Harada, M.
Osaka J. Math.
28 (1991), 793–809

ALMOST HEREDITARY RINGS

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(Received February 8, 1991)

A. Tozaki and the author defined almost relative projective modules in [5], and the author has given a new concept of almost projective modules in [6]. On the other hand, we know that an artinian ring R is hereditary if and only if the Jacobson radical of R is projective, namely the Jacobson radical of P is projective for any finitely generated R-projective module P. Analogously we call R a right almost hereditary ring if the Jacobson radical of P is almost projective. In the first section, we shall show the following two theorems: 1) R is right almost hereditary if and only if R is a direct sum of i) hereditary rings, ii) serial (two-sided Nakayama) rings and iii) special tri-angular matrix rings over hereditary rings and serial rings in the first category; 2) R is two-sided almost hereditary if and only if R is a direct sum of hereditary rings and serial rings.

We shall give a proof of the second theorem in the third section. In the fourth section, we shall study more strong rings such that every submodule of P is again almost projective (resp. the Jacobson radidal of Q is almost projective for any finitely generated and almost projective module Q).

1. Main theorems

In this paper every ring R is an artinian ring with identity and every module M is a unitary right R-module. By $|M|$, J(M), E(M) and Soc$_k(M)$ we denote *the length, the Jacobson radical, the injective hull and the* k^{th}*-lower Loewy series* of M, respetively. $\bar{M}$ means $M/\mathrm{J}(M)$. We shall denote J(R) by J. As is well known, if J is R-projective, then R is called a hereditary ring [1]. Analogously if J is almost projective as a right R-module [5], then we call R *a right almost hereditary ring*. We can define similarly a left almost hereditary ring. The above definition is equivalent to the following: J(P) is almost projective for any finitely generated projective module P. Therefore the definition of almost hereditary ring is Morita equivalent, and hence we may assume that R is a basic artinian ring, when we study the structure of R.

If R is hereditary, every submodule of P is again projective. However if R is right almost hereditary, then every submodule of P is not necessarily almost projective (see § 4).

From now on, R is a basic right artinian ring and $\{e_i\}$ is a set of mutually orthogonal primitive idempotents with $1=\Sigma e_i$. In this paper following [8], we call two-sided Nakayama rings serial rings. Consider a sequence $\{e_1, e_2, \cdots, e_s\}$ (or $\{e_1R, e_2R, \cdots, e_sR\}$). If $\bar{e}_i\bar{J}\sim\bar{e}_{i+1}\bar{R}$ for $1\leqq i\leqq s-1$ (and $\bar{e}_s J\sim\bar{e}_1\bar{R}$), then we call this sequence *a* (*cyclic*) *Kupisch series*, Let D be a division ring. Consider a factor ring of tri-angular matrix ring over D by an ideal:

$$\begin{array}{l} n_1\left\{\begin{array}{l}\\ \\ \end{array}\right. \\ \\ n_2\left\{\begin{array}{l}\\ \\ \end{array}\right. \\ \vdots \\ n_s\left\{\begin{array}{l}\\ \\ \end{array}\right. \end{array}\left(\begin{array}{cccc|cccc|c} D & D & D\cdots & D & 0 & \cdots\cdots & & 0 & \\ & D & D\cdots & D & 0 & \cdots\cdots & & 0 & \\ \hline & & & D & D & D\cdots D & & 0\cdots 0 & \\ & & & & D & D\cdots D & & 0\cdots 0 & \\ & O & & & & & & \cdots & \\ & & & & & & & & \\ & & & & & & & & D \end{array}\right)$$

By $\mathrm{T}(n_1, n_2, \cdots, n_s\colon D)$ we denote the above ring. It is well known that the above ring is a serial (two-sided Nakayama) ring. We call this ring a serial ring in the first category and a serial ring with cyclic Kupisch series is called a serial ring in the second category [8].

Lemma 1. *Let R be a right almost hereditary ring.* 1)*: Assume that e_1R is not injective and $e_1J\neq 0$. Then $e_1 J$ does not contain a direct summand X isomorphic to e_1R/A_1 for any $A_1\subset e_1R$.* 2)*: Assume that e_2R is injective and $0\neq e_2J\sim e_iR/A_i$ for some e_i and some A_i in e_iR. Then e_3J does not contain a direct summand Y isomorphic to e_iR/B_i for any B_i in e_iR, provided $e_3R\not\sim e_2R$.*

Proof. The first half is clear from [6], Theorem 1, for $|e_1R|>|e_1J|$. Assume that e_3J contains a direct summand Y isomorphic to e_iR/B_i. Then $B_i\supset A_i$ or $B_i\subset A_i$ by [6], Theorem 1. If $B_i\supset A_i$, $e_iR/A_i\sim e_2J$ is injective by [6], Theorem 1, a contradiction. Similarly we obtain the same result for $B_i\subset A_i$, since e_3R ($\supset e_3J$) is indecomposable. Hence $B_i=A_i$. Consider a diagram

$$\begin{array}{ccccc} e_3R & \longleftarrow & e_3J & \longleftarrow & 0 \\ & \searrow & \downarrow p & & \\ & g & Y & & \\ & & \wr\wr & & \\ & & e_2J & & \\ & & \cap & & \\ & & e_2R & & \end{array}$$

where p is the projection.

Since e_2R is injective, there exists $g\colon e_3R\to e_2R$, which makes the above diagram commutative. Since $e_3R \nsim e_2R$, $g(e_3R)=e_2J=g(e_2J)$. Hence e_3R being local, $e_3R=e_3J$, a contradiction,

The following two lemmas are well known (cf. [4], Proposition 1).

Lemma 2. *Let R be a basic ring and $1=g_1+g_2+\cdots+g_t+f_1+f_2+\cdots+f_s$, where $\{g_i, f_j\}$ is a set of mutually orthogonal primitive idempotents. Assume $g_iJ=0$ or $\bar{g}_i\bar{J}\sim\Sigma_j\oplus\bar{g}_{j(i)}\bar{R}$ for all $i\leqq t$ and $f_jJ=0$ or $\bar{f}_j\bar{J}\sim\Sigma_i\oplus\bar{f}_{i(j)}\bar{R}$ for all $j\leqq s$. Then $R=(g_1+\cdots+g_t)R(g_1+\cdots+g_t)\oplus(f_1+\cdots+f_s)R(f_1+\cdots+f_s)$ as rings, where $\{g_{j(i)}\}\subset\{g_k\}$ and $\{f_{i(j)}\}\subset\{f_s\}$.*

Proof. We assume that $\bar{g}_i\bar{J}^p\sim\Sigma\oplus\bar{g}_{j(i,p)}\bar{R}$, where $g_{j(i,p)}\in\{g_1,\cdots,g_t\}$. Then $g_iJ^p=a_1R+a_2R+\cdots+a_nR+g_iJ^{p+1}$, where $a_q=a_qg_{q(i,p)}$. Then $g_iJ^{p+1}=a_1g_{1(i,p)}J+\cdots+a_ng_{n(i,p)}J+g_iJ^{p+2}$. Hence $\bar{g}_i\bar{J}^{p+1}$ is a homomorphic image of $\Sigma_{i,j}\oplus\bar{g}_{j(i)}\bar{J}$ by assumption. As a consequence $\bar{g}_i\bar{J}^{p+1}\sim\Sigma_j\oplus\bar{g}_{j(i,p+1)}\bar{R}$ and $g_{j(i,p+1)}\in\{g_1,\cdots,g_t\}$. Thus any simple factor module in the composition series of $g_1R\oplus\cdots\oplus g_tR$ is isomorphic to some $\bar{g}_q\bar{R}$, Therefore $R=(g_1+\cdots+g_t)\cdot R(g_1+\cdots+g_t)\oplus(f_1+\cdots+f_s)R(f_1+\cdots+f_s)$ as rings.

From the above argument we obtain

Lemma 3. *Assume $\bar{e}_1\bar{J}\sim\bar{e}_2\bar{R}$ and $\bar{e}_2\bar{J}\sim\bar{e}_1\bar{R}$. Then e_1R and e_2R are uniserial and the simple factor modules in the composition series of e_1R (resp. e_2R) is $\{\bar{e}_1\bar{R}, \bar{e}_1\bar{J}\sim\bar{e}_2\bar{R}, \bar{e}_1\bar{J}^2\sim\bar{e}_1\bar{R}, \bar{e}_1\bar{J}^3\sim\bar{e}_2\bar{R},\cdots\}$ ($\{\bar{e}_2\bar{R}, \bar{e}_2\bar{J}\sim\bar{e}_1\bar{R}, \bar{e}_2\bar{J}^2\sim\bar{e}_2\bar{R},\cdots\}$). We obtain the similar result for a cyclic Kupisch series $\{e_1, e_2,\cdots, e_n\}$.*

Lemma 4. *If R is a hereditary ring or a serial ring, then R is a (right) almost hereditary ring.*

Proof. If R is hereditary, then R is clearly right almost hereditary. We assume that R is serial. We take a cyclic Kupisch series $\{e_1, e_2,\cdots, e_n\}$. Then $e_iJ\sim e_{i+1}R/A_{i+1}$ for some A_i, provided $e_iJ\neq 0$. Hence

$$|e_iR|-1\leqq|e_{i+1}R|,$$

Further $$|e_iR|\sim|e_jR|\leqq n-1 \text{ for any } i \text{ and } j.$$

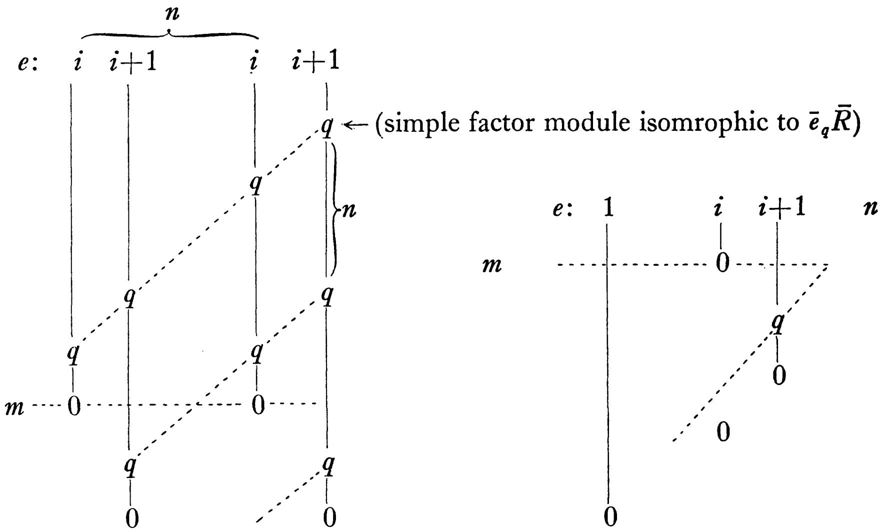
Put $m=|e_iR|$. If $|e_{i+1}R|=m-1$, $e_iJ\sim e_{i+1}R$, and hence e_iJ is projective. Assume $|e_{i+1}R|\geqq m$, and take any k such that $|e_{i+1}R/\mathrm{Soc}_k(e_{i+1}R)|\geqq m$. Let $\mathrm{Soc}_{k+1}(e_{i+1}R)/\mathrm{Soc}_k(e_{i+1}R)\sim\bar{e}_q\bar{R}$ for some q, and put $E=E(\bar{e}_q\bar{R})$. Since R is serial, $E\sim e_sR/A_s$ and $\mathrm{Soc}(e_sR/A_s)\sim\bar{e}_q\bar{R}$ for some s and A_s. Further

$$(1)\qquad |e_sR/A_s| \text{ is the largest one among } |e_{s'}R/A_{s'}|, \text{ with } \mathrm{Soc}(e_{s'}R/A_{s'})\sim\bar{e}_q\bar{R}.$$

On the other hand, by the structure of e_iR (see Figer (2) below) $|e_{i+1}R/\mathrm{Soc}_k(e_{i+1}R)|$ is a unique largest one among $|e_tR/A_t|$ with $\mathrm{Soc}(e_tR/A_t)\sim\bar{e}_q\bar{R}$. Hence $e_{i+1}R/\mathrm{Soc}_k(e_{i+1}R)$ is injective. Furthermore $J\,\mathrm{Soc}_k(e_{i+1}R)=0$ from (2).

Hence e_iJ is almost projective by [6], Theorem 1. We obtain the similar result for a non cyclic Kupisch series.

(2)



Lemma 5. *Let R be a basic ring. Assume that e_1R is injective and $\{e_1, e_2, \cdots, e_n\}$ is a Kupisch series such that $0 \neq e_iJ \sim e_{i+1}R$ for $i \leqq n-1$, $e_nJ=0$ and $\Sigma_i e_i = 1$. Then $R \sim \mathrm{T}(n: D)$.*

Proof. R is right Nakayama by Lemma 2 and its proof, and R is hereditary, Hence R has the following form:

$$(3) \qquad \begin{pmatrix} D_1 & M_{12} \cdots M_{1n} \\ 0 & D_2 \ \cdots M_{2n} \\ & \cdots \\ 0 & \\ & & D_n \end{pmatrix}$$

where the D_i are division rings and the M_{ij} are D_i-D_j bimodules (cf. [3], Theorem 1. We denote the above ring by $T(D_1, D_2, \cdots, D_n)$). We shall show the lemma by induction on n. Let $\{e_{ij}\}$ be the set of matrix units. Then we may suppose $e_i = e_{ii}$. Now e_1R is a two-sided ideal in R. Put $\tilde{R} = R/e_1R$. Since e_1J is characteristic in e_1R and $e_1J \sim e_2R$, $\tilde{e}_2\tilde{R}$ is injeative as an $\tilde{R}$-module by Bare's criterion. Furthermore $\{\tilde{e}_2, \cdots, \tilde{e}_n\}$ is a Kupisch series in $\tilde{R}$, which satisfies the condition in the lemma. Put $E = e_2 + \cdots + e_n$. Then $\tau: \tilde{R} \sim ERE$ and $\theta: \mathrm{T}(n-1: D) \sim \tilde{R}$ as rings by induction. Let $u_j = \tau\theta(g_{jj})$, where the g_{jj} are the matrix units in $\mathrm{T}(n-1: D)$. Then $u_j \sim e_j$, $E = \Sigma_{j \geqq 2} u_j$, and hence $\{e_1, u_2, \cdots, u_n\}$ is a set of mutually orthogonal primitive idempotents in R with $1 = e_1 + \Sigma_{j \geqq 2} u_j$. Hence we can suppose that R is of the form

$$\begin{pmatrix} D_1 & M_{12} & \cdots M_{1n} \\ & D & D \cdots D \\ & 0 & D \cdots \\ & & 0 \quad \ddots \\ & & \qquad D \end{pmatrix}$$

Since $e_1 J \sim e_2 R$, $M_{1i} = v_{1i} D$. Consider any element d in $D \sim \mathrm{Hom}_R(e_2 R, e_2 R)$. Since $e_1 R$ is injective and v_{12}: $e_2 R \sim e_1 J$, $(v_{12} d_{12}^{-1}) | e_1 J$ is given by a left mutiplication of an element d_1 in D_1. Then $d_1 v_{12} = v_{12} d v_{12}^{-1} v_{12} = v_{12} d$, and hence $d_1 v_{12} = v_{12} d$. On the other hand, $D_1 v_{12} \subset v_{12} D$. Hence there exists an isomorphism g of D to D_1 such that $g(d) v_{12} = v_{12} d$. We may assume $v_{1i} = v_{12} e_{2i}$. Then $g(d) v_{1i} = v_{1i} d$. Hence we obtain an isomrophism G of $T(n: D)$ to R:

$$G \begin{pmatrix} d_1 x_{12} & \cdots & x_{1n} \\ & d_2 x_{23} \cdots & x_{2n} \\ & \cdots & \\ & & d_n \end{pmatrix} = \begin{pmatrix} g(d_1) v_{12} x_{12} & \cdots & v_{1n} x_{1n} \\ & d_2 x_{23} \cdots & x_{2n} \\ & \cdots & \\ & & d_n \end{pmatrix}$$

where the d_i and the x_{ij} are elements in D (cf. [3], Lemma 13).

Lemma 6. *Let R be right almost hereditary. If any $e_i R$ is never injective or all i, then R is hereditary.*

Proof. This is clear from by [6], Theorem 1.

Let R be a basic and (right) artinian ring and $1 = \sum_{i=1}^{n} e_i$ as before. We assume that R is right almost hereditary. If R is not hereditary, then there exists an injective module $e_j R$ for some j from Lemma 6. From now on we shall denote an injective module $e_i R$ by $f_t R$ and a non-injective module $e_j R$ by $g_j R$. Hence $1 = \Sigma f_i + \Sigma g_j$. We start from f_1. If $f_1 J \neq 0$, then since $f_1 J$ is uniform, $f_1 J \sim f_2 R / A_2$ or $f_1 J \sim g_{11} R$ by [6], Theorem 1. In either case $f_2 R$ and $g_{11} R$ are uniform. Hence $g_{11} J \sim g_{12} R$ or $g_{11} J \sim f_2 R / A_2$, provided $g_{11} J \neq 0$ (resp. $f_2 J \sim g_{21} R$ or $f_2 J \sim f_3 R / A_3$, provided $f_2 J \neq 0$). Furthermore we have monomorphisms θ_2: $g_{12} R \to g_{11} R$ and θ_1: $g_{11} R \to f_1 R$. Repeating this procedure, we can get a Kupisch series $f_1 R, g_{11} R, \cdots, g_{1r^1} R, f_2 R, \cdots, f_s R, \cdots, g_{sr^s} R$, each of which is uniform.

Lemma 7. *Let R be a basic and right almost hereditary ring. We assume that there exists a cyclic Kupisch series $f_1 R, g_{11} R, \cdots, g_{1r^1} R, f_2 R, \cdots, f_s R, \cdots, g_{sr^s} R$ such that $1 = \Sigma f_i + \Sigma g_{kj} = \Sigma e_i$. Then R is serial.*

Proof. By n_1 (resp. n_{ij}) we denote $|f_i R|$ (resp. $|g_{ij} R|$). Then $n_{jr^j} = n_j - r_j$ for any j. Since $g_{jr^j} J \sim f_{j+1} R / A_{j+1}$, $f_{j+1} R / (\mathrm{Soc}_{n^{j+1} - (n^j - r^j)}(f_{j+1} R)$, $f_{j+1} R / \mathrm{Soc}_{n^{j+1} - (nj - rj) - 1}(f_{j+1} R)$, $\cdots$, $f_{j+1} R / \mathrm{Soc}(e_{j+1} R)$, $f_{j+1} R$ are injeative by [6], Theorem 1. Clearly $f_j R / \mathrm{Soc}_p(f_j R) \not\sim f_k R / \mathrm{Soc}_q(f_k R)$ for $j \neq k$. Therefore there exist

$\Sigma(n_{j+1}-n_j+r_j+1)=s+\Sigma r_j=n$ distinct injective modules. Since the f_iR, the $g_{ij}R$ are uniserial by Lemma 3, R is right Nakayama and right co-Nakayama. As a consequence R is serial by [2], Theorem 5.4.

Lemma 8. *Let R be as above. Assume that $f_1R, g_{11}R, \cdots, g_{1r^1}R, f_2R, \cdots, g_{2r^2}R, \cdots, f_sR, \cdots, g_{sr^s}R$ is a cyclic Kupisch series. Then $R=FRF\oplus(1-F)R(1-F)$ and FRF is serial, where $F=\Sigma f_i+\Sigma g_{ij}$.*

Proof. From the stucture of $\{f_iR, g_{jk}R\}$, every simple factor module in the composition series of FR is isomorphic to some $\bar{f}_i\bar{R}$ or $\bar{g}_{jk}\bar{R}$. Let h be an idempotent in $\{e_i\}-\{f_i, g_{jk}\}$. Suppose that hJ contains a direct summand X isomorphic to $g_{jk}R$. Since $g_{jk}R\subset f_jR$ (isomorphically), there exists a homomorphism $\theta: hR\to f_jR$ since f_jR injective, which is a contradiction from the above observation. Next assume $X\sim f_iR/A_i$. Then $f_{i-1}R\supset g_{i-1r^{i-1}}J\sim f_iR/A_i$ by [6], Theorem 1 (cf. the proof of Lemma 1). Hence we have the same result. Accordingly $\bar{h}\bar{J}$ does not contain a simple component isomrophic to $\bar{f}_i\bar{R}$ or $\bar{g}_{jk}\bar{R}$ again by [6], Theorem 1. Therefore $R=FRF\oplus(1-F)R(1-F)$ by Lemma 2, and FRF is serial by Lemma 7.

Lemma 9. *Let R be a basic and right almost hereditary ring. Assume that R is two-sided indecomposable, not hereditary and $e_iJ\neq 0$ for all i. Then R is serial.*

Proof. Since R is not hereditary, there exists some injective module e_tR by Lemma 6. Let $\{f_iR\}$ be the set of injective modules e_jR and $\{g_{ij}R\}$ the set of non-injective modules $e_{j'}R$ as before. Since $e_kJ\neq 0$ for all k, extending a Kupisch series from f_1R as long as possible, we obtain finally a Kupisch series $\{f_1R, g_{11}R, \cdots, f_sR, g_{s1}R, \cdots, g_{sr^s}R, e_xR\}$ such that

ii) $e_xR=f_aR$ $(a\leqq s)$ or

ii) $e_xR=g_{ab}R$ $(a\leqq s)$.

i) $\{f_eR, g_{a1}R, \cdots, e_xR\}$ is cyclic.

ii) $\mathrm{Soc}(f_aR)\sim\mathrm{Soc}(e_xR)=\mathrm{Soc}(g_{ab}R)\sim\mathrm{Soc}(f_aR)$, and hence $s=a$. Then $\{g_{sb}((R, g_{sb+1}R, \cdots, g_{ar^s}R\}$ is a cyclic Kupisch series. Since $g_{sj}J\sim g_{sj+1}R$ by [6], Theorem 1, $|g_{sj}R|>|g_{sj+1}R|$, which is a contradiction to a cyclic series. Hence we obtain always a cyclic Kupisch series in i). Therefore by Lemmas 7 and 8 $a=1$ and R is serial, since R is two-sided indecomposable.

From Lemma 9 we may suppose that there do not exist any cyclic Kupisch series. Hence we study the structure of R in case of there exists a simple module e_jR, i.e. $e_jJ=0$ for some j. We note that if $f_1R, g_{11}R, \cdots, g_{1r^1}R, f_2R, g_{21}R, \cdots, g_{2r^2}R$ is a Kupisch series with $g_{2r^2}R$ simple, then $g_{1k}R\neq g_{2t}R$ for any k and t, for $f_1R\nsim f_2R$. Now we obtain Kupisch series

$$(4)\qquad \begin{array}{l} f_1R,\ g_{11}R,\ \cdots, f_2R,\ \cdots, f_sR,\ \cdots, g_{sr^s}R \text{ and } g_{sr^s}J=0\,,\\ f'_1R,\ g'_{11}R,\ \cdots, f'_2R,\ g'_{s'r'^{s'}}R \text{ and } g'_{s'r'^{s'}}J=0\,,\\ \cdots\cdots\cdots\cdots \end{array}$$

If $g_{tj}=g'_{t'j'}$, for some (t, j) and (t', j'), then $\mathrm{Soc}(f_tR)\sim\mathrm{Soc}(f'_{t'}R)$ and hence $f_t=f'_{t'}$. If $t=1$, or $t'=1$ one series is a part of the other. Hence we assume $t>1$, $t'>1$, Since $g_{t-1r^{t-1}}J\sim f_tR/A_t$, $g'_{t'-1r'^{t'-1}}J\sim f'_{t'}R/A'_{t'}$, $A_t=A'_{t'}$ by [6], Theorem 1. Accordingly $\mathrm{Soc}(g_{t-1r^{t-1}}R)\sim\mathrm{Soc}(g'_{t'-1r'^{t'-1}}R)$, and hence $f_{t-1}=f'_{t'-1}$. If $f_q=f'_{q'}$, $\{f_q, g_{q1}, \cdots\}=\{f'_{q'}, g'_{q'1}\cdots\}$. Therefore $\{f_1, g_{11}, \cdots, g_{sr^s}\}\subset\{f'_1, \cdots, g'_{s'r'^{s'}}\}$ or $\{f_1, g_{11}, \cdots, g_{sr^s}\}\supset\{f'_1, \cdots, g'_{s'r'^{s'}}\}$, provided they have a common component. Hence we may assume that the Kupisch series in (4) are the longest series and they are disconnected and contains all f_kR. We put $\{h_1, h_2, \cdots, h_p\}=\{e_i\}-\{f_1, g_{11}, \cdots, g_{sr^s}, f_1', \cdots, g'_{s'r'^{s'}}, \cdots \text{ in (4)}\}$ and $H=h_1+h_2+\cdots+h_p$.

Lemma 10. *Let R and H be as above. Then HRH is a hereditary ring,* $(1-H)RH=0$ *and* $(1-H)R(1-H)=\Sigma\oplus_i\mathrm{T}(n_{i(1)}, n_{i(2)}, \cdots, n_{i(q^i)}: D_i)$.

Proof. From the proof of Lemma 8 we know that h_iJ does not contain a direct summand X isomorphic to $g_{jt}{}^{(k)}R$ (or $f_i{}^{(k)}R/A_i{}^{(k)}$ $i\neq1$), where $f_i{}^{(k)}R$ and $g_{jt}{}^{(k)}R$ are in (4). Since $\{h_q\}\cap\{f_i{}^{(k)}, g_{jt}{}^{(k)}\}=\phi$, $(1-H)RH=0$ and

$$(5)\qquad f_i{}^{(k)}R=f_i{}^{(k)}(1-H)R(1-H),\ g_{jt}{}^{(k)}R=g_{jt}{}^{(k)}(1-H)R(1-H)$$

from Lemma 2 and the structure of $\{f_i{}^{(k)}R, g_{jt}{}^{(k)}R\}$. On the other hand, from the above we have

$$(6)\qquad \begin{array}{l} h_iJ\sim\Sigma\oplus_{j\neq i}(h_jR)^{(m^{(i,j)})}\oplus(f_1R/A_1)^{(m^{(i1)})}\\ \oplus(f'_1R/A'_1)^{(n^{(i1)})}\oplus\cdots,\ A_1\neq0,\ A'_1\neq0\cdots\,, \end{array}$$

Accordingly $h_iJH\sim\Sigma\oplus_{j\neq i}(h_jHRH)^{(m^{(i,j)})}$, and hence HRH is hereditary. Next we shall show $(1-H)R(1-H)\sim T(n_1, n_2, \cdots, n_q: D)$ by induction on $n=\#\{f^{(1)}{}_1, g^{(1)}{}_{11}, \cdots\}$. From (5) we may replace R by $(1-H)R(1-H)$. Since $e_kRf_1=0$ for all $e_k\not\sim f_1$, f_1R is a two-sided ideal. Put $\tilde{R}=R/f_1R$. Then $\tilde{g}_{11}\tilde{R}$ is injective and $\{\tilde{g}_{11}, \cdots, \tilde{f}_2, \cdots, \tilde{g}_{2r^2}, \cdots\}$ is a Kupisch series with the same property as $\{f_{11}, \cdots, g_{1r^1}, \cdots\}$ (cf. the proof of Lemma 5). Hence $\tilde{R}\sim\mathrm{T}(n'_1, n_2, \cdots, n_q: D)$ by induction. Further f_1R is injective and $f_1J\sim g_{11}R$, and hence we can show in the manner given in the proof of Lemma 5 that $R\sim T(n'_1+1, n_2, \cdots, n_q: D)$.

We shall discuss the structure of $HR(1-H)$. Since HRH is hereditary, HRH is of the form (3). We may assume $h_i=e_{ii}$ in (3). We shall first rewrite (4) in more detail.

$$(7)\qquad \begin{array}{l} f_{1,1}R,\ g_{1,11}R,\ \cdots, g_{1,1r^{11}}R,\ f_{1,2}R,\ \cdots, f_{1,s^1}R,\ \cdots, g_{1,s^1,r^{1s1}}R\\ f_{2,1}R,\ g_{2,11}R,\ \cdots, g_{2,1r^{21}}R,\ f_{2,2}R,\ \cdots, f_{2,s^2}R,\ \cdots, g_{2,s^2,r^{2s2}}R\\ \cdots\cdots\cdots\cdots\\ f_{x,1}R,\ g_{x,11}R,\ \cdots, g_{x,r^{x1}}R,\ f_{x,2}R,\ \cdots, f_{x,s^x}R,\ \cdots, g_{x,s^x,r^{xsx}}R \end{array}$$

Since HRH is hereditary, if $e_{ii}JH \sim (e_{i+1i+1}RH)^{(m(i,i+1))} \oplus (e_{i+2i+2}RH)^{(m(i,i+2))} \oplus \cdots \oplus (e_{nn}RH)^{(m(i,n))}$, then we have from (6)

$$h_i J \sim (h_{i+1}R)^{(m(i,i+1))} \oplus (h_{i+2}R)^{(m(i,i+2))} \oplus \cdots \oplus (h_n R)^{(m(i,n))} \oplus (f_{1,1}R/A_{1,1})^{(n(i,1))} \oplus (f_{2,1}R/A_{2,1})^{(n(i,2))} \oplus \cdots,$$

where $m(i, a)$, $n(i, b)$ are non-negative integers, i.e.,

$$h_i R(1-H) \sim (h_{i+1}R(1-H))^{(m(i,i+1))} \oplus (h_{i+2}R(1-H))^{(m(i,i+2))} \oplus \cdots \oplus (f_{1,1}R/A_{1,1})^{(n(i,1))} \oplus (f_{2,1}R/A_{2,1})^{(n(i,2))} \oplus \cdots, \tag{8}$$

where the $A_{i,1} \neq 0$, $\{m(i, i+1), m(i, i+2), \cdots\}$ are the integers given in the above and $h_i R(1-H) = (f_{1,1}R/A_{1,1})^{(t_1)} \oplus (f_{2,1}R/A_{2,1})^{(t_2)} \oplus \cdots$.
Summarizing the above we have

Theorem 1. *Let R be a (basic) artinian ring. Then R is right almost hereditary if and only if R is a direct sum of the following rings:*

1) *Hereditary rings.*
2) *Serial rings.*
3) $$\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix} \tag{9}$$

where T_1 is a hereditary ring and T_2 is a serial ring in the first category and X is a $T_1 - T_2$ bimodule given by (8) (see the example bleow).

Proof. We have shown the first half. Assume that R is of the form (9). Then for each primitive idempotent e in T_2, any simple module in the composition series of eR is never isomrophic to other one. Hence we can easily show $f_{1,1}R/\mathrm{Soc}_q(f_{1,1}R)$, $f_{1,1}R/\mathrm{Soc}_{q-1}(f_{1,1}R)$, $\cdots, f_{1,1}R$ are injective by Baer's criterion and $JA_{11}=0$, where $\mathrm{Soc}_{q+1}(f_{1,1}R)=A_{1,1}$. Accordingly $f_{11}R/A_{11}$ is almost projective by [6], Theorem 1. We have the same results for $f_{j,k}R$ and $g_{j,k-1r^{k-1}}J$. Further $f_{j,k}J$ and $g_{j,kt}J$ $(t \neq r_k)$ are projective from the structure of T_2. Therefore R in (9) is right almost hereditary.

Theorem 2. *Let R be as above. R is right and left almost hereditary if and only if R is a direct sum of hereditary rings and serial rings.*

We shall give a proof in the third seation.

2. Corollaries

In this section we shall give some corollaries to Theorem 1.

Corollary 1. *Let R be a basic ring with $1=\sum_{i=1}^{n} e_i$. R is right almost hereditary if and only if*

R is a direct sum of hereditary rings and serial rings, provided $n\leqq 2$,

If $n=3$, *R is either a direct sum of hereditary rings and serial rings or the following form:*

$$\begin{pmatrix} D_1 & X & 0 \\ 0 & D_2 & D_2 \\ 0 & 0 & D_2 \end{pmatrix}$$

where D_1, D_2 *are division rings and* X *is a* D_1-D_2 *bimodule.*

Proof. This is clear from Theorem 1.

We note that the above ring is right almost hereditary, but not left almost hereditary if $[X:D_1]>1$.

Corollary 2. *If R is right almost hereditary ring, then R is a direct sum of serial rings and factor rings of hereditary rings.*

Proof. This is clear from Theorem 1 and [3], Theorem 5.

Corollary 3. *Let R be a hereditary and two-sided indecomposable ring. If* e_iR *is injective for some* i, *then R is serial, i.e.* $R=\mathrm{T}(n: D)$. *Hence there do not exist two non-isomorphic injective modules* e_iR *and* e_jR.

Proof. From Lemma 8 and [3], Theorem 1 we obtain a Kupisch series $g_1R, g_2R, \cdots, g_nR$ and $g_nJ=0$, where $g_1=e_i$. Let h be a primitive idempotent not in $\{g_i\}$. Since g_1R is injective, hJ does not contain a direct summand X isomorphic to g_1R. If $X\sim g_iR$ for some $i\neq 1$, $hR\sim g_kR$ for some k as in the proof of Lemma 8. Hence $(g_1+\cdots+g_n)R(g_1+\cdots+g_n)$ is a direct summand of R as rings by Lemma 2. Therefore $R=(g_1+\cdots+g_n)R(g_1+\cdots+g_n)=\mathrm{T}(n: D)$ Lemma 5.

Corollary 4 ([8]). *Let R be a serial and two-sided indecomposable ring. If* e_iR *is simple for some* i, *then* $R=\mathrm{T}(n_1, n_2, \cdots, n_r: D)$.

Proof. Taking a Kupisch series, we obtain the corollary from the proof of Lemma 10.

It is well known for a hereditary ring R that R is a QF-ring if and only if R is semisimple. If we replace "hereditary" by "almost hereditary", we obtain

Corollary 5. *Let R be a (non-semisimple) two-sided indecomposable basic and artinian ring. Let* $\{e_i\}_{i=1}^n$ *be a set of mutually orthogonal primitive idempotents with* $1=\Sigma_i e_i$. *Consider the following conditions:*

1) *R is a QF-ring.*
2) *R is a right almost hereditary ring.*

3) *R is a (right Nakayama) ring with the following properties: i) $\bar{e}_i\bar{J}\sim\bar{e}_{i+1}\bar{R}$ for $1\leqq i\leqq n-1$ and $e_n\bar{J}\sim\bar{e}_1\bar{R}$, i.e., $\{e_1, e_{,2}\cdots, e_n\}$ is a cyclic Kupisch series. ii) $|e_iR|=|e_1R|$ for all i.*
Then any two of the above conditions imply the remainder.

Proof. This is clear from Theorem 1, [2], Theorem 5.4 and [6], Theorem 1.

3. Proof of Theorem 2

We shall give here a proof of Theorem 2. "If" part is clear from Lemma 4. Assume that R is a two-sided almost hereditary ring.

Let T_1, T_2 be the hereditary ring and the serial ring in (9), respectively. We assume first that T_1 and T_2 are two-sided indecomposable rings. We put $T_1=\mathrm{T}(H_1, H_2, \cdots, H_n)$ and we may assume $T_2=\mathrm{T}(m: D)$ (cf. the proof of Lemma 10), where the H_i and the D are division rings. Further we set $T_1=\sum_{i=1}^{n}\oplus h_iR$, $T_2=\sum_{j=1}^{m}\oplus d_jR$ and $E=\Sigma\, d_j$, where $\{h_i\}$ and $\{d_j\}$ are sets of orthogonal primitive idempotents.
Put

$$(10)\qquad R=\left(\begin{array}{ccccc|cccccc}
H_1 & M_{12} & \cdots\cdots & & M_{1n} & N_{11} & N_{12} & \cdots & N_{1c} & & \\
 & H_2 & M_{23} & \cdots & M_{2n} & N_{21} & N_{22} & \cdots & N_{2c} & & \\
 & & \cdots\cdots & & & & \cdots\cdots & & & & \\
 & & & H_k\cdots & M_{kn} & N_{k1} & N_{k2} & \cdots & N_{kc} & & \\
 & & & \cdots & H_n & N_{n1} & N_{n2} & \cdots & N_{nc} & & \\
\hline
 & & & & & D_1 & D_1 & \cdots & D_1 & D_1\cdots & D_1 \\
 & & 0 & & & & D_1 & \cdots & D_1 & D_1\cdots & D_1 \\
 & & & & & & & & D_1 & D_1\cdots & D_1 \\
 & & & & & & & & & \cdots & \\
 & & & & & & & & & & D_1
\end{array}\right)$$

where $(N_{k1}, N_{k2}, \cdots, N_{kc})\sim(d_1R/A_1)^{t^k}$ and $A_1=d_1T_2d_{c+1}+d_1T_2d_{c+2}+\cdots+d_1T_2d_m=(0, \cdots, 0, D_1, \cdots, D_1)$.
We note the following fact. Since R is a generalized tri-angular matrix ring, $Rh_i\supset(M_{1i}, \cdots, M_{i-1i})^t\supset\cdots\supset(M_{1i}, 0\cdots, 0)^t$ is a chain of submodules in Rh_i, where $(\ )^t$ is the transpose matrix of $(\ )$. We have a similar result for Rd_j. From (10) Jd_{c+1} is not projective ($X\neq 0$). Hence since R is left almost hereditary, we have by [6], Theorem 1

(11) $\qquad Rd_c/(N_{1c}, \cdots, N_{kc})^t$ is injective for any $k<n$, provided $N_{k'c}\neq 0$ for some $k'>k$.

Now we suppose

(12) $$X \sim (h_{a^1}RE)^{(x^1)} \oplus (h_{a^2}RE)^{(x^2)} \oplus \cdots \oplus (h_{a^p}RE)^{(x^p)}$$

where $a_1 < a_2 < \cdots < a_p$ and $x_i > o$, $h_{a^i}RE \sim d_1T_2/A_1$ for all i.
Suppose $M_{a^ij} \neq 0$ for some $j \notin \{a_1, a_2, \cdots, a_p\} = Z$. We note that $M_{a^ij} = (M_{1j}, \cdots, M_{a^ij})^t/(M_{1j}, \cdots, M_{a^i-1j})^t (\subset Rh_{a^j}/(M_{1j}, \cdots, M_{a^i-1j})^t)$ and $N_{a^ic} \sim (N_{a^1c}, \cdots, N_{a^ic})^t/(N_{a^1c}, \cdots, N_{a^{i-1}c})^t (\subset Rd_c/(N_{a^1c}, \cdots, N_{a^{i-1}c}))$ are semisimple H_{a^i}-modules. Since $Rd_c/(N_{1c}, \cdots, N_{a^{i-1}c})^t$ is injective from (11), there exists a non-zero homomorphism of Rh_{a^j} to $Rd_c/(N_{1c}, \cdots, N_{a^i-1c})^t$. Hence $N_{jc} \neq 0$ from the structure (10), a contradiction to (12). Hence

$$M_{a^ij} = 0 \qquad \text{for all } j \notin Z \text{ and all } a_i\,.$$

Now Rd_c is injective from (11). If Rd_t is injective for some $t < c$, then since $N_{a^1t} \neq 0$ and $N_{a^1c} \neq 0$, there exists a non-zero homomorphism of Rh_c to Rh_t. However $h_cRh_t = 0$ from (10). Accordingly Rd_t is not injective for all $t < c$. Hence $\{Rd_c, Rd_{c-1}, \cdots, Rd_1\}$ is a Kupisch series such that $Jd_k \sim Rd_{k-1}$ and the Rd_k are uniform by the initial remark after (10) and [6], Theorem 1. Next we consider Rh_{a^p}. First we assume that

any Rh_i is not injective.

Since Rh_{a^p} is not injective by assumption and Jd_1 is uniform, $Jd_1 \sim Rd_{a^p}$ by the initial remark and [6], Theorem 1. Then noting $Jd_1 = (0, \cdots, N_{a^11}, 0, \cdots, N_{a^21}, 0, \cdots, N_{a^p1}, \cdots, 0, \cdots, 0)^t$, we have similarly $Jh_{a^p} \sim Rh_{a^p-1}, \cdots, Jh_{a^2} \sim Rh_{a^1}$ and $Jh_{a^1} = 0$, since Rh_i is not injective by assumption. Hence.

$$M_{ja^i} = 0 \qquad \text{for all } j \notin Z \text{ and all } a_i\,.$$

Therefore $(a_1 + \cdots + a_p)T_1(a_1 + \cdots + a_p)$ is a direct summand of T_1 as rings, and hence $\{a_1, \cdots, a_p\} = \{1, 2, \cdots, n\}$ since T_1 is indecomposable. Then $\{Rd_m, \cdots, Rd_1, Rh_n, \cdots, Rh_1\}$ is a Kupisch series from the above. Hence

(13) $$R = \mathrm{T}(n+c, (m-c)\colon D)$$

by Lemma 10. Next suppose that

some Rh_i is injective.

Then $i = n$ by Corollary 3, and $T_1 = \mathrm{T}(n\colon D)$. We note as above that $\{Rd_c, Rd_{c-1}, \cdots, Rd_1\}$ is a Kupisch series such that $Jd_k \sim Rd_{k-1}$ and the Rd_k are unifrom. Let $N_{k1} \neq 0$ for some k. Then $Rh_n/(M_{1n}, \cdots, M_{k-1n})^t$ is isomorphic to a submodule of the injective module $Rd_c/(N_{1c}, \cdots, N_{k-1c})^t$ by (11). Hence $N_{k'1} \neq 0$ for all $k' \geq k$, and so $Jd_1 = (0, 0, 0, N_{r1}, \cdots, N_{n1})^t$ for some r and $N_{q1} \neq 0$ for all $q \geq r$. Then $\{Rd_m, \cdots, Rd_{c+1}, Rd_c, \cdots, Rd_1, Rh_n, \cdots, Rh_1\}$ is a Kupisch series. Hence

(14) $$R = \mathrm{T}(n, c, m-c: D)\,.$$

Finally we study the general form in (9). Let $T_{11}, \cdots, T_{1a}$ be two-sided indecomposable and hereditary rings and $T_{21}, \cdots, T_{2b}$ two-sided indecomposable and serial rings. Then

$$R = \begin{pmatrix} \begin{array}{|cc|}\hline T_{11} & \\ & 0 \\ & T_{1a} \\ \hline\end{array} & X \\ 0 & \begin{array}{|cc|}\hline T_{21} & \\ & 0 \\ & T_{2b} \\ \hline\end{array} \end{pmatrix}$$

Let $\{h_{ij}\}_j$ (resp. $\{d_{ij}\}_j$) be the matrix units in the diagonal of T_{1i} (resp. T_{2i}), and $E_i=\Sigma_j d_{ij}$. Put $F_j=\Sigma_i h_{ji}$. Now

$$X \sim \Sigma\oplus X_{jk}, \quad \text{where} \quad X_{jk} = F_j X E_k\,.$$

First we consider the following ring:

$$\begin{pmatrix} \begin{array}{|cc|}\hline T_{11} & \\ & 0 \\ & T_{1a} \\ \hline\end{array} & \begin{array}{c} X_{11} \\ \\ X_{a1} \end{array} \\ 0 & \begin{array}{|c|}\hline T_{21} \\ \hline\end{array} \end{pmatrix}$$

It is clear from the above structure and (9) that $Rd_{1c^1}\supset\Sigma\oplus_p X_{p1}d_{1c^1}$ and $X_{j1}\neq 0$ if and only if $X_{j1}d_{1c^1}\neq 0$. On the other hand, since R is left almost hereditary, Rd_{1c^1} is injective. Therefore $X_{i1}\neq 0$ only for one i and $X_{j1}=0$ for all $j\neq i$. Similarly $X_{i'k}\neq 0$ only for one i' and $X_{j'k}=0$ for all $j'\neq i'$ and any k. Next we consider the following ring:

$$\begin{pmatrix} \begin{array}{|c|}\hline T_{11} \\ \hline\end{array} & \begin{array}{c} N^{(1)}{}_{11}\cdots N^{(1)}{}_{1c^1} \\ \cdots \\ N^{(1)}{}_{n1}\cdots N^{(1)}{}_{nc^1} \end{array} & \begin{array}{c} N^{(2)}{}_{11}\cdots N^{(2)}{}_{1c^2} \\ \cdots \\ N^{(2)}{}_{n1}\cdots N^{(2)}{}_{nc^2} \end{array} \\ 0 & \begin{array}{|c|}\hline T_{21} \\ \hline\end{array} & 0 \\ & & \begin{array}{|c|}\hline T_{22} \\ \hline\end{array} \end{pmatrix}$$

Assume $N^{(1)}{}_{kc^1}\neq 0$ and $N^{(2)}{}_{kc^2}\neq 0$ for some k. Then $Rd_{1c^1}/(N^{(1)}{}_{1c^1}, \cdots, N^{(1)}{}_{k-1c^1})^t$ $(=M^{(1)})$, $Rd_{2c^2}/(N^{(2)}{}_{1c^2}, \cdots, N^{(2)}{}_{k-1c^2})^t(=M^{(2)})$ are injective. Hence there exists a non-zero homomorphism of $M^{(2)}$ to $M^{(1)}$, a contradiction. As a consequence if $N^{(1)}{}_{kc^1}\neq 0$ for some k, then $N^{(2)}{}_{ij}=0$ for all i, j, (if $N^{(2)}{}_{kc^2}\neq 0$, then $N^{(1)}{}_{ij}=0$ for all i, j), because if $N^{(1)}{}_{kc^1}\neq 0$, $N^{(1)}{}_{nc^1}\neq 0$ from the argument to obtain (13) and (14). Thus we have shown

$$\begin{pmatrix} T_{11} & X_{11} \\ 0 & T_{21} \end{pmatrix} \left(\text{resp. } \begin{pmatrix} T_{11} & X_{21} \\ 0 & T_{22} \end{pmatrix}\right)$$

is a direct summand of R, provided $X_{11}\neq 0$ (resp. $X_{21}\neq 0$). We obtain the same result for any set (T_{1i}, T_{2k}, T_{2s}). Therefore R is serial from (13) and (14).

4. Strongly almost hereditary rings

Among right almost hereditary rings, we shall determine the structure of such rings with

(15) Every submodule of finitely generated projective module is again almost projective.

Lemma 11. *If a simple module eR/eJ is almost projective, then either eR is uniserial and $eR/eJ^2, \cdots, eR/eJ^n$, eR are injective or eR is simple, where $eJ^{n+1}=0$.*

Proof. This is clear from [6], Theorem 1.

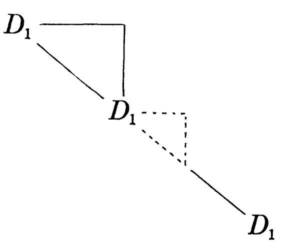
First we suppose that R is a right almost hereditary ring with (15). Then we may study the following rings from Theorem 1:

α) Hereditary rings.

β) Serial rings in the first category.

Then $R=\mathrm{T}(n_1, \cdots, n_r: D)$. Let $\{e^{(i)}{}_j\}_{i=1,j=1}^{r\ \ n_i}$ be matrix units in the diagonal of R. Take $\mathrm{Soc}(e^{(r-1)}{}_1R)$. Then it is almost projective and isomorphic to $e^{(r)}{}_kR/e^{(r)}{}_kJ$ for some k from the structure of R. Hence $k=1$ or $e^{(r)}{}_kR$ is simple from Lemma 11. However we do not have the latter. Hence $\mathrm{Soc}(e^{(r-1)}{}_1R)\sim e^{(r)}{}_1R/e^{(r)}{}_1J$. Similarly we obtain $\mathrm{Soc}(e^{(i)}{}_1R)\sim e^{(i+1)}{}_1R/e^{(i+1)}{}_1J$ for $n>i\geq 1$. (See the diagram below.)

(16)



γ) Serial rings in the second category.

Since R has a cyclic Kupisch series, $eR/eJ\sim\mathrm{Soc}(fR)$ for some f. Hence

eR is injective by Lemma 11 for each e. Then $|eR|$ is same for all e (cf. the proof of Lemma 4), say $|eR|=n$ and eR/eJ^k is injective for all $k>1$ by Lemma 11. Therefore there exist $m(n-1)$ distinct injectives, where $1=\sum_{i=1}^m e_i$. Accordingly $m(n-1)\leqq m$ (R is basic), and hence $n\leqq 2$.

δ) $\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$

From γ) T_2 is a direct sum of serial rings in the first category as in β). For the sake of simplicity, we study R in which T_2 is indecomposable. Since $X=\Sigma\oplus f_{11}R/A_{11}$ as in (8), we have $A_{11}=e_{11}J$ by β) and applying Lemma 11 to $\mathrm{Soc}(f_{11}R/A_{11})$. Hence X is semisimple.

Conversely we shall show that the above rings with stated properties satisfy the condition (15).

α) This is clear.

β) and γ) Let A be a submodule of projective module $\sum_{i=1}^n\oplus e_iR$. If $n=1$, we can easily see that A is almost projective by the structure of e_iR and [6], Theorem 1. Assume $n>1$. Since R is a serial ring, $A\sim\Sigma_K\oplus e'_iR/B_i$. e'_iR/B_i being uniserial, e'_iR/B_i is monomorphic to a submodule of some e_iR. Hence A is almost projective by the initial remark.

δ) First we shall show that every submodule in $\Sigma_K\oplus h'_iR$ is almost projective, where the h'_i are the h_i in (8). Put

$$C=R-\begin{pmatrix} T_1 & X \\ 0 & f_{11}R/e_{11}J \end{pmatrix}.$$

(We may assume, from the proof below, that T_2 is indecomposable.)
Then C is a two-sided ideal. Set $\tilde{R}=R/C=\sum_{i=1}^n\oplus\tilde{h}'_i\tilde{R}\oplus D_1$, where $D_1=f_{11}Rf_{11}$ $(\sim f_{11}R/f_{11}J)$. Then A is an $\tilde{R}$-module. Since X is semisimple, R is hereditary by (8) and (9). Hence $A\sim\Sigma_{K'}\oplus\tilde{h}'_j\tilde{R}\oplus\Sigma_L\oplus f_1R/f_1J$, where $f_1=f_{11}$. Those modules being R-modules via the natural map $R\to\tilde{R}$ and $\tilde{h}'_j\tilde{R}=h'_jR$, A is an almost projective R-module from Theorem 1. Let A be any submodule of projective module $\Sigma\oplus_K h'_iR\oplus\Sigma_J\oplus f_{1i}R\oplus\Sigma_L\oplus g_{ij}R$ (the same notations given in §1). Put $P_1=\Sigma_K\oplus h'_iR$, $P_2=\Sigma_{J'}\oplus f_1R$, $P_3=\Sigma_{L'}\oplus e_iR$, where $L'=\{f_{1i}, g_{ij}\}-f_{11}$. Set $A_1=A_L\cap P_1$, $A_{23}=A\cap(P_2\oplus P_3)$. Then taking a suitable decomposition of $P_2\oplus P_3$, we may assume that $P_2=\Sigma_{J'}\oplus f'_1R$, $P_3=\Sigma_{L'}\oplus e'_jR$ and $A_{23}=\Sigma_{J'}\oplus A_{2k}\oplus\Sigma_{L'}\oplus A_{3s}$, where $f'_1\sim f_1$, $e'_j\sim e_j$, $A_{2k}\subset f'_1R$ and $A_{3s}\subset e'_sR$ (note that T_2 is serial, cf. [7], Lemma 2). We remark that the common simple sub-factor module between P_1 and $P_2\oplus P_3$ is f_1R/f_1J. Since $A^1/A_1\sim A^{23}/A_{23}$, $(A^1/A_1)f_1=A^1/A_1$ from the structure of R, where A^1 (resp. A^{23}) is the projection of A to P_1 (resp. $P_2\oplus P_3$). Further $P_1f_1R\subset\mathrm{Soc}(P_1)$. Hence A^1/A_1 is a semisimple module whose simple component is isomorphic to f_1R/f_1J. As a consequence $A^1=X'\oplus A_1$, where $\mathrm{Soc}(P_1)=X'\oplus X''$. Put τ: $A^{23}/A_{23}\sim A^1/A_1=X'$. Let $\tilde{P}=(P_2\oplus P_3)/A_{23}=\Sigma_{J'}\oplus f'_1R/A_{2k}\oplus\Sigma_{L'}\oplus e'_jR/A_{3s}$. Since $(A^1/A_1\sim)A^{23}/A_{23}$ is a semisimple module as

above, $\tilde{A}^{23} \subset \Sigma \oplus_{J'} f'_1 R / f'_1 J$. Since $f_1 R$ is uniserial, after taking a suitable decomposition $\Sigma_{J'} \oplus f'_1 R = \Sigma_{J'} \oplus f''_i R$ ($f''_i \sim f_1$), we may assume that there exists a subset J'' of J' such that $A_{2p} = f''_1 J$ for $p \in J''$ and $\tilde{A}^{23} = \Sigma_{J''} \oplus f''_1 R / f''_1 J$. Using the natural epimorphism θ: $\Sigma \oplus_{J''} f''_1 R \to \Sigma \oplus f''_1 R / f''_1 R$, we obtain $P = P_1 \oplus P'_2(\tau\theta) \oplus P''_2 \oplus P_3$ and $A = A_1 \oplus P'_2(\tau\theta) \oplus \Sigma_{J'-J''} \oplus A_{2s} \oplus \Sigma_{L'} \oplus e'_s A_{3s}$, which is almost hereditary by the initial remark, where $P'_2 = \Sigma_{J''} \oplus f''_1 R$ and $P''_2 = \Sigma_{J'-J''} \oplus f''_1 R$.

Summarizing the above, we have

Theorem 3. *Let R be an artinian ring. Then R is a right almost hereditary ring with* (15) *if and only if R is a direct sum of the following rings:*

1) *Hereditary rings.*
2) *Serial rings in the first category with the structure* (16).
3) *Serial rings in the second category with $J^2 = 0$.*
4) *Rings given in* (9), *where T_2 is a direct sum of serial rings in* 2) *and $A_{i1} = e_{i1} J$ in* (8).

Next we study the second stronger condition than that of almost hereditary rings.

(17) $\mathrm{J}(Q)$ is almost projective for any finitely generated almost projective module Q.

(18) Every submodule of Q is again almost projective.

Lemma 12. *Let R be a ring in* (9) *and $X \neq 0$. If R is two-sided indecomposable and not serial, then $h_i R$ is never injective for any h_i in* (8).

Proof. R is right almost hereditary by Theorem 1. First assume that $T_1 = \mathrm{T}(D_1, D_2, \cdots, D_n)$ and T_2 are two-sided indecomposable. Further we may assume $T_2 = \mathrm{T}(m: D)$ (see the proof below). Let $A_1 = f_2 T_2 f_{k+1} \oplus \cdots \oplus f_1 T_2 f_m = (0, \cdots, 0, D, \cdots, D)$, where the f are the idempotents in the diagonal of T_2. Put $E_1 = 1_{T1}$ and $C = R(f_{k+1} + \cdots + f_m)$. Then C is a two-sided ideal. Set $\tilde{R} = R/C$ (cf. the proof of Theorem 2), and $\tilde{R}$ is hereditary by (8). Assume that $h_i R$ is injective for some i. Then $h_i R$ is also injective as an $\tilde{R}$-module. Hence $\tilde{R} = \tilde{R}_1 \oplus \tilde{R}_2$ by the proof of Corollary 3, where $\tilde{R}_1 = T(n_*: D^*)$ is given from the Kupisch series $\{h_1, \cdots\}$. Now $T_1 = E_1 \tilde{R} E_1 = E_1 \tilde{R}_1 E_1 \oplus E_1 \tilde{R}_2 E_1$, and hence $T = E_1 \tilde{R}_1 E_1$ for $E_1 h_i E_1 \neq 0$. As a consequence $T_1 \tilde{R} \subset \mathrm{T}(n_*: D_*) = \Sigma h_{\rho(k)} \tilde{R} \oplus \Sigma f_{\mu(k)} \tilde{R}$, where $\{h_{\rho(k)}\} \subset \{h_j\}$ and $\{f_{\mu(k)}\} \subset \{f_j\}$. Therefore $h_i \tilde{R} = h_1 \tilde{R}$ and $\{h_1, h_2, \cdots, h_n, f_1, \cdots, f_m\}$ is a Kupisch series in R from the structure (9). Accordingly $R = T(n+k, m-k: D)$ is serial, a contradiction. Hence any h_i is never injective. In general case we can use the same agrument given in the proof of Theorem 2.

Theorem 4. *Let R be a basic artinian ring. Then R satisfies* (17) *if and* *only if R is a direct sum of the following rings:*

1) *Hereditary rings which ane not serial.*
2) *Serial rings with $J^2=0$.*
3) *The ring in* (9) *with* $\mathrm{J}(T_2)^2=0$.

In this case R satisfies (18).

Proof. We may assume that R is two-sided indecomposable. The ring with (17) is almost hereditary.

i) R is hereditary. If some e_iR is injective, then R is a simple ring or $R=\mathrm{T}(n: D)$ by Corollary 3. Suppose $R=\mathrm{T}(n: D)$ ($n\geqq 2$). Then $e_{11}R/\mathrm{Soc}(e_{11}R)$ is almost projective by [6], Theorem 1, where the e_{ii} are matrix units in R, and $e_{11}J/\mathrm{Soc}(e_{11}R)\sim e_{22}R/\mathrm{Soc}(e_{22}R)$ by the structure of $T(n: D)$, provided $n>2$. Then $e_{22}R$ is injective by [6], Theorem 1, a contradiction. Therefore $R=T(2: D)$. Next if any e_iR is not injective, then R is not serial.

ii) R is a serial ring in the first category. Then we can see $J^2=0$ in the above manner.

iii) R is a serial ring in the second category. Then $J^2=0$ from Theorem 3.

iv) R is the ring in (9), Then $\mathrm{J}(T_2)^2=0$ as above. Conversely if R is non-serial and hereditary, then every almost projective module is projeative by Corollary 3 and [6], Theorem 1. Hence (17) holds true. Next if R is a serial ring with $J^2=0$, then $\mathrm{J}(Q)=0$ for every non-projective, almost projective module Q, and so (17) holds true from Theorem 3. Finally let R be of the form (3) and $\tilde{R}=R/C$ as in the proof of Theorem 2. Then $\tilde{R}$ is hereditary and any $h_iR=\tilde{h}_i\tilde{R}$ is not injective by Lemma 12. Then there are no almost projective modules isomorphic to h_iR/A_i ($A_i\neq 0$). Accordinagly non-projective almost projective modules are given from T_2. Therefore $Q'=\Sigma_I\oplus h_iR\oplus\Sigma_J\oplus f_jR\oplus\Sigma_K\oplus f_jR/f_jJ$ for a finitely generated and almost projective module Q', where the f_j are primitive idempotents in T_2, and hence $\mathrm{J}(Q)$ is almost projective from the above. Thus we have obtained the equivalence in the theorem from Theorem 1. Noting that if h_iR is injective in the ring (3), then R is serial from the above, we can show (18) in the similar manner to the proof of Theorem 3.

We remain the last case:

Every submodule of Q is projective.

We can easily see that R has the above property if and only if R is hereditary and not serial. If we assume (15), (17) and (18) for left R-modules as well as right R-modules, then we obtain a characterization of hereditary rings and serial rings satisfying the conditions in Theorems 3 and 4, respectively.

REMARK. In the above proof we used (17) and (18) only for indecomposable projective modules and indecomposable almost projective modules, respeatively. Hence if every submodule of indecomposable and projective (resp. almost projective) module is almost projective, then (17) (resp. (18)) holds ture.

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