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ON SUBFIELD SYMMETRIC SPACES OVER A FINITE FIELD*

Dedicated to Professor Nobuhiko Tatsuuma on his sixtieth birthday

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1. Introduction

Let $\boldsymbol{G}$ be a connected reductive linear algebraic group defined over a finite field $\boldsymbol{F}_{q'}$. We put

$$G=\boldsymbol{G}(\boldsymbol{F}_q)\,, \quad q=q'^2\,.$$

Then the q'-th power Frobenius map $\sigma\colon \boldsymbol{G}\to\boldsymbol{G}$ induces on G an involutory automorphism $\tau\colon g\to{}^{\tau}g$ with the fixed point set

$$G_\tau=\boldsymbol{G}(\boldsymbol{F}_{q'})\,.$$

We are concerned with the irreducible representations of the Hecke algebra $H(G, G_\tau)$, or, almost equivalently, with the zonal spherical functions on the *subfield symmetric space* G/G_τ. (A similar object, in the category of real Lie groups, is also being studied; see, e.g., [9], [24].) In the present paper, we take up the following problem:

(A) Classify the irreducible representations of $H(G, G_\tau)$, and determine their dimensions.

Since the classification of the irreducible representations of G is well-understood by works of G. Lusztig (see [21]), we can reduce problem (A) to the following one:

(A′) For each irreducible character χ of G determine the multiplicity $m_\tau(\chi)=\langle 1_{G_\tau}^G, \chi\rangle$ with which χ appears in the induced character $1_{G_\tau}^G$.

For an irreducible character χ of G, let $c_\tau(\chi)$ be the *twisted Frobenius-Schur indicator* [13]:

$$c_\tau(\chi)=|G|^{-1}\sum_{g\in G}\chi({}^{\tau}gg)\,.$$

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We extend $m_\tau(\cdot)$ and $c_\tau(\cdot)$ to linear functions on the space $Cl(G)$ of class functions on G. The fundamental relation between $c_\tau(\cdot)$ and $m_\tau(\cdot)$ is:

$$m_\tau(\chi) = c_\tau(t^*(\chi)), \tag{1.1}$$

where $t^*\colon Cl(G)\to Cl(G)$ is the twisting operator, introduced in [11] and studied intensively by T. Asai [2], [3] and F. Digne and J. Michel [8]. We also have [13]

$$c_\tau(\chi) = \begin{cases} \pm 1 & \text{if } {}^\tau\chi = \bar{\chi}, \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

for an irreducible character χ of G.
If χ is a uniform function, i.e. a linear combination of Deligne-Lusztig virtual characters [6], then, by Asai [3], we know:

$$t^*(\chi) = \chi. \tag{1.3}$$

By (1.1)–(1.3), we have

$$m_\tau(\chi) = \begin{cases} 1 & \text{if } {}^\tau\chi = \bar{\chi}, \\ 0 & \text{otherwise} \end{cases}$$

for a uniform irreducible character χ of G. Since almost all irreducible characters are uniform, we might say that the induced character $1_{G_\tau}^G$ is "almost" multiplicity-free. For a not necessarily uniform irreducible character χ, the calculation of $m_\tau(\chi)$ is reduced to solving the following problems (A'a), (A'b):

(A'a) Determine $\langle t^*(\chi), \eta\rangle$ for any irreducible character η of G.

(A'b) Determine $c_\tau(\eta)$ for any irreducible character η of G such that $\langle t^*(\chi), \eta\rangle \neq 0$.

Thanks to the works of Asai [2] [3] and Digne and Michel [8], we already know quite a lot concerning problem (A'a). So we concentrate on problem (A'b). First we take up the case when χ is unipotent. Assume that $\boldsymbol{G}$ is simple modulo its center. If $\boldsymbol{G}$ is of exceptional type, we further assume that the characteristic is good. Then we can determine $c_\tau(\chi)$ for unipotent irreducible characters χ of G. Once we know these values, Asai's results [2] combined with (1.1) allow us to compute $m_\tau(\chi)$ for unipotent χ's. See the formula (5.1.2). The values $c_\tau(\chi)$ and $m_\tau(\chi)$ for unipotent χ's are given in 5.3 for exceptional groups, and in 5.4 and 6.2–6.3 for classical groups. A part of our result can be stated as follows. (Recall [21] that unipotent characters are partitioned into "families" and that each family contains a unique "special" unipotent character.)

Theorem 1.4. (i) *Let $\boldsymbol{G}$ be a connected reductive group wihch is simple modulo its center and is split over a finite field, and $\sigma\colon \boldsymbol{G}\to\boldsymbol{G}$ the corresponding*

Frobenius map. When $\boldsymbol{G}$ is of exceptional type, we assume that the characteristic is good. Let χ be a unipotent irreducible character of $G=\boldsymbol{G}_{\sigma^2}$, and λ_χ the root of 1 *associated with χ by Lusztig* [21; Ch. 11] *in connection with the eigenvalues of σ^2 on l-adic cohomology spaces of "Deligne-Lusztig varieties $X(w)$". Then we have*

$$c_\tau(\chi) = \begin{cases} \lambda_\chi & \text{if } \lambda_\chi = \pm 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Under the same notation as in (i), let χ_0 be a special unipotent character of G, and Γ_{χ_0} the finite group associated* [21] *with χ_0. Then $m_\tau(\chi_0)$ is equal to the number of conjugate classes of Γ_{χ_0} contained in $\{a\in\Gamma_{\chi_0} \mid a^2=1\}$, if χ_0 is "not exceptional", i.e. if χ_0 is not in the families corresponding to* 512 *dimensional representations of the Weyl group of type E_7 or to* 4096 *dimensional representations of the Weyl group of type E_8. If χ_0 is exceptional, then $m_\tau(\chi_0)=1$.*

Moreover, when G is of classical type, we have

$$m_\tau(\chi) = 0$$

for any non-special unipotent character χ.

In determining $c_\tau(\chi)$ and $m_\tau(\chi)$ for non-unipotent characters χ, we must restrict ourselves to the case when $\boldsymbol{G}$ is a classical group with connected center (see (6.4.1)). (In order to be able to treat a more general case, Asai's result [3] must be generalized.) Then, using results of Lusztig [21] and Asai [3], we show in Theorem 6.4.3 that the problem can be reduced to the case of unipotent characters already mentioned. Thus pioblem (A) is solved completely when $\boldsymbol{G}$ is a classical gruop with connected center.

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NOTATION. For a set X, $|X|$ denotes its cardinality. If τ is a transformation of X, we put

$$X_\tau = \{x\in X \mid {}^\tau x = x\}.$$

If the inverse element $x^{-1}\in X$ is defined for any $x\in X$, we put

$$X_{-\tau} = \{x\in X \mid {}^\tau(x^{-1}) = x\}.$$

Let Y be a subset of X, and f a map from X to another set. Then $f \mid Y$ denotes the restriction of f to Y. Let G be a group, g an element of G, and S a subset of G. Then

$${}^gS = \{gsg^{-1} \mid s\in S\}.$$

If G is a finite group, $\hat{G}$ denotes the set of irreducible complex characters of G. Let H be a subgroup of G, and ϕ a class function on H. Then ϕ^G is the class function on G induced from ϕ, and, for $g \in G$, ${}^g\phi$ is the class function on ${}^gH$ defined by

$$ {}^g\phi(ghg^{-1}) = \phi(h), \quad h \in H. $$

2. Preliminaries

2.1. Let $\tilde{G}$ be a finite group containing a normal subgroup G of index 2. Let

$$ \tau \in \tilde{G} - G. $$

For $\chi \in \hat{G}$, we define the twisted Frobenius-Schur indicator $c_\tau(\chi)$ by

$$ (2.1.1) \qquad c_\tau(\chi) = |G|^{-1} \sum_{g \in G} \chi({}^\tau g g) = |G|^{-1} \sum_{x \in \tilde{G}-G} \chi(x^2), $$

or, equivalently, by

$$ (2.1.2) \qquad \sum_{\chi \in \hat{G}} c_\tau(\chi)\, \chi(g) = |\{h \in G \mid (\tau h)^2 = g\}|, \quad g \in G. $$

We have the following generalization of a theorem of Frobenius and Schur.

Theorem 2.1.3 ([13]). (i) *We have*

$$ c_\tau(\chi) = \begin{cases} \pm 1 & \text{if } {}^\tau\chi = \bar{\chi}, \\ 0 & \text{otherwise.} \end{cases} $$

where the bar means the complex conjugation.

(ii) *Assume that $\tau^2 = 1$. Let M_χ be a G-module affording $\chi \in \hat{G}$. Then $c_\tau(\chi) = 1$ (resp. -1) if and only if there exists a non-zero symmetric (resp. skew symmetric) bilinear form $B(\cdot, \cdot)$ on M_χ such that*

$$ B(g \cdot m_1, {}^\tau g \cdot m_2) = B(m_1, m_2), \quad g \in G,\ m_1, m_2 \in M_\chi. $$

We also have the following generalization of a result of G. W. Mackey.

Theorem 2.1.4 ([13]). *Let H be a sjubgroup of G such that $\tau^2 \in H$, and $D_{-\tau}$ a set of representatives of the double cosets ${}^\tau HxH$, $x \in G$, such that ${}^\tau({}^\tau HxH)^{-1} = {}^\tau HxH$. Let α be a (possibly reducible) character of H. For $x \in D_{-\tau}$, let ${}^{\tau x}\alpha \cdot \alpha$ be the character $h \to \alpha(x^{-1}\tau^{-1}h\tau x)\, \alpha(h)$ of ${}^{\tau x}H \cap H$. Choose an element z_x of $xH \cap {}^\tau H \cdot {}^\tau x^{-1}$. Then:*

(i) *$H(\tau x, \tau z_x) = \langle \tau z_x, {}^{\tau x}H \cap H \rangle$ contains ${}^{\tau x}H \cap H$ as a normal subgroup of index 2.*

(ii) *There exist characters $({}^{\tau x}\alpha \cdot \alpha)^\pm$ of $H(\tau x, \tau z_x)$ such that*

$$ ({}^{\tau x}\alpha \cdot \alpha)^\pm \mid {}^{\tau x}H \cap H = {}^{\tau x}\alpha \cdot \alpha $$

and that

$$({}^{\tau x}\alpha\cdot\alpha)^{\pm}(y)=\pm\alpha(y)^2\,,\quad y\in\tau z_x({}^{\tau x}H\cap H)\,.$$

(iii) *We have*

$$\begin{aligned}c_\tau(\alpha^G)&=\sum_{x\in D_{-\tau}}|H(\tau x,\tau z_x)|^{-1}\sum_{y\in H(\tau x,\tau z_x)}\{({}^{\tau x}\alpha\cdot\alpha)^+-({}^{\tau x}\alpha\cdot\alpha)^-\}\,(y)\\&=\sum_{x\in D_{-\tau}}c_{\tau z_x}(\alpha|{}^{\tau x}H\cap H)\,.\end{aligned}$$

2.2. Let $\boldsymbol{G}$ be a connected linear algebraic group over an algebraically closed field K. Let σ be a surjective endomorphism of $\boldsymbol{G}$ such that

$$|\boldsymbol{G}_{\sigma^2}|<\infty\,.$$

We put

$$G=\boldsymbol{G}_{\sigma^2}\,.$$

Then the cyclic group $\langle\tau\rangle$ of order 2 acts on G by

$${}^\tau x={}^\sigma x\,,\quad x\in G\,.$$

Hence

$$G_\tau=\boldsymbol{G}_\sigma\,.$$

We also put

$$\tilde{G}=\langle\tau\rangle\,G\quad\text{(semi-direct product).}$$

By a theorem of Lang and Steinberg, any $g\in\boldsymbol{G}$ can be written as

$$g={}^{\sigma^2}(\alpha_g^{-1})\,\alpha_g \tag{2.2.1}$$

with some $\alpha_g\in\boldsymbol{G}$.

Lemma 2.2.2. (i) *Let g be an element of G, and α_g as in* (2.2.1). *Then*

$$t(g)=\alpha_g\cdot{}^{\sigma^2}(\alpha_g^{-1})$$

is again an element of G, and its G-conjugacy class does not depend on the choice of α_g. Moreover, the transformation

$$t^*\colon\chi\to\chi\circ t$$

on the space of class functions on G is unitary with respect to the standard hermitian inner product $\langle\ ,\ \rangle$.

(ii) *Let g be an element of G. Then $t({}^\tau gg)$ is an element of G_τ, and its G_τ-conjugacy class $Cl_{G_\tau}(t({}^\tau gg))$ depends only on the $\tilde{G}$-conjugacy class $Cl_{\tilde{G}}(\tau g)$ of τg. The correspondence*

$$Cl_{\tilde{G}}(\tau g)\to Cl_{G_\tau}(t({}^\tau gg))$$

gives a bijection between the conjugacy classes of $\tilde{G}$ contained in the coset τG and the conjugacy classes of G_τ. Moreover, we have

$$|G|^{-1}|Cl_{\tilde{G}}(\tau g)| = |G_\tau|^{-1}|Cl_{G_\tau}(t({}^\tau gg))|$$

for any $g \in G$.

Both parts (i) and (ii) are special cases of [11; Lem. 2.2]. In fact, by putting $m=1$ and replacing σ with σ^2 (resp. by putting $m=2$) in [loc. cit.], we get (i) (resp. (ii)).

For a class function χ on G, we put

$$m_\tau(\chi) = \langle 1_{G_\tau}^G, \chi\rangle = |G_\tau|^{-1}\sum_{g\in G_\tau}\chi(g)\,.$$

We also define the number $c_\tau(\chi)$ by the formula (2.1.1). The main result of this section is:

Theorem 2.2.3. (i) *Let χ be a class function on G. Then*

$$m_\tau(\chi) = c_\tau(t^*(\chi))\,.$$

(ii) *Let $\chi \in \hat{G}$ be such that $\chi = t^*(\chi)$. Then*

$$m_\tau(\chi) = c_\tau(\chi) = \begin{cases} 1 & \text{if } {}^\tau\chi = \bar{\chi}\,, \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *Let $\chi \in \hat{G}$. Put*

$$n(\chi) = |\{\eta \in \hat{G} \mid \langle t^*(\chi), \eta\rangle \neq 0\}|\,.$$

Then

$$m_\tau(\chi) \leq \sqrt{n(\chi)}\,.$$

(iv) *Let $\chi \in \hat{G}$. If $m_\tau(\chi) \neq 0$, then there exists an $\eta \in \hat{G}$ such that $\langle t^*(\chi), \eta\rangle \neq 0$, $c_\tau(\eta) \neq 0$ and $\eta^\tau = \bar{\eta}$.*

(v) *Let $\chi \in \hat{G}$. If $c_\tau(\chi) \neq 0$, then there exists an $\eta \in \hat{G}$ such that $\langle (t^*)^{-1}(\chi), \eta\rangle \neq 0$ and $m_\tau(\eta) \neq 0$.*

Proof. Part (i) follows from Lemma 2.2.2 and the definitions of $m_\tau(\cdot)$ and $c_\tau(\cdot)$. Part (ii) follows from part (i) and Theorem 2.1.3. To prove part (iii), we write

$$t^*(\chi) = \sum_{\eta\in\hat{G}} a_\eta \eta$$

with $a_\eta \in \boldsymbol{C}$. Since t^* is unitary, we have

$$\sum_\eta |a_\eta|^2 = 1\,. \tag{2.2.4}$$

By part (i), we have

$$m_\tau(\chi) = \sum_\eta a_\eta c_\tau(\eta) . \tag{2.2.5}$$

Hence, by Theorem 2.1.3, (2.2.4), and the Cauchy inequality,

$$m_\tau(\chi) \leq \sum_\eta |a_\eta| \leq \sqrt{n(\chi)} .$$

Thus we get part (iii). Part (iv) follows from (2.2.5). Part (v) is similar.

REMARK 2.2.6. By Theorem 2.2.3, for the calculation of $m_\tau(\chi)$ for a given $\chi \in \hat{G}$, it is enough to solve the following two problems:

(a) Determine $\langle t^*(\chi), \eta \rangle$ for any $\eta \in \hat{G}$.

(b) Determine $c_\tau(\eta)$ $(=1, -1$ or $0)$ for any $\eta \in \hat{G}$ such that $\langle t^*(\chi), \eta \rangle \neq 0$.

3. Deligne-Lusztig virtual characters

In this section, we apply Theorem 2.2.3 to the case when G is a finite reductive group, and χ is a Deligne-Lusztig virtual character of G.

Henceforth, $\boldsymbol{G}$ denotes a connected reductive linear algebraic group over an algebraically closed field K of positive characteristic p. Let q be a positive integral power of p. Let σ be an endomorphism of $\boldsymbol{G}$ whose square σ^2 is a q-th power Frobenius map of $\boldsymbol{G}$ with respect to an $\boldsymbol{F}_q$-rational structure of $\boldsymbol{G}$. As in Section 2.2, we put $G = \boldsymbol{G}_{\sigma^2}$, and consider the involutive automorphism $\tau: x \to {}^\tau x = {}^\sigma x$ of G.

3.1. Let $\boldsymbol{T}$ be a σ^2-stable maximal tours of $\boldsymbol{G}$, and θ a character of $T = \boldsymbol{T}_{\sigma^2}$. Let $r_{\boldsymbol{T}}[\theta] = r_{\boldsymbol{T}}^{\boldsymbol{G}}[\theta]$ be the character of the Deligne-Lusztig virtual representation $R_{\boldsymbol{T}}^{\boldsymbol{G}}[\theta]$ of G [6].

Lemma 3.1.1. (i) $\overline{r_{\boldsymbol{T}}[\theta]} = r_{\boldsymbol{T}}[\bar{\theta}]$.

(ii) ${}^\tau(r_{\boldsymbol{T}}[\theta]) = r_{{}^\tau\boldsymbol{T}}[{}^\tau\theta]$.

Proof. Part (i) follows from [6; 4.2] and the fact [6; p. 123] that $r_{\boldsymbol{T}}[\theta]$ is integer valued on the unipotent elements. Part (ii) follows from the definition of $R_{\boldsymbol{T}}^{\boldsymbol{G}}[\theta]$ and standard properties of l-adic cohomology.

The following result is of fundmaental importance for us.

Theorem 3.1.2 (Asai [3; 2.4.1]). *For any σ^2-stable maximal torus $\boldsymbol{T}$ and any character θ of $\boldsymbol{T}_{\sigma^2}$, we have*

$$t^*(r_{\boldsymbol{T}}[\theta]) = r_{\boldsymbol{T}}[\theta] .$$

By Theorem 3.1.2 and Theorem 2.2.3, we get:

Theorem 3.1.3. *Let χ be a uniform function on G, i.e. a linear combina-*

tion of Deligne-Lusztig virtual characters.

(i) *We have*

$$m_\tau(\chi) = c_\tau(\chi) .$$

(ii) *If* $\chi \in \hat{G}$, *then*

$$m_\tau(\chi) = c_\tau(\chi) = \begin{cases} 1 & \text{if } {}^\tau\chi = \bar{\chi} , \\ 0 & \text{otherwise.} \end{cases}$$

(iii) *If* $m_\tau(\chi) \neq 0$ (*resp.* $c_\tau(\chi) \neq 0$), *then there exists an* $\eta \in \hat{G}$ *such that* $\langle \eta, \chi \rangle \neq 0$ *and* $c_\tau(\eta) \neq 0$ (*resp.* $m_\tau(\eta) \neq 0$).

3.2. Let $S = S_G$ be the set of σ^2-stable semisimple conjugacy classes of the dual group [6] $\boldsymbol{G}^*$ of $\boldsymbol{G}$. Then S can be identified with the set of geometric conjugacy classes of pairs $(\boldsymbol{T}, \theta)$ of σ^2-stable maximal tori $\boldsymbol{T}$ and characters θ of $T = \boldsymbol{T}_{\sigma^2}$. For $(s) \in S$, let $\hat{G}_{(s)}$ be the set of irreducible characters of G contained in some virtual character $r_{\boldsymbol{T}}[\theta]$ such that $(\boldsymbol{T}, \theta) \in (s)$. Then [6; 10.1] we have a partition

$$\hat{G} = \bigcup_{(s) \in S} \hat{G}_{(s)} .$$

We denote by $(G/G_\tau)^\wedge$ the set of irreducible characters of G contained in $1_{G_\tau}^G$.

Theorem 3.2.1. *Let* $(s) \in S = S_G$. *Then* $\hat{G}_{(s)}$ *contains an element of* $(G/G_\tau)^\wedge$ *if and only if*

$$(s) \in S_{-\sigma} = \{(s) \in S \mid {}^\sigma(s) = (s)^{-1}\} .$$

In other words, we have a partition

$$(G/G_\tau)^\wedge = \bigcup_{(s) \in S_{-\sigma}} (G/G_\tau)^\wedge_{(s)}$$

of $(G/G_\tau)^\wedge$ *into non-empty parts*:

$$(G/G_\tau)^\wedge_{(s)} = (G/G_\tau)^\wedge \cap \hat{G}_{(s)} .$$

Proof. Let α be an element of $(G/G_\tau)^\wedge$, and (s) an element of S such that $\alpha \in \hat{G}_{(s)}$. Put

$$\alpha_{(s)} = \sum_{\mu \in \hat{G}_{(s)}} \mu(1)\, \mu .$$

Then $\alpha_{(s)}$ is a linear combination of $\{r_{\boldsymbol{T}}[\theta] \mid (\boldsymbol{T}, \theta) \in (s)\}$ by [6; 7.5]. Since

$$m_\tau(\alpha_{(s)}) \geq \mu(1)\, m_\tau(\mu) > 0 ,$$

we have

$$c_\tau(\mu) \neq 0$$

for some $\mu \in \hat{G}_{(s)}$ by Theorem 3.1.3 (iii). By Theorem 2.1.2 (i), such μ satisfies

$$ {}^{\tau}\mu = \bar{\mu} . $$

Hence, by Lemma 3.1.1, we have

$$ (s) \in S_{-\sigma} . $$

This proves the only-if-part. To prove the converse, we embed $\boldsymbol{G}$ in a group $\boldsymbol{G}_1$ with connected center and the same derived group. Let $(s) \in (S_G)_{-\sigma}$. Then there exists an element $(s_1) \in (S_{G_1})_{-\sigma}$ which corresponds to (s) under the canonical map $\boldsymbol{G}_1^* \to \boldsymbol{G}^*$. Since the if-part of (i) holds for $\boldsymbol{G}$ replaced by $\boldsymbol{G}_1$ (this follows from Theorem 3.2.2 below), there exists an irreducible character χ_1 of $G_1 = (\boldsymbol{G}_1)_{\sigma^2}$ contained in $(G_1/(G_1)_\tau)^\wedge \cap \hat{G}_{1(s_1)}$. The irreducible components of $\chi_1 | G$ are contained in $\hat{G}_{(s)}$, and at least one of them is contained in $(G/G_\tau)^\wedge$. Hence $(G/G_\tau)^\wedge \cap \hat{G}_{(s)}$ is non-empty. This proves the if-part.

Theorem 3.2.2. *Assume that the center of $\boldsymbol{G}$ is connected. Let $(s) \in S_{-\sigma}$. Let $\rho_{(s)}$ and $\rho'_{(s)}$ be the elements of $\hat{G}_{(s)}$ defined by Deligne and Lusztig* [6; 10.7]. *Then*

$$ m_\tau(\rho_{(s)}) = c_\tau(\rho_{(s)}) = 1 $$

and

$$ m_\tau(\rho'_{(s)}) = c_\tau(\rho'_{(s)}) = 1 . $$

Proof. By the definition of $\rho_{(s)}$, $\rho'_{(s)}$ and Lemma 3.1.1, we see that $\rho_{(s)}$ and $\rho'_{(s)}$ are linear combinations of $r_T[\theta]$, $(\boldsymbol{T}, \theta) \in (s)$, and satisfy ${}^{\tau}\rho_{(s)} = \bar{\rho}_{(s)}$ and ${}^{\tau}\rho'_{(s)} = \bar{\rho}'_{(s)}$. Hence the theorem follows from Theorem 3.1.3 (ii).

Let $\boldsymbol{T}$ be a σ-stable maximal torus, and $T = \boldsymbol{T}_{\sigma^2}$. Then the subgroup $T_{-\tau}$ of G will be called a τ-anisotropic maximal torus of G, or a maximal torus of the symmetric space G/G_τ. The "small" Weyl group

$$ W^{(1)}(\boldsymbol{T}) = (N_G(\boldsymbol{T})/\boldsymbol{T})_\sigma $$

(as compared to the "big" Weyl group

$$ W^{(2)}(\boldsymbol{T}) = (N_G(\boldsymbol{T})/\boldsymbol{T})_{\sigma^2}) $$

acts on $T_{-\tau}$, hence on $(T_{-\tau})^\wedge$. An element φ of $(T_{-\tau})^\wedge$ is said to be in general position if the stabilizer of φ in $W^{(1)}(\boldsymbol{T})$ is trivial.

Lemma 3.2.3. *Let T and φ be as above. We define $\theta_\varphi \in \hat{T}$ by*

$$ \theta_\varphi(t) = \varphi({}^{\tau}t^{-1} \cdot t) , \quad t \in T . $$

(i) *The following conditions for $\theta \in \hat{T}$ are equivalent.* (a) ${}^{\tau}\theta = \bar{\theta}$, (b) $\theta | T_\tau = 1$,

(c) $\theta=\theta_\varphi$ *for some* $\varphi\in(T_{-\tau})^\wedge$.

(ii) *Assume that the center of* $\boldsymbol{G}$ *is connected Then* $\varphi\in(T_{-\tau})^\wedge$ *is in general position if and only if* θ_φ *is in general position in the sense of* [6] (*i.e. the stabilizer of* θ_φ *in* $W^{(2)}(\boldsymbol{T})$ *is trivial*).

Proof (i) We have $T_\tau=\{{}^\tau tt|t\in T\}$ (resp. $T_{-\tau}=\{{}^\tau t^{-1}t|t\in T\}$) Hence (a) and (b) (resp. (b) and (c)) are equivalent.

(ii) If $\theta=\theta_\varphi$ is not in general position, then, by [6; 5.13], the stabilizer (in the sense of [loc. cit.]) $\boldsymbol{Z}_\theta$ of θ in $N_G(\boldsymbol{T})/\boldsymbol{T}$ is a non-trivial subgroup generated by reflections. Since ${}^\tau\theta=\theta^{-1}$, $\boldsymbol{Z}_\theta$ is σ-stable. Hence, by [6; 5.17], $(\boldsymbol{Z}_\theta)_\sigma$ is non-trivial. Since $(\boldsymbol{Z}_\theta)_\sigma(\subset W^{(1)}(\boldsymbol{T}))$ stabilizes φ, we see that φ is not in general position. This proves the only-if-part. The converse, which is true even if the center of $\boldsymbol{G}$ is not connected, is easy.

Theorem 3.2.4. *Assume that the center of* $\boldsymbol{G}$ *is connected. Let* $\boldsymbol{T}$ *be a* σ-*stable maximal torus, and* φ *a character of* $\boldsymbol{T}_{-\sigma}$. *If* φ *is in general position, then* $r_T[\theta_\varphi]$ *is an irreducible character of G, and*

$$m_\tau(r_T[\theta_\varphi])=1\,.$$

Proof. By Lemma 3.2.3, $\varepsilon(\boldsymbol{T})\,r_T[\theta_\varphi]=\rho_{(s)}$ for some $(s)\in S_{-\sigma}$ and some sign $\varepsilon(\boldsymbol{T})$. But by [6; Prop. 7.4] and the fact that $\boldsymbol{T}$ is σ-stable, we have $\varepsilon(\boldsymbol{T})=1$. Hence the theorem follows from Theorem 3.2.2.

REMARK 3.2.5. The above result confirms the conjecture in [4; 6.7] in this particular case.

Theorem 3.2.6. *Assume that* q *is large* (*see Remark* 3.2.7 (*i*) *below*). *Let* $\boldsymbol{T}$ *be a* σ^2-*stable maximal torus of* $\boldsymbol{G}$, *and* θ *a character of* $T=\boldsymbol{T}_{\sigma^2}$. *Let* $\sigma(\boldsymbol{G},\boldsymbol{T})$ *be the set of* σ-*stable maximal tori of* $\boldsymbol{G}$ *which are G-conjugate to* $\boldsymbol{T}$. *Then we have the following.*

(i) $m_\tau(r_T[\theta])=c_\tau(r_T[\theta])=|G_\tau|^{-1}\sum_{\boldsymbol{A}\in\sigma(\boldsymbol{G},\boldsymbol{T})}|\{w\in W^{(2)}(\boldsymbol{A})|{}^{\tau w}\theta_A={}^w\bar{\theta}_A\}|\,|A_\tau|$,

where for $\boldsymbol{A}\in\sigma(\boldsymbol{G},\boldsymbol{T})$, $A=\boldsymbol{A}_{\sigma^2}$ *and* θ_A *is the character of* A *defined by*

$$\theta_A(a)=\theta(g_A^{-1}ag_A)\,,\quad a\in A$$

using a fixed element g_A *of* G *such that* $g_A\boldsymbol{T}g_A^{-1}=\boldsymbol{A}$.

(ii)

$$\sum_{\substack{u\in G_\tau\\ \text{unipotent}}} r_T[\theta](u)=|W^{(2)}(\boldsymbol{T})|\,|\sigma(\boldsymbol{G},\boldsymbol{T})|\,.$$

REMARK 3.2.7. (i) In [22], Lusztig proved formulas for

$$|\boldsymbol{K}_\delta|^{-1}\sum_{x\in \boldsymbol{K}_\delta} r_T^H[\theta](x)\,,$$

and for

$$\sum_{\substack{x \in K_\delta \\ \text{unipoten}}} r_T^H[\theta](x),$$

where δ is a Frobenius map of a connected reductive group $\boldsymbol{H}$ over a field of odd charaateristic, $\boldsymbol{T}$ is a δ-stable maximal torus of $\boldsymbol{H}$, $\theta \in (\boldsymbol{T}_\delta)$, and $\boldsymbol{K}$ is the fixed points of an involutory automorphism α of $\boldsymbol{G}$ commuting with δ. For example, we can take

$$\boldsymbol{H} = \boldsymbol{G} \times \boldsymbol{G},$$

$$\delta : (x, y) \rightarrow ({}^\sigma y, {}^\sigma x),$$

and

$$\alpha : (x, y) \rightarrow (y, x), \quad (x, y) \in \boldsymbol{H},$$

where $\boldsymbol{G}$ is a connected reductive group and σ is a Frobenius map of $\boldsymbol{G}$. In this case, we have

$$\boldsymbol{H}_\delta \cong \boldsymbol{G}_{\sigma^2} \subset \boldsymbol{G}_\sigma \cong \boldsymbol{K}_\delta .$$

Applying Lusztig's result to this particular case, we get (i) (ii) of Theorem 3.2.6 without using the assumption $q \gg 0$. (The odd characteristic assumption is not needed in this case.)

(ii) If we apply Theorem 3.2.6 (ii) to the case when $\boldsymbol{G} = GL_n(\boldsymbol{F}_q)$ and σ is the q-th power Frobenius map, we obtain a set of formulas for Green polynomials. Using [23; III, 7], it can be translated to the one for symmetric functions as follows. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, let $p_\lambda(x)$ be the symmetric function in $(x) = (x_1, x_2, x_3, \cdots)$ defined by

$$p_\lambda(x) = \prod_i (x_1^{\lambda_i} + x_2^{\lambda_i} + x_3^{\lambda_i} + \cdots),$$

and $P_\lambda(x; t)$ the Hall-Littlewood symmetric function. (We are following the notations in [23].) Then, for any positive integer n,

$$\begin{aligned} &\sum_\lambda b_\lambda(t^2)\, b_\lambda(t)^{-1} P_\lambda(x, t^2) \\ &= \sum_\mu z_\mu^{-1} \{\prod_j (1 - t^{2j})^{m_j(\rho(\mu))}\} \, \{\prod_j (1 - t^j)^{m_j(\mu)}\}^{-1} p_{\rho(\mu)}(x) \end{aligned}$$

where λ and μ run over the set of partitions of n, $m_j(\mu)$ denotes the number of times j occurs as a part of μ, $\rho(\mu)$ is the partition of n defined by

$$m_j(\rho(\mu)) = \begin{cases} 2m_{2j}(\mu) & \text{if } j \text{ is even,} \\ m_j(\mu) + 2m_{2j}(\mu) & \text{if } j \text{ is odd,} \end{cases}$$

and $b_\lambda(t)$ and z_μ are defined by

$$b_\lambda(t) = \prod_j (1-t)(1-t^2)\cdots(1-t^{m_j(\lambda)})$$

and

$$z_\mu = \prod_i \mu_i \prod_j m_j(\mu)! \quad (\mu = (\mu_1, \mu_2, \cdots)).$$

We now sketch our proof of Theorem 3.2.6. The proof is quite similar to that of [6; 6.8, 6.9] and is done by induction on dim $\boldsymbol{G}$. We need:

Lemma 3.2.8. *Assume that Theorem* 3.2.6 (ii) *holds, for* $\boldsymbol{G}$ *replaced by* $Z_{\boldsymbol{G}}(s)^0$, *where s is any semisimple element of* G_τ *not contained in the center* $\boldsymbol{Z}$ *of* $\boldsymbol{G}$. *Then*

$$|G_\tau|\, m_\tau(r_{\boldsymbol{T}}[\theta]) = \alpha+\beta\,,$$

where

$$\alpha = \sum_{s\in \boldsymbol{T}_\tau \cap \boldsymbol{Z}} \theta(s)\Big(\sum_{\substack{u\in G_\tau \\ \text{unipotent}}} r_{\boldsymbol{T}}[\theta](u) - |W^{(2)}(\boldsymbol{T})|\,|\sigma(\boldsymbol{G}, \boldsymbol{T})|\Big)$$

and

$$\beta = \sum_{\boldsymbol{A}\in{}^{\sigma}(\boldsymbol{G}, \boldsymbol{T})} |\{w\in W^{(2)}(\boldsymbol{A})\,|\, {}^{\tau w}\theta_A = {}^{w}\bar{\theta}_A\}|\,|A_\tau|\,.$$

In particular, part (ii) *of Theorem* 3.2.6 *implies part* (i).

This lemma, which is a counterpart of [6; 6.10], can be proved using [6; 4.2]. We omit the details. To prove part (ii) of Theorem 3.2.6, we may assume by induction, that this is true for $\boldsymbol{G}$ replaced by $\boldsymbol{Z}(s)^0$, where s is an arbitrary semisimple element in $G_\tau - \boldsymbol{Z}$, and for a σ^2-stable maximal torus of $\boldsymbol{Z}(s)^0$. Let $\boldsymbol{T}^*$ be the maximal torus of the dual group $\boldsymbol{G}^*$ corresponding to $\boldsymbol{T}$, and $T^* = \boldsymbol{T}^*_{\sigma^2}$ Then, for large q, there exists a σ^2-stable conjugacy class (s) of $\boldsymbol{G}^*$ such that

$$(3.2.9) \qquad (s)\cap T^* \quad \text{is non-empty and} \quad {}^{\sigma}(s)^{-1} \neq (s)\,.$$

Let $\theta\in\hat{T}$ be such that $(\boldsymbol{T}, \theta)\in(s)$. Then, by Theorem 3.2.1, $m_\tau(r_{\boldsymbol{T}}[\theta])=0$. Moreover, by (3.2.9) and Lemma 3.2.3, we have ${}^{\tau w}\theta_A \neq {}^{w}\bar{\theta}_A$ for any $\boldsymbol{A}\in\sigma(\boldsymbol{G}, \boldsymbol{T})$ and any $w\in W^{(2)}(\boldsymbol{A})$. Hence, by Lemma 3.2.8, we get $\alpha=0$. This proves part (ii) of Theorem 3.2.6. (We did not try to weaken the assumption of Theorem 3.2.6 because of the reason mentioned in Remark 3.2.7 (i).)

4. Induced characters

Let $\boldsymbol{G}, \sigma, G, \tau, \cdots$ be as in Section 2. Let $\boldsymbol{P}$ be a σ^2-stable parabolic subgroup of $\boldsymbol{G}$, and $\boldsymbol{L}$ a σ^2-stable Levi subgroup of $\boldsymbol{P}$. We put $P=\boldsymbol{P}_{\sigma^2}$ and $L=\boldsymbol{L}_{\sigma^2}$. Let ϕ_P be a cuspidal irreducible character of L lifted to P. In this section, we are concerned with the calculation of $c_\tau(\chi)$ for irreducible components χ of ϕ_P^G. For simplicity, we assume that σ^2 is an $\boldsymbol{F}_q$-split Frobenius map of

$\boldsymbol{G}$. The main result in this section is Theorem 4.5.9.

4.1. Let $\boldsymbol{B}$ be a σ-stable Borel subgroup of $\boldsymbol{G}$, $\boldsymbol{T}_1$ a σ-stable maximal torus contained in $\boldsymbol{B}$, $T_1=(\boldsymbol{T}_1)_{\sigma^2}$, and $W=N_G(\boldsymbol{T}_1)/\boldsymbol{T}_0=N_G(\boldsymbol{T}_1)/T_1$. Then τ acts on W in such a way that the τ^2-action is trivial. Let $\boldsymbol{P}$ be a σ^2-stable parabolic subgroup containing $\boldsymbol{B}$, $\boldsymbol{L}$ the Levi subgroup of $\boldsymbol{P}$ containing $\boldsymbol{T}_1$, and $\boldsymbol{U}$ the unipotent radical of $\boldsymbol{P}$. We put $B=\boldsymbol{B}_{\sigma^2}$, $P=\boldsymbol{P}_{\sigma^2}$, $L=\boldsymbol{L}_{\sigma^2}$, and $U=\boldsymbol{U}_{\sigma^2}$. Let ϕ be a cuspidal irreducible character of L, and R: $L\to GL(V)$ a representation of L affording ϕ. The lifts of ϕ and R to P are denoted by ϕ_P and R_P, respectively. The representation R_P^G of G induced from R_P is realized on the space $F(G, P, R)$ $=F(P, R)$ of functions f: $G\to V$ satisfying

$$f(pg)=R_P(p)f(g)\,, \quad p\in P, g\in G\,.$$

The group G acts on $F(P, R)$ by right translation. We put

$$W(L)=\{w\in W\,|\,{}^{w}(L\cap B)=L\cap B\}\cong N_G(\boldsymbol{L})/L$$

and

$$W(\phi)=\{w\in W(L)\,|\,{}^{w}\phi=\phi\}\,.$$

4.2. We briefly recall results of R.B. Howlett and G.I. Lehrer [10]. Let Σ be the set of roots of $\boldsymbol{G}$ with respect to $\boldsymbol{T}_1$, Σ^+ (resp. π) the set of positive (resp. simple) roots corresponding to $\boldsymbol{B}$, and $\pi(\boldsymbol{L})$ the subset of π corresponding to $\boldsymbol{B}\cap\boldsymbol{L}$. For $\alpha\in\Sigma-\pi(\boldsymbol{L})$ such that

$$\text{(4.2.1)} \qquad \pi(\boldsymbol{L})\cup\{\alpha\}\subset w\pi \qquad \text{for some} \quad w\in W\,,$$

we put

$$w_{\bar\alpha}=w_0(\pi(\boldsymbol{L})\cup\{\alpha\})\,w_0(\pi(\boldsymbol{L}))$$

where $w_0(\pi(\boldsymbol{L})\cup\{\alpha\})$ and $w_0(\pi(\boldsymbol{L}))$ are the longest elements of the finite reflection groups with simple root systems $\pi(\boldsymbol{L})\cup\{\alpha\}$ and $\pi(\boldsymbol{L})$ respectively. Assume that

$$\text{(4.2.2)} \qquad w_{\bar\alpha}^2=1 \quad \text{and} \quad w_{\bar\alpha}\in W(\phi)\,.$$

Let $\boldsymbol{M}$ be the subgroup of $\boldsymbol{G}$ generated by $\boldsymbol{T}_1$ and the root subgroups corresponding to the roots in

$$\Sigma\cup\boldsymbol{R}(\pi(\boldsymbol{L})\cup\{\alpha\})\,,$$

and $M=\boldsymbol{M}_{\sigma^2}$. Then $P\cap M$ is a parabolic subgroup of M, and L is a Levi subgroup of $P\cap M$. The induced character $\phi_{P\cap M}^M$ splits into exactly two irreducible components η_α and η'_α whose degrees are related by

$$\eta_\alpha(1)/\eta'_\alpha(1)=q^{m(\alpha)}\geq 1$$

with a non-negative integer $m(\alpha)$. Now we define $\Sigma(\phi)$ to be the set of $\alpha\in\Sigma-\pi(\boldsymbol{L})$ satisfying (4.2.1), (4.2.2) and

$$m(\alpha)>0 .$$

For $\alpha\in\Sigma(\phi)$, let $\bar{\alpha}$ be its image in $\boldsymbol{R}\pi/\boldsymbol{R}\pi(\boldsymbol{L})$. Then $\bar{\Sigma}(\phi)=\{\bar{\alpha}\mid\alpha\in\Sigma(\phi)\}$ is a root system, and $\bar{\Sigma}^+(\phi)=\{\bar{\alpha}\mid\alpha\in\Sigma(\phi)\cap\Sigma^+\}$ is a set of positive roots in $\bar{\Sigma}(\phi)$. Let $\bar{\pi}(\phi)$ be the corresponding set of simple roots, and $W'(\phi)=\langle w_{\bar{\alpha}}\mid\alpha\in\Sigma(\phi)\rangle$ the Weyl group of $\Sigma(\bar{\phi})$. We denote by $l(\cdot)$ the length function on $W'(\phi)$ with respect to the set $\bar{S}(\phi)=\{w_{\bar{\alpha}}\mid\bar{\alpha}\in\bar{\pi}(\phi)\}$ of simple reflections. For each $w\in W'(\phi)$, there corresponds a canonically defined element t_w of $\mathrm{End}_G(F(P,R))$ with the following properties: (a) t_1=identity, (b) t_w, $w\in W'(\phi)$, are linearly independent, (c) $t_w t_{w'}=t_{ww'}$ if $l(ww')=l(w)+l(w')$, (d) $(t_s-1)(t_s+q^{m(\alpha)})=0$ if $s=w_{\bar{\alpha}}$ with $\bar{\alpha}\in\bar{\pi}(\phi)$. We put

$$\mathrm{End}'_G(F(P,R))=\bigoplus_{w\in W'(\phi)}\boldsymbol{C}t_w .$$

4.3. Let $W(\phi,\tau)$ be the set of elements w of W satisfying

$${}^{\tau w}(L\cap B)=L\cap B \quad (\text{hence, } {}^{\tau w}L=L)$$

and

$${}^{\tau w}\phi=\bar{\phi} .$$

For $v\in W(\phi,\tau)$, we consider the vector space isomorphism

$$f\to{}^{\tau v}\bar{f}$$

from $F(P,R)$ onto $F({}^{\tau v}P,{}^{\tau v}\bar{R})$. This induces an algebra isomorphism from $\mathrm{End}'_G(F({}^{\tau v}P,{}^{\tau v}\bar{R}))$ onto $\mathrm{End}'_G(F(P,R))$. If we identify $\mathrm{End}'_G(F({}^{\tau v}P,R^{\tau v}\bar{R}))$ and $\mathrm{End}'_G(F(P,R))$ using their canonical basis elements t_w, $w\in W'(\phi)=W'({}^{\tau v}\bar{\phi})$, then the induced map gives the automrophsim

$$t_w\to t_{w_1},\quad w_1={}^{\tau v}w,\quad w\in W'(\phi)$$

of the algebra $\mathrm{End}'_G(F(P,R))$.

4.4. Let $D(L,\tau)$ be the set of elements w of W satisfying

$${}^{w}(L\cap B)\subset B$$

and

$${}^{\tau}(L\cap B)\subset{}^{w}B .$$

Then

$$G=\bigcup_{w\in D(L,\tau)}{}^{\tau}Pn_w P \quad (\text{disjoint}),$$

where, for $w \in W$, n_w is a representative of w in the normalizer $N_G(\boldsymbol{T}_1)$ of $\boldsymbol{T}_1$ in G. The set $W(\phi, \tau)$ defined in 4.3 is contained in $D(L, \tau)$. For $v \in W(\phi, \tau)_{-\tau}$, one can define the twisted Frobenius-Schur indicator

$$c_{\tau v}(\phi) = c_{\tau n_v}(\phi) \quad (= \pm 1)\,.$$

We note that the representative $n_v \in N_G(\boldsymbol{T}_1)$ of $v \in W(\phi, \tau)_{-\tau}$ can be so chosen that $(\tau n_v)^2 = 1$. In fact, $(\tau n_v)^2 \in T_1$ is fixed by the map $t \to {}^{\tau v}t$, $t \in T_1$. Hence, by Lang's theorem, there exists an element s of T_1 such that $(\tau n_v)^2 = {}^{\tau v}s \cdot s$. Then the new representative $n'_v = n_v\, s^{-1}$ of v satisfies $(\tau n'_v)^2 = 1$.

Lemma 4.4.1. *Let ϕ be a cuspidal irreducible character of L. Then*

$$c_\tau(\phi_P^G) = \sum_{v \in W(\phi,\tau)_{-\tau}} c_{\tau v}(\phi)\,.$$

Proof. First, note that, for $w \in D(L, \tau)$,

$${}^\tau({}^\tau P n_w\, P)^{-1} = {}^\tau P n_w\, P$$

if and only if $w \in D(L, \tau)_{-\tau}$. For $v \in D(L, \tau)_{-\tau}$, we consider the character

$${}^{\tau v}\phi_P \cdot \phi_P \colon p \to \phi_P(v^{-1}\, \tau^{-1} p \tau v)\, \phi_P(p)$$

of ${}^{\tau v}P \cap P$. Then, by a standard argument in the theory of cuspidal characters (see, e.g. [5; Prop. 9.1.5]), we see that

$$\langle 1_{({}^{v\tau}P \cap P)}, {}^{\tau v}\phi_P \cdot \phi_P \rangle = \sum_{p \in {}^{\tau v}P \cap P} ({}^{\tau v}\phi_P \cdot \phi_P\rangle\, (p) = 0$$

unless $v \in W(\phi, \tau)_{-\tau}$. Hence, if we define the characters $({}^{\tau v}\phi_P \cdot \phi_P)^{\pm}$ of $P(\tau v, \tau v)$ $= \langle \tau v, {}^{\tau v}P \cap P \rangle$ as in Theorem 2.1.4, then

$$\sum_{y \in P(\tau v, \tau v)} ({}^{\tau v}\phi_P \cdot \phi_P)^{\pm}(y) = \langle 1_{P(\tau v, \tau v)}, ({}^{\tau v}\phi_P \cdot \phi_P)^{\pm} \rangle = 0$$

unless $v \in W(\phi, \tau)_{-\tau}$. Hence, by Theorem 2.1.4, we have

$$c_\tau(\phi_P^G) = \sum_{v \in W(\phi,\tau)_{-\tau}} |{}^{\tau t}P \cap P|^{-1} \{ \sum_{p \in {}^{\tau v}P \cap P} \phi_P((\tau n_v\, p)^2) \}\,. \tag{4.4.2}$$

Since, for $v \in W(\phi, \tau)_{-\tau}$, we have

$${}^{\tau v}P \cap P = L({}^{\tau v}U \cap U)\,,$$

$p \in {}^{\tau v}P \cap P$ can be written as $p = lu$ with $l \in L$ and $u \in {}^{\tau v}U \cap U$. Then

$$\phi_P((\tau n_v\, p)^2) = \phi((\tau n_v\, l)^2)\,.$$

Hence the statement follows from (4.4.2).

4.5. Let $\boldsymbol{L}$, L and ϕ be as in Lemma 4.4.1. We make the following

Assumption 4.5.1.

(i) $W(\phi)=W'(\phi)$.

(ii) Let γ and δ be two Frobenius maps of $\boldsymbol{L}$ such that $L(=\boldsymbol{L}_{\sigma^2})=\boldsymbol{L}_{\gamma^2}=\boldsymbol{L}_{\delta^2}$. The involutive automrophism of L induced from γ and δ are also denoted by γ and δ, respectively. If ${}^{\gamma}\phi=\bar{\phi}$ and ${}^{\delta}\phi=\bar{\phi}$, then

$$c_{\gamma}(\phi)=c_{\delta}(\phi)\,.$$

In particular, the sings $c_{\tau v}(\phi)$, $v\in W(\phi,\tau)_{-\tau}$ in Lemma 4.4.1 are independent of v.

Lemma 4.5.2. *If $W(\phi,\tau)$ is non-empty, then $W(\phi,\tau)_{-\tau}$ is also non-empty. In that case, we have*

$$W(\phi,\tau)_{-\tau}=vW(\phi)_{-\tau v}\,,\quad v\in W(\phi,\tau)_{-\tau}\,.$$

Proof. By the definition of $W(\phi,\tau)$, we have

$$W(\phi,\tau)=uW(\phi)\,,\quad u\in W(\phi,\tau)$$

if $W(\phi,\tau)$ is non-empty. So if we prove the first statement, the second one follows. We use the notations in 4.2. Let $u\in W(\phi,\tau)$. Since $\sigma u(\pi(\boldsymbol{L}))=\pi(\boldsymbol{L})$ and ${}^{\sigma u}\phi=\bar{\phi}$, we have

$$\sigma u(\Sigma(\phi))=\Sigma(\phi)\,.$$

Let $\Sigma^+(\phi)=\Sigma(\phi)\cap\Sigma^+$. We show the existence of $v\in W(\phi,\tau)$, such that

$$\sigma v(\Sigma^+(\phi))\subset\Sigma^+\,. \tag{4.5.3}$$

If $\sigma u(\Sigma^+(\phi))\not\subset\Sigma^+$, then the set

$$\bar{\Sigma}^+(\phi,\sigma u)=\{\bar{\alpha}\,|\,\alpha\in\Sigma^+(\phi)\ \sigma u(\alpha)<0\}\quad(\subset\bar{\Sigma}^+(\phi))$$

is non-empty. Hence there exists an element $\bar{\alpha}$ of $\bar{\Sigma}^+(\phi,\sigma u)$ such that $\bar{\alpha}\in\bar{\pi}(\phi)$. Consider the element $uw_{\bar{\alpha}}\in W(\phi,\tau)$. Then

$$\bar{\Sigma}^+(\phi,\sigma uw_{\bar{\alpha}})=w_{\bar{\alpha}}(\bar{\Sigma}^+(\phi,\sigma u)-\{\bar{\alpha}\})\,.$$

Repeating this process a number of times, we eventually find an element $v=uw_{\bar{\alpha}}\,w_{\bar{\alpha}'}\cdots\in W(\phi,\tau)$ satisying (4.5.3). Since $(\sigma u)^2\in W(\phi)$ for any $u\in W(\phi,\tau)$, we see that

$$(\sigma v)^2\in W(\phi)=W'(\phi)\quad\text{(Assumption 4.5.1 (i))}$$

and

$$(\sigma v)^2(\bar{\Sigma}^+(\phi))\subset\bar{\Sigma}^+(\phi)\,.$$

Hence we must have $(\sigma v)^2=1$, which is equivalent to $v\in W(\phi,\tau)_{-\tau}$.

By Assumption 4.5.1 (i), we see that

$$\mathrm{End}_G(F(P, R)) = \mathrm{End}'_G(F(P, R))$$

is isomorphic to the group algebra $\boldsymbol{C}W(\phi)$ of the finite reflection group $W(\phi)$. Hence there exists a $1-1$ correspondence

$$\mu \to \chi_\mu$$

between the irreducible characters μ of $W(\phi)$ and the irreducible components χ_μ of ϕ_P^G such that

$$\phi_P^G = \sum_{\mu \in \hat{W}(\phi)} \mu(1)\, \chi_\mu \,, \tag{4.5.4}$$

and that

$$^{\tau}\chi_\mu (= {}^{\tau u}\chi_\mu) = \bar{\chi}_{(^{\tau u}\mu)}\,, \qquad \mu \in \widehat{W}(\phi) \quad u \in W(\phi, \tau)_{-\tau}\,. \tag{4.5.5}$$

By Assumption 4.5.1 (ii), Lemma 4.4.1, Lemma 4.5.2, and (4.5.4), we have

$$|W(\phi)_{-\tau u}|\, c_{\tau u}(\phi) = \sum_{\mu \in \hat{W}(\phi)} \mu(1)\, c_\tau(\chi_\mu) \tag{4.5.6}$$

for any $u \in W(\phi, \tau)_{-\tau}$. On the other hand, by (2.1.2), we have

$$|W(\phi)_{-\tau u}| = \sum_{\mu \in \hat{W}(\phi)} \mu(1)\, c_{\tau u}(\mu)\,. \tag{4.5.7}$$

Lemma 4.5.8. *Let $u \in W(\phi, \tau)_{-\tau}$. and let $\mu \in \widehat{W}(\phi)$. We have*

$$c_{\tau u}(\mu) = \begin{cases} 1 & \text{if } \ {}^{\tau u}\mu = \mu\,, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It is enough to prove this in the case when $P=B$, $\phi_P = 1_B$ and $u=1$. Then $W(\phi) = W(\phi, \tau) = W$. We can assume that W is an irreducible Weyl group. Since it is known that any $\mu \in \hat{W}$ is afforded by a real representation, the (classical) Frobenius-Schur indicator $c(\mu)$ is equal to 1. Hence we may asusme that the τ-action on W is non-trivial. When W is of type A_n or E_6, then such τ-action is given by

$$\tau_w = w_0 w w_0^{-1}\,, \quad w \in W\,,$$

where w_0 is the longest element of W with respect to the simple reflections corresponding to $\boldsymbol{B}$. Hence, for any $\mu \in \hat{W}$, we have $^{\tau}\mu = \mu$ and

$$c_\tau(\mu) = |W|^{-1} \sum_{w \in W} \mu((w_0 w)^2) = c(\mu) = 1\,.$$

When W is of type D_n, then the semi-direct porduct $\langle\tau\rangle W$ is isomrophic to the Weyl group of type B_n. Moreover, if $\mu \in \hat{W}$ is fixed by τ, and $\tilde{\mu} \in (\langle\tau\rangle W)^\wedge$

is taken so that $\tilde{\mu}\,|\,W=\mu$, then

$$2c(\tilde{\mu}) = c_\tau(\mu)+c(\mu)\,.$$

Since $c(\tilde{\mu})=c(\mu)=1$, we have $c_\tau(\mu)=1$ in this case. If ${}^\tau\mu\neq\mu$, then ${}^\tau\mu\neq\bar{\mu}(=\mu)$. Hence $c_\tau(\mu)=0$ by Theorem 2.1.3. Finally, when W is of type B_2, G_2 or F_4 (i.e. G_τ is a group of Suzuki or Ree), we can directly verify that

$$\sum_{\substack{\mu\in\hat{W} \\ {}^\tau\mu=\mu}} \mu(1) = |\{w\in W\,|\,{}^\tau ww = 1\}|$$

using the generating relations and character table (see [14]) of W. Hence, in this case, the lemma follows from (2.1.2).

Theorem 4.5.9. *Let L be a Levi subgroup of a parabolic subgroup of G, and ϕ a cuspidal irreducible character of G.*

(i) *If $W(\phi, \tau)$ (see 4.3) is empty, then, for any irreducible component χ of ϕ_P^G, we have*

$$c_\tau(\chi) = 0\,.$$

(ii) *If $W(\phi, \tau)$ is non-empty, then under Assumption 4.5.1 (i) (ii), we have*

$$c_\tau(\chi_\mu) = \begin{cases} c_{\tau v}(\phi) & \text{if } {}^{\tau v}\mu = \mu\,, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu\in\widehat{W(\phi)}$, v is an element of $W(\phi, \tau)_{-\tau}$ (which is non-empty), and χ_μ is the irreducible component of ϕ_P^G corresponding to μ.

Proof. (i) Let (s) be the σ^2-stable semisimple class of $\boldsymbol{L}^*(\subset\boldsymbol{G}^*)$ such that $\phi\in\hat{L}_{(s)}$, and $((s))$ the class of $\boldsymbol{G}^*$ containing (s). Then the emptyness of $W(\phi, \tau)$ implies that ${}^\sigma((s))\neq((s))^{-1}$. Since $\chi\in\hat{G}_{((s))}$, this implies that $c_\tau(\chi)=0$ by Theorem 3.2.1.

(ii) By (4.5.5) and Theorem 2.1.3, we have

$$c_\tau(\chi_\mu) = 0\,, \qquad \text{if } {}^{\tau v}\mu\neq\mu\,.$$

Hence, part (ii) follows from (4.5.6), (4.5.7) and Lemma 4.5.8.

Lusztig [17] [21] showed that Assumption 4.5.1 (i) is true if ϕ is unipotent or if the center of $\boldsymbol{G}$ is connected. His result also implies that, when $\boldsymbol{G}$ modulo its center is simple and ϕ is unipotent, we always have

$${}^{\tau v}\mu = \mu\,,\quad \mu\in\widehat{W(\phi)}\,,\quad v\in W(\phi, \tau)_{-\tau}$$

unless:

$\boldsymbol{G}$ is of type D_{2n}, σ is a twisted Frobenius map, $P=B$ (hence $\phi_P=1_B$), and

μ is a "degenerate" (see 6.3) irreducible character of $W=W(\phi)$.

5. Unipotent characters of exceptional groups

Let $\boldsymbol{G}, \sigma, G, \tau, \cdots$ be as in Section 3. In this section, we restrict our attention to the unipotent irreducible characters of G, i.e. the elements of $\hat{G}_{(1)}$. By [6; 7.10], we may assume that $\boldsymbol{G}$ modulo its center is simple. Then σ^2 is a split Frobenius map unless $\boldsymbol{G}_\sigma$ is of type ${}^3D_4$. Since the ${}^3D_4$-case is easy (see 5.4), we shall assume, unless otherwise stated, that σ^2 is split.

5.1. Let $\boldsymbol{T}_1$ be a σ-stable maximal torus of $\boldsymbol{G}$ contained in a σ-stable Borel subgroup $\boldsymbol{B}$. Then, by the assumption made above, $\boldsymbol{T}_1$ is σ^2-split. For $w\in W=N_{\boldsymbol{G}}(\boldsymbol{T}_1)/\boldsymbol{T}_1$, let $\boldsymbol{T}_w$ be a σ^2-stable maximal torus of $\boldsymbol{G}$ whose G-conjugacy class corresponds to the conjugacy class of w in W in the standard manner. We put $T_w=(\boldsymbol{T}_w)_{\sigma^2}$. For $\mu\in\hat{W}$, we define

$$r_\mu=|W|^{-1}\sum_{w\in W}\mu(w)\,r_{T_w}[1]\,.$$

For $\mu, \mu'\in\hat{W}$, we write $\mu\sim\mu'$ if there exists a series $\mu=\mu_0, \mu_1, \cdots, \mu_n=\mu'$ of elements of $\hat{W}$ such that $\langle r_{\mu_{i-1}}, r_{\mu_i}\rangle\neq 0$ for $1\leq i\leq n$. Equivalence classes of $\hat{W}$ under $\sim$ are called [21] families, or two-sided cells, in $\hat{W}$. For each family F in $\hat{W}$, we put

$$U_F=\{\chi\in\hat{G}_{(1)}\,|\,\langle\chi, r_\mu\rangle\neq 0\quad\text{for some}\quad\mu\in F\}\,,$$

which is called a family of unipotent characters.

F. Digne and J. Michel [8], and T. Asai [2], [3] discovered that there exists a beautiful connection between the operator t^* and Lusztig's nonabelian Fourier transfomration [21]. We summarize a part of Asai's results in the following:

Theorem 5.1.1. *Assume that $\boldsymbol{G}$ modulo its cneter is simple. If $\boldsymbol{G}$ is of exceptional type, we also assume that the characteristic is good. Let F be a family in $\hat{W}$, and $U=U_F$ the corresponding family in $\hat{G}_{(1)}$. When $\boldsymbol{G}$ is of type E_6 (resp. E_8), we further assume that F does not contain a character μ such that $\mu(1)=512$* (resp. 4096). *Let $\chi=\chi_{(x,\alpha)}$ be an element of U corresponding to $(x, \alpha)\in M_U=M(\Gamma)$ under Lusztig's parametrization* [21]. *($\Gamma=\Gamma_U$ is a finite group associated* [21] *with the family U, and $M(\Gamma)$ is the set of pairs (x, α) with $x\in\Gamma$ and $\alpha\in Z_\Gamma(x)$ taken modulo Γ-conjugacy, where $Z_\Gamma(x)$ is the centralizer of x in Γ.) Then*

$$t^*(\chi_{(x,\alpha)})=(\alpha(x)/\alpha(1))\sum_{(y,\beta)\in M(\Gamma)}\{(x, \alpha), (y^{-1}, \beta)\}\,(\beta(y)/\beta(1))\,\chi_{(y,\beta)}$$

where the pairing $\{\cdot, \cdot\}$ on $M(\Gamma)$ is defined by

$$\{(x, \alpha), (y, \beta)\}=\sum_{\substack{z\in\Gamma\\ zyz^{-1}\in Z_\Gamma(x)}}|Z_\Gamma(x)|^{-1}|Z_\Gamma(y)|^{-1}\,\alpha(zyz^{-1})\,\beta(z^{-1}x^{-1}z)\,.$$

This result is proved in Asai [2; 6.2.1]. Notice that, in the terminology of [2; p. 2763], our t is equal to $(t_{1,\sigma^2,id})^{-1}$ with a split Frobenius map σ^2. Combining Theorem 5.1.1 and Theorem 2.2.3 (i) we get:

$$(5.1.2)\quad m_\tau(\chi_{(x,\alpha)}) = (\alpha(x)/\alpha(1)) \sum_{(y,\beta)\in M(\Gamma)} \{(x,\alpha),(y^{-1},\beta)\}\,(\beta(y)/\beta(1))\,c_\tau(\chi_{(y,\beta)})$$

for any $\chi_{(x,\alpha)}\in U_F$, if F is not one of the families excluded in Theorem 5.1.1.

5.2. Following the general program described in Remark 2.2.4, we now turn to the determination of the twisted Frobenius-Schur indicators $c_\tau(\chi)$, $\chi\in\hat{G}_{(1)}$. We first consider the case when χ is a component of the induced character 1_B^G with $B=\boldsymbol{B}_{\sigma^2}$. As a special case of Theorem 4.5.9, we have

Lemma 5.2.1. *Let χ_μ be an irreducible component of 1_B^G corresponding to $\mu\in\hat{W}$. Then*

$$c_\tau(\chi_\mu) = \begin{cases} 1 & \text{if } {}^\tau\mu=\mu\,, \\ 0 & \text{otherwise.} \end{cases}$$

We also have the following

Lemma 5.2.2. *For $\mu\in\hat{W}$, let r_μ be as in* 5.1. *Then*

$$c_\tau(r_\mu) = m_\tau(r_\mu) = \begin{cases} 1 & \text{if } {}^\tau\mu=\mu\,, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The G_τ-conjugacy classes of σ-stable maximal tori of $\boldsymbol{G}$ are in 1—1 corresponence with the σ-twisted conjugacy classes of W. Let $\boldsymbol{T}_v^{(1)}$ be a σ-stable maximal torus corresponing to the σ-twisted class of $v\in W$. Then, as a σ^2-stable maximal torus, $\boldsymbol{T}_v^{(1)}$ is G-conjugate to $\boldsymbol{T}_w$ with $w={}^\tau vv$. Using this fact and Theorem 3.2.6 (and Remark 3.2.7 (i)), we see that

$$c_\tau(r_{T_w}[1]) = m_\tau(r_{T_w}[1]) = |\{v\in W\,|\,{}^\tau vv=w\}|\,.$$

Hence we have, for $\mu\in\hat{W}$,

$$c_\tau(r_\mu) = m_\tau(r_\mu) = c_\tau(\mu)\,.$$

The lemma now follows from Lemma 4.5.9.

Combining Lemma 5.2.2 with the formula of Lusztig [21] for the multiplities $\langle r_\mu, \chi\rangle$, $\chi\in\hat{G}_{(1)}$, we get the following.

Lemma 5.2.3. *Let F be a family in $\hat{W}$, and U the corresponding family in $\hat{G}_{(1)}$. For $\chi\in U$ and $\mu\in F$, we denote by $(x,\alpha)_\chi$ and $(y,\beta)_\mu$ the elements of M_U corresponding to χ and χ_μ (which is known to be contained in U), respectively. Then, for any $\mu\in F$, we have*

$$\sum_{\chi\in U} c_\tau(\chi)\,\Delta(\chi)\,\{(x,\alpha)_\chi,(y,\beta)_\mu\} = \begin{cases} 1 & \text{if } {}^\tau\mu=\mu\,, \\ 0 & \text{otherwise.}\end{cases}$$

where $\Delta(\cdot)\colon U\to\{\pm 1\}$ *is a certain function which is identically* 1 *unless* F *is one of the families excluded in Theorem* 5.1.1.

5.3. Let F be a family in $\hat{W}$, U the family in $\hat{G}_{(1)}$ associated with F, and Γ the finite group associated with U. It is known [21] that, when $\boldsymbol{G}$ is an exceptional simple group,

(5.3.1) $\qquad \Gamma\cong S_i$ (the i-th symmetric group)

for some $1\le i\le 5$. In this subsection, we determine $c_\tau(\chi)$ and $m_\tau(\chi)$, $\chi\in U$, under the assumption (5.3.1). In the case when $\boldsymbol{G}$ is of exceptional type, we also need to assume the characteristic is good. In the calculation below, we use the explicit values $\{(x,\alpha),(y,\beta)\}$ $((x,\alpha),(y,\beta)\in M(S_i),\ 2\le i\le 5)$. For $2\le i\le 4$ (resp. $i=5$), these can be found (resp. partly found) in [5; 13.6].

(1) $\Gamma\cong S_1=\{1\}$. In this case, F consists of a single element μ, $U=\{\chi_\mu\}$, and χ_μ is a uniform function. Hence, by Theorem 3.1.3 (ii) and Lemma 5.2.1,

$$m_\tau(\chi_\mu)=c_\tau(\chi_\mu)=\begin{cases} 1 & \text{if } {}^\tau\mu=\mu\,, \\ 0 & \text{otherwise.}\end{cases}$$

In particular, for unipotent characters χ of a group of type A_n, we always have $m_\tau(\chi)=c_\tau(\chi)=1$.

(2) $\Gamma\cong S_2$. In this case, $M(\Gamma)$ consists of four elements $\{(1,1),(1,\varepsilon),(g_2,1),(g_2,\varepsilon)\}$, where ε and g_2 are non-trivial elements of S_2 and $\hat{S}_2$, respectively. If F is not one of the families excluded in Theorem 5.1.1, then $\chi_{(1,1)}$, $\chi_{(1,\varepsilon)}$ and $\chi_{(g_2,1)}$ are contained in 1_B^G. Hence, by Lemma 5.2.1 and Lemma 5.2.3, we get

$$c_\tau(\chi_{(1,1)})=c_\tau(\chi_{(g_2,\varepsilon)})=1\,,\quad c_\tau(\chi_{(1,\varepsilon)})=c_\tau(\chi_{(g_2,1)})=0\,,$$

if $\boldsymbol{G}$ is of type B_2, and the τ-action on W is non-trivial (i.e. if $G_\tau=\boldsymbol{G}_\sigma$ is a Suzuki group), and

$$c_\tau(\chi_{(1,1)})=c_\tau(\chi_{(1,\varepsilon)})=c_\tau(\chi_{(g_2,1)})=1\,,\quad c_\tau(\chi_{(g_2,\varepsilon)})=-1$$

otherwise. Hence, by (5.1.2), we get

$$m_\tau(\chi_{(1,1)})=m_\tau(\chi_{(g_2,\varepsilon)})=0\,,\quad m_\tau(\chi_{(1,\varepsilon)})=m_\tau(\chi_{(g_2,1)})=1\,,$$

if G_τ is a Suzuki group, and

$$m_\tau(\chi_{(1,1)})=2\,,\quad m_\tau(\chi_{(1,\varepsilon)})=m_\tau(\chi_{(g_2,1)})=m_\tau(\chi_{(g_2,\varepsilon)})=0\,,$$

otherwise. Even if the family F is one of those excluded in Theorem 5.1.1, a similar argument as above leads to:

(5.3.2) $$c_\tau(\chi_{(1,1)}) = c_\tau(\chi_{(1,\varepsilon)}) = 1\,, \quad c_\tau(\chi_{(g_2,1)}) + c_\tau(\chi_{(g_2,\varepsilon)}) = 0\,.$$

Moreover, according to Asai [2; 6.2.1 (iv)], we have either

$$t^*(\hat{\chi}_{(g_2,1)}) = \xi\hat{\chi}_{(g_2,1)}$$

or

$$t^*(\hat{\chi}_{(g_2,\varepsilon)}) = \xi\hat{\chi}_{(g_2,1)}$$

with a primitive 4-th root ξ of 1, where the root means the Lusztig's non-abelian Fourier transformation. Hence, by Theorem 2.2.3 (i), we have either

$$m_\tau(\hat{\chi}_{(g_2,1)}) = \xi c_\tau(\hat{\chi}_{(g_2,1)})$$

or

$$m_\tau(\hat{\chi}_{(g_2,\varepsilon)}) = \xi c_\tau(\hat{\chi}_{(g_2,1)})\,.$$

Hence, in any case, we must have $c_\tau(\hat{\chi}_{(g_2,1)})=0$ because $m_\tau(\hat{\chi}_{(g_2,1)})$ and $m_\tau(\hat{\chi}_{(g_2,\varepsilon)})$ are real mumbers. This and (5.3.2) imply

$$c_\tau(\chi_{(g_2,1)}) = c_\tau(\chi_{(g_2,\varepsilon)}) = 0\,.$$

Analogously, we also have

$$m_\tau(\chi_{(1,1)}) = m_\tau(\chi_{(1,\varepsilon)}) = 1\,, \quad m_\tau(\chi_{(g_2,1)}) = m_\tau(\chi_{(g_2,\varepsilon)}) = 0\,.$$

(3) $\Gamma \cong S_3$. In this case $M(\Gamma)$ consists of 8 elements, namely, $(1, 1)$, $(g_2, 1)$, $(1, r)$, $(g_3, 1)$, $(1, \varepsilon)$, (g_2, ε), (g_3, θ), and (g_3, θ^2) in the notation of [21; Ch. 4]. We denote the corresponding elements in U by $\chi_1, \chi_2, \cdots, \chi_8$, respectively. We also put $c_i=c_\tau(\chi_i)$ and $m_i=m_\tau(\chi_i)$ for $1\leq i\leq 8$. By Lemma 5.2.1, we have

$$c_i = 1\,, \quad 1\leq i\leq 4\,.$$

Hence, by Lemma 5.2.3, we get

(5.3.3) $$c_5+3c_6+2c_7+2c_8 = -2\,, \quad -c_5-c_6 = 0\,.$$

Moreover, by (5.1.2), we have

$$\begin{aligned} 6m_1 &= 8+c_5-3c_6+2\theta c_7+2\theta^2 c_8\,, \\ 3m_7 &= -2\theta+2\theta c_5-2\theta^2 c_7+4c_8\,. \end{aligned}$$

Since m_1 and m_7 are real and θ is a primitive 3rd root of 1, we must have

(5.3.4) $$c_7-c_8 = 0\,, \quad -1+c_5+c_7 = 0\,.$$

Solving (5.3.3) and (5.3.4), we have

$$(c_5, c_6, c_7, c_8) = (1, -1, 0, 0)\,.$$

Hence, by (5.1.2),

$$(m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8) = (2, 0, 1, 1, 0, 0, 0, 0)\,.$$

(4) $\Gamma \cong S_4$. $M(\Gamma)$ consists of 21 elements, namely, $(1, 1)$, $(1, \lambda^1)$, $(g_2', 1)$, $(1, \lambda^2)$, (g_2', ε''), $(g^4, 1)$, (g_2, ε''), (g_2', ε'), $(g_3, 1)$, $(1, \sigma)$, $(g_2, 1)$, (g_2', r), (g_2, ε'), $(g_4, -1)$, (g_3, θ), (g_3, θ^2), (g_4, i), $(g_4, -i)$, (g_2', ε), $(1, \lambda^3)$, and (g_2, ε), in the notation of [21; Ch. 4]. We denote the corresponding elements of U by $\chi_1, \chi_2, \cdots, \chi_{21}$, respectively. We also put $c_i = c_\tau(\chi_i)$ and $m_i = m_\tau(\chi_i)$ for $1 \leq i \leq 21$. By Lemma 5.2.1, we have

$$c_i = 1\,, \quad 1 \leq i \leq 11\,.$$

By an argument similar to the one used in case (3), we get

$$(c_{12}, c_{13}, \cdots, c_{21}) = (-1, -1, -1, 0, 0, 0, 0, 1, 1, -1)\,.$$

Hence, by (5.1.2),

$$\begin{aligned} &(m_1, m_2, m_3) = (3, 1, 1); \qquad && m_i = 0\,, \quad 4 \leq i \leq 8;\\ &(m_9, m_{10}) = (1, 2); && m_i = 0\,, \quad 11 \leq i \leq 18;\\ &(m_{19}, m_{20}, m_{21}) = (1, 0, 0)\,. \end{aligned}$$

(5) $\Gamma \cong S_5$. $M(\Gamma)$ consists of 39 elements, namely, $(1, 1)$, $(g_3, 1)$, $(g_2', 1)$, $(1, \nu)$, $(1, \lambda^1)$, $(g_5, 1)$, (g_3, ε), $(1, \nu')$, (g_2', ε''), $(1, \lambda^2)$, (g_2', ε'), $(1, \lambda^3)$, $(g_2, 1)$, $(g_4, 1)$, $(g_6, 1)$, (g_2, r), (g_2, ε), $(g_4, -1)$, $(g_6, -1)$, $(g_2, -r)$, $(g_2, -1)$, (g_2', r), (g_3, θ), (g_6, θ), (g_3, θ^2), (g_6, θ^2), (g_5, ζ), (g_5, ζ^2), (g_5, ζ^3), (g_5, ζ^4), $(g_6, -\theta)$, $(g_6, -\theta^2)$, $(g_3, \varepsilon\theta)$, $(g_3, \varepsilon\theta^2)$, (g_4, i), $(g_4, -i)$, (g_2', ε), $(1, \lambda^4)$, and $(g_2, -\varepsilon)$ in the notation of [21; Ch. 4]. We denote the corresponding elements of U by $\chi_1, \chi_2, \cdots, \chi_{39}$, respectively. We also put $c_i = c_\tau(\chi_i)$ and $m_i = m_\tau(\chi_i)$ for $1 \leq i \leq 39$. By Lemma 5.2.1, we have

$$c_i = 1\,, \quad 1 \leq i \leq 17\,.$$

By an argument similar to the one used in case (3), we get

$$c_i = -1,\ 18 \leq i \leq 22; \quad c_i = 0,\ 23 \leq i \leq 36; \quad (c_{37}, c_{38}, c_{39}) = (1, 1, -1)\,.$$

Hence, by (5.1.2),

$$\begin{aligned} &(m_1, m_2, \cdots, m_8) = (3, 2, 1, 2, 2, 1, 0, 1);\\ &m_i = 0,\ 9 \leq i \leq 36; \quad (m_{73}, m_{38}, m_{39}) = (1, 0, 0)\,. \end{aligned}$$

REMARK 5.3.5. (i) Assume that $\boldsymbol{G}$ modulo its center is simple, and the characteristic is good if $\boldsymbol{G}$ is of exceptional type. Let σ be a split Frobenius map of $\boldsymbol{G}$. For any unipotent character χ of $G = \boldsymbol{G}_{\sigma_2}$, we can consider

(a) the twisted Frobenius-Schur indicator

$$c_\tau(\chi)\ (= 1, -1 \text{ or } 0)\,,$$

and

(b) the root λ_χ of 1 defined in terms of the eigenvalues of σ^2 on l-adic cohomology spaces of Deligne-Lusztig vareites (see [21; Ch. 11]). Comparing the results on $c_\tau(\chi)$ obtained in this section with the explicit calculation of λ_χ given in [21; Ch. 11], we see that

$$c_\tau(\chi) = \begin{cases} \lambda_\chi & \text{if } \lambda_\chi = \pm 1\,, \\ 0 & \text{otherwise,} \end{cases}$$

when $\boldsymbol{G}$ is of type A_n or of exceptional type. For groups of type B_n, C_n or D_n, this will be proved in the next section.

(ii) Let $\boldsymbol{G}$ be the adjoint group of type B_2 (in any characteristic) or the adjoint group of type G_2 in good characteristic. Then non-unipitent irreducible characters of $G=\boldsymbol{G}_{\sigma^2}$ are uniform. Hence the calculation given in this subsection (case (1)-case (3)) together with Theorem 3.1.3 determines $c_\tau(\chi)$ and $m_\tau(\chi)$ for any $\phi\in\hat{G}$. (R. Lawther and J. Saxl [15] determined $m_\tau(\chi)$, $\chi\in\hat{G}$, when $\boldsymbol{G}$ is the adjoint group of type B_2 in characteristic 2 using a method different from ours.)

5.4. In this subsection, we consider the $^3D_4$-case. Let $\boldsymbol{G}$ be an adjoint group of type D_4, and σ a surjective endomorphism of $\boldsymbol{G}$ such that $|\boldsymbol{G}_{\sigma^2}|<\infty$ and that the σ-action on the root system of $\boldsymbol{G}$ is of order 3. Then G $(=\boldsymbol{G}_{\sigma^2})$ and $G_\tau(=\boldsymbol{G}_\sigma)$ are isomorphic to the groups $^3D_4(q^2)$ and $^3D_4(q)$, respectively, for a power q of a prime.

Lemma 5.4.1. *Let $U=\{[\rho_1], [\rho_2], {}^3D_4[1], {}^3D_4[-1]\}$ be the unique four element family (see [18]) of unipotent characters of G. Then*

$$t^*\begin{pmatrix} [\rho_1] \\ [\rho_2] \\ {}^3D_4[1] \\ {}^3D_4[-1] \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} [\rho_1] \\ [\rho_2] \\ {}^3D_4[1] \\ {}^3D_4[-1] \end{pmatrix},$$

i.e. the t^*-action on U is the same as in case (2) in 5.3. Moreover any $\chi\in\hat{G}-U$ is t^*-invariant.

Proof. Let A be the space of class functions on G. Then A can be written as:

$$A = A_0 \oplus \boldsymbol{C}\eta \quad \text{(orthogonal direct sum)},$$

where A_0 is the space of uniform functions, and $\eta=[\rho_1]-[\rho_2]-{}^3D_4[1]+{}^3D_4[-1]$. See [18], [7]. Since t^* acts trivially on A_0 by Theorem 3.1.2, and t^* acts involutively on A by [12; I, (2.2)], we must have

$$t^*(\eta) = -\eta .$$

The lemma follows from this.

The following result can now be obtained by an argument similar to the one used in case (2) in 5.3.

Proposition 5.4.2. *We have*

$$c_\tau([\rho_1]) = c_\tau([\rho_2]) = c_\tau({}^3D_4[1]) = 1 , \quad c_\tau({}^3D_4[-1]) = -1$$

and

$$m_\tau([\rho_1]) = 2 , \quad m_\tau([\rho_2]) = m_\tau({}^3D_4[1]) = m_\tau({}^3D_4[-1]) = 0 .$$

If χ is a non-unipotent irreducible character of G, then

$$c_\tau(\chi) = m_\tau(\chi) = \begin{cases} 1 & \textit{if } {}^\tau\chi = \chi , \\ 0 & \textit{otherwise.} \end{cases}$$

6. Classical groups

In this section, we consider the case when $\boldsymbol{G}$ is of type A_n, B_n, C_n or D_n. Since we already treated the ${}^2B_2$-and ${}^3D_4$-cases in Section 5, we may assume that $\sigma: \boldsymbol{G} \to \boldsymbol{G}$ is the Frobenius map for a rational structure of $\boldsymbol{G}$ over a finite field $\boldsymbol{F}_{q'}$, and that the Frobenius map σ^2 is split over $\boldsymbol{F}_q$, $q = q'^2$.

6.1. Recall [16] [17], [19]-[21] that a (reduced) symbol is an unordered pair $\Lambda = (S, T)$ of finite subsets of $\{0, 1, 2, \cdots\}$ such that $0 \notin S \cap T$. The rank and defect of a symbol $\Lambda = (S = (\lambda_1, \lambda_2, \cdots, \lambda_a), T = (\mu_1, \mu_2, \cdots, \mu_b))$ are defined by

$$\operatorname{rank}(\Lambda) = \sum_i \lambda_i + \sum_j \mu_j - [(a+b-1)^2/4]$$

and

$$\operatorname{def}(\Lambda) = |a-b| ,$$

respectively, when $[z]$ denotes the largest integer m such that $m \leq z$.

6.2. Let $\boldsymbol{G}$ be of type $D_n (n \geq 2)$. Let $S(D, n)$ be the set of symbols of rank n and defect divisible by 4. A symbol $\Lambda = (S, T) \in S(D, n)$ is called non-degenerate (resp. degenerate) if $S \neq T$ (resp. $S = T$). By Lusztig [16] [20], one can associate with each non-degenerate (resp. degenerate) symbol Λ a unipotent character χ_Λ (resp. two unipotent characters $\chi_{\Lambda,1}$, $\chi_{\Lambda,2}$) of G. For $\Lambda \in S(D, n)$, we put

$$s = s_\Lambda = \operatorname{def}(\Lambda)/2 .$$

Then $n \geq s^2$. Let P be a parabolic subgroup of G with a Levi subgroup L of type D_{s^2}. Let ϕ be the unique cuspidal unipotent character of L. Then

$$\langle \chi_\Lambda, \phi_P^G \rangle \neq 0 \tag{6.2.1}$$

if Λ is non-degenerate, and

$$\langle \chi_{\Lambda,i}, \phi_P^G \rangle \neq 0\,, \quad i=1,2$$

if Λ is degenerate. (In the latter case, $P=B$ and $\phi_P=1_B$). For non-degenerate elements Λ and Λ' of $S(D,n)$, χ_Λ and $\chi_{\Lambda'}$ lie in the same family if and only if $S \amalg T$ and $S' \amalg T'$ contain the same integers with the same multiplicities. For degenerate $\Lambda \in S(D,n)$, the sets $\{\chi_{\Lambda,1}\}$ and $\{\chi_{\Lambda,2}\}$ are 1-element families. Let

$$Z=\begin{pmatrix}(z_1, z_2, \cdots, z_m)\\ (z'_1, z'_2, \cdots, z'_m)\end{pmatrix}$$

be a non-degenerate symbol of rank n and defect 0. We arrange z's and z''s in such a way that $z_1<z_2<\cdots<z_m$, $z'_1<z'_2<\cdots<z'_m$. We assume that χ_Z is special [21], i.e. $z_1 \leq z'_1 \leq z_2 \leq z'_2 \leq \cdots \leq z_m \leq z'_m$ or $z'_1 \leq z_1 \leq z'_2 \leq z_2 \leq \cdots \leq z'_m \leq z_m$. Let Z_1 (resp. Z_2) be the set of integers which appear exactly once (resp. twice) in Z. Then $|Z_1|=2d$ for some positive integer d. Let $U(Z)$ be the family containing χ_Z, and let

$$S(D,Z)=\{\Lambda \in S(D,n) \mid \chi_\Lambda \in U(Z)\}\,.$$

Then any $\Lambda \in S(D,Z)$ can be written uniquely as

$$\Lambda=\begin{pmatrix}Z_2 \amalg (Z_1-M(\Lambda))\\ Z_2 \amalg M(\Lambda)\end{pmatrix}$$

with some $M(\Lambda) \subset Z_1$ such that $|M(\Lambda)| \equiv d \pmod 2$. The finite group Γ associated with $U(Z)$ is isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^{d-1}$. Let $(x_\Lambda, \alpha_\Lambda)$ be an element of $M(\Gamma)$ corresponding to $\chi_\Lambda \in U(Z)$. Then $(x_Z, \alpha_Z)=(1,1)$, and, for $\Lambda \in S(D,Z)$,

$$\alpha_\Lambda(x_\Lambda)=(-1)^{(|M(\Lambda)|-d)/2}\,, \tag{6.2.2}$$

which can also be interpreted in terms of Frobenius eigenvalues associated with χ_Λ (see [20; 3.18], [21; 11.2]). If Λ is degenerate, then the groups associated with $\{\chi_{\Lambda,i}\}$ $(i=1,2)$ are $\{1\}$. As a special case of the formula (5.1.2), we have

Lemma 6.2.3. *Let $\Lambda \in S(D,n)$ be non-degenerate, and Z the special symbol such that $\Lambda \in S(D,Z)$. Then*

$$m_T(\chi_Z)=2^{-(d-1)} \sum_{\Lambda \in S(D,Z)} \alpha_\Lambda(x_\Lambda)\, c_T(\chi_\Lambda)\,.$$

Theorem 6.2.4. *Let G be of type $D_n (n \geq 2)$.*

(i) *Let Λ be a degenerate symbol of rank n and defect divisible by 4, and*

$\chi_{\Lambda,1}$, $\chi_{\Lambda,2}$ *the corresponding unipotent characters of* $G=G_{\sigma^2}$. *Then*

$$m_\tau(\chi_{\Lambda,i}) = c_\tau(\chi_{\Lambda,i}) = \begin{cases} 1 & \text{if } \sigma \text{ is untwisted,} \\ 0 & \text{otherwise,} \end{cases}$$

for $i=1, 2$.

(ii) *Let* Λ *be a non-degenerate symbol of rank* n *and defect divisible by* 4, *and* χ_Λ *the corresponding unipotent character of* G. *Then*

$$c_\tau(\chi_\Lambda) = \alpha_\Lambda(x_\Lambda)\,.$$

(iii) *Under the same notation as in (ii), we have*

$$m_\tau(\chi_\Lambda) = \begin{cases} 2^{d-1} & \text{if } \chi_\Lambda \text{ is special,} \\ 0 & \text{otherwise,} \end{cases}$$

where $2d$ *is the number of integers appearing exactly once in* Λ.

Proof. (i) This is a special case of case (1) in 5.3.

(ii) Let P be a parabolic subgroup of G with a Levi subgroup L of type D_r, $r<n$. Since the case $n=2$ is easy, we can assume that the statement (i) (with G replaced by L) and Assumption 4.5.1 (ii) (for any cuspidal unipotent character ϕ of any L of type D_r, $r<n$) are both true. If χ_Λ is not cuspidal, i.e. if $n>s_\Lambda^2$, then we can apply Lemma 4.5.7 (with $u=1$) to the ϕ and P in (6.2.1). Hence we have

$$c_\tau(\chi_\Lambda) = c_\tau(\phi)\,.$$

This and (6.2.2) imply that the statement (ii) is true for non-cuspidal χ_Λ. It only remains to show that (ii) is true for cuspidal χ_Λ, because then Assumption 4.5.1 (ii) is true for L of type D_n. If χ_Λ is cuspidal, and take the symbol Z such that χ_Z, is special and that $\Lambda\in S(D, Z)$. Then, by Lemma 6.2.3 and the statement (ii) for non-cuspidal characters, we see that

$$m_\tau(\chi_Z) = 2^{-(d-1)}\,\{\alpha_\Lambda(x_\Lambda)\,c_\tau(\chi_\Lambda)+(2^{2(d-1)}-1\})$$

must be an integer. If $d\geq 3$, this is the case only when $c_\tau(\chi_\Lambda)=\alpha_\Lambda(x_\Lambda)$. If $d=1$ or 2, then the group Γ associated with $U(Z)$ is isomorphic to $\{1\}$ or S_2. This case has already been treated in 5.3. This proves part (ii).

(iii) Let Z be as in Lemma 6.2.3. Then by part (ii) and Lemma 6.2.3, we have

$$m_\tau(\chi_Z) = 2^{d-1}\,.$$

On the other hand, by (2.2.4) and (2.2.5), we have

$$\sum_\Lambda m_\tau(\chi_\Lambda)^2 = \sum_\Lambda c_\tau(\chi_\Lambda)^2 = 2^{2(d-1)}\,,$$

where the both sums are taken over $S(D, Z)$. The statement (iii) now follows.

6.3. Let G be of type B_n or $C_n (n \geq 1)$. Let $S(BC, n)$ denote the set of symbols of rank n and odd defect. By Lusztig [16] [19], there is a 1–1 correspondence

$$\Lambda \to \chi_\Lambda$$

between the elements Λ of $S(BC, n)$ and the unipotent characters χ_Λ of G. For $\Lambda \in S(BC, n)$, we put

$$s = s_\Lambda = (\mathrm{def}(\Lambda) - 1)/2 .$$

Then $n \geq s^2 + s$. Let P be a parabolic subgroup of G with a Levi subgroup L of type B_{s^2+s} or C_{s^2+s} (according as G is of type B_n or C_n). Let ϕ be the unique cuspidal unipotent character of L. Then

$$\langle \chi_\Lambda, \phi_P^G \rangle \neq 0 .$$

For elements $\Lambda = (S, T)$ and $\Lambda' = (S', T')$ of $S(BC, n)$, χ_Λ and $\chi_{\Lambda'}$ lie in the same family if and only if $S \amalg T$ and $S' \amalg T'$ contain the same integers with the same multiplicities. Let

$$Z = \begin{pmatrix} (z_0, z_2, \cdots, z_{2m}) \\ (z_1, z_3, \cdots, z_{2m-1}) \end{pmatrix}$$

be a symbol of rank n and defect 1. We arrange z's in such a way that $z_2 < z_2 < \cdots < z_{2m}$, $z_1 < z_3 < \cdots < z_{2m-1}$. We assume that χ_Z is special [21], i.e. $z_0 \leq z_1 \leq z_2 \leq \cdots \leq z_{2m-1} \leq z_{2m}$. Let Z_1 (resp. Z_2) be the set of integers which appear exactly once (resp. twice) in Z. Then $|Z_1| = 2d+1$ for some non-negative integer d. Let $U(Z)$ be the family containing χ_Z, and let

$$S(BC, Z) = \{\Lambda \in S(BC, n) \,|\, \chi_\Lambda \in U(Z)\} .$$

Then, any $\Lambda \in S(BC, Z)$ can be written uniquely as

$$\Lambda = \begin{pmatrix} Z_2 \amalg (Z_1 - M)\Lambda)) \\ Z_2 \amalg M(\Lambda) \end{pmatrix}$$

with some $M(\Lambda) \subset Z_1$ such that $|M(\Lambda)| \equiv d \pmod 2$. The finite group Γ associated with $U(Z)$ is isomorphic to $(\boldsymbol{Z}/2\boldsymbol{Z})^d$. Let $(x_\Lambda, \alpha_\Lambda)$ be an element of $M(\Gamma)$ corresponding to $\chi_\Lambda \in U(Z)$ under Lusztig's parametrization [21]. Then $(x_Z, \alpha_Z) = (1,1)$, and for $\Lambda \in S(BC, Z)$,

$$\alpha_\Lambda(x_\Lambda) = (-1)^{(|M(\Lambda)| - d)/2} ,$$

which can aslo be interpreted in terms of Frobenius eigenvalues associated with χ_Λ (see [19; 6.6], [21; 11.2]). Thus, as a special case of the formula (5.1.2), we have

Lemma 6.3.1. *Under the above notation, we have for $\Lambda \in S(BC, Z)$,*

$$m_\tau(\chi_Z) = 2^{-d} \sum_{\Lambda \in S(BC,Z)} \alpha_\Lambda(x_\Lambda)\, c_\tau(\chi_\Lambda)\,.$$

The proof of the following theorem is similar to that of Theorem 6.2.4.

Theorem 6.3.2. *Let $\boldsymbol{G}$ be of type B_n or C_n ($n \geq 1$). Let Λ be a symbol of rank n and odd defect, and χ_Λ the corresponding unipotent character of $G = \boldsymbol{G}_{\sigma^2}$.* Then

(i) $$c_\tau(\chi_\Lambda) = \alpha_\Lambda(x_\Lambda)\,.$$

(ii) $$m_\tau(\chi_\Lambda) = \begin{cases} 2^d & \text{if } \chi_\Lambda \text{ is special,} \\ 0 & \text{otherwise,} \end{cases}$$

where $2d+1$ is the number of integers appearing exactly once in Λ.

6.4. Let k be an algebraically closed field of characteristic p. In what follows, we assume that $\boldsymbol{G}$ is one of the following groups defined over $\boldsymbol{F}_{q'}$ (see [16]):

$$(6.4.1)\qquad \begin{cases} GL_{n+1}(k) \quad (n \geq 1); & Sp_{2n}(k) \quad (n \geq 1, p = 2)\,; \\ SO^{\pm}_{2u}(k) \quad (n \geq 2, p = 2)\,; & SO_{2n+1}(k) \quad (n \geq 1, p \neq 2)\,; \\ CSp_{2n}(k) \quad (n \geq 1, p \neq 2)\,; & CO^{\pm}_{2n}{}^{0}(k) \quad (n \geq 2, p \neq 2)\,. \end{cases}$$

Let σ be the corresponding Frobenius map. We would like to calculate $c_\tau(\chi)$ and $m_\tau(\chi)$ for any irreducible character χ of $G = \boldsymbol{G}_{\sigma^2}$. This will be done by reducing the problem to the case of unipotent characters considered in 6.1–6.3. For that purpose, we need:

Theorem 6.4.2 (Lusztig [16] [21], Asai [3]; see also Asai [1]). *Let $\boldsymbol{G}$ be one of the groups listed in* (6.4.1). *For a semisimple element s of the dual group $G^* = \boldsymbol{G}^*_{\sigma^2}$ of G, we put $\boldsymbol{G}[s] = Z_{\boldsymbol{G}^*}(s)^*$ (the dual of the centralizer of s in $\boldsymbol{G}^*$), and $G[s] = \boldsymbol{G}[s]_{\sigma^2}$. Then there exists a* 1–1 *correspondence*

$$\chi \to \chi_{\text{unip}}$$

from $\hat{G}_{(s)}$ onto $\hat{G}[s]_{(1)}$ with the following porperties:

(a) *If χ is cuspidal, then χ_{unip} is cuspidal.*

(b) *Let L be a Levi subgroup of a parabolic subgroup of G, and s a semisimple element of L^*. Let R be a cuspidal irreducible representation whose character ϕ is contained in $\hat{L}_{(s)}$, and R_{unip} that of $L[s]$ whose character if ϕ_{unip}. Let $P[s]$ be a parabolic subgroup of $G[s]$ with Levi subgroup $L[s]$. Then there exists a natural isomorphism*

$$\mathrm{End}_G(F(G, P, R)) \cong \mathrm{End}_{G[s]}(F(G[s], P[s] R_{\mathrm{unip}}))$$

(*see* 4.1) *by which each irreducible component* χ *of* ϕ_P^G *is mapped to* χ_{unip}.

(c) *Let* $\boldsymbol{T}$ *be a* σ^2*-stable maximal torus of* $\boldsymbol{G}$*, and* θ *a character of* $T=\boldsymbol{T}_{\sigma^2}$. *Let* (s) *be the semisimple conjugacy class of* G^* *corresponding to the geometric conjugacy class of* $(\boldsymbol{T}, \theta)$. *Then, for any* $\chi \in \hat{G}$,

$$\langle \chi, r_T^G[\theta] \rangle = \varepsilon(s) \langle \chi_{\mathrm{unip}}, r_{T'}^{G[s]}[1] \rangle ,$$

where $\varepsilon(s)=\pm 1$*, and* $\boldsymbol{T}'$ *is the torus of* $\boldsymbol{G}[s]$ *corresponding to* $\boldsymbol{T}$.

(d) *For any* $\chi \in \hat{G}$,

$$\chi(1) = |G|_{p'} |G[s]|_{p'}^{-1} \chi_{\mathrm{unip}}(1) ,$$

where $|\cdot|_{p'}$ *is the part of the integer* $|\cdot|$ *prime to* p.

(e) *Let* $\chi \in \hat{G}_{(s)}$ *for a semisimple element* s *of* G^*. *Let* $a_\eta \in \boldsymbol{Q}$, $\eta \in \hat{G}_{(s)}$, *be the coefficients in:*

$$t^*(\chi_{\mathrm{unip}}) = \sum_{\eta \in \hat{G}_{(s)}} a_\eta \cdot \eta_{\mathrm{unip}}$$

(*see Theorem* 5.1.1). *Then we have*

$$t^*(\chi) = \sum_{\eta \in \hat{G}_{(s)}} a_\eta \cdot \eta .$$

Now we can prove the main result of this section.

Theorem 6.4.3. *Let* $\boldsymbol{G}$ *be as in* (6.4.1). *Let* χ *be an irreducible character of* $G=\boldsymbol{G}_{\sigma^2}$*, and* (s) *the semisimple conjugacy class of* G^* *such that* $\chi \in \hat{G}_{(s)}$. *If* ${}^\tau(s) \neq (s)^{-1}$*, then* $c_\tau(\chi)=m_\tau(\chi)=0$. *If* ${}^\tau(s)=(s)^{-1}$*, then* $c_\tau(\chi)=c_\tau(\chi_{\mathrm{unip}})$ *and* $m_\tau(\chi) =m_\tau(\chi_{\mathrm{unip}})$.

Combining this with Theorem 6.2.4 and Theorem 6.3.2 (and also case (1) in 5.3), we can determine $c_\tau(\chi)$ and $m_\tau(\chi)$ for any $\chi \in \hat{G}$. It is extremely likely that the statement of Thoerem 6.4.3 is true for any connected reductive group with connected center.

Proof of Theorem 6.4.3. The first statement is already proved in Theorem 3.2.1. If ${}^\tau(s)=(s)^{-1}$, we can take a representative s in the conjugacy class (s) so that ${}^\tau s=s^{-1}$ by [6; 5.23]. Then σ preserves $\boldsymbol{G}[s]=Z_{G^*}(s)^*$. Hence we can consider $c_\tau(\eta)$ and $m_\tau(\eta)$ for $\eta \in \widehat{G[s]}$. By Theorem 6.4.2 (e) and Theorem 2.2.3 (i), it is enough to prove

$$c_\tau(\chi) = c_\tau(\chi_{\mathrm{unip}}) . \tag{6.4.4}$$

We show, by induction on n, that (6.4.4) and Assumption 4.5.1 (ii) are true. For $n=1$ or 2, they are easy to verify, For a larger n, we first assume that χ is not cuspidal. Then there exists a proper parabolic subgroup P of G, and a cuspidal irreducible character ϕ of a Levi subgroup L of P such that

$$\langle \chi, \phi_P^G \rangle \neq 0 .$$

If we write G as $G(n)$ to indicate its matrix size, the group L is isomorphic to

$$GL_{m_1}(\boldsymbol{F}_q) \times \cdots \times GL_{m_f}(\boldsymbol{F}_q) \times G(n')$$

for some f and some $m_i, n' < n$. Let (t) be the semisimple class of $L^* \subset G^*$ such that $\phi \in \hat{L}_{(t)}$, and $((t))$ the conjugacy class of G^* containing it. Then $((t)) = (s)$. Hence there exists a $w \in W$ such that ${}^{\tau w}L = L$ and ${}^{\tau w}(t) = (t)^{-1}$. Since ϕ is the unique cuspidal character contained in $\hat{L}_{(t)}$ (Lusztig [16]), we have

$${}^{\tau w}\phi = \bar{\phi} .$$

This implies that the set $W(\phi, \tau)$ defined in 4.3 is non-empty. Hence, by the induction assumption, Theorem 4.5.9, and the property (b) in Theorem 6.4.2, we have

$$c_\tau(\chi_\mu) = c_{\tau v}(\phi) = c_{\tau v}(\phi_{\mathrm{unip}}) = c_\tau((\chi_\mu)_{\mathrm{unip}})$$

for $v \in W(\phi, \tau)_{-\tau}$ and $\mu \in \widehat{W(\phi)}$. It only remains to prove (6.4.4) when χ is cuspidal, because then Assumption 4.5.1 (ii) follows from Theorem 6.2.4 and Theorem 6.3.2. We can do this by an argument similar to the one used in the proof of Theorem 6.2.4. Here we shall employ another method. By [16; p. 159],

$$\sum_{\rho \in \widehat{G[s]}_{(1)}} \rho(1)\,\rho = \frac{|G[s]|_{p'}}{|G[s]|} \sum_{T \subset G[s]} \varepsilon(\boldsymbol{G}[s], \boldsymbol{T})\, r_T^{G[s]}[1] \tag{6.4.5}$$

and

$$\sum_{\eta \in \hat{G}_{(s)}} \eta(1)\,\eta = \frac{|G|_{p'}}{|G[s]|} \sum_{T \subset G[s]} \varepsilon(\boldsymbol{G}, \boldsymbol{T}')\, r_{T'}^G[\theta_s] , \tag{6.4.6}$$

where $\boldsymbol{T}'$ is the maximal torus of $\boldsymbol{G}$ corresponding to $\boldsymbol{T}$, θ_s is a character of $T'^{\,2}_\sigma$ such that the pair $(\boldsymbol{T}', \theta_s)$ corresponds to (s), and $\varepsilon(\boldsymbol{G}[s], \boldsymbol{T})$ (resp. $\varepsilon(\boldsymbol{G}, \boldsymbol{T}')$) are signs which are equal to 1 if $\boldsymbol{T}$ (resp. $\boldsymbol{T}'$) is σ-stable. By (6.4.6) and Theorem 6.4.2 (d),

$$\sum_{\eta \in \hat{G}_{(s)}} \eta_{\mathrm{unip}}(1)\,\eta = \frac{|G[s]|}{|G[s]|_{p'}} \sum_{T \subset G[s]} \varepsilon(\boldsymbol{G}, \boldsymbol{T}')\, r_{T'}^G[\theta_s] . \tag{6.4.7}$$

On the other hand, by Theorem 3.2.6 (i), we see that

$$c_\tau(r_{T'}^G[\theta_s]) = c_\tau(r_T^{G[s]}[1])$$

if $\boldsymbol{T}'$ corresponds to $\boldsymbol{T}$. Hence, by (6.4.5) and (6.4.7), we have

$$\sum_{\rho\in\hat{G}[s]_{(1)}} \rho(1)\, c_\tau(\rho) = \sum_{\eta\in\hat{G}_{(s)}} \eta_{\mathrm{unip}}(1)\, c_\tau(\eta)\,. \tag{6.4.8}$$

Since we already know that, for $\eta \neq \chi$, $c_\tau(\eta)=c_\tau(\eta_{\mathrm{unip}})$, we conclude

$$c_\tau(\chi_{\mathrm{unip}}) = c_\tau(\chi)$$

from (6.4.8). This completes the proof of Theorem 6.4.3.

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