, x_1, \dots, x_{m-1}, x)\| \\
 &\leq \sum_{k_1 + \dots + k_m = k} \frac{k!}{k_1! \cdots k_m!} \\
 &\times \int_{(\mathbb{R}^{2n})^{m-1}} dx_1 \cdots dx_{m-1} \left\| E \left[ \left( \frac{S_0(1)}{m} \right)^{k_1} e^{i \lambda S_0(1)} M_m^e(1) \delta_{\sqrt{2m} x_1}(x(1)) \right] \right\| (2m)^n \\
 &\times \left\| E \left[ \left( \frac{S_{x_1}(1)}{m} \right)^{k_2} e^{i \lambda S_{x_1}(1)} M_m^e(1) \delta_{\sqrt{2m} x_2}(\sqrt{2m} x_1 + x(1)) \right] \right\| (2m)^n \\
 &\times \cdots \times \left\| E \left[ \left( \frac{S_{x_{m-2}}(1)}{m} \right)^{k_{m-1}} e^{i \lambda S_{x_{m-2}}(1)} M_m^e(1) \right. \right. \\
 &\times \left. \left. \delta_{\sqrt{2m} x_{m-1}}(\sqrt{2m} x_{m-2} + x(1)) \right] \right\| (2m)^n \\
 &\times \left\| x^\beta \partial^\alpha E \left[ \left( \frac{S_{x_{m-1}}(1)}{m} \right)^{k_m} e^{i \lambda S_{x_{m-1}}(1)} M_m^e(1) \delta_x \left( x_{m-1} + \frac{x(1)}{\sqrt{2m}} \right) \right] \right\| \\
 &= \sum_{k_1 + \dots + k_m = k} \frac{k! m^k}{k_1! \cdots k_m!} \\
 &\times \int_{(\mathbb{R}^{2n})^{m-1}} dx_1 \cdots dx_{m-1} \left\| E \left[ \left( \frac{S_0(1)}{m^2} \right)^{k_1} e^{i \lambda S_0(1)} M_m^e(1) \delta_{x_1}(x(1)) \right] \right\| \\
 &\times \left\| E \left[ \left( \frac{S_{x_1/\sqrt{2m}}(1)}{m^2} \right)^{k_2} e^{i \lambda S_{x_1/\sqrt{2m}}(1)} M_m^e(1) \delta_{x_2}(x_1 + x(1)) \right] \right\|
 \end{aligned}$$



$$\begin{aligned} & \times \cdots \times \left\| E \left[ \left( \frac{S_{x_{m-2}/\sqrt{2m}}(1)}{m^2} \right)^{k_{m-1}} e^{i\lambda S_0(1)} M_m^e(1) \delta_{x_{m-1}}(x_{m-2} + x(1)) \right] \right\| \\ & \times \left\| x^\beta \partial^\alpha E \left[ \left( \frac{S_{x_{m-1}/\sqrt{2m}}(1)}{m^2} \right)^{k_m} e^{i\lambda S_{x_{m-1}/\sqrt{2m}}(1)} M_m^e(1) \delta_x \left( \frac{x_{m-1} + x(1)}{\sqrt{2m}} \right) \right] \right\|. \end{aligned}$$

Since

$$(1 + |x_1|) \cdot (1 + |x_2 - x_1|) \cdots (1 + |x_j - x_{j-1}|) \geq 1 + |x_j|$$

for any  $j \geq 1$ , if we set  $R_n = \int_{\mathbb{R}^{2n}} (1 + |x|)^{-2n-1} dx$ ,

the right hand side of (4.13)

$$\begin{aligned} & \leq \sum_{k_1 + \cdots + k_m = k} \frac{k! m^k}{k_1! \cdots k_m!} \int_{(\mathbb{R}^{2n})^{m-1}} dx_1 \cdots dx_{m-1} \\ & \times \left\| (1 + |x_1|)^{2n+2k+|\alpha|+|\beta|+1} E \left[ \left( \frac{S_0(1)}{m^2} \right)^{k_1} e^{i\lambda S_0(1)} M_m^e(1) \delta_{x_1}(x(1)) \right] \right\| \\ & \times \left\| (1 + |x_2 - x_1|)^{2n+2k+|\alpha|+|\beta|+1} E \left[ \left( \frac{S_{x_1/\sqrt{2m}}(1)}{m^2(1 + |x_1|)^2} \right)^{k_1} e^{i\lambda S_0(1)} \right. \right. \\ & \quad \left. \left. \times M_m^e(1) \delta_{x_2}(x_1 + x(1)) \right] \right\| \\ (4.14) \quad & \times \cdots \times \left\| (1 + |x_{m-1} - x_{m-2}|)^{2n+2k+|\alpha|+|\beta|+1} E \left[ \left( \frac{S_{x_{m-2}/\sqrt{2m}}(1)}{m^2(1 + |x_{m-2}|)^2} \right)^{k_{m-1}} \right. \right. \\ & \quad \left. \left. \times e^{i\lambda S_0(1)} M_m^e(1) \delta_{x_{m-1}}(x_{m-2} + x(1)) \right] \right\| \\ & \times \left\| \frac{x^\beta}{(1 + |x_{m-1}|)^{|\alpha|+|\beta|}} \partial^\alpha E \left[ \left( \frac{S_{x_{m-1}/\sqrt{2m}}(1)}{m^2(1 + |x_{m-1}|)^2} \right)^{k_m} e^{i\lambda S_{x_{m-1}/\sqrt{2m}}(1)} \right. \right. \\ & \quad \left. \left. \times M_m^e(1) \delta_x \left( \frac{x_{m-1} + x(1)}{\sqrt{2m}} \right) \right] \right\| \\ & \times \frac{1}{(1 + |x_1|)^{2n+1}} \cdot \frac{1}{(1 + |x_2 - x_1|)^{2n+1}} \cdots \frac{1}{(1 + |x_{m-1} - x_{m-2}|)^{2n+1}} \\ & \leq \sum_{k_1 + \cdots + k_m = k} \frac{k! m^k}{k_1! \cdots k_m!} \\ & \times \left\{ \prod_{j=1}^{m-1} R_n \sup_{x, x'} \left\| (1 + |x' - x|)^{2n+2k+|\alpha|+|\beta|+1} E \left[ \left( \frac{S_{x/\sqrt{2m}}(1)}{m^2(1 + |x|)^2} \right)^{k_j} \right. \right. \right. \\ & \quad \left. \left. \times e^{i\lambda S_0(1)} M_m^e(1) \delta_{x'}(x + x(1)) \right] \right\| \left. \right\} \\ & \times \sup_{x, x'} \left\| \frac{x'^\beta}{(1 + |x|)^{|\alpha|+|\beta|}} \partial_{x'}^\alpha E \left[ \left( \frac{S_{x/\sqrt{2m}}(1)}{m^2(1 + |x|)^2} \right)^{k_m} e^{i\lambda S_{x/\sqrt{2m}}(1)} \right. \right. \\ & \quad \left. \left. \times M_m^e(1) \delta_{x'} \left( \frac{x + x(1)}{\sqrt{2m}} \right) \right] \right\|. \end{aligned}$$

The middle factor under the summation in the right hand side of (4.14) is estimated as follows:

$$\begin{aligned}
 & \| (1 + |x' - x|)^{2n+2k+|\alpha|+|\beta|+1} E \left[ \left( \frac{S_x / \sqrt{2m}(1)}{m^2(1+|x|)^2} \right)^l e^{i\lambda S_0(1)} \right. \\
 & \quad \times \left. M_m^e(1) \delta_{x'}(x+x(1)) \right] \| \\
 & \leq \sum_{h=0}^l \frac{l! \cdot (1 + |x' - x|)^{2n+2k+|\alpha|+|\beta|+1}}{h!(l-h)!} \left( \frac{\sum_{j=1}^n |x^{n+j} x'^j - x^j x'^{n+j}|}{\sqrt{m} m^2(1+|x|)^2} \right)^h \\
 (4.15) \quad & \times \left\| E \left[ \left( \frac{S_0(1)}{m^2(1+|x|)^2} \right)^{l-h} e^{i\lambda S_0(1)} M_m^e(1) \delta_{x'}(x+x(1)) \right] \right\| \\
 & \leq \left( \frac{4}{m} \right)^l (1 + |x' - x|)^{2n+3k+|\alpha|+|\beta|+1} \\
 & \quad \times \sum_{h=0}^l \left\| E \left[ \left( \frac{S_0(1)}{m} \right)^h e^{i\lambda S_0(1)} M_m^e(1) \delta_{x'}(x+x(1)) \right] \right\|.
 \end{aligned}$$

On the other hand, since

$$\partial_{x'}^{\alpha} E[\Phi \delta_{ax'}(x+x(1))] = (-a)^{|\alpha|} E[\Phi \partial^{\alpha} \delta_{ax'}(x+x(1))]$$

for any  $a > 0$  and  $\alpha \in \mathbf{Z}_+^{2n}$ ,

$$\begin{aligned}
 & \partial_{x'}^{\alpha} E \left[ \left( \frac{S_x / \sqrt{2m}(1)}{m^2(1+|x|)^2} \right)^l e^{i\lambda S_x / \sqrt{2m}(1)} M_m^e(1) \delta_{x'} \left( \frac{x+x(1)}{\sqrt{2m}} \right) \right] \\
 & = \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \sum_{h=0}^l \frac{l!}{h!(l-h)!} \partial_{x'}^{\alpha_1} \left( \frac{\sqrt{2} \sum_{j=1}^n (x^{n+j} x'^j - x^j x'^{n+j})}{m^2(1+|x|)^2} \right)^h \\
 & \quad \times \partial_{x'}^{\alpha_2} \exp \left( i\lambda \sqrt{2} \sum_{j=1}^n (x^{n+j} x'^j - x^j x'^{n+j}) \right) (-1)^{|\alpha_3|} (2m)^{n+|\alpha_3|/2} \\
 & \quad \times E \left[ \left( \frac{S_0(1)}{m^2(1+|x|)^2} \right)^{l-h} e^{i\lambda S_0(1)} M_m^e(1) \partial^{\alpha_3} \delta_{\sqrt{2m}x'}(x+x(1)) \right].
 \end{aligned}$$

Thus the last factor under the summation of the right hand side of (4.4) is estimated as follows:

$$\begin{aligned}
 & \left\| \frac{x'^{\beta}}{(1+|x|)^{|\alpha|+|\beta|}} \partial_{x'}^{\alpha} E \left[ \left( \frac{S_x / \sqrt{2m}(1)}{m^2(1+|x|)^2} \right)^l e^{i\lambda S_x / \sqrt{2m}(1)} M_m^e(1) \delta_{x'} \left( \frac{x+x(1)}{\sqrt{2m}} \right) \right] \right\| \\
 & \leq \sum_{\alpha_1+\alpha_2+\alpha_3=\alpha} \sum_{h=0}^l \frac{l!}{h!(l-h)!} \left( \frac{|x'|}{1+|x|} \right)^{|\beta|} \left( \frac{2\sqrt{2}n|x||x'|}{m^2(1+|x|)^2} \right)^h \left( \frac{\sqrt{2}\lambda|x|}{1+|x|} \right)^{|\alpha_2|} \\
 (4.16) \quad & \times (2m)^{n+|\alpha_3|/2} \left\| E \left[ \left( \frac{S_0(1)}{m^2(1+|x|)^2} \right)^{l-h} e^{i\lambda S_0(1)} M_m^e(1) \partial^{\alpha_3} \delta_{\sqrt{2m}x'}(x+x(1)) \right] \right\| \\
 & \leq 2^{|\alpha|+2k+n} m^{n+|\alpha|/2} n^k \lambda^{|\alpha|} (1 + |\sqrt{2m}x' - x|)^{|\beta|+k} \\
 & \quad \times \sum_{\substack{h \leq l \\ \alpha' \leq \alpha}} \left\| E \left[ \left( \frac{S_0(1)}{m} \right)^h e^{i\lambda S_0(1)} M_m^e(1) \partial^{\alpha'} \delta_{\sqrt{2m}x'}(x+x(1)) \right] \right\|.
 \end{aligned}$$

By combining (4.14), (4.15) and (4.16), we obtain

$$\sup_x \int_{(\mathbb{R}^{2n})^{m-1}} dx_1 \cdots dx_{m-1} \|\Phi_{h,\alpha,\beta}^e(m, \lambda, x_1, \dots, x_{m-1}, x)\|$$

$$\begin{aligned}
 &\leq \sum_{k_1+\dots+k_m=k} \frac{k!m^k}{k_1!\dots k_m!} \left(\frac{4}{m}\right)^k 2^{|\alpha|+2k+n} m^{n+|\alpha|/2} n^k \lambda^{|\alpha|} \\
 &\quad \times \left\{ \prod_{j=1}^{m-1} R_n \sum_{h=0}^k \sup_x \left\| (1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E \left[ \left(\frac{S_0(1)}{m}\right)^h \right. \right. \right. \\
 &\quad \quad \times e^{i\lambda S_0(1)} M_m^e(1) \delta_x(x(1)) \left. \left. \left. \right] \right\| \right\} \\
 (4.17) \quad &\quad \times \left\{ \sum_{\substack{0 \leq h \leq k \\ \alpha' \leq \alpha}} \sup_x \left\| (1+|x|)^{|\beta|+k} E \left[ \left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)} M_m^e(1) \right. \right. \right. \\
 &\quad \quad \times \partial^{\alpha'} \delta_x(x(1)) \left. \left. \left. \right] \right\| \right\} \\
 &\leq 2^{|\alpha|+4k+n} m^{n+k|\alpha|/2} \lambda^{|\alpha|} n^k \\
 &\quad \times \left\{ R_n \sum_{h=0}^k \sup_x \left\| (1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E \left[ \left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)} \right. \right. \right. \\
 &\quad \quad \times M_m^e(1) \delta_x(x(1)) \left. \left. \left. \right] \right\| \right\}^{m-1} \\
 &\quad \times \left\{ \sum_{\substack{0 \leq h \leq k \\ \alpha' \leq \alpha}} \sup_x \left\| (1+|x|)^{|\beta|+k} E \left[ \left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)} \right. \right. \right. \\
 &\quad \quad \times M_m^e(1) \partial^{\alpha'} \delta_x(x(1)) \left. \left. \left. \right] \right\| \right\} \\
 &\leq 2^{|\alpha|+4k+n} m^{n+k|\alpha|/2} \lambda^{|\alpha|} n^k \\
 &\quad \times \{ C \sup_x |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda S_0(1)} \delta_x(x(1))] | \\
 &\quad + C(p, s) \|e^{i\lambda S_0(1)} (M_m^e(1) - I)\|_{p,s} \\
 &\quad \quad \times \sup_x (1+|x|)^{2n+3k+|\alpha|+|\beta|+1} \|\delta_x(x(1))\|_{q,-s} \\
 &\quad + C(p, s) \sum_{h=1}^k \left\| \left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)} M_m^e(1) \right\|_{p,s} \\
 &\quad \times \sup_x (1+|x|)^{2n+3k+|\alpha|+|\beta|+1} \|\delta_x(x(1))\|_{q,-s} \}^{m-1} \\
 &\quad \times \left\{ C(p, s) \sum_{h=0}^k \left\| \left(\frac{S_0(1)}{m}\right)^h e^{i\lambda S_0(1)} M_m^e(1) \right\|_{p,s} \right. \\
 &\quad \quad \times \left. \sum_{\alpha' \leq \alpha} \sup_x (1+|x|)^{|\beta|+k} \partial^{\alpha'} \|\delta_x(x(1))\|_{q,-s} \right\}
 \end{aligned}$$

where  $p, q > 1, 1/p + 1/q = 1, s \in \mathbb{N}$ . Furthermore,

$$\begin{aligned}
 &\|e^{i\lambda S_0(1)} (M_m^e(1) - I)\|_{p,s} \\
 &\leq \|e^{i\lambda S_0(1)}\|_{2p,s} \|M_m^e(1) - I\|_{2p,s} \\
 &\leq C(p) \lambda^{N(s)} \|M_m^e(1) - I\|_{2p,s}
 \end{aligned}$$

where  $N(s)$  is an integer which is determined by the number  $s$ . Similarly,

$$\|S_0(1)^h e^{i\lambda S_0(1)} M_m^e(1)\|_{p,s} \leq C(p, h) \lambda^{N(s)} \|M_m^e(1)\|_{p,s}.$$

Thus by Lemma 4.2, we have

$$(4.18) \quad \|e^{i\lambda S_0(1)} (M_m^\varepsilon(1) - I)\|_{p,s} \leq \frac{\varepsilon}{\sqrt{m}} \lambda^{N(s)} C(p, s) \exp \left\{ \left( \frac{\varepsilon}{\sqrt{m}} \right)^p C(p, s) \right\}$$

and

$$(4.19) \quad \|S_0(1)^k e^{i\lambda S_0(1)} M_m^\varepsilon(1)\|_{p,s} \leq \lambda^{N(s)} C(p, s, h) \exp \left\{ \left( \frac{\varepsilon}{\sqrt{m}} \right)^p C(p, s) \right\}.$$

By applying lemma 4.3, (4.18) and (4.19) to (4.17), we have

$$(4.20) \quad \begin{aligned} & \sup_{x \in \mathbb{R}^{2n}} \int_{(\mathbb{R}^{2n})^{m-1}} dx_1 \cdots dx_{m-1} \|\Phi_{k,\alpha,\beta}^\varepsilon(m, \lambda, x_1, \dots, x_{m-1}, x)\| \\ & \leq 2^{|\alpha|+4k+n} m^{n-k+|\alpha|/2} \lambda^{|\alpha|} n^k \\ & \times \{C \sup_{x \in \mathbb{R}^{2n}} |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda S_0(1)} \delta_x(x(1))]| \\ & \quad + \frac{\varepsilon}{\sqrt{m}} \lambda^{N(s)} C(p, s, \alpha, \beta, k) \exp \left( \left( \frac{\varepsilon}{\sqrt{m}} \right)^p C(p, s) \right) \\ & \quad + \frac{1}{m} \lambda^{N(s)} C(p, s, \alpha, \beta, k) \exp \left( \left( \frac{\varepsilon}{\sqrt{m}} \right)^p C(p, s) \right)\}^{m-1} \\ & \times \{\lambda^{N(s)} C(p, s, \alpha, \beta, k) \exp \left( \left( \frac{\varepsilon}{\sqrt{m}} \right)^p C(p, s) \right)\}. \end{aligned}$$

Since the right hand side of (4.20) is finite, the exchanging of the order of differentiations and integrations in (4.11) is justified and we obtain

$$(4.21) \quad \begin{aligned} & \sup_{x \in \mathbb{R}^{2n}, 0 < \varepsilon \leq \varepsilon_0} \|x^\beta \partial^\alpha E[S_0(1)^k e^{i\lambda S_0(1)} M^\varepsilon(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right)]\| \\ & \leq 2^{|\alpha|+4k+n} m^{n+k+|\alpha|/2} \lambda^{|\alpha|} n^k \\ & \times \{C \sup_{x \in \mathbb{R}^{2n}} |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda S_0(1)} \delta_x(x(1))]| \\ & \quad + \frac{1}{\sqrt{m}} \lambda^{N(s)} C(p, s, \alpha, \beta, k, \varepsilon_0)\}^{m-1} \\ & \times \{\lambda^{N(s)} C(p, s, \alpha, \beta, k, \varepsilon_0)\}. \end{aligned}$$

If we replace  $(\lambda, m)$  with  $(\lambda \zeta / [\lambda], [\lambda])$  where  $[\lambda]$  is the integral part of  $\lambda$ ,

$$(4.22) \quad \begin{aligned} & \sup_{x \in \mathbb{R}^{2n}, 0 < \varepsilon \leq \varepsilon_0} \|x^\beta \partial^\alpha E[S_0(1)^k e^{i\lambda S_0(1)} M^\varepsilon(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right)]\| \\ & \leq 2^{|\alpha|+4k+n} [\lambda]^{n+k+|\alpha|/2} \left( \frac{\lambda}{[\lambda]} \zeta \right)^{|\alpha|} n^k \\ & \times \{C \sup_{x \in \mathbb{R}^{2n}} |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda S_0(1)} \delta_x(x(1))]| \\ & \quad + \frac{1}{\sqrt{[\lambda]}} \left( \frac{\lambda}{[\lambda]} \zeta \right)^{N(s)} C(p, s, \alpha, \beta, k, \varepsilon_0)\}^{[\lambda]-1} \end{aligned}$$

$$\times \left\{ \left( \frac{\lambda}{[\lambda]} \zeta \right)^{N(s)} C(p, s, \alpha, \beta, k, \varepsilon_0) \right\}.$$

Hence we have

$$\begin{aligned} (4.23) \quad & \overline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \sup_{x \in \mathbf{R}^{2n}} \|x^\beta \partial^\alpha E[S_0(1)^h e^{i\lambda \zeta S_0(1)} M^\varepsilon(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right)]\| \\ &= \overline{\lim}_{\lambda \rightarrow \infty} \log \left\{ C \sup_{x \in \mathbf{R}^{2n}} |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda \zeta S_0(1)/[\lambda]} \delta_x(x(1))]| \right. \\ & \quad \left. + \frac{1}{\sqrt{[\lambda]}} \left( \frac{\lambda}{[\lambda]} \zeta \right)^{N(s)} C(p, s, \alpha, \beta, k, \varepsilon_0) \right\} \\ &= \log \overline{\lim}_{\lambda \rightarrow \infty} \left\{ C \sup_{x \in \mathbf{R}^{2n}} |(1+|x|)^{2n+3k+|\alpha|+|\beta|+1} E[e^{i\lambda \zeta S_0(1)/[\lambda]} \delta_x(x(1))]| \right\}. \end{aligned}$$

Since  $\lambda \coth \lambda \geq 1$  for  $\forall \lambda \in \mathbf{R}$ ,

$$\begin{aligned} (4.24) \quad & \sup_{x \in \mathbf{R}^{2n}} |(1+|x|)^h E[e^{i\lambda S_0(1)} \delta_x(x(1))]| \\ &= \sup_{x \in \mathbf{R}^{2n}} |(1+|x|)^h \left( \frac{1}{2\pi} \right)^n \left( \frac{\lambda/2}{\sinh(\lambda/2)} \right)^n \exp \left( -\frac{|x|^2 \lambda}{4} \coth \frac{\lambda}{2} \right)| \\ &\leq C(h) \left( \frac{\lambda}{\sinh(\lambda)} \right)^n, \end{aligned}$$

and so

$$\begin{aligned} & \overline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \sup_{x \in \mathbf{R}^{2n}, 0 < \varepsilon \leq \varepsilon_0} \|x^\beta \partial^\alpha E[S_0(1)^h e^{i\lambda S_0(1)} M^\varepsilon(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right)]\| \\ &\leq \log C(k, \alpha, \beta) \overline{\lim}_{\lambda \rightarrow \infty} \left( \frac{\lambda \zeta / [\lambda]}{\sinh \lambda \zeta / [\lambda]} \right)^n \\ &= \log C(k, \alpha, \beta) \left( \frac{\zeta}{\sinh \zeta} \right)^n. \end{aligned}$$

Consequently we obtain

$$\begin{aligned} (4.25) \quad & \overline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\lambda} \log \sup_{x \in \mathbf{R}^{2n}, 0 < \varepsilon \leq \varepsilon_0} \|x^\beta \partial^\alpha E[S_0(1)^h e^{i\lambda S_0(1)} M^\varepsilon(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right)]\| \\ &\leq \overline{\lim}_{\zeta \rightarrow \infty} \overline{\lim}_{\lambda \rightarrow \infty} \frac{1}{\zeta \lambda} \log \sup_{x \in \mathbf{R}^{2n}, 0 < \varepsilon \leq \varepsilon_0} \|x^\beta \partial^\alpha E[S_0(1)^h e^{i\zeta \lambda S_0(1)} M^\varepsilon(1) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right)]\| \\ &\leq \overline{\lim}_{\zeta \rightarrow \infty} \frac{1}{\zeta} \log C(k, \alpha, \beta) \left( \frac{\zeta}{\sinh \zeta} \right)^n \\ &= \overline{\lim}_{\zeta \rightarrow \infty} \frac{n}{\zeta} \log C(k, \alpha, \beta) \left( \frac{\zeta}{\sinh \zeta} \right) \\ &= -n \end{aligned}$$

which completes the proof of (i).

For the proof of (ii), we use the following (4.26), instead of (4.1): for any  $m \in \mathbb{N}$  and  $\varepsilon > 0$ ,

$$\begin{aligned}
 & E[S_0(1)^k e^{i m \lambda S_0(1)} (M^\varepsilon(1) - I) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right)] \\
 &= \sum_{k_1 + \dots + k_m = k} \sum_{l=1}^m \frac{k!}{k_1! \dots k_m!} \\
 & \quad \times \int_{\mathbb{R}^{2n}} dx_1 E \left[ \left( \frac{S_0(1)}{m} \right)^{k_1} e^{i \lambda S_0(1)} M_m^\varepsilon(1) \delta_{x_1} \left( \frac{x(1)}{\sqrt{2m}} \right) \right] \\
 (4.26) \quad & \times \dots \times \int_{\mathbb{R}^{2n}} dx_{l-1} E \left[ \left( \frac{S_{x_{l-2}}(1)}{m} \right)^{k_{l-1}} e^{i \lambda S_{x_{l-2}}(1)} M_m^\varepsilon(1) \right. \\
 & \quad \times \delta_{x_{l-1}} \left( x_{l-2} + \frac{x(1)}{\sqrt{2m}} \right) \left. \right] \\
 & \times \int_{\mathbb{R}^{2n}} dx_l E \left[ \left( \frac{S_{x_{l-1}}(1)}{m} \right)^{k_l} e^{i \lambda S_{x_{l-1}}(1)} (M_m^\varepsilon(1) - I) \right. \\
 & \quad \times \delta_{x_l} \left( x_{l-1} + \frac{x(1)}{\sqrt{2m}} \right) \left. \right] \\
 & \times \int_{\mathbb{R}^{2n}} dx_{l+1} E \left[ \left( \frac{S_{x_l}(1)}{m} \right)^{k_{l+1}} e^{i \lambda S_{x_l}(1)} \delta_{x_{l+1}} \left( x_l + \frac{x(1)}{\sqrt{2m}} \right) \right] \\
 & \times \dots \times \int_{\mathbb{R}^{2n}} dx_{m-1} E \left[ \left( \frac{S_{x_{m-2}}(1)}{m} \right)^{k_{m-1}} e^{i \lambda S_{x_{m-2}}(1)} \right. \\
 & \quad \times \delta_{x_{m-1}} \left( x_{m-2} + \frac{x(1)}{\sqrt{2m}} \right) \left. \right] \\
 & \times E \left[ \left( \frac{S_{x_{m-1}}(1)}{m} \right)^{k_m} e^{i \lambda S_{x_{m-1}}(1)} \delta_x \left( x_{m-1} + \frac{x(1)}{\sqrt{2m}} \right) \right].
 \end{aligned}$$

This identity is easily proven from (4.1).  $\square$

### 5. Proof of Theorem 2.1 and properties of the heat kernels

In this section, we prove Theorem 2.1. Furthermore we prove some properties of the kernel. We name the right hand side of (2.12) as follows:

$$\begin{aligned}
 & h_i((x, u, r), (x', u', r')) \\
 &= \mathbf{p}_i^{n-2q+2}((x, u)^{-1}(x', u')) e_i^-(r, r') P + \mathbf{p}_i^{n-2q}((x, u)^{-1}(x', u')) e_i^+(r, r') Q \\
 (5.1) \quad & + \mathbf{q}_i^{n-2q}((x, u)^{-1}(x', u'), r+r') Q \\
 & =: h_i^{(1)}((x, u, r), (x', u', r')) + h_i^{(2)}((x, u, r), (x', u', r')) \\
 & \quad + h_i^{(3)}(x, u, r), (x', u', r').
 \end{aligned}$$

First we prove that  $h_i^{(i)}$  has the following good regularity properties.

**Proposition 5.1.** *If  $i \neq 3$  or  $q \neq 0$  (resp.  $i=3$  and  $q=0$ ),  $h_i^{(i)}(X, X')$  is the restriction to  $\bar{D}$  of a rapidly decreasing function of  $X$  for each fixed  $(t, X') \in (0, \infty)$*

$\times \bar{D}$  (resp.  $(0, \infty) \times D$ ) and the restriction to  $\bar{D}$  of a rapidly decreasing function of  $X'$  for each fixed  $(t, X) \in (0, \infty) \times \bar{D}$  (resp.  $(0, \infty) \times D$ ) uniformly with respect to  $t \in [t_0, t_1]$  for any  $0 < t_0 \leq t_1$ . Moreover  $h_i^{(i)}(X, X')$  is a smooth function of  $(t, X, X') \in (0, \infty) \times D \times D$ .

Proof. For any  $\beta, \gamma \in \mathbf{Z}_+^{2n}$  and  $k_1, k_2, k_3, k_4 \in \mathbf{Z}_+$ , we exchange the order of differentiations and integrate by parts in  $\mathbf{q}_t$  formally:

$$\begin{aligned}
 & x^\gamma u^{k_1} r^{k_2} \partial_x^\beta \frac{\partial^{k_3}}{\partial u^{k_3}} \frac{\partial^{k_4}}{\partial r^{k_4}} \mathbf{q}_t^\alpha(x, u, r) \\
 (5.2) \quad & = \text{a linear combination of } \left\{ \int_{-\infty}^{\infty} d\lambda \cdot \lambda^{l_1} \exp(i\lambda u - t\lambda\alpha) \right. \\
 & \quad \times x^\gamma \partial_x^\beta E[S_0(t)^{l_2} e^{-i\lambda S_0(t)} M(t) \delta_x\left(\frac{x(t)}{\sqrt{2}}\right)] r^{l_3} e^{r\lambda} \int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu e^{-\mu^2}, \\
 & \quad \int_{-\infty}^{\infty} d\lambda \cdot \lambda^{l_1} \exp(i\lambda u - t\lambda\alpha) x^\gamma \partial_x^\beta E[S_0(t)^{l_2} e^{-i\lambda S_0(t)} M(t) \delta_x\left(\frac{x(t)}{\sqrt{2}}\right)] \\
 & \quad \left. \times r^{l_3} e^{r\lambda} \exp\left(-\frac{t}{2} \left(\frac{r}{t} + \lambda\right)^2\right) \right\}_{0 \leq l_0, l_2, l_3 \leq k} \\
 & =: \underline{\mathbf{q}}_t(x, u, r),
 \end{aligned}$$

where  $k = 1 + k_1 + k_2 + k_3 + k_4$ . By using (2.21), the right hand side of (4.2) is estimated as follows:

$$\begin{aligned}
 & \sup_{\substack{(x,u) \in \mathbf{R}^{2n+1} \\ r \geq r_0 \\ t \in [t_0, t_1]}} \|\underline{\mathbf{q}}_t(x, u, r)\| \\
 (5.3) \quad & \leq C(k, t_0, t_1) \sum_{0 \leq l_1, l_2, l_3 \leq k} \int_{-\infty}^{\infty} d\lambda \cdot |\lambda^{l_1}| \sup_{t \in [t_0, t_1]} \{\exp(-t\lambda\alpha) \\
 & \quad \times \sup_{x \in \mathbf{R}^{2n}} \|x^\gamma \partial_x^\beta E[S_0(t)^{l_2} e^{-i\lambda S_0(t)} M(t) \delta_x\left(\frac{x(t)}{\sqrt{2}}\right)]\| \\
 & \quad \times \sup_{r \geq r_0} |r|^{l_3} e^{r\lambda} \left( \int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu e^{-\mu^2} + \exp\left(-\frac{t}{2} \left(\frac{r}{t} + \lambda\right)^2\right) \right\} \\
 & \leq C(k, \gamma, \beta, t_0, t_1, \varepsilon, r_0) \sum_{0 \leq l_1, l_2, l_3 \leq 2k} \left[ \int_0^{\infty} d\lambda \cdot \lambda^{l_1} \sup_{t \in [t_0, t_1]} \{e^{-i\lambda\alpha} \right. \\
 & \quad \left. \times e^{\lambda(-n+\varepsilon)t} \exp\left(-\frac{t}{4} \lambda^2\right) \right] \\
 & \quad + \int_{-\infty}^0 d\lambda \cdot (-\lambda)^{l_1} \sup_{t \in [t_0, t_1]} \{e^{-i\lambda\alpha} e^{r_0\lambda} e^{-\lambda(-n+\varepsilon)t}\} \Big]
 \end{aligned}$$

for any  $r_0 \geq 0$  and  $\varepsilon > 0$ . The right hand side of (5.3) is finite if and only if  $\alpha < n$  or  $\alpha \leq n$  and  $r_0 > 0$ . In this case the formal computations in (5.2) are justified and  $\mathbf{q}_t^\alpha(x, u, r)$  is the restriction to  $\mathbf{R}^{2n+1} \times [r_0, \infty)$  of a rapidly decreasing function uniformly with respect to  $t \in [t_0, t_1]$  for any  $0 < t_0 \leq t_1$  and a smooth function of  $(t, x, u, r) \in (0, \infty) \times \mathbf{R}^{2n+1} \times [r_0, \infty)$ . By similar methods, we can prove that

$\mathcal{P}_i^\alpha(x, u)$  is a rapidly decreasing function of  $(x, u) \in \mathbf{R}^{2n+1}$  and a smooth function of  $(t, x, u) \in (0, \infty) \times \mathbf{R}^{2n+1}$ .  $\square$

Now, for any  $f \in L_2^{\mathcal{L},q}(D)$ , we define

$$(5.4) \quad \begin{aligned} (H_i^{(i)} f)(x, u, r) &:= \int_{\bar{D}} h_i^{(i)}((x, u, r), (x', u', r')) f(x', u', r') \, dx' \, du' \, dr' \\ (x, u, r) &\in \bar{D}, i = 1, 2, 3. \end{aligned}$$

For this  $H_i^{(i)}$ , we have the following (cf. [10] [11]):

**Theorem 5.1.**  $H_i^{(i)}, i=1, 2, 3$ , are bounded operators on  $L_2^{\mathcal{L},q}(D)$ . Furthermore, for fixed  $f \in \Lambda_0^{\mathcal{L},q}(D)$ , the following holds:

- (i) for  $T > 0$ , there is a constant  $C$  depending only on  $f$  and  $T$  such that for  $t \in (0, T]$ ,  $\|H_i^{(i)} f\| \leq C, i=1, 2, 3$ ;
- (ii)  $H_i^{(1)} f \rightarrow Pf, H_i^{(2)} f \rightarrow Qf, H_i^{(3)} f \rightarrow 0$  in  $L_2^{\mathcal{L},q}(D)$  and uniformly as  $t \rightarrow 0$ ;
- (iii)  $H_i^{(i)} f, i=1, 2, 3$ , are the restrictions of rapidly decreasing forms;
- (iv)  $H_i^{(i)} f, i=1, 2, 3$ , are differentiable in  $t$ ;
- (v)  $H_t f := H_i^{(1)} f + H_i^{(2)} f + H_i^{(3)} f \in \text{Dom } \square$ ;
- (vi)  $(\partial/\partial t + \square) H_t f = 0$ ;
- (vii)  $\square H_t f = H_t \square f$ .

Proof. For any  $r_0 \geq 0$ , we set  $D_{r_0} := \{(x, u, r) \in D; r \geq r_0\}$ . Then, for any  $r_0 \geq 0$  and  $f \in L_2^{\mathcal{L},q}(D)$ , we have

$$(5.5) \quad \begin{aligned} &\int_{D_{r_0}} dX \| (H_i^{(i)} f)(X) \|^2 \\ &\leq \int_{D_{r_0}} dX \int_D dY \| h_i^{(i)}(X, Y) \|^2 \int_D dY \| h_i^{(i)}(X, Y) \|^2 \| f(Y) \|^2 \\ &\leq \left( \sup_{x \in D_{r_0}} \int_D dY \| h_i^{(i)}(X, Y) \|^2 \right) \int_D dY \| f(Y) \|^2 \int_{D_{r_0}} dX \| h_i^{(i)}(X, Y) \|^2 \\ &\leq \left( \sup_{x \in D_{r_0}} \int_D dY \| h_i^{(i)}(X, Y) \|^2 \right) \left( \sup_{x \in D_{r_0}} \int_{D_{r_0}} dX \| h_i^{(i)}(X, Y) \|^2 \right) \int_D dY \| f(Y) \|^2 \\ &\leq \left( \int_{D_{r_0}} dY \| h_i^{(i)}(0, Y) \|^2 \right)^2 \int_D dY \| f(Y) \|^2. \end{aligned}$$

Hence  $H_i^{(i)}$  is a bounded linear operator from  $L_2^{\mathcal{L},q}(D)$  to  $L_2^{\mathcal{L},q}(D_{r_0})$  if  $r_0 > 0$  or  $(i, q) \neq (3, 0)$ . Especially, if  $(i, q) \neq (3, 0)$ ,  $H_i^{(i)}$  is a bounded linear operator on  $L_2^{\mathcal{L},q}(D)$ . In the followings, let  $f \in \Lambda_0^{\mathcal{L},q}(D)$ . We will prove (i)–(vii) in order.

(i): From (5.5), it is enough to show that

$$(5.6) \quad \sup_{t \in (0, T_1]} \int_D \| h_{t^2}^{(i)}(0, Y) \|^2 dY < \infty.$$

To do this for  $i=1$  and 2, it is enough to show that

$$(5.7) \quad \sup_{t \in (0, T_1]} \int_{\mathbf{R}^{n+1}} \| \mathcal{P}_{t^2}^\alpha(x, u) \|^2 dx du < \infty$$



for any  $\alpha \in \mathbf{Z}$ . We use scaling property of the Brownian motion and make the change of the variables  $(x \rightarrow \varepsilon x, u \rightarrow \varepsilon u, \varepsilon \lambda \rightarrow \lambda)$ :

$$\begin{aligned}
 & \int_{\mathbf{R}^{2n+1}} \|p_{\varepsilon^2}^\alpha(x, u)\| \, dx du \\
 &= \int_{\mathbf{R}^{2n+1}} dx du \left\| \int_{-\infty}^{\infty} d\lambda \frac{1}{2\pi} \exp(i\lambda u - \varepsilon \lambda \alpha - \frac{\lambda^2}{2}) \right. \\
 & \quad \times E[e^{-i\varepsilon \lambda S_0(1)} M^\varepsilon(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] \left. \right\| \\
 (5.8) \quad & \leq C \sup_{(x,u) \in \mathbf{R}^{2n+1}} (1 + |x|^2)^{2n} (1 + u^2) \left\| \int_{-\infty}^{\infty} d\lambda \exp(i\lambda u - \varepsilon \lambda \alpha - \frac{\lambda^2}{2}) \right. \\
 & \quad \times E[e^{-i\varepsilon \lambda S_0(1)} M^\varepsilon(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] \left. \right\| \\
 &= C \sup_{(x,u) \in \mathbf{R}^{2n+1}} (1 + |x|^2)^{2n} \left\| \int_{-\infty}^{\infty} d\lambda e^{i\lambda u} \left(1 - \frac{\partial^2}{\partial \lambda^2}\right) \right. \\
 & \quad \times \left\{ \exp\left(-\varepsilon \lambda \alpha - \frac{\lambda^2}{2}\right) E[e^{-i\varepsilon \lambda S_0(1)} M^\varepsilon(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] \right\} \left. \right\| \\
 & \leq C(\alpha) (1 + \varepsilon^2) \int_{-\infty}^{\infty} d\lambda (1 + |\lambda|^2) \exp\left(-\varepsilon \lambda \alpha - \frac{\lambda^2}{2}\right) \\
 & \quad \times \sum_{k=0}^2 \sup_{x \in \mathbf{R}^{2n}} (1 + |x|^2)^{2n} \|E[S_0(1)^k e^{-i\varepsilon \lambda S_0(1)} M^\varepsilon(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]\| \\
 & \leq C(\alpha, T).
 \end{aligned}$$

The last inequality follows from Theorem 2.2 (i). From (5.8), we obtain (5.7). For  $i=3$ , we use the following inequality instead of (5.5): if  $R$  is the distance from  $\text{supp } f$  to  $bD$ ,

$$(5.9) \quad \|H_i^{(3)}f\|_2 \leq C \left( \int_{\mathbf{R}^{2n+1}} dx du \int_R^\infty dr \|q_i^{n-2q}(x, u, r)\| \right) \|Qf\|_2.$$

As in (5.8), we use scaling property and make the change of variables  $(x \rightarrow \varepsilon x, u \rightarrow \varepsilon u, r \rightarrow \varepsilon r, \varepsilon \lambda \rightarrow \lambda)$ :

$$\begin{aligned}
 & \int_{\mathbf{R}^{2n+1}} dx du \int_R^\infty dr \|q_{\varepsilon^2}^\alpha(x, u, r)\| \\
 &= \frac{1}{\pi^{n+3/2}} \int_{\mathbf{R}^{2n+1}} dx du \int_{R/\varepsilon}^\infty dr \left\| \int_{-\infty}^{\infty} d\lambda \cdot \lambda \exp(i\lambda u - \varepsilon \alpha) \right. \\
 & \quad \times E[e^{-i\varepsilon \lambda S_0(1)} M^\varepsilon(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] e^{r\lambda} \int_{(r+\lambda)/\sqrt{2}} d\mu e^{-\mu^2} \left. \right\| \\
 & \leq C \int_{R/\varepsilon}^\infty dr \sup_{(x,u) \in \mathbf{R}^{2n+1}} (1 + |x|^2)^{2n} (1 + u^2) \left\| \int_{-\infty}^{\infty} d\lambda \cdot \lambda \exp(i\lambda u - \varepsilon \lambda \alpha) \right. \\
 & \quad \times E[e^{-i\varepsilon \lambda S_0(1)} M^\varepsilon(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] e^{r\lambda} \int_{(r+\lambda)/\sqrt{2}} d\mu e^{-\mu^2} \left. \right\| \\
 (5.10) \quad &= C \int_{R/\varepsilon}^\infty dr \sup_{(x,u) \in \mathbf{R}^{2n+1}} (1 + |x|^2)^{2n} \left\| \int_{-\infty}^{\infty} d\lambda e^{i\lambda u} \left(1 - \frac{\partial^2}{\partial \lambda^2}\right) \right\| \lambda e^{-\varepsilon \lambda \alpha}
 \end{aligned}$$

$$\begin{aligned}
& \times E[e^{-i\varepsilon\lambda S_0(1)} M^\varepsilon(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] e^{r\lambda} \int_{(r+\lambda)^{\sqrt{2}}} d\mu e^{-\mu^2} \Big\| \\
& \leq C(\alpha)(1+\varepsilon^2) \int_{R/\varepsilon}^\infty dr(1+r^2) \int_{-\infty}^\infty d\lambda(1+\lambda^2) e^{-\varepsilon\alpha} \\
& \quad \times \sum_{k=0}^2 \sup_{x \in \mathbb{R}^{2n}} (1+|x|^2)^{2n} \|E[S_0(1)^k e^{-i\varepsilon\lambda S_0(1)} M^\varepsilon(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]\| \\
& \quad \times e^{r\lambda} \left( \int_{(r+\lambda)^{\sqrt{2}}}^\infty d\mu e^{-\mu^2} + \exp\left(-\frac{1}{2}(r+\lambda)^2\right) \right) \\
& \leq C(\alpha, T, \eta) \int_{R/\varepsilon}^\infty dr(1+r^2) \int_{-\infty}^\infty d\lambda(1+\lambda^2) \exp(-\varepsilon\lambda\alpha - \varepsilon|\lambda|(n-\eta) \\
& \quad + r\lambda) \left( \int_{(r+\lambda)^{\sqrt{2}}}^\infty d\mu e^{-\mu^2} + \exp\left(-\frac{1}{2}(r+\lambda)^2\right) \right)
\end{aligned}$$

for any  $\eta > 0$ . The last inequality follows from Theorem 2.2 (i). Furthermore,

$$\begin{aligned}
& \int_{R/\varepsilon}^\infty dr(1+r^2) \int_{-\infty}^\infty d\lambda(1+\lambda^2) \exp(-\varepsilon\lambda\alpha - \varepsilon|\lambda|(n-\eta) + r\lambda) \\
& \quad \times \exp\left(-\frac{1}{2}(r+\lambda)^2\right) \\
& \leq C(\alpha, T, \eta) \exp\left(-\frac{R^2}{4\varepsilon^2}\right),
\end{aligned}$$

and

$$\begin{aligned}
& \int_{R/\varepsilon}^\infty dr(1+r^2) \int_0^\infty d\lambda(1+\lambda^2) \exp(-\varepsilon\lambda\alpha - \varepsilon|\lambda|(n-\eta) + r\lambda) \\
& \quad \times \int_{(r+\lambda)^{\sqrt{2}}}^\infty d\mu e^{-\mu^2} \\
& \leq C(\alpha, T, \eta) \exp\left(-\frac{R^2}{4\varepsilon^2}\right).
\end{aligned}$$

For  $0 < \eta < R/T^2$  and  $\alpha \leq n$ , we have

$$\begin{aligned}
& \int_{R/\varepsilon}^\infty dr(1+r^2) \int_{-\infty}^0 d\lambda(1+\lambda^2) \exp(-\varepsilon\lambda\alpha - \varepsilon|\lambda|(n-\eta) + r\lambda) \int_{(r+\lambda)^{\sqrt{2}}}^\infty d\mu e^{-\mu^2} \\
& = \int_{R/\varepsilon}^\infty dr(1+r^2) \exp\left(-\frac{r^2}{2}\right) \int_0^\infty d\mu \exp(-\mu^2 - \sqrt{2}r\mu) \\
& \quad \times \int_0^\infty d\lambda(1+\lambda^2) \exp\left(-\frac{\lambda^2}{2} + \lambda(\sqrt{2}\mu + \varepsilon(\alpha - n + \eta))\right) \\
& \leq C(\alpha, T, \eta) \int_{R/\varepsilon}^\infty dr(1+r^2) \exp\left(-\frac{r^2}{2}\right) \int_0^\infty d\mu \exp(\sqrt{2}\mu(\varepsilon(\alpha - n + \eta) - r)) \\
& \quad + \frac{\varepsilon^2}{2}(\alpha - n + \eta)^2 \\
& \leq C(\alpha, T, \eta) \int_{R/\varepsilon}^\infty dr(1+r^2) \exp\left(-\frac{r^2}{2}\right) \int_0^\infty d\mu \exp\left(-\sqrt{2}\varepsilon\mu\left(\frac{R}{T^2} - \eta\right)\right)
\end{aligned}$$

$$\begin{aligned} &\leq C(\alpha, T, \eta, R) \frac{1}{\varepsilon} \int_{R/\varepsilon}^{\infty} dr(1+r^2) \exp\left(-\frac{r^2}{2}\right) \\ &\leq C(\alpha, T, \eta, R) \exp\left(-\frac{R^2}{4\varepsilon^2}\right). \end{aligned}$$

Hence if  $\alpha \leq n$ , we have

$$(5.11) \quad \int_{\mathbb{R}^{2n+1}} dxdu \int_{\mathbb{R}} dr \|\mathbf{q}_{\varepsilon^2}^{\alpha}(x, u, r)\| \leq C(\alpha, T, R) \exp\left(-\frac{R^2}{4\varepsilon^2}\right)$$

for any  $\varepsilon \in [0, T]$ . This lead to (i) for  $i=3$ .

(ii) To prove (ii) for  $i=1$  and 2, it is enough to show that

$$(5.12) \quad \begin{aligned} &\int_D \mathbf{P}_i^{\alpha}((x, u)^{-1}(x', u')) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(r-r')^2}{2t}\right) f(x', u', r') dx' du' dr' \\ &\rightarrow f(x, u, r) \end{aligned}$$

and

$$(5.13) \quad \begin{aligned} &\int_D \mathbf{P}_i^{\alpha}((x, u)^{-1}(x', u')) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(r+r')^2}{2t}\right) f(x', u', r') dx' du' dr' \\ &\rightarrow 0 \end{aligned}$$

in  $L_2^{p,q}(D)$  and uniformly as  $t \rightarrow 0$ . We set

$$\begin{aligned} \mathbf{p}_i^{0,\alpha}(x, u) &:= \int_{-\infty}^{\infty} d\lambda f_i^{0,\alpha}(x, u, \lambda) \exp\left(-\frac{t}{2} \lambda^2\right), \\ \mathbf{P}_i^{-,\alpha}(x, u) &:= \mathbf{P}_i^{\alpha}(x, u) - \mathbf{p}_i^{0,\alpha}(x, u) I. \end{aligned}$$

As in (5.5), we have

$$(5.14) \quad \begin{aligned} &\int_D \left\| \int_D \mathbf{P}_i^{-,\alpha}((x, u)^{-1}(x' u')) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(r \pm r')^2}{2t}\right) \right. \\ &\quad \left. \times f(x' u' r') dx' du' dr' \right\|^2 dx du dr \\ &\leq C \left( \int_{\mathbb{R}^{2n+1}} \|\mathbf{P}_i^{-,\alpha}(x, u)\| dx du \right)^2 \|f\|_2^2. \end{aligned}$$

On the other hand, it is easy to see that

$$(5.15) \quad \begin{aligned} &\sup_{(x,u,r) \in D} \left\| \int_D \mathbf{P}_i^{-,\alpha}((x, u)^{-1}(x' u')) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(r \pm r')^2}{2t}\right) \right. \\ &\quad \left. \times f(x', u', r') dx' du' dr' \right\| \\ &\leq \int_{\mathbb{R}^{2n+1}} \|\mathbf{P}_i^{-,\alpha}(x, u)\| dxdu \|f\|_{\infty}. \end{aligned}$$

If we set  $t = \varepsilon^2$ , as in (5.8), we have

$$\int_{\mathbb{R}^{2n+1}} \|\mathbf{P}_{\varepsilon^2}^{-,\alpha}(x, u)\| dxdu$$

$$\begin{aligned}
 (5.16) \quad &\leq C(T, \alpha) \sum_{k=0}^2 \sup_{\lambda \in \mathbb{R}, t \in \mathbb{R}^{2n}} (1 + |x|^2)^{2n} \| E[S_0(1)^k e^{-i\lambda S_0(1)} \\
 &\quad \times (M^\varepsilon(1) - I) \delta_x \left( \frac{x(1)}{\sqrt{2}} \right) ] \| \\
 &\leq C(T, \alpha) \varepsilon
 \end{aligned}$$

for  $0 < \varepsilon \leq T$ . Here we used Theorem 2.2 (ii). Hence

$$\begin{aligned}
 (5.17) \quad &\int_D \mathbf{p}_i^{-\alpha}((x, u)^{-1}(x', u')) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(r \pm r')^2}{2t}\right) f(x', u', r') dx' du' dr' \\
 &\rightarrow 0
 \end{aligned}$$

in  $L^{\frac{1}{2}, q}(D)$  and uniformly as  $t \rightarrow 0$ . The following facts are proved in Stanton [10]:

$$\begin{aligned}
 (5.18) \quad &\int_D \mathbf{p}_i^{0, \alpha}((x, u)^{-1}(x', u')) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(r - r')^2}{2t}\right) f(x', u', r') dx' du' dr' \\
 &\rightarrow f(x, u, r),
 \end{aligned}$$

and

$$\begin{aligned}
 (5.19) \quad &\int_D \mathbf{p}_i^{0, \alpha}((x, u)^{-1}(x', u')) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(r + r')^2}{2t}\right) f(x', u', r') dx' du' dr' \\
 &\rightarrow 0
 \end{aligned}$$

in  $L^{\frac{1}{2}, q}(D)$  and uniformly as  $t \rightarrow 0$ . From (5.16–18), we conclude (5.12) and (5.13).

Next we will consider  $H_i^{(3)}f$ . If  $R$  is the distance from  $\text{supp } f$  to  $bD$ ,

$$\begin{aligned}
 (5.20) \quad &\sup_{(x, u, r) \in D} \|(H_i^{(3)}f)(x, u, r)\| \\
 &\leq \left( \int_{\mathbb{R}^{2n+2}} dx du \int_R^\infty dr \|q_i^{(n-2q)}(x, u, r)\| \right) \sup_{(x, u, r) \in D} \|Qf(x, u, r)\|.
 \end{aligned}$$

Thus, by (5.9) and (5.20), to prove (ii) for  $i=3$ , it is enough to show that

$$(5.21) \quad \int_{\mathbb{R}^{2n+1}} dx du \int_R^\infty dr \|q_i^\alpha(x, u, r)\| \rightarrow 0 \quad \text{as } t \rightarrow 0$$

for  $\alpha \leq n$ . We set  $t = \varepsilon^2$  and calculate like (5.10). Then for  $\alpha \leq n$  and  $0 < \varepsilon \leq T$ , we have

$$(5.22) \quad \int_{\mathbb{R}^{2n+1}} dx du \int_R^\infty dr \|q_{\varepsilon^2}^\alpha(x, u, r)\| \leq C(\alpha, T, R) \exp\left(-\frac{R^2}{4\varepsilon^2}\right),$$

which proves (5.21).

(iii) (iv): Since  $f$  has compact support, we can easily prove (iii) and (iv) by using Proposition 5.1.

(v): From (5.1), we easily see that

$$(5.23) \quad h_i^{(i)}((x, u, 0), (x', u', r')) = 0.$$

On the other hand,

$$(5.24) \quad \begin{aligned} & \left( \frac{\partial}{\partial r} + i \frac{\partial}{\partial u} \right) \left[ \frac{1}{\sqrt{2\pi t}} \left\{ \exp\left(-\frac{(r-r')^2}{2t}\right) + \exp\left(-\frac{(r+r')^2}{2t}\right) \right\} \right. \\ & \quad \times \int_{-\infty}^{\infty} d\lambda \cdot f_i^t(x, u, r) \exp\left(-\frac{t}{2} \lambda^2\right) \\ & \quad \left. - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\lambda \cdot \lambda f_i^{\alpha}(x, u, \lambda) e^{(r+r')\lambda} \int_{\sqrt{t/2}((r+r')/t+\lambda)}^{\infty} d\mu e^{-\mu^2} \right] \Big|_{r=0} \\ & = 0 \end{aligned}$$

Hence we have

$$(5.25) \quad \left( \frac{\partial}{\partial r} - i \frac{\partial}{\partial u} \right) (h_i^{(2)} + h_i^{(3)})((x, u, 0), (x', u', r')) = 0.$$

(vi): By Feynman-Kac formula, we see that

$$(5.26) \quad \frac{\partial}{\partial t} f_i^{\alpha}((x, u)^{-1}(x', u'), \lambda) = -\square_{\lambda}^{\alpha} f_i^{\alpha}((x, u)^{-1}(x', u'), \lambda)$$

where

$$\begin{aligned} \square_{\lambda}^{\alpha} &= -\frac{1}{4} \sum_{j=1}^n \left\{ \left( \frac{\partial}{\partial x^j} - 2i\lambda x^{n+j} \right)^2 + \left( \frac{\partial}{\partial x^{n+j}} + 2i\lambda x^j \right)^2 \right\} \\ & \quad - \sum_{j=1}^n \frac{A_j}{\sqrt{2}} \left( \frac{\partial}{\partial x^j} - 2i\lambda x^{n+j} \right) - \sum_{j=1}^n \frac{A_{n+j}}{\sqrt{2}} \left( \frac{\partial}{\partial x^{n+j}} + 2i\lambda x^j \right) \\ & \quad + B + \lambda \alpha. \end{aligned}$$

By using the fact that  $f_i^{\alpha}((x, u), \lambda)$  has good regularity, we have

$$(5.27) \quad \begin{aligned} & \frac{\partial}{\partial t} h_i^{(i)}((x, u, r), (x', u', r')) \\ & = -\square h_i^{(i)}(x, u, r), (x', u', r'), \quad t > 0, \quad (x, u, r) \in \bar{D}, \end{aligned}$$

for each  $(x', u', r') \in \bar{D}$  and  $i=1, 2, 3$  ( $(x', u', r') \in D$  for  $q=0$  and  $i=3$ ).

(vii): By using Markovian property of the Brownian motion, we can prove that

$$(5.28) \quad \begin{aligned} & \square_{\lambda}^{\alpha} \int_{H_n} f_i^{\alpha}((x, u)^{-1}(x', u'), \lambda) g(x', u') dx' du' \\ & = \int_{H_n} f_i^{\alpha}((x, u)^{-1}(x', u'), \lambda) \square_{\lambda}^{\alpha} g(x', u') dx' du' \end{aligned}$$

for any rapidly decreasing section on  $H_n$  to  $\Lambda^{p,q}(T^*D) \Big|_{H_n}$  where  $\square_{\lambda}^{\alpha}$  is that of (5.26). Then we can easily prove (vii).

Finally we prove that  $H_t^{(3)}$  is a bounded operator on  $L_2^{\beta,0}(D)$ . By the above (i-vii) and the proof of Proposition 2.1, we see that

$$(5.29) \quad H_t f = e^{-t\Box} f \quad \text{for } f \in \Lambda^{\beta,q}(\bar{D}).$$

For any  $r_0 > 0$ , since both  $H_t$  and  $e^{-t\Box}$  are bounded operators from  $L_2^{\beta,0}(D)$  to  $L_2^{\beta,0}(D_{r_0})$ , we have

$$(H_t f)(X) = (e^{-t\Box} f)(X) \quad \text{a.e. } X \in D_{r_0}$$

for any  $f \in L_2^{\beta,q}(D)$ . Hence we have

$$(5.30) \quad H_t f = e^{-t\Box} f \quad \text{for any } f \in L_2^{\beta,q}(D).$$

Since  $e^{-t\Box}$  is a bounded operator on  $L_2^{\beta,0}(D)$ ,  $H_t$  is also a bounded operator on  $L_2^{\beta,0}(D)$ .  $\square$

By Proposition 5.1 and Theorem 5.1, we see that the integral operator involved by the kernel  $h_t(X, Y)$  satisfies the conditions of a fundamental solution in Section 2. Hence we conclude Theorem 2.1.

### 6. Short time asymptotic behavior

In this section, we examine the behaviour of  $h_t$  on the diagonal of  $\bar{D} \times \bar{D}$  as  $t \rightarrow 0$  (cf. Stanton [11] § 6, Beals-Stanton [2]). By (5.1), we have

$$(6.1) \quad \begin{aligned} & h_t((x, u, r), (x, u, r)) \\ &= p_t^{n-2q+2}(0) \frac{1}{\sqrt{2\pi t}} P - p_t^{n-2q+2}(0) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{2r^2}{t}\right) P \\ &+ p_t^{n-2q}(0) \frac{1}{\sqrt{2\pi t}} Q + p_t^{n-2q}(0) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{2r^2}{t}\right) Q \\ &+ q_t^{n-2q}(0, 2r) Q \\ &=: k_t^{n-2q+2}(0) P - k_t^{n-2q+2}(r) P \\ &+ k_t^{n-2q}(0) Q + k_t^{n-2q}(r) Q + q_t^{n-2q}(0, 2r) Q \end{aligned}$$

where

$$k_t^\alpha(r) = p_t^\alpha(0) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{2r^2}{t}\right).$$

We examine the behaviour of  $k_t^\alpha(r)$  for  $\alpha \in \mathbb{Z}$  and  $q_t^\alpha(0, 2r)$  for  $\alpha \leq n$  as  $t \rightarrow 0$ .

**Proposition 6.1.** *For any  $\alpha \in \mathbb{Z}$ ,*

$$(6.2) \quad \lim_{t \rightarrow 0} t^{n+1} k_t^\alpha(0) = \frac{1}{2\pi^{n+1}} I.$$

*Proof.*  $k_t^\alpha(0)$  is decomposed as follows:

$$\begin{aligned}
 (6.3) \quad k_i^\alpha(0) &= p_i^\alpha(0) \frac{1}{\sqrt{2\pi t}} \\
 &= p_i^{0,\alpha}(0) \frac{1}{\sqrt{2\pi t}} I + p_i^{-,\alpha}(0) \frac{1}{\sqrt{2\pi t}}.
 \end{aligned}$$

As in Stanton [11] Proposition 6.1,

$$(6.4) \quad \lim_{t \rightarrow 0} t^{n+1} p_i^{0,\alpha}(0) \frac{1}{2\pi t} = \frac{1}{2\pi^{n+1}}.$$

For the second term of the right hand side of (6.3), we take  $t = \varepsilon^2$  and use scaling property of the Brownian motion and Theorem 2.2 (ii): for any  $\varepsilon \in (0, T]$ ,

$$\begin{aligned}
 (6.5) \quad & \| \varepsilon^{2(n+1)} p_{\varepsilon^2}^{-,\alpha}(0) \frac{1}{\sqrt{2\pi\varepsilon^2}} \| \\
 &= \| \varepsilon^2 \int_{-\infty}^{\infty} d\lambda \frac{1}{2\pi} \exp\left(-\varepsilon^2 \lambda \alpha - \frac{\varepsilon^2}{2} \lambda^2\right) \\
 &\quad \times E[e^{-i\varepsilon^2 \lambda S_0(1)} (M^\varepsilon(1) - I) \delta_0\left(\frac{x(1)}{\sqrt{2}}\right)] \| \\
 &\leq C(T) \varepsilon^2.
 \end{aligned}$$

Hence we have

$$(6.6) \quad \lim_{t \rightarrow 0} t^{n+1} p_i^{-,\alpha}(0) \frac{1}{\sqrt{2\pi t}} = 0.$$

By (6.4) and (6.6), we conclude (6.2).  $\square$

Next we consider the boundary correction term:

**Proposition 6.2.** *For any  $\alpha \in \mathbf{Z}$ ,  $r > 0$  and  $k \in \mathbf{Z}$ ,*

$$(6.7) \quad \lim_{t \rightarrow 0} t^k k_i^\alpha(r) = 0.$$

*For any  $\alpha \leq n$ ,  $r > 0$  and  $k \in \mathbf{Z}$ ,*

$$(6.8) \quad \lim_{t \rightarrow 0} t^k q_i^\alpha(0, 2r) = 0.$$

*Proof.* We easily see that

$$(6.9) \quad k_i^\alpha(r) = k_i^\alpha(0) \exp\left(-\frac{2r^2}{t}\right)$$

and  $\lim_{t \rightarrow 0} t^k \exp(-2r^2/t) = 0$ . Thus we conclude (6.7) from (6.2). Since

$$e^{2r\lambda} \int_{\sqrt{t/2}(2r/t+\lambda)}^{\infty} d\mu e^{-\mu^2} \leq \begin{cases} \exp\left(-\frac{2r^2}{t}\right) \int_{\sqrt{t/2}\lambda}^{\infty} d\mu e^{-\mu^2}, \\ \exp\left(-\frac{r^2}{t} + \sqrt{2}r\lambda\right) \int_{\sqrt{t/2}(r/\sqrt{2t}+\lambda)}^{\infty} d\mu e^{-\mu^2}, \end{cases}$$

we have

$$\begin{aligned}
 & \|q_i^\alpha(0, 2r)\| \\
 & \leq C \{ \exp(-2r^2/t) \int_0^\infty d\lambda \cdot \lambda \|f_i^\alpha(0, \lambda)\| \int_{\sqrt{t/2}\lambda}^\infty d\mu e^{-\mu^2} \\
 (6.10) \quad & + \exp\left(-\frac{r^2}{t}\right) \int_{-\infty}^0 d\lambda \cdot (-\lambda) \|f_i^\alpha(0, \lambda)\| \exp(\sqrt{2r}\lambda) \\
 & \quad \times \int_{\sqrt{t/2}(r/\sqrt{2t}+\lambda)}^\infty d\mu e^{-\mu^2} \} \\
 & \leq C' \exp\left(-\frac{r^2}{t}\right) \{ \int_0^\infty d\lambda \cdot \lambda \|f_i^\alpha(0, \lambda)\| \exp\left(-\frac{t\lambda^2}{4}\right) \\
 & \quad + \int_{-\infty}^0 d\lambda \cdot (-\lambda) \|f_i^\alpha(0, \lambda)\| e^{\sqrt{2}r\lambda} \}.
 \end{aligned}$$

As in (5.15), we take  $t = \varepsilon^2 (\leq T^2)$  and use scaling property of the Brownian motion and Theorem 2.2 (i): for any  $\varepsilon \in (0, T]$ ,

$$\begin{aligned}
 & \int_0^\infty d\lambda \cdot \lambda \|f_{\varepsilon^2}^\alpha(0, \lambda)\| \exp\left(-\frac{\varepsilon^2 \lambda^2}{4}\right) \\
 (6.11) \quad & = C\varepsilon^{-2n} \int_0^\infty d\lambda \cdot \lambda e^{-\varepsilon^2 \lambda \alpha} \|E[e^{-i\varepsilon^2 \lambda S_0(1)} M^\varepsilon(1) \delta_0\left(\frac{x(1)}{\sqrt{2}}\right)]\| \exp\left(-\frac{\varepsilon^2 \lambda^2}{4}\right) \\
 & \leq C(T, \alpha) \varepsilon^{-2(n+1)}.
 \end{aligned}$$

For  $0 < \eta < \sqrt{2}r/T^2$  and  $\alpha \leq n$ ,

$$\begin{aligned}
 & \int_{-\infty}^0 d\lambda \cdot (-\lambda) \|f_{\varepsilon^2}^\alpha(0, \lambda)\| e^{\sqrt{2}r\lambda} \\
 (6.12) \quad & = C\varepsilon^{-2n} \int_{-\infty}^0 d\lambda \cdot (-\lambda) e^{-\varepsilon^2 \lambda \alpha} \|E[e^{-i\varepsilon^2 \lambda S_0(2)} M^\varepsilon(1) \delta_0\left(\frac{x(1)}{\sqrt{2}}\right)]\| e^{\sqrt{2}r\lambda} \\
 & \leq C(\alpha, T, \eta) \varepsilon^{-2(n+2)}.
 \end{aligned}$$

Hence, if  $\alpha \leq n$ , for any  $k \in \mathbb{Z}$ ,

$$\begin{aligned}
 & \|t^k q_i^\alpha(0, 2r)\| \\
 (6.13) \quad & \leq C(T, \alpha) \exp\left(-\frac{r^2}{t}\right) t^{-(n+2)+k} \rightarrow 0 \quad \text{as } t \rightarrow 0.
 \end{aligned}$$

□

Next result suggest the short time asymptotic behaviour of the trace of the heat semigroup by Beals and Stanton [2]:

**Proposition 6.3.** (i) For any  $\alpha \in \mathbb{Z}$ ,

$$(6.14) \quad \lim_{t \rightarrow 0} t^{n+1/2} \int_0^\infty dr K_i^\alpha(r) = \frac{1}{4\sqrt{2}\pi^{n+1/2}} I.$$

(ii) For any  $\alpha < n$ ,



$$(6.15) \quad \begin{aligned} & \lim_{t \rightarrow 0} t^{n+1} \int_0^\infty dr \mathbf{q}_t^\alpha(0, 2r) \\ &= \frac{1}{2\pi^{n+1}} \int_0^\infty d\tau \left( \frac{\tau}{\sinh \tau} \right)^n e^{\omega\tau} I. \end{aligned}$$

Proof. (i) follows from (6.2) and (6.9). We consider (ii),  $\mathbf{q}_t^\alpha(0, 2r)$  is decomposed as follows:

$$(6.16) \quad \mathbf{q}_t^\alpha(0, 2r) = q_t^{0,\alpha}(0, 2r) I + \mathbf{q}_t^{-,\alpha}(0, 2r).$$

As is shown in Stanton [11] Theorem 2.2 (i),

$$(6.17) \quad \lim_{t \rightarrow 0} t^{n+1} \int_0^\infty dr q_t^{0,\alpha}(0, 2r) = \frac{1}{2\pi^{n+1}} \int_0^\infty d\tau \left( \frac{\tau}{\sinh \tau} \right)^n e^{\omega\tau}.$$

On the other hand,

$$(6.18) \quad \begin{aligned} & \left\| \int_0^\infty dr \mathbf{q}_t^{-,\alpha}(0, 2r) \right\| \\ & \leq C \int_{-\infty}^\infty d\lambda |\lambda| \|\mathbf{f}_t^{-,\alpha}(0, \lambda)\| \\ & \quad \times \int_{\sqrt{t/2}\lambda}^\infty d\mu e^{-\mu^2} \int_0^{\sqrt{t/2}\mu - t\lambda/2} dr e^{2r\lambda} \\ & = C \int_{-\infty}^\infty d\lambda \|\mathbf{f}_t^{-,\alpha}(0, \lambda)\| \\ & \quad \times \left| \frac{\sqrt{\pi}}{2} \exp\left(-\frac{t}{2}\lambda^2\right) - \int_{\sqrt{t/2}\lambda}^\infty d\mu e^{-\mu^2} \right| \\ & \leq C' \left\{ \int_0^\infty d\lambda \cdot \exp\left(-\frac{t}{4}\lambda^2\right) \|\mathbf{f}_t^{-,\alpha}(0, \lambda)\| \right. \\ & \quad \left. + \int_{-\infty}^0 d\lambda \|\mathbf{f}_t^{-,\alpha}(0, \lambda)\| \right\}. \end{aligned}$$

Now we calculate as in (6.11) and (6.12) by using Theorem 2.2 (ii): for  $\varepsilon \in [0, T]$ ,

$$(6.19) \quad \int_0^\infty d\lambda \cdot \exp\left(-\frac{\varepsilon_2}{4}\lambda^2\right) \|\mathbf{f}_{\varepsilon_2}^{-,\alpha}(0, \lambda)\| \leq C(T) \varepsilon^{-2n}$$

and

$$(6.20) \quad \int_{-\infty}^0 d\lambda \|\mathbf{f}_{\varepsilon_2}^{-,\alpha}(0, \lambda)\| \leq C(T) \varepsilon^{-2n-1}.$$

Hence we have

$$(6.21) \quad \left\| t^{n+1} \int_0^\infty dr \mathbf{q}_t^{-,\alpha}(0, 2r) \right\| \leq C(T) \sqrt{t} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

From (6.17) and (6.21), we conclude (6.15).  $\square$

**7. The analogue of the Siegel domain with a nondegenerate indefinite Levi form**

For  $0 \leq \kappa \leq n$ , let  $D_\kappa = \{(z, w) \in \mathbf{C}^n \times \mathbf{C} : \text{Im } w > \sum_{j=1}^n \varepsilon_j |z^j|^2\}$  where  $\varepsilon_j = 1$  for  $1 \leq j \leq \kappa$  and  $\varepsilon_j = -1$  for  $\kappa < j \leq n$ . This is an analogue of the Siegel domain  $D$  with a nondegenerate indefinite Levi form having  $\kappa$  positive and  $n - \kappa$  negative eigenvalues. In [10], Stanton states that her results also apply to this domain in the  $(0, q)$ -form case. In the general  $(p, q)$ -form case, we need a few remarks to apply our methods (cf. Remark 7.1 (i) below). Our main theorem is Theorem 7.2 below.

We consider the Hermitian metric for which  $\{Z_1, Z_2, \dots, Z_{n+1}\}$  is an orthonormal basis of  $T^{(1,0)}(D_\kappa)$ , where

$$(7.1) \quad Z_j = \frac{\partial}{\partial z^j} + 2i\varepsilon_j \bar{z}^j \frac{\partial}{\partial w}, \quad j = 1, 2, \dots, n, \quad Z_{n+1} = i\sqrt{2} \frac{\partial}{\partial w}.$$

The dual basis is given by

$$(7.2) \quad \omega^j = dz^j, \quad \omega^{n+1} = \frac{1}{i\sqrt{2}}(dw - 2i \sum \varepsilon_j \bar{z}^j dz^j).$$

Then  $\bar{\partial}$  and  $\bar{\partial}^*$  can be represented as follows:

$$(7.3) \quad \begin{aligned} \bar{\partial} &= \sum_{j=1}^{n+1} \text{ext}(\bar{\omega}^j) \bar{Z}_j + \sqrt{2} \sum_{j=1}^n \varepsilon_j \text{ext}(\omega^j) \text{ext}(\bar{\omega}^j) \text{int}(\bar{\omega}^{n+1}), \\ \bar{\partial}^* &= -\sum_{j=1}^{n+1} \text{int}(\omega^j) \bar{Z}_j + \sqrt{2} \sum_{j=1}^n \varepsilon_j \text{ext}(\omega^{n+1}) \text{int}(\omega^j) \text{int}(\bar{\omega}^j), \end{aligned}$$

on  $S^{p,q}(\bar{D}_\kappa)$ .

We use as coordinates on  $D_\kappa(z, u, r)$  where  $u = \text{Re } w$  and  $r = \text{Im } w - \sum_{j=1}^n \varepsilon_j |z^j|^2$ . Then we can regard  $D_\kappa$  as product of the boundary  $bD_\kappa$  and  $\mathbf{R}^+$ . We identify  $bD_\kappa$  with the Heisenberg group  $H_n$ . The group law on  $H_n$  is

$$(7.4) \quad \begin{aligned} (z', u')(z, u) &= (z' + z, u' + u + 2 \sum_{j=1}^n \varepsilon_j (x'^{n+j} x^j - x'^j x^{n+j})) \\ &\text{for } (z', u'), (z, u) \in H_n, \end{aligned}$$

where  $z^j = x^j + ix^{n+j}$ . The metric we gave is the product metric of an invariant metric on  $H_n$  and the standard metric on  $\mathbf{R}^+$ . In terms of these coordinates,

$$(7.5) \quad Z_j = \frac{\partial}{\partial z^j} + i\varepsilon_j \bar{z}^j \frac{\partial}{\partial u}, \quad j = 1, 2, \dots, n, \quad Z_{n+1} = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial r} + i \frac{\partial}{\partial u} \right).$$

Then we have

$$Z_j = 1/2(X_j - iX_{n+j})$$

where

$$(7.6) \quad X_j = \frac{\partial}{\partial x^j} + 2\varepsilon_j x^{n+j} \frac{\partial}{\partial u}, \quad X_{n+j} = \frac{\partial}{\partial x^{n+j}} - 2\varepsilon_j x^j \frac{\partial}{\partial u}, \quad j = 1, 2, \dots, n.$$

If we define the projections  $P$  and  $Q$  by

$$P = \text{ext}(\bar{\omega}^{n+1}) \text{int}(\omega^{n+1}) \quad \text{and} \quad Q = \text{int}(\omega^{n+1}) \text{ext}(\bar{\omega}^{n+1}),$$

the  $\bar{\partial}$ -Neumann boundary condition can be rewritten as

$$(7.7) \quad Pf \upharpoonright_{bD_\kappa} = Q\left(\frac{\partial}{\partial r} - i\frac{\partial}{\partial u}\right)f \upharpoonright_{bD_\kappa} = 0 \quad \text{for } f \in S^{p,q}(\bar{D}_\kappa).$$

The  $\bar{\partial}$ -Laplacian  $\square$  is expressed on  $S^{p,q}(\bar{D}_\kappa)$  as follows:

$$(7.8) \quad \square = -\frac{1}{4} \sum_{j=1}^{2n} \underline{X}_j^2 - \frac{1}{2} \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial r^2} \right) + iU \frac{\partial}{\partial u} - \sum_{j=1}^{2n} \frac{A_j}{\sqrt{2}} X_j - B$$

where

$$\begin{aligned} U &= \sum_{j=1}^n \varepsilon_j [\text{int}(\omega^j), \text{ext}(\bar{\omega}^j)], \\ A_j &= \varepsilon_j (\text{ext}(\omega^{n+1}) \text{int}(\bar{\omega}^j) - \text{ext}(\omega^j) \text{int}(\bar{\omega}^{n+1})), \quad j = 1, 2, \dots, n, \\ A_{n+j} &= i\varepsilon_j (\text{ext}(\omega^{n+1}) \text{int}(\bar{\omega}^j) - \text{ext}(\omega^j) \text{int}(\bar{\omega}^{n+1})), \quad j = 1, 2, \dots, n, \\ B &= -2 \sum_{j,k=1}^n \varepsilon_j \varepsilon_k \text{ext}(\omega^j) \text{ext}(\bar{\omega}^j) \text{int}(\omega^k) \text{int}(\bar{\omega}^k) \\ &\quad - 2 \sum_{j=1}^n (\text{int}(\omega^j) \text{ext}(\bar{\omega}^j) - \text{ext}(\omega^j) \text{int}(\bar{\omega}^j)) \text{ext}(\omega^{n+1}) \text{int}(\bar{\omega}^{n+1}). \end{aligned}$$

REMARK 7.1. (i) The coefficient  $U$  of  $\partial/\partial u$  in (7.8) does not commute with  $A_j$  and  $B$ . This is the only difficult point of the problem in the domain  $D_\kappa$  (cf. Lemma 7.2 and Theorem 7.1).

(ii) As in Remark 2.2,  $\square$  acts diagonally on  $(0, q)$ -forms and generally preserves the orthogonal decomposition  $\Lambda^{p,q}(T^*D) = \text{Ran } P \oplus \text{Ran } Q$ .

Now we consider the heat equation (1.1) for  $D_\kappa$ . Let  $(W_0^{2(n+1)}, P)$  be a  $2(n+1)$ -dimensional Wiener space as in Section 3. We consider the following diffusion process  $(X(t), U(t), R(t))$ :

$$(7.9) \quad \begin{cases} X^j(t) = x^j + \frac{x^j(t)}{\sqrt{2}}, & j = 1, 2, \dots, 2n, \\ U(t) = u + u(t) + S_x^c(t) - i\phi(t), \\ R(t) = r + B(t) + \phi(t) \end{cases}$$

where

$$S_x^\kappa(t) = \sum_{j=1}^n \varepsilon_j \int_0^t \sqrt{2}(X^{n+j}(s) \circ dx^j(s) - X^j(s) \circ dx^{n+j}(s)),$$

$$B(t) = \int_0^t \operatorname{sgn}(r+r(s))dr(s)$$

and

$$\phi(t) = \lim_{\eta \searrow 0} \frac{1}{2\eta} \int_0^t I_{(-\eta, \eta)}(r+r(s))ds .$$

We define the End  $(\Lambda^{p,q}(T^*D_\kappa))$ -valued process  $M(t, x(\cdot), u(\cdot))$  by the solution of the SDE

$$(7.10) \quad \begin{cases} dM(t) = M(t)d\Xi(t), \\ M(0) = I \end{cases}$$

where

$$\Xi(t) = -iUu(t) + \sum_{j=1}^{2n} A_j x^j(t) + Bt .$$

Let  $K(t)$  be the End  $(\Lambda^{p,q}(T^*D_\kappa))$ -valued process as in (3.5).

Then the heat kernel  $h_t$  is expected to be represented as follows

$$(7.12) \quad h_t(X, X') = \int_{-\infty}^{\infty} d\lambda H_{t,\lambda}(X, X')$$

where

$$(7.13) \quad \begin{aligned} & H_{t,\lambda}((x, u, r), (x', u', r')) \\ &= \frac{1}{2\pi} e^{i\lambda(u'-u)} E^x[e^{-i\lambda S_x^\kappa(t)} E^u[e^{-i\lambda u(t)} M(t, x(\cdot), u(\cdot))]] \delta_{x'}(X(t)) \\ & \quad \times E[e^{-\lambda \phi(t)} K(t) | R(t) = r'] r_t(r, r') \end{aligned}$$

where  $E^u$  is the expectation with respect to the 1-dimensional Brownian motion  $u(\cdot)$  and  $E^x$  is the generalized expectation of the generalized Wiener functional with respect to the  $2n$ -dimensional Brownian motion  $x(\cdot)$ .

For any  $\varepsilon > 0$ ,  $m \in \mathbf{N}$  and  $\lambda \in \mathbf{R}$ , we define the End  $(\Lambda^{p,q}(T^*D_\kappa))$ -valued processes  $m_m^\lambda(t)$  and  $M_m^{\lambda,\varepsilon}(t)$  by the solutions of

$$(7.14) \quad \begin{cases} dm_m^\lambda(t) = m_m^\lambda(t) \left( -\frac{\lambda}{m} U \right) dt, \\ m_m^\lambda(0) = I \end{cases}$$

and

$$(7.15) \quad \begin{cases} dM_m^{\lambda,\varepsilon}(t) = M_m^{\lambda,\varepsilon}(t) d\Xi_m^{\lambda,\varepsilon}(t), \\ M_m^{\lambda,\varepsilon}(0) = I \end{cases}$$

respectively, where

$$\Xi_m^{\lambda, \varepsilon}(t) = \sum_{j=1}^{2n} \frac{\varepsilon}{\sqrt{m}} A_j x^j(t) + \frac{\varepsilon^2}{m} Bt - \lambda \frac{\varepsilon^2}{m} Ut.$$

When  $m=1$ , we omit the subscript  $m$ :  $\Xi^{\lambda, \varepsilon}(t) =: \Xi^{\lambda, \varepsilon}(t)$ ,  $M_1^{\lambda, \varepsilon}(t) =: M^{\lambda, \varepsilon}(t)$ ,  $m_1^{\lambda}(t) =: m^{\lambda}(t)$ . When  $m=\varepsilon=1$ , we also omit the superscript  $\varepsilon$ :  $\Xi_1^{\lambda, 1}(t) =: \Xi^{\lambda}(t)$ ,  $M_1^{\lambda, 1}(t) =: M^{\lambda}(t)$ .

Then  $H_{t, \lambda}(X, X')$  is rewritten as follows:

**Lemma 7.2.**  $H_{t, \lambda}(X, X')$  is expressed as follows :

$$\begin{aligned} & H_{t, \lambda}((x, u, r), (x', u', r')) \\ (7.16) \quad & = f_i((x, u)^{-1}(x', u'), \lambda) \exp\left(-\frac{t}{2}\lambda^2\right) e_{\bar{i}}(r, r') P \\ & + f_i((x, u)^{-1}(x', u'), \lambda) \exp\left(-\frac{t}{2}\lambda^2\right) e_i^{\dagger}(r, r') Q \\ & - \frac{2}{\sqrt{\pi}} \lambda f_i((x, u)^{-1}(x', u'), \lambda) e^{r\lambda} \int_{\sqrt{t/2}(r/t+\lambda)}^{\infty} d\mu e^{-\mu^2} Q \end{aligned}$$

where

$$f_i(x, u, \lambda) = \frac{1}{2\pi} e^{i\lambda u} E[e^{-i\lambda S_0^{\varepsilon}(t)} M^{\lambda}(t) \delta_x\left(\frac{x(t)}{\sqrt{2}}\right)].$$

Proof. Generally for a square integrable continuous  $\sigma(x(s), u(s): s \leq t)$ -adapted process  $\Phi(\cdot)$ , it holds that

$$E^u \left[ \int_0^t \Phi(s) dx^j(s) \right] = \int_0^t E^u [\Phi(s)] dx^j(s), \quad 1 \leq j \leq 2n.$$

Then, by the uniqueness of the solution of (7.15), we obtain

$$(7.17) \quad M^{\lambda}(t) = E^u [e^{-i\lambda u(t)} M(t, x(\cdot), u(\cdot))] \exp\left(\frac{t}{2}\lambda^2\right).$$

Hence we have

$$\begin{aligned} & E^x [e^{-\lambda S_2^{\varepsilon}(t)} E^u [e^{-i\lambda u(t)} M(t, x(\cdot), u(\cdot))] \delta_{x'}(x(t))] \\ (7.18) \quad & = \exp\left(-i\lambda \sum_{j=1}^n 2\varepsilon_j (x^j x'^j - x^j x'^{n+j}) - \frac{t}{2}\lambda^2\right) \\ & \times E [e^{-i\lambda S_0^{\varepsilon}(t)} M^{\lambda}(t) \delta_{x'-x}\left(\frac{x(t)}{\sqrt{2}}\right)]. \end{aligned}$$

By combining (7.13), (7.18) and (3.14), we conclude (7.16).  $\square$

We first note the following theorem (cf. Theorem 2.2).

**Theorem 7.1.** We take  $\alpha, \beta \in \mathbf{Z}_+^{2n}, k \in \mathbf{Z}_+$  and  $\varepsilon_0 > 0$  arbitrary.

(i) If  $q \neq n - \kappa$ ,

$$(7.19) \quad \overline{\lim}_{\lambda \rightarrow -\infty} \frac{1}{|\lambda|} \log \sup_{\substack{x \in \mathbf{R}^{2n} \\ 0 < \varepsilon \leq \varepsilon_0}} \left\| x^\beta \partial_x^\alpha \frac{\partial^k}{\partial \lambda^k} E[e^{-i\lambda S_0^\varepsilon(1)} M^{\lambda/\varepsilon^2, \varepsilon}(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] Q \right\| \leq -2.$$

Generally

$$(7.20) \quad \overline{\lim}_{\lambda \rightarrow \pm\infty} \frac{1}{|\lambda|} \log \sup_{\substack{x \in \mathbf{R}^{2n} \\ 0 < \varepsilon \leq \varepsilon_0}} \left\| x^\beta \partial_x^\alpha \frac{\partial^k}{\partial \lambda^k} E[e^{-i\lambda S_0^\varepsilon(1)} M^{\lambda/\varepsilon^2, \varepsilon}(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] \right\| \leq 0.$$

(ii) If  $q \neq n - \kappa$ ,

$$(7.21) \quad \overline{\lim}_{\lambda \rightarrow -\infty} \frac{1}{|\lambda|} \log \sup_{\substack{x \in \mathbf{R}^{2n} \\ 0 < \varepsilon \leq \varepsilon_0}} \left\| x^\beta \partial_x^\alpha \frac{\partial^k}{\partial \lambda^k} E[e^{-i\lambda S_0^\varepsilon(1)} (M^{\lambda/\varepsilon^2, \varepsilon}(1) - m^\lambda(1)) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)] Q/\varepsilon \right\| \leq -2.$$

Generally,

$$(7.22) \quad \overline{\lim}_{\lambda \rightarrow \pm\infty} \frac{1}{|\lambda|} \log \sup_{\substack{x \in \mathbf{R}^{2n} \\ 0 < \varepsilon \leq \varepsilon_0}} \left\| x^\beta \partial_x^\alpha \frac{\partial^k}{\partial \lambda^k} E[e^{-i\lambda S_0^\varepsilon(1)} (M^{\lambda/\varepsilon^2, \varepsilon}(1) - m^\lambda(1)) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]/\varepsilon \right\| \leq 0.$$

**Corollary.** We take  $\alpha, \beta \in \mathbf{Z}_+^{2n}$  and  $\varepsilon_0$  arbitrary. If  $q \neq n - k$ ,

$$(7.23) \quad \overline{\lim}_{\lambda \rightarrow -\infty} \sup_{t_0 \leq t \leq t_1} \frac{1}{t|\lambda|} \log \sup_{x \in \mathbf{R}^{2n}} \left\| x^\beta \partial_x^\alpha \frac{\partial^k}{\partial \lambda^k} E[e^{-i\lambda S_0^\varepsilon(t)} M^\lambda(t) \delta_x\left(\frac{x(t)}{\sqrt{2}}\right)] Q \right\| \leq -2.$$

Generally,

$$(7.24) \quad \overline{\lim}_{\lambda \rightarrow \pm\infty} \sup_{t_0 \leq t \leq t_1} \frac{1}{t|\lambda|} \log \sup_{x \in \mathbf{R}^{2n}} \left\| x^\beta \partial_x^\alpha \frac{\partial^k}{\partial \lambda^k} E[e^{-i\lambda S_0^\varepsilon(t)} M^\lambda(t) \delta_x\left(\frac{x(t)}{\sqrt{2}}\right)] \right\| \leq 0.$$

The proof of Theorem 7.1 proceeds just like the proof of Theorem 2.2, using Lemmas 7.3 and 7.4 below: we omit the details.

**Lemma 7.3.** For any  $m \in \mathbf{N}$  and  $\varepsilon > 0$ ,

$$E[e^{i\lambda S_0^\varepsilon(1)} M^{\lambda/\varepsilon^2, \varepsilon}(1) \delta_x\left(\frac{x(1)}{\sqrt{2}}\right)]$$

$$\begin{aligned}
(7.25) \quad &= \int_{\mathbf{R}^{2n}} dx_1 E[e^{i\lambda S_0^k(1)/m} M_m^{\lambda/\varepsilon^2, \varepsilon}(1) \delta_{x_1} \left( \frac{x(1)}{\sqrt{2m}} \right)] \\
&\times \int_{\mathbf{R}^{2n}} dx_2 E[e^{i\lambda S_1^k(1)/m} M_m^{\lambda/\varepsilon^2, \varepsilon}(1) \delta_{x_2} \left( x_1 + \frac{x(1)}{\sqrt{2m}} \right)] \\
&\times \cdots \times \int_{\mathbf{R}^{2n}} dx_{m-1} E[e^{i\lambda S_{x_{m-2}}(1)/m} M_m^{\lambda/\varepsilon^2, \varepsilon}(1) \\
&\quad \times \delta_{x_{m-1}} \left( x_{m-2} + \frac{x(1)}{\sqrt{2m}} \right)] \\
&\times E[e^{i\lambda S_{x_{m-1}}(1)/m} M_m^{\lambda/\varepsilon^2, \varepsilon}(1) \delta_x \left( x_{m-1} + \frac{x(1)}{\sqrt{2m}} \right)].
\end{aligned}$$

**Lemma 7.4.** (i) For any  $p > 1$ ,  $s \in \mathbf{Z}_+$ ,  $h \in \mathbf{Z}_+$  and  $\lambda > 0$ ,

$$(7.26) \quad \left\| \frac{\partial^h}{\partial \lambda^h} M_m^{\lambda/\varepsilon^2, \varepsilon}(1) \right\|_{p,s} \leq C(p, s, h) m^{-h} \exp \left\{ C(p, s) \left( \left( \frac{\varepsilon^2}{m} \right)^{p/2} + \left( \frac{|\lambda|}{m} \right) \right) \right\}$$

and

$$\begin{aligned}
(7.27) \quad &\left\| \frac{\partial^h}{\partial \lambda^h} (M_m^{\lambda/\varepsilon^2, \varepsilon}(1) - m_m^\lambda(1)) \right\|_{p,s} \\
&\leq C(p, s, h) m^{-h-1/2} \exp \left\{ C(p, s, h) \left( \left( \frac{\varepsilon^2}{m} \right)^{p/2} + \left( \frac{|\lambda|}{m} \right)^p + \frac{|\lambda|}{m} \right) \right\}.
\end{aligned}$$

(ii) If  $q \neq n - \kappa$ ,

$$(7.28) \quad \left\| \frac{\partial^h}{\partial \lambda^h} m_m^\lambda(1) Q \right\| \leq C m^{-h} \exp \left\{ \frac{|\lambda|(n-2)}{m} \right\}$$

for any  $\lambda \leq 0$ . Generally,

$$(7.29) \quad \left\| \frac{\partial^h}{\partial \lambda^h} m_m^\lambda(1) \right\| \leq C m^{-h} \exp \left\{ \frac{|\lambda|n}{m} \right\}.$$

By using Theorem 7.1, we can prove the following theorem:

**Theorem 7.2.** The unique fundamental solution  $H_t$  of the heat equation for the  $\bar{\partial}$ -Neumann problem on  $(p, q)$ -forms on  $D$  has the following smooth kernel:

$$\begin{aligned}
(7.30) \quad &h_t((x, u, r), (x', u', r')) \\
&= \mathbf{p}_t((x, u)^{-1}(x', u')) e_{\bar{t}}(r, r') P + \mathbf{p}_t((x, u)^{-1}(x', u')) e_t^+(r, r') Q \\
&\quad + \mathbf{q}_t((x, u)^{-1}(x', u'), r+r') Q
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{p}_t(x, u) &= \int_{-\infty}^{\infty} d\lambda \mathbf{f}_t(x, u, \lambda) \exp\left(-\frac{t}{2} \lambda^2\right), \\
\mathbf{q}_t((x, u), r) &= -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\lambda \cdot \lambda \mathbf{f}_t(x, u, \lambda) e^{r\lambda} \int_{\sqrt{t/2}(r/|\lambda|)}^{\infty} d\mu e^{-\mu^2},
\end{aligned}$$

and  $f_t(x, u, \lambda)$  is defined in (7.16).

As in Section 5, we can state properties of the heat kernel more precisely. We set

$$\begin{aligned}
 (7.31) \quad & h_i^{(1)}((x, u, r), (x', u', r')) = \mathbf{p}_t((x, u)^{-1}(x', u')) e_t^-(r, r') P, \\
 & h_i^{(2)}((x, u, r), (x', u', r')) = \mathbf{p}_t((x, u)^{-1}(x', u')) e_t^+(r, r') Q, \\
 & h_i^{(3)}((x, u, r), (x', u', r')) = \mathbf{q}_t((x, u)^{-1}(x', u'), r+r') Q.
 \end{aligned}$$

Then we have the following:

**Proposition 7.1.** *If  $i \neq 3$  or  $q \neq n - \kappa$  (resp.  $i = 3$  and  $q = n - \kappa$ ),  $h_i^{(i)}(X, X')$  is the restriction to  $\bar{D}_\kappa$  of a rapidly decreasing function of  $X$  for each fixed  $(t, X') \in (0, \infty) \times \bar{D}_\kappa$  (resp.  $(0, \infty) \times D_\kappa$ ) and the restriction to  $\bar{D}_\kappa$  of a rapidly decreasing function of  $X'$  for each fixed  $(t, X) \in (0, \infty) \times \bar{D}_\kappa$  (resp.  $(0, \infty) \times D_\kappa$ ) uniformly with respect to  $t \in [t_0, t_1]$  for any  $0 < t_0 < t_1$ . Moreover  $h_i^{(i)}(X, X')$  is a smooth function of  $(t, X, X') \in (0, \infty) \times D_\kappa \times D_\kappa$ .*

For any  $f \in L_2^{q, q}(D_\kappa)$ , we define

$$(H_i^{(i)} f)(X) := \int_{D_\kappa} h_i^{(i)}(X, Y) f(Y) dY, \quad X \in D_\kappa, \quad i = 1, 2, 3.$$

For this  $H_i^{(i)}$ , we have the following theorem:

**Theorem 7.3.**  *$H_i^{(i)}, i = 1, 2, 3$ , are bounded operators on  $L_2^{q, q}(D_\kappa)$ . Furthermore, for fixed  $f \in L_2^{q, q}(D_\kappa)$ , the followings hold:*

- (i) *for  $T > 0$ , there is a constant  $C$  depending only on  $f$  and  $T$  such that for  $t \in (0, T]$ ,  $\|H_i^{(i)} f\| \leq C, i = 1, 2, 3$ ;*
- (ii)  *$H_i^{(1)} f \rightarrow Pf, H_i^{(2)} f \rightarrow Qf, H_i^{(3)} f \rightarrow 0$  in  $L_2^{q, q}(D_\kappa)$  and uniformly as  $t \rightarrow 0$ ;*
- (iii)  *$H_i^{(i)} f, i = 1, 2, 3$ , are the restrictions of rapidly decreasing forms;*
- (iv)  *$H_i^{(i)} f, i = 1, 2, 3$ , are differentiable in  $t$ ;*
- (v)  *$H_t f := H_t^{(1)} f + H_t^{(2)} f + H_t^{(3)} f \in \text{Dom } \square$ ;*
- (vi)  *$(\partial/\partial t + \square) H_t f = 0$ ;*
- (vii)  *$\square H_t f = H_t \square f$ .*

Finally we investigate the behavior of  $h_t(X, X')$  on the diagonal of  $\bar{D}_\kappa \times \bar{D}_\kappa$  as  $t \rightarrow 0$ . As in (6.1), we have

$$(7.32) \quad h_t((x, u, r), (x, u, r)) = \mathbf{k}_t(0) - \mathbf{k}_t(r) P + \mathbf{k}_t(r) Q + \mathbf{q}_t(0, 2r) Q$$

where

$$\mathbf{k}_t(r) = \mathbf{p}_t(0) \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{2r^2}{t}\right).$$

Let  $L$  be the Levi form on  $bD_\kappa$  and  $L^q$  be its extension to  $\Lambda^q(T^{(1,0)} D_\kappa \cap CT_z(bD_\kappa))$  (cf. Beals-Stanton [2] p. 407):



$$L_X^q(Z_{j_1} \wedge \cdots \wedge Z_{j_q}, Z_{k_1} \wedge \cdots \wedge Z_{k_q}) = \left( \sum_{i=1}^q 2\varepsilon_{j_i} \right) \delta_{j_1 k_1} \cdots \delta_{j_q k_q}$$

for  $1 \leq j_1 < \cdots < j_q \leq n$  and  $1 \leq k_1 < \cdots < k_q \leq n$ . Then we have the followings:

**Proposition 7.3.** (i)

$$(7.33) \quad \lim_{t \rightarrow 0} t^{n+1} k_t(0) = \frac{1}{2\pi^{n+1}} I.$$

(ii) For any  $r > 0$  and  $k \in Z$ ,

$$(7.34) \quad \lim_{t \rightarrow 0} t^k k_t(r) = 0$$

and

$$(7.35) \quad \lim_{t \rightarrow 0} t^k q_t(0, 2r) Q = 0.$$

(iii)

$$(7.36) \quad \lim_{t \rightarrow 0} t^{n+1/2} \int_0^\infty dr k_t(r) = \frac{1}{4\sqrt{2\pi^{n+1/2}}} I.$$

(iv) When  $q \neq n - \kappa$ , we have

$$(7.37) \quad \begin{aligned} & \lim_{t \rightarrow 0} t^{n+1} \int_0^\infty dr \cdot \text{tr} [q_t(0, 2r) Q] \\ &= \binom{n+1}{p} \frac{1}{2\pi^{n+1}} \int_0^\infty d\tau \text{tr} [\exp(-\tau L^q)] \\ & \quad \times \exp(-\tau \text{tr}^- L) \det \{ \tau |L| (1 - e^{-\tau |L|})^{-1} \}. \end{aligned}$$

where  $\text{tr}^- L$  is the sum of the negative parts of the eigenvalues of  $L$  and  $|L| := (L^* L)^{1/2} = (L^2)^{1/2}$ .

Proof. The proofs of (i), (ii) and (iii) are similar to the corresponding results in Section 6. As in Proposition 6.3 (ii), we can prove the following: when  $q \neq n - \kappa$ ,

$$(7.38) \quad \begin{aligned} & \lim_{t \rightarrow 0} t^{n+1} \int_0^\infty dr q_t(0, 2r) Q \\ &= \sum_{0 \leq i \leq \kappa \wedge q} \frac{1}{2\pi^{n+1}} \int_0^\infty d\tau \left( \frac{\tau}{\sinh \tau} \right)^n e^{-(n-2\kappa-2q+4i)\tau} Q_i \end{aligned}$$

where  $Q_i$  is the projection onto the space

$$\begin{aligned} & \text{span} \{ \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_i} \wedge \bar{\omega}^{k_{i+1}} \wedge \cdots \wedge \bar{\omega}^{k_q} \\ & \quad | j_1, \dots, j_i \leq \kappa < k_{i+1}, \dots, k_q \leq n \}. \end{aligned}$$

By taking the trace of the both hand sides in the equality (7.38), we have (7.37).

□

### Acknowledgement

The author would like to express his gratitude to Professor N. Ikeda for many helpful advices.

---

### References

- [1] H. Airault: *Perturbations singulières et solutions stochastique de problèmes de D. Neumann-Spencer*, J. Math. Pures Appl. **55** (1976), 233–268.
- [2] R. Beals and N.K. Stanton: *The heat equation for the  $\bar{\partial}$ -Neumann problem I*, Comm. Partial Differential Equations **12**(4) (1987), 351–413.
- [3] G.B. Folland and J.J. Kohn: *The Neumann problem for the Cauchy-Riemann complex*, Princeton Univ. Press, Princeton, N.J., 1972.
- [4] B. Gaveau: *Principe de moindre action, propagation de la chaleur, estimées sous elliptiques sur certains groupes nilpotents*, Acta Math. **139** (1977), 1–63.
- [5] N. Ikeda and S. Watanabe: *Stochastic differential equations and diffusion processes*, second edition, Kodansha/North-Holland, Tokyo/Amsterdam, 1989.
- [6] K. Itô and H.P. McKean, Jr.: *Diffusion processes and their sample paths*, Springer, Berlin, 1965.
- [7] P. Malliavin: *Sur une résolution stochastique de certains problèmes de dérivée oblique dans le semi-espace*, C.R. Acad. Sc. Paris **283** (1976), 515–518.
- [8] G. Metivier: *Spectral asymptotics for the  $\bar{\partial}$ -Neumann problem*, Duke Math. J. **48** (1981), 779–806.
- [9] I. Shigekawa, N. Ueki and S. Watanabe: *A probabilistic proof of the Gauss-Bonnet-Chern theorem for manifolds with boundary*, Osaka J. Math. **26** (1989), 897–930.
- [10] N.K. Stanton: *The heat equation for the  $\bar{\partial}$ -Neumann problem in a strictly pseudoconvex Siegel domain*, J. Analyse Math. **38** (1980), 67–112.
- [11] N.K. Stanton: *The heat equation for the  $\bar{\partial}$ -Neumann problem in a strictly pseudoconvex Siegel domain II*, J. Analyse Math. **39** (1981), 189–202.
- [12] N. Ueki: *A probabilistic approach to the heat equation for the  $\bar{\partial}$ -Neumann problem*, Diffusion processes and related problems in analysis, vol. 1, ed. by M.A. Pinsky, Progress in Probability **22**, pp. 235–244, Birkhäuser, Boston-Basel-Berlin, 1990.
- [13] S. Watanabe: *Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels*, Ann. Probab. **15** (1987), 1–39.

Department of Mathematics  
 Faculty of Science  
 Osaka University  
 Toyonaka, Osaka  
 560 Japan

Present address:  
Department of Mathematics  
Faculty of Science  
Himeji Institute of Technology  
Shosha 2167, Himeji  
671-22 Japan

