ON THE FACTORIZATION OF THE WHITEHEAD PRODUCT $[\iota_{2n+1}, \iota_{2n+1}]$

Kouyemon IRIYE and Kaoru MORISUGI

1. Introduction

As is well-known, the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ is of order 2 if $n \neq 0$, 1 or 3. It is commonly recognized that the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ plays a significant role for studying the homotopy groups of spheres.

In this paper we investigate the following problem: For a given finite complex X, when can the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ be factorized as $\alpha \circ \beta$, where $\beta \colon S^{4n+1} \to X$ and $\alpha \colon X \to S^{2n+1}$?

Two extreme cases are known; First, when X=one point, then the classical theorem of J.F. Adams gives the answer, that is, in this case, the above factorization happens if and only if $n=0, 1$ or 3. Second, let F be one of the fields R(real), C(complex) or H(quaternion). Let $Q^n(F)$ be the quasi F-projective space [3]. We take X as $\Sigma^{d(n+1)-1}Q^n(F)$, $d(n+1)-1$ fold suspension of the space $Q^n(F)$, where d is the dimension of F over R. Let $\alpha \colon \Sigma^{2d(n+1)-3} \to \Sigma^{d(n+1)-1}Q^n(F)$ be the $d(n+1)-1$ fold suspension of the attaching map of the top cell of $Q^{n+1}(F)$ and $\beta \colon \Sigma^{d(n+1)-1}Q^n(F) \to S^{d(n+1)-1}$ be the unstable representative of the S^{d-1} transfer map. For example, we can take β as the adjoint map of the following composite;

$$Q^n(F) \xrightarrow{r} G_F(n) \xrightarrow{J} \Omega^{dn}S^{dn} \xrightarrow{\Sigma^{d-1}} \Omega^{d(n+1)-1}S^{d(n+1)-1},$$

where r is the reflection map [3], $G_F(n)$ is the orthogonal F-linear group, and J is the J-map. Then, for any n, $\alpha \circ \beta = [\iota_{d(n+1)-1}, \iota_{d(n+1)-1}]$. This result is due to James and Whitehead [4].

There is another known example. Let $Q_{n+1}^{2n+1}(F)$ be the stunted quasi-projective space $Q^{2n+1}(F)/Q^n(F)$. There is a canonical cofibration;

$$S^{d(n+1)-1} \to Q_{n+1}^{2n+1}(F) \to Q_{n+2}^{2n+1}(F) \xrightarrow{\partial} S^{d(n+1)} \to \cdots,$$

where the first map is the inclusion of the bottom sphere and the second is the pinching map of the bottom sphere. It is easy to see that there exist a complex

X and a map $\alpha: X \to S^{d(n+1)-1}$ such that $\Sigma X = Q_{n+2}^{2n+1}(F)$ and $\Sigma \alpha = \partial$. So we can ask whether there exists a map $\beta: S^{2d(n+1)-3} \to X$ such that $\alpha \circ \beta = [\iota_{d(n+1)-1}, \iota_{d(n+1)-1}]$. In this case, if $n+1 \neq 2^t$, it is known that there is no such a map β. This result is due to Oshima [8] (See also [6]). However in case that $n+1 = 2^t$, almost nothing is known except the case that $(n, d) = (7, 2)$ or $(3,4)$: in the both cases, there is a factorization as above.

In this paper we deal with the cases that X is a bouquet of spheres and that X is a two cell complex or their bouquet. When X is a sphere, the following theorem is already known [7].

Theorem A. *Assume* $n \neq 0, 1$ *or* 3. *The Whitehead product* $[\iota_{2n+1}, \iota_{2n+1}]$ *can be represented by a product of elements of positive stem in the homotopy groups of spheres if and only if* $n = 2, 5$ *or* 7.

Main results of this paper are the following results:

Theorem B. *Assume* $n \neq 0, 1$ *or* 3. *The Whitehead product* $[\iota_{2n+1}, \iota_{2n+1}]$ *is decomposable by elements of positive stem in the homotopy groups of spheres if and only if* $n = 2, 4, 5$ *or* 7.

Theorem C. *Assume* $n \neq 0, 1$ *or* 3. *The Whitehead product* $[\iota_{2n+1}, \iota_{2n+1}]$ *belongs to such a Toda bracket as* $\langle \alpha_0, \gamma, \beta_0 \rangle$ *if and only if* $n = 2, 4, 5, 6, 7$ *or* 11, *where* α_0, β_0 *and* γ *are some elements of positive stem in the homotopy groups of spheres.*

Theorem D. *Assume* $n \neq 0, 1$ *or* 3. *The Whitehead product* $[\iota_{2n+1}, \iota_{2n+1}]$ *can be represented as such a sum of Toda brackets as* $\sum_i \langle \alpha_i, \gamma_i, \beta_i \rangle$ *if and only if* $n = 2, 4, 5, 6, 7, 8, 9$ *or* 11, *where* α_i, β_i *and* γ_i *are some elements of positive stem of the homotopy groups of spheres.*

This paper is organized as follows; In section 2, we recall from [6] a necessary condition for the existence of factorization above. In section 3, we list up examples of "deocmposables". In section 4–5 we prove Theorem B. In section 6–7, we give the outline of the proofs of Theorem C and D and list up the needed algebraic lemmas. The proofs of those algebraic lemmas are omitted, because they are very similar to those of the algebraic lemmas which are needed in the proof of Theorem B.

2. A necessary condition of a factorization

For a given finite complex X and a map $\alpha: X \to S^{2n+1}$, we denote the mapping cone of α by Y.

Proposition 2.1. *Let* $i: S^{2n+1} \to Y$ *be the inclusion map. Assume that* X *is*

$2n+1$ *connected. Then the following are equivalent.*

(1) *There exists a map* $\beta\colon S^{4n+1}\to X$ *such that* $[\iota_{2n+1}, \iota_{2n+1}]=\alpha\circ\beta$.

(2) *The Whitehead product* $[i,i]$ *is zero in* $\pi_{4n+1}(Y)$.

(3) *There exists a map* $f\colon S^{4n+3}\to\Sigma Y$ *whose Hopf invariant is one.*

(4) *There exists a map* $f\colon S^{4n+3}\to\Sigma Y$ *whose Hopf invariant is odd.*

Proof. Consider the exact sequence of homotopy groups of the pair (S^{n+1}, X). Then (1)$\Leftrightarrow$(2) follows from the Blakers-Massey theorem and the fact that $i_*[\iota_{2n+1}, \iota_{2n+1}]=[i, i]$. (2)$\Leftrightarrow$(3) follows from the EHP-sequences and their naturality with respect to the spaces Y, S^{2n+1} and the inclusion $i\colon S^{2n+1}\to Y$. (3)$\Leftrightarrow$(4) follows from the fact that for any $n\geq 0$ there exists a map $g\colon S^{4n+3}\to S^{2n+2}$ whose Hopf invariant is 2.

REMARK. Generally the Hopf invariant $H\colon \pi_*(\Sigma Y)\to\pi_*(\Sigma Y\wedge Y)$ is induced by the map $\Omega\Sigma Y\to\Omega\Sigma Y\wedge Y$. In our case, since $\pi_{4n+3}(\Sigma Y\wedge Y)$ is isomorphic to $\mathbf{Z}$, the Hopf invariant $H(f)$ takes its value in $\mathbf{Z}$ for $f\in\pi_{4n+3}(\Sigma Y)$.

In this section, we observe a necessary condition (Corollary 2.5) for the existence of a map whose Hopf invariant is one, under some conditions of a space Y. Calculating this necessary condition, we prove main theorems. The content of this section is a slight generalization of the appendix of [6].

Throughout this section, for technical reason, the complex Y is assumed to satisfy the following conditions.

0) Y is a connected finite complex with a base point.

1) Y is $2n$-connected.

2) dim. $Y\leq 4n+1$.

3) $H_{2n+1}(Y; \mathbf{Z})\cong\mathbf{Z}$.

4) $H^*(\Sigma Y; \mathbf{Z})$ is free and $H^{\mathrm{even}}(\Sigma Y; \mathbf{Z})$ is generated by $\{u_1, u_2, \cdots, u_l\}$ with dim. $u_i=2(n+n_i+1)$ and $n_i\leq n_{i+1}$ ($n_1=0$).

Under these conditions, the reduced K-theory of ΣY, $K(\Sigma Y)$, is also free. We can choose a basis $\{x_1, x_2, \cdots, x_l\}$ of $K(\Sigma Y)$ so that there exist rational numbers $c_{i,j}$ for $1\leq i, j\leq l$ such that

$$ch(x_j) = \sum_{i=1}^{l} c_{i,j}u_i, \quad \text{with} \quad c_{i,i} = 1 \quad \text{and} \quad c_{i,j} = 0 \quad \text{if} \quad i<j,$$

where ch is the Chern character, $K(\Sigma Y)\to H^{\mathrm{even}}(\Sigma Y; \mathbf{Q})\cong H^{\mathrm{even}}(\Sigma Y; \mathbf{Z})\otimes\mathbf{Q}$.

We denote the matrix $(c_{i,j})$ by C. For an integer k, let $\mathcal{A}(k)$ be the diagonal matrix with diagonal entries, $\{k^{n+n_1+1}, k^{n+n_2+1}, \cdots, k^{n+n_l+1}\}$. Then the following proposition holds by virtue of the Adams operations in K-theory and their relations with the Chern character.

Proposition 2.2. *For any* $k\in\mathbf{Z}$, *all entries of the matrix* $C^{-1}\mathcal{A}((k)C$ *are*

integral.

We denote the j-th column vector of the matrix $k^{2n+2}\mathcal{E}-C^{-1}\mathcal{A}(k)C$ by $\boldsymbol{a}^j(k)$, where $\mathcal{E}$ is the unit matrix. Especially we denote $\boldsymbol{a}^1(2)$ by $\boldsymbol{h}$ which we call the *Hopf vector.*

Let $f\in\pi_{4n+3}(\Sigma Y)$. The *e-invarinat vector* of f, $\boldsymbol{e}(f)$, is defined by

$$\boldsymbol{e}(f) = \begin{pmatrix} e_C(f)(x_1) \\ e_C(f)(x_2) \\ \vdots \\ e_C(f)(x_l) \end{pmatrix},$$

where e_C is the Adams-Toda e_C-invariant, that is,

$$e_C\colon \pi_{4n+3}(\Sigma Y) \to \mathrm{Hom}\,(K(\Sigma Y),\ \boldsymbol{Q}/\boldsymbol{Z})\,.$$

The following theorem gives a relation between the Hopf invariant and the e-invariant. It is a slight generalization of Adams or Toda's observation in case $Y=S^{2n+1}$ (Cf. [2]).

Theorem 2.3. *Under the same assumption, let $f\in\pi_{4n+3}(\Sigma Y)$. Then for any $k\in\boldsymbol{Z}$, the inner product of the vector $\boldsymbol{a}^j(k)$ and the e-invariant vector, $(\boldsymbol{e}(f),\boldsymbol{a}^j(k))$ is always integer. And the mod 2 Hopf invariant of f, $H_{(2)}(f)$, is equal to the mod 2 reduction of the integer $(\boldsymbol{e}(f),\boldsymbol{h})$.*

Remark that the properties mentioned above are independent of the choices of bases of $H^{\mathrm{even}}(\Sigma Y)$ and $K(\Sigma Y)$.

Let $E_{4n+2}(Y)$ be the image of $e_C\colon \pi_{4n+3}^s(\Sigma Y)\to(\boldsymbol{Q}/\boldsymbol{Z})^l$ under our choice of basis of $K(\Sigma Y)$. Then we have,

Theorem 2.4. *There exists a map $f\in\pi_{4n+3}(\Sigma Y)$ whose Hopf invariant is one if and only if there exists an element $\boldsymbol{e}\in E_{4n+2}(Y)$ such that $(\boldsymbol{e},\boldsymbol{h})$ is odd.*

The following corollary gives a necessary condition for existence of a map with Hopf invariant one.

Corollary 2.5. *Under the same assumption, if $H_{(2)}\colon \pi_{4n+3}(\Sigma Y)\to\boldsymbol{Z}/2$ is onto, then, there exists a row vector $\boldsymbol{x}\in\boldsymbol{Q}^l$ which satisfies the following:*
(1) for any $k\in\boldsymbol{Z}$, the vector $\boldsymbol{x}(k^{2n+2}\mathcal{E}-\mathcal{A}(k))C$ is an integral row vector.
(2) The first component of the row vector $\boldsymbol{x}(2^{2n+2}\mathcal{E}-\mathcal{A}(2))C$ is an odd integer.

Proof. Take $\boldsymbol{x}$ as ${}^t\boldsymbol{e}(f)C^{-1}$. Then the proof follows easily from Theorem 2.3. Here the symbol ^{t} means the transposition of a vector.

3. Examples of decomposabilities

In this section we freely use Toda's notation [9] for the 2-component of

the homotopy groups of spheres. The following proposition holds:

Proposition 3.1.

1) $[\iota_1, \iota_1] = 0,$

    $[\iota_3, \iota_3] = 0,$

    $[\iota_7, \iota_7] = 0,$

2) $[\iota_5, \iota_5] = \nu_5 \circ \eta_8,$

    $[\iota_{11}, \iota_{11}] = \sigma_{11} \circ \nu_{18},$

    $[\iota_{15}, \iota_{15}] = 2\sigma_{15}^2,$

3) $[\iota_9, \iota_9] = \sigma_9 \circ \eta_{16} + \eta_9 \circ \sigma_{10},$

4) $[\iota_{13}, \iota_{13}] = \Sigma\theta \in \Sigma\langle \sigma_{12}, \nu_{19}, \eta_{22} \rangle,$

    $[\iota_{23}, \iota_{23}] \in \Sigma^7 \langle \sigma_{16}, 2\sigma_{23}, \sigma_{30} \rangle,$

5) $[\iota_{17}, \iota_{17}] \in \Sigma\langle \sigma_{16}, 2\sigma_{23}, \eta_{30} \rangle + \Sigma^2 \langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle,$

    $[\iota_{19}, \iota_{19}] \in \Sigma^3 \langle \sigma_{16}, 2\sigma_{23}, \nu_{30} \rangle + \Sigma^7 \langle \sigma_{12}, \nu_{19}, \sigma_{22} \rangle.$

Proof. All above statements except the first of 5) in Proposition 3.1 are already known [9] (For the second of 4), see Lemma 8.3 in [5]). It should be noted that the above 2) and 3) follow from a general formula (Proposition 3.2 in [9]). Now we shall prove the first of 5). According to [9],

$$[\iota_{17}, \iota_{17}] = \eta^* + \omega_{17} \quad \mathrm{mod} \ \sigma_{17} \circ \mu_{24} \quad \text{and} \quad \eta^* \in \Sigma\langle \sigma_{16}, 2\sigma_{23}, \eta_{30} \rangle.$$

Thus it is enough to show that

Lemma 3.2. $\omega_{17} \in \Sigma^2 \langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle.$

Proof. The bracket $\langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle$ is clearly contained in $\pi_{31}(S^{15})$, which is isomorphic to $\mathbf{Z}_2\{\eta^*\} \oplus \mathbf{Z}_2\{\omega_{15}\} \oplus \mathbf{Z}_2\{\sigma_{15} \circ \mu_{22}\}$ (By Theorem 12.16 in [9]). From page 160 in [9], the element $\Sigma^2\eta^{*\prime}$ belongs to $\Sigma^3\pi_{30}(S^{14})$, which is equal to $\mathbf{Z}_2\{\omega_{17}\} \oplus \mathbf{Z}_2\{\sigma_{17} \circ \mu_{24}\}$. Thus, $\Sigma^2\langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle \subseteq \mathbf{Z}_2\{\omega_{17}\} \oplus \mathbf{Z}_2\{\sigma_{17} \circ \mu_{24}\}$. Take any element $x \in \langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle$. Then there exist numbers a and $b \in \mathbf{Z}_2$, which are uniquely determined by the element x, such that

$$\Sigma^2 x = a\omega_{17} + b\sigma_{17} \circ \mu_{24}.$$

Now we claim that $a = 1$. If $a = 0$, then $b\sigma_{17} \circ \mu_{24} = \Sigma^2 x \in \Sigma^2 \langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle$. Since $\sigma_{17} \circ \mu_{24} = \Sigma^2 \eta_{15} \circ \rho_{16}$ (Proposition 12.20 in [9]), this implies that $0 \in \Sigma^2 \langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle$. This contradicts the fact that the stable Toda bracket $\langle \eta, 2\sigma, \sigma \rangle = \langle \sigma, 2\sigma, \eta \rangle$ does not contain zero. Since $\sigma_{17} \circ \mu_{24}$ belongs to the indeterminacy of the bracket $\Sigma^2 \langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle$ we conclude that

$$\omega_{17} \in \Sigma^2 \langle \eta_{15}, 2\sigma_{16}, \sigma_{23} \rangle.$$

Proposition 3.1 clearly implies that if $n \leq 12$ and $n \neq 10$, then the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ is "decomposable" in the sense of the main theorems

A−D.

4. Proof of Theorem B

Suppose that the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ can be expressed as $\sum_{i=1}^{s} \alpha_i \circ \beta_i$ by elements of the homotopy groups of spheres with stems ≥ 1. Let Y be the mapping cone of the bouguet of $\{\alpha_i\}$. Then, by Proposition 2.1 there exists a map $f \in \pi_{4n+3}(\Sigma Y)$ whose Hopf invariant is one. It is obvious that the complex Y satisfies the conditions in §2. We shall apply Corollary 2.5. Since odd dimensional cells of ΣY have no relation with our necessary condition, we may asume that the stem of α_i is odd for all i. (If the stem of α_i is even for all i, then it follows that $n=0, 1$ or 3.) Let the stem of α_i be $2m_i-1$ and $e_C(\alpha_i)= a_i \in \boldsymbol{Q}/\boldsymbol{Z}$, where e_C stands for the complex e-invariant. In this situation we can choose bases of $H^{\mathrm{even}}(\Sigma Y; \boldsymbol{Z})$ and $K(\Sigma Y)$ so that the matrix of the Chern character of ΣY is

$$C = \begin{pmatrix} 1 & 0 & 0 \cdots\cdots 0 \\ a_1 & 1 & 0 \cdots\cdots 0 \\ a_2 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & 0 \\ \vdots & \vdots & & 0 & \ddots \\ a_s & 0 & & & 1 \end{pmatrix}.$$

So the matrix $k^{2n+2}\mathcal{E}-C^{-1}\mathcal{A}(k)C$ is given by

$$\begin{pmatrix} k^{2n+2}-k^{n+1} & & 0 & & \\ -k^{n+1}(k^{m_1}-1)a_1 & k^{n+1}(k^{n+1}-k^{m_1}) & & & \\ \vdots & & 0 & \ddots & 0 \\ -k^{n+1}(k^{m_s}-1)a_s & & & & k^{n+1}(k^{n+1}-k^{m_s}) \end{pmatrix}.$$

Therefore by Proposition 2.2, for any $k \in \boldsymbol{Z}$ and $1 \leq i \leq s$,

(4.1) $k^{n+1}(k^{m_i}-1)a_i \in \boldsymbol{Z}$.

And by Corollary 2.5, there exist rational numbers $x_0, x_1, \cdots, x_s$ which satisfy that for any integer $k \in \boldsymbol{Z}$,

(4.2) $\begin{cases} x_0 \cdot k^{n+1}(k^{n+1}-1)+\sum_{i=1}^{s} x_i \cdot a_i k^{n+1}(k^{n+1}-k^{m_i}) \in \boldsymbol{Z} , \\ x_i \cdot k^{n+1}(k^{n+1}-k^{m_i}) \in \boldsymbol{Z} \quad \text{for} \quad 1 \leq i \leq s , \\ x_0 \cdot 2^{n+1}(2^{n+1}-1)+\sum_{i=1}^{s} x_i \cdot a_i 2^{n+1}(2^{n+1}-2^{m_i}) \quad \text{is an odd integer} . \end{cases}$

Let q be a rational number. We denote the 2-adic valuation of q by $v_2(q)$. For convenience, from now on we always assume that all spaces are localized at 2. Therefore the e-invariant e_C takes its value in $\mathbf{Q}/\mathbf{Z}_{(2)}$. Recall that the stem of α_i is $2m_i - 1$ and $a_i = e_C(\alpha_i)$. Note that $m_i \leq n$. From (4.1) and taking $k = 3$ (see Lemma 5.1), we have

Lemma 4.3. *Under the notation above, for each i, it holds that*

$$v_2(a_i) \geqq -v_2(m_i) - 2 .$$

Especially if $2^p \leq n < 2^{p+1}$, then $v_2(a_i) \geqq -p - 2$.

Now the main part of the proof of Theorem B follows from the following lemma:

Lemma 4.4. *If $n \geq 8$ or $n = 6$, then the above equation (4.2) has no solution.*

The proof of the above lemma is given in §5.

Proposition 3.1 and Lemma 4.4 clearly imply Theorem B.

5. The proof of the algebraic lemma 4.4

This section is devoted to the proof of Lemma 4.4. The following numerical lemma is well-known:

Lemma 5.1.

(1) $\quad v_2(3^n - 1) = \begin{cases} v_2(n) + 2 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$

(2) For a positive integer t, $t \geq v_2(t) + 1$. Moreover, $t \geq v_2(t) + 3$ unless $t = 1, 2$ or 4.

We need the following lemmas:

Lemma 5.2. *Let $p = 2$ and $n = 6$ or $p \geq 3$ and $2^p \leq n < 2^{p+1}$. Then*

$$n + 1 - v_2(3^{n+1} - 1) - p - 2 \geqq 1 .$$

Proof. For $p = 2$ and $n = 6$, the assertion is easily proved. Assume that $n + 1$ is odd. Then by Lemma 5.1,

$$
\begin{aligned}
n + 1 - v_2(3^{n+1} - 1) - p - 2 &= n + 1 - p - 3 \\
&\geq 2^p - p - 2 \geqq 1 , \quad \text{if} \quad p \geqq 3 .
\end{aligned}
$$

Assume that $n + 1$ is even. Then by Lemma 5.1,

$$n + 1 - v_2(3^{n+1} - 1) - p - 2 = n + 1 - v_2(n+1) - 2 - p - 2 .$$

If $2^p < n+1 < 2^{p+1}$, then $n+1 \geqq 2^p+2$ and $\nu_2(n+1) \leqq p-1$. Thus in this case,

$$n+1-\nu_2(n+1)-2-p-2 \geqq 2^p-2p-1$$
$$\geqq 1, \quad \text{if } p \geqq 3.$$

If $n+1 = 2^{p+1}$, then

$$n+1-\nu_2(n+1)-2-p-2 = 2^{p+1}-2p-5$$
$$> 0, \quad \text{if } p \geqq 3.$$

This completes the proof.

Lemma 5.3. *Let $n \geqq 5$. If $1 \leqq m \leqq n$, then*

$$n+1+m-\nu_2(3^{n+1-m}-1)-\nu_2(m)-2 \geqq 1.$$

Proof. Assume that $n+1-m$ is odd. Then, by Lemma 5.1,

$$n+1+m-\nu_2(3^{n+1-m}-1)-\nu_2(m)-2 = n+m-\nu_2(m)-2$$
$$\geqq n-1 \geqq 1, \quad \text{if } n \geqq 2.$$

Assume that $n+1-m$ is even. Then by Lemma 5.1,

$$n+1+m-\nu_2(3^{n+1-m}-1)-\nu_2(m)-2$$
$$= \{(n+1-m)-\nu_2(n+1-m)-1\} + \{m-\nu_2(m)-1\} + m-2.$$

If $m \geqq 3$, then clearly the assertion is right by (2) of Lemma 5.1. It is easily checked that for $m=1$ or 2, the assertion still holds. This completes the proof.

Now we shall prove Lemma 4.4. In the equation (4.2) in §4, take $k=3$. Then we have

(5.4) $$x_0 \cdot 3^{n+1}(3^{n+1}-1) + \sum_{i=1}^{s} x_i \cdot a_i 3^{n+1}(3^{n+1}-3^{m_i}) \in \mathbf{Z}_{(2)},$$

(5.5) $$x_i \cdot 3^{n+1}(3^{n+1}-3^{m_i}) \in \mathbf{Z}_{(2)} \quad \text{for } 1 \leqq i \leqq s,$$

where $\mathbf{Z}_{(2)}$ is the set of integers localized at (2). In order to prove Lemma 4.4, we shall prove that the 2-exponent of the each term in the last equation of (4.2) in §4 is positive. First, the equation (5.5) implies:

(5.6) $$\nu_2(x_i) + \nu_2(3^{n+1-m_i}-1) \geqq 0.$$

If $a_i \in \mathbf{Z}_{(2)}$, then using (5.6) and Lemma 5.3, it is easy to check that $\nu_2(x_i \cdot a_i \cdot 2^{n+1}(2^{n+1}-2^{m_i})) \geqq 1$. So we can neglect such terms. Therefore we may assume that the number $a_i \not\equiv 0$ in $\mathbf{Q}/\mathbf{Z}_{(2)}$ for all i. From (5.4) and (5.5) it follows that

$$\nu_2(x_0) + \nu_2(3^{n+1}-1) \geqq \min_{1 \leqq i \leqq s} \{\nu_2(a_i)\}.$$

Now assume that $n=6$ and $p=2$ or $n\geq 8$ and $2^p\leq n<2^{p+1}$. Then,

$$
\begin{aligned}
\nu_2(x_0\cdot 2^{n+1}\cdot(2^{n+1}-1)) &= n+1+\nu_2(x_0) \\
&\geq n+1-\nu_2(3^{n+1}-1)+\min_{1\leq i\leq s}\{\nu_2(a_i)\} \\
&\geq n+1-\nu_2(3^{n+1}-1)-p-2 \qquad \text{from Lemma 4.3} \\
&\geq 1 \qquad\qquad\qquad\qquad\qquad \text{from Lemma 5.2}
\end{aligned}
$$

$$
\begin{aligned}
\nu_2(x_i\cdot a_i\cdot 2^{n+1}(2^{n+1}-2^{m_i})) &= \nu_2(x_i)+\nu_2(a_i)+n+1+m_i \\
&\geq n+1+m_i+\nu_2(a_i)-\nu_2(3^{n+1-m_i}-1) \qquad\qquad \text{from (5.6)} \\
&\geq n+1+m_i-\nu_2(m_i)-2-\nu_2(3^{n+1-m_i}-1) \text{ from Lemma 4.3} \\
&\geq 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{from Lemma 5.3}
\end{aligned}
$$

This completes the proof of Lemma 4.4.

6. The proof of Theorem C

Suppose that there exist maps $\alpha_0: S^{2n+1+l_1}\to S^{2n+1}$, $\gamma: S^{2n+1+l_2}\to S^{2n+1+l_1}$, and $\beta_0: S^{4n}\to S^{2n+1+l_2}$ such that $2n-2\geq l_2>l_1\geq 1$ and the unstable Toda bracket $\langle\alpha_0\gamma, \beta_0\rangle$ is defined. Assume that the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ belongs to this bracket.

Let X be the mapping cone of γ. Then from the definition of Toda bracket, it is clear that there exists a map $\alpha: X\to S^{2n+1}$ which is an extension of α_0, and a map $\beta: S^{4n+1}\to X$ which is a coextension of $\Sigma\beta_0$, such that $\alpha\circ\beta=[\iota_{2n+1}, \iota_{2n+1}]$. Let Y be the mapping cone of α. Then $\Sigma Y=S^{2n+2}\cup e^{2n+2+l_1+1}\cup e^{2n+2+l_2+2}$. Since $2n-2\geq l_2>l_1\geq 1$, this space ΣY satisfies the conditions of §2. We shall apply Corollary 2.5. If $l_1\equiv l_2 \bmod 2$, then the matrix C of the Chern character is of type $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ and if l_1 is even and l_2 is odd, then $C=(1)$. Thus in those cases, from the similar argument in the proof of Theorem B it follows that $n\leq 5$ or $n=7$ [7].

From now on we assume that l_1 is odd and l_2 is even. Put $2m_1=l_1+1$ and $2m_2=l_2+2$. Then $n\geq m_2>m_1\geq 1$. In this case we can take the matrix of the Chern character as

$$
C = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}.
$$

Here, from Proposition 2.2, it follows that for any $k\in\mathbf{Z}$

$$
(6.1) \qquad
\begin{cases}
ak^{n+1}(k^{m_1}-1)\in\mathbf{Z}, \\
bk^{n+1}(k^{m_2}-k^{m_1})\in\mathbf{Z}, \\
abk^{n+1}(k^{m_1}-1)-ck^{n+1}(k^{m_2}-1)\in\mathbf{Z}.
\end{cases}
$$

Note that $a=e_C(\alpha_0)$ and $b=e_C(\gamma)$.

Now from Corollary 2.5 there exists a rational row vector (x_0, x_1, x_2) which satisfies the following conditions: for any $k \in \mathbf{Z}$

$$(6.2) \quad \begin{cases} x_0 k^{n+1}(k^{n+1}-1)+x_1 a k^{n+1}(k^{n+1}-k^{m_1})+x_2 c k^{n+1}(k^{n+1}-k^{m_2}) \in \mathbf{Z}, \\ x_1 k^{n+1}(k^{n+1}-k^{m_1})+x_2 b k^{n+1}(k^{n+1}-k^{m_2}) \in \mathbf{Z}, \\ x_2 k^{n+1}(k^{n+1}-k^{m_2}) \in \mathbf{Z}. \end{cases}$$

Lemma 6.3. *Let* $n \geq 10$ *and* $\neq 11$. *Then under the conditions* (6.1) *and* (6.2), *each term of the sum* $x_0 2^{n+1}(2^{n+1}-1)+x_1 a 2^{n+1}(2^{n+1}-2^{m_1})+x_2 c 2^{n+1}(2^{n+1}-2^{m_2})$ *is even in* $\mathbf{Z}_{(2)}$. *Especially there is no such vector* (x_0, x_1, x_2) *as in Corollary 2.5.*

The proof of Lemma 6.3 follows from the following observations. From (6.1) and (6.2) taking $k=3$, we have

Lemma 6.4.
(1) $\nu_2(a) \geq -\nu_2(3^{m_1}-1)$.
(2) $\nu_2(b) \geq -\nu_2(3^{m_2-m_1}-1)$.
(3) $\nu_2(c) \geq -\nu_2(3^{m_2}-1)-\nu_2(3^{m_2-m_1}-1)$.

Lemma 6.5.
(1) $\nu_2(x_2) \geq -\nu_2(3^{n+1-m_2}-1)$.
(2) $\nu_2(x_1) \geq -\nu_2(3^{n+1-m_1}-1)-\nu_2(3^{m_2-m_1}-1)$.
(3) $\nu_2(x_0) \geq -\nu_2(3^{n+1}-1)-\nu_2(3^{m_2-m_1}-1)-\max. \{\nu_2(3^{m_1}-1), \nu_2(3^{m_2}-1)\}$.

Lemma 6.6. *Let* $n \geq 10$ *and* $n \neq 11$. *Then for* $n \geq m_2 > m_1 \geq 1$,

$$n+1-\nu_2(3^{n+1}-1)-\nu_2(3^{m_2-m_1}-1)-\max. \{\nu_2(3^{m_2}-1), \nu_2(3^{m_2}-1)\} \geq 1,$$

which implies that $\nu_2(x_0 2^{n+1}(2^{n+1}-1)) \geq 1$.

Lemma 6.7. *Under the conditions* (6.1) *and* (6.2),
(1) $\nu_2(x_1 a 2^{n+1}(2^{n+1}-2^{m_1})) \geq 1$.
(2) $\nu_2(x_2 c 2^{n+1}(2^{n+1}-2^{m_2})) \geq 1$.

From Lemma 6.3 and Proposition 3.1, Theroem C has been proved unless $n=8$ or 9. For $n=8$ or 9, we need the following lemma;

Lemma 6.8. *Let* $n=8$ *or* 9. *There is no stable 4 cell-complex* Z *which satisfies;*

$$Z = S^0 \cup e^{2i} \cup e^{2j} \cup e^{2n+2} \quad with \quad 0<i<j<n+1$$

and

$$Sq^{2n+2} x_0 = x_{n+1},$$

where $x_k \in H^{2k}(Z; \mathbf{Z}/2) \cong \mathbf{Z}/2$ *is the generator corresponding to the* $2k$-*dimensional cell of Z.*

Proof. We shall prove the case $n=8$. Since $H^{\mathrm{odd}}(Z; Z/2)=0$, calculating the Adem relation of Sq^2Sq^{16} and Sq^4Sq^{14}, we have

$$Sq^{18}x_0 = Sq^2Sq^{16}x_0 = Sq^{16}Sq^2x_0 \quad \text{on} \quad H^*(Z; Z/2).$$

This implies that $i=1$ and $j=8$. Since $Sq^{2^k}x_1=0$ for $k=1, 2$ or 3, usig the Admas decomposition of Sq^{16},

$$Sq^{16} = \sum_{0 \leq k \leq l \leq 3} a_{k,l}\Phi_{k,l} \qquad (l \neq k+1),$$

it follows that $17-2^k-2^l=2$ for some k and l. But this is apparently impossible. The proof of the case $n=9$ is almost the same as $n=8$.

Lemma 6.8 implies that for $n=8$ or 9, the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ can not belong to such a Toda bracket as stated in Theorem B.

This completes the proof of Theorem C.

7. The proof of Theorem D

Suppose that there exist maps for $1 \leq i \leq s$, $\alpha_i: S^{2n+1+l(i,1)} \to S^{2n+1}$, $\gamma_i: S^{2n+1+l(i,2)} \to S^{2n+1+l(i,1)}$, and $\beta_i: S^{4n} \to S^{2n+1+l(i,2)}$ such that $2n-2 \geq l(i,2) > l(i,1) \geq 1$ and the unstable Toda bracket $\langle \alpha_i, \gamma_i, \beta_i \rangle$ is defined. Assume that the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ belongs to the sum of those brackets.

Let X be the wedge of the mapping cones of γ_i. Then from the definition of Toda bracket, it is clear that there exist a map $\alpha: X \to S^{2n+1}$ which is an extension of α_i on each of bottom spheres, and a map $\beta: S^{4n+1} \to X$ which is a coextension to each of top cells of $\Sigma\beta_i$, such that $\alpha \circ \beta = [\iota_{2n+1}, \iota_{2n+1}]$. Let Y be the mapping cone of α. Then $\Sigma Y = S^{2n+2} \cup \bigvee_{i=1}^{s} e^{2n+3+l(i,1)} \cup e^{2n+4+l(i,2)}$. Since $2n-2 \geq l(i,2) > l(i,1) \geq 1$, this space ΣY satisfies the conditions of §2. We shall apply Corollary 2.5. Odd dimensional cells of ΣY have no relation with our calculation, we may assume that the space ΣY has the following cell structure:

$$S^{2n+2} \cup \left(\bigvee_{i=1}^{t} (e^{2n+2+2m(i,1)} \cup e^{2n+2+2m(i,2)}) \right) \vee e^{2n+2+2m(t+1)} \vee \cdots \vee e^{2n+2+2m(s)},$$

where $n \geq m(i,2) > m(i,1) \geq 1$ for each i, $n \geq m(l) \geq 1$ for each l, and $m(i,1) \leq m(j,1)$ if $i < j$.

Corresponding to the above ordering of cells, we can take the representative matrix of the Chern character as follows:

$$
C = \left(
\begin{array}{c|cccccccc|c}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
a_1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \\
c_1 & b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & \\
a_2 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \\
c_2 & 0 & 0 & b_2 & 1 & \cdots & 0 & 0 & \\
\vdots & & & & \cdots & & & & \mathcal{O} \\
\vdots & & & & \cdots & & & & \\
a_t & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \\
c_t & 0 & 0 & 0 & 0 & \cdots & b_t & 1 & \\
\hline
d_1 & & & & & & & & \\
\vdots & & & & O & & & & \mathcal{E}_{s-2t} \\
d_{s-t} & & & & & & & &
\end{array}
\right).
$$

Then the inverse matrix of C is given by

$$
C^{-1} = \left(
\begin{array}{c|cccccccc|c}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
-a_1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \\
a_1 b_1 - c_1 & -b_1 & 1 & 0 & 0 & \cdots & 0 & 0 & \\
-a_2 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & \\
a_2 b_2 - c_2 & 0 & 0 & -b_2 & 1 & \cdots & 0 & 0 & \\
\vdots & & & & \cdots & & & & \mathcal{O} \\
\vdots & & & & \cdots & & & & \\
-a_t & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & \\
a_t b_t - c_t & 0 & 0 & 0 & 0 & \cdots & -b_t & 1 & \\
\hline
-d_1 & & & & & & & & \\
\vdots & & & & \mathcal{O} & & & & \mathcal{E}_{s-2t} \\
-d_{s-t} & & & & & & & &
\end{array}
\right).
$$

Under the ordering of cells, $\mathcal{A}(k)$ is the diagonal matrix with diagonal entries $\{k^{n+1},\ k^{n+1+m(1,1)},\ k^{n+1+m(1,2)},\ \cdots,\ k^{n+1+m(t,1)},\ k^{n+1+m(t,2)},\ k^{n+1+m(t+1)},\ \cdots,\ k^{n+1+m(s)}\}$. Since the matrix $C^{-1}\mathcal{A}(k)C$ is integral for any $k \in \mathbf{Z}$, calculating this matrix, we see that for any i, j, k such that $k \in \mathbf{Z}$, $1 \le i \le t$ and $t+1 \le j \le s$,

$$(7.1) \quad \begin{cases} a_i k^{n+1}(k^{m(i,1)}-1)\in Z, \\ b_i k^{n+1}(k^{m(i,2)}-k^{m(i,1)})\in Z, \\ a_i b_i k^{n+1}(k^{m(i,1)}-1)-c_i k^{n+1}(k^{m(i,1)}-1)\in Z, \\ d_i k^{n+1}(k^{m(j)}-1)\in Z. \end{cases}$$

Corollary 2.5 implies that there exist rational numbers $x_0, x_1, y_1, \cdots, x_t, y_t, z_{t+1}, \cdots, z_s$ such that for any $k \in Z$,

$$(7.2) \quad \begin{cases} x_0 k^{n+1}(k^{n+1}-1)+\sum\limits_{i=1}^{t} x_i a_i k^{n+1}(k^{n+1}-k^{m(i,1)}) \\ \quad +\sum\limits_{i=1}^{t} y_i c_i k^{n+1}(k^{n+1}-k^{m(i,2)})+\sum\limits_{j=t+1}^{s} z_j d_j k^{n+1}(k^{n+1}-k^{m(j)})\in Z, \\ x_i k^{n+1}(k^{n+1}-k^{m(i,1)})+y_i b_i k^{n+1}(k^{n+1}-k^{m(i,2)})\in Z \quad \text{for} \quad 1\leqq i\leqq t, \\ y_i k^{n+1}(k^{n+1}-k^{m(i,2)})\in Z \quad \text{for} \quad 1\leqq i\leqq t, \\ z_j k^{n+1}(k^{n+1}-k^{m(j)})\in Z \quad \text{for} \quad t+1\leqq j\leqq s. \end{cases}$$

Now the rest of the proof of Theorem D is almost the same as Theorem C.

References

[1] J.F. Admas: *On the groups $J(X)$ IV*, Topology **5** (1966), 21–71.

[2] J.F. Adams and M.F. Atiyah: *K-theory and the Hopf invariant*, Quart. J. Math. (Oxford) **17** (1966), 31–38.

[3] I.M. James: *The topology of Stiefel manifolds*, L.M.S. Lecture Note Series 24 (1976).

[4] I.M. James and J.H.C. Whitehead: *The homotopy theory of sphere bundles over spheres (I)*, Proc. London Math Soc., **4** (1954), 196–218.

[5] M. Mimura: *On the generalized Hopf homomorphism and the higher composition Part II, $\pi_{n+i}(S^n)$ for $i=21$ and 22*, J. Math. Kyoto Univ., **6** (1965), 301–325.

[6] K. Morisugi: *On the homotopy group $\pi_{8n+4}(Sp(n))$ and the Hopf invariant*, J. Math. Kyoto Univ. **31** (1991), 121–132.

[7] K. Morisugi: *On the Hopf invariant and its application*, Q. and Z. in Gen. Top., (Special issue), Proceedings of the second Soviet-Japan joint symposium of topology **8** (1990), 185–192.

[8] H. Oshima: *Whitehead products in Stiefel manifolds and Samelson products in classical groups*, Advanced Studies in Pure Math. 9 (1986), Homotopy theory and related topics, 237–258.

[9] H. Toda: *Composition methods in homotopy groups of spheres*, Ann. Math. Studies, **49** (1962).

Kouyemon IRIYE
Department of Applied Mathematics,
Osaka Women's University
Sakai 590, Japan

Kaoru MORISUGI
Department of Mathematics,
Wakayama University
Wakayama 640, Japan