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THREE-FOLD IRREGULAR BRANCHED COVERINGS OF SOME SPATIAL GRAPHS

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1. Introduction

A *spatial graph* is a graph embedded in a 3-sphere S^3. In this paper, we consider three-fold irregular branched coverings of some spatial graphs. In particular, we investigate those of some of θ-curves and handcuff graphs in S^3 and prove that there exists at least one three-fold irregular branched covering of these graphs. Further, we identify these branched coverings. Hilden [4] and Montesinos [6] independently showed that every orientable closed 3-manifold is a three-fold irregular covering of S^3, branched along a link.

Let L be a spatial graph and $G=\pi_1(S^3-L)$. Then there is a one-to-one correspondence between n-fold unbranched coverings of S^3-L and conjugacy classes of transitive representations of G into S_n, the symmetric group with n letters $\{0, 1, \cdots, n-1\}$. Let μ be such a representation, called a *monodromy map*, and $T=\mu(G)$. Define T_0 as the subgroup of T that fixes letter 0. Then $\mu^{-1}(T_0)$ is the fundamental group of the unbranched covering associated with μ. To each unbranched covering of S^3-L there exists the unique completion $\tilde{M}_\mu(L)$ called the associated branched covering (see Fox [1]).

In this paper we investigate a monodromy map $\mu: G\to S_3$ which is surjective, i.e. the covering is irregular. We call μ an *S_3-representation* of L. Further we only consider the case that the branched covering associated with μ is an orientable 3-manifold.

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2. Three-fold branched coverings of spatial θ-curves

In this section, let L denote a spatial θ-curve that consists of three egdes e_1, e_2 and e_3, each of which has distinct endpoints A and B. Suppose that each of e_1, e_2 and e_3 is oriented from A to B. Then $G=\pi_1(S^3-L)$ is generated by $x_1, \cdots, x_l$; $y_1, \cdots, y_m$; $z_1, \cdots, z_n$, where each of x_i, y_j and z_k corresponds to a meridian of each of e_1, e_2 and e_3, respectively. Note that every element of S_3 can be expressed as $a^\delta b^\varepsilon$, where $a=(01)$, $b=(012)$; $\delta=0, 1$, $\varepsilon=0, 1, 2$. We assume that

$\mu(x_i)=a^{\alpha_{1i}}b^{\alpha_{2i}}$, $\mu(y_j)=a^{\beta_{1j}}b^{\beta_{2j}}$, $\mu(z_k)=a^{\gamma_{1k}}b^{\gamma_{2k}}$. Let $r_1=x_1y_1z_1=1$ be the relation corresponding to A. By applying $ba=ab^{-1}$ to $r_1=1$, we have $\alpha_{11}+\beta_{11}+\gamma_{11}\equiv 0$ (mod 2). We put $\alpha_{11}=\beta_{11}=1$ and $\gamma_{11}=0$ without loss of generality. Since $\mu(x_i)$ is a conjugation of $\mu(x_{i-1})$ with $a^{\delta}b^{\varepsilon}$, we have $\alpha_{1i}=1$. Similarly we have $\beta_{1j}=1$ and $\gamma_{1k}=0$. Hence we have

$$(1)\qquad \begin{cases} \mu(x_i) = ab^{\alpha_i}, & i = 1, \cdots, l, \\ \mu(y_j) = ab^{\beta_j}, & j = 1, \cdots, m, \quad \text{and} \\ \mu(z_k) = b^{\gamma_k}, & k = 1, \cdots, n. \end{cases}$$

Let F be the free group generated by $x_1, \cdots, x_l; y_1, \cdots, y_m; z_1, \cdots, z_n$ and ϕ the canonical projection from F to G. Further let ψ: $G\to H=\langle t\rangle$, where $\psi(x_i)=t$, $\psi(y_j)=t^{-1}$ and $\psi(z_k)=1$. Then *the Jacobian matrix* $A(G, \psi)$ *of* G *at* ψ is defined as follows (see Kinoshita [5]): Let r be the p-th relation of G. Then the p-th row of $A(G, \psi)(t)$ can be expressed as

$$\left(\left(\frac{\partial r}{\partial x_i}\right)^{\psi\phi} \left(\frac{\partial r}{\partial y_j}\right)^{\psi\phi} \left(\frac{\partial r}{\partial z_k}\right)^{\psi\phi}\right),$$

where $\partial/\partial x_i$, $\partial/\partial y_j$, and $\partial/\partial z_k$ are the Fox's free derivatives. Let ν be the nullity of $A(G, \psi)(-1)$ in Z_3-coefficients. Note that $\nu\geq 1$. Then we have

Theorem 2.1. *The number of conjugacy classes of* S_3*-representations of* L*, each of which satisfies* (1), *is equal to* $(3^{\nu}-3)/3!$.

Since one of the relations of G is a consequence of the others, the deficiency of G is equal to two. Hence $\nu\geq 2$. Therefore we have

Collorary 2.2. *There exists at least one* S_3*-representation of* L *which satisfies* (1).

Proof of Theorem 2.1. We may deform a diagram of any spatial θ-curve so that there is no crossing on e_3 (see Figure 2.1). In Figure 2.1 let T be a 2-string tangle. Then G has generators $x_1, \cdots, x_l; y_1, \cdots, y_m$; z and relations,

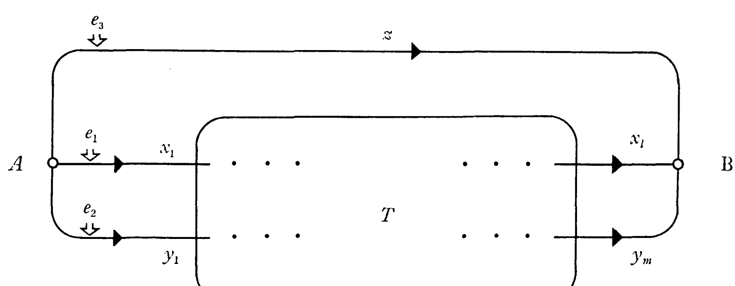

Fig. 2.1

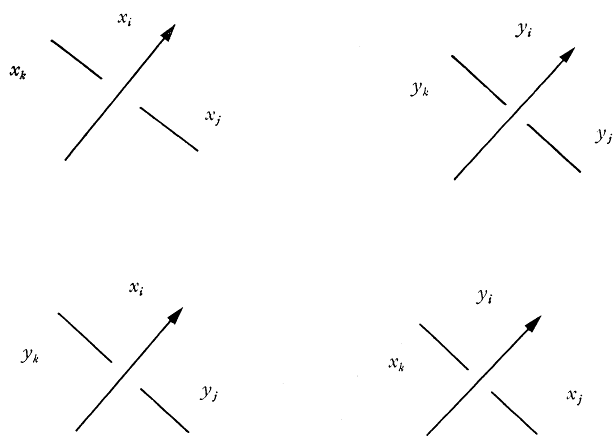

Fig. 2.2

each of which can be expressed as one of the following six types: $r_1=x_1y_1z$, $r_2=x_ly_mz$, $r_3=x_ix_jx_i^{-1}x_k^{-1}$, $r_4=y_iy_jy_i^{-1}y_k^{-1}$, $r_5=x_iy_jx_i^{-1}y_k^{-1}$ and $r_6=y_ix_jy_i^{-1}x_k^{-1}$, where r_1 and r_2 correspond to vertices A and B, and r_3, r_4, r_5 and r_6 correspond to four types of crossings as shown in Figure 2.2, respectively. Since $\mu(r_i)=1$, $i=1, \cdots, 6$, we have the following equations which correspond to r_i, $i=1, \cdots, 6$, respectively:

(2.1) $$\alpha_1-\beta_1-\gamma\equiv 0 \pmod 3\,,$$

(2.2) $$\alpha_l-\beta_m-\gamma\equiv 0 \pmod 3\,,$$

(2.3) $$2\alpha_i-\alpha_j-\alpha_k\equiv 0 \pmod 3\,,$$

(2.4) $$2\beta_i-\beta_j-\beta_k\equiv 0 \pmod 3\,,$$

(2.5) $$2\alpha_i-\beta_j-\beta_k\equiv 0 \pmod 3\,,$$

(2.6) $$2\beta_i-\alpha_j-\alpha_k\equiv 0 \pmod 3\,.$$

On the other hand, for six types of relations of G we have

(3.1) $$\left(\frac{\partial r_1}{\partial x_1}\right)^{\psi\phi}=1\,,\qquad \left(\frac{\partial r_1}{\partial y_1}\right)^{\psi\phi}=t\,,\qquad \left(\frac{\partial r_1}{\partial z}\right)^{\psi\phi}=1\,;$$

(3.2) $$\left(\frac{\partial r_2}{\partial x_l}\right)^{\psi\phi}=1\,,\qquad \left(\frac{\partial r_2}{\partial y_m}\right)^{\psi\phi}=t\,,\qquad \left(\frac{\partial r_2}{\partial z}\right)^{\psi\phi}=1\,;$$

(3.3) $$\left(\frac{\partial r_3}{\partial x_i}\right)^{\psi\phi}=1-t\,,\qquad \left(\frac{\partial r_3}{\partial x_j}\right)^{\psi\phi}=t\,,\qquad \left(\frac{\partial r_3}{\partial x_k}\right)^{\psi\phi}=-1\,;$$

(3.4) $$\left(\frac{\partial r_4}{\partial y_i}\right)^{\psi\phi}=1-t^{-1}\,,\qquad \left(\frac{\partial r_4}{\partial y_j}\right)^{\psi\phi}=t^{-1}\,,\qquad \left(\frac{\partial r_4}{\partial y_k}\right)^{\psi\phi}=-1\,;$$

(3.5) $\left(\frac{\partial r_5}{\partial x_i}\right)^{\psi\phi} = 1-t^{-1}, \quad \left(\frac{\partial r_5}{\partial y_j}\right)^{\psi\phi} = t, \quad \left(\frac{\partial r_5}{\partial y_k}\right)^{\psi\phi} = -1;$

(3.6) $\left(\frac{\partial r_6}{\partial y_i}\right)^{\psi\phi} = 1-t, \quad \left(\frac{\partial r_6}{\partial x_j}\right)^{\psi\phi} = t^{-1}, \quad \left(\frac{\partial r_6}{\partial x_k}\right)^{\psi\phi} = -1.$

Therefore we have the following equation:

$$(4) \qquad A(G,\psi)(-1)\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_l\\ \beta_1\\ \vdots\\ \beta_m\\ -\gamma\end{pmatrix} \equiv \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix} \pmod 3.$$

Since the nullity of $A(G, \psi)(-1)$ is ν, there are 3^ν solutions for (4). In order to count the number of S_3-representations, we must omit three solutions $\alpha_i=\beta_j=0$, $\gamma=0$; $\alpha_i=\beta_j=1$, $\gamma=0$; $\alpha_i=\beta_j=2$, $\gamma=0$, since each of the corresponding monodromy maps is not surjective. The monodromy map corresponding to any other solution is surjective. Hence, by taking into account the six inner automorphisms of S_3, the number of solutions corresponding to S_3-representations (up to conjugation) is $(3^\nu-3)/3!$.

EXAMPLES. (1) Let L be a θ-curve illustrated in Figure 2.3, where T is a 1-string tangle. Let K be a constituent knot $e_1 \cup e_2$ of L.

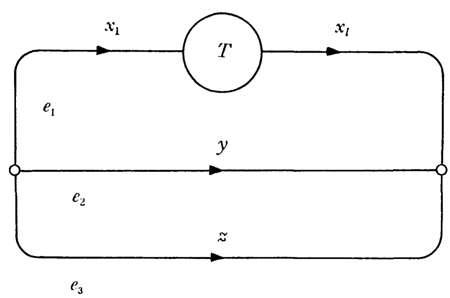

Fig. 2.3

Case 1. Suppose that $\mu(z)=b^\gamma$, where γ is equal to 0, 1 or 2. Let $\tilde{M}_2(K)$ be the two-fold branched covering of K and $\tilde{M}_3(K)$ the three-fold irregular branched covering of K. If we denote the Betti number of $H_1(\tilde{M}_2(K); Z_3)$ by λ, then $\nu=\lambda+2$. Note that the number of conjugacy classes of S_3-representations of K is equal to $(3^{\lambda+1}-3)/3!$. By Theorem 2.1, the number of conjugacy classes of μ is equal to $(3^{\lambda+2}-3)/3!$. Actually, the set of $\tilde{M}_\mu(L)$ consists of one

$\tilde{M}_2(K)$, $(3^{\lambda+1}-3)/3!$ $\tilde{M}_3(K)$'s and $2\,(3^{\lambda+1}-3)/3!$ $\tilde{M}_3(K)\#(S^2\times S^1)$'s.

Case 2. Suppose that $\mu(x_i)=b^{\alpha_i}$, where α_i is equal to 1 or 2, $i=1, \cdots, l$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is the three-fold cyclic branched covering of K.

(2) Let L be a rational θ-curve $\theta(p, q)$ illustrated in Figure 2.4, where

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{2n}}}}$$

(see Harikae [2]). Note that L has the symmetry for e_1 and e_2.

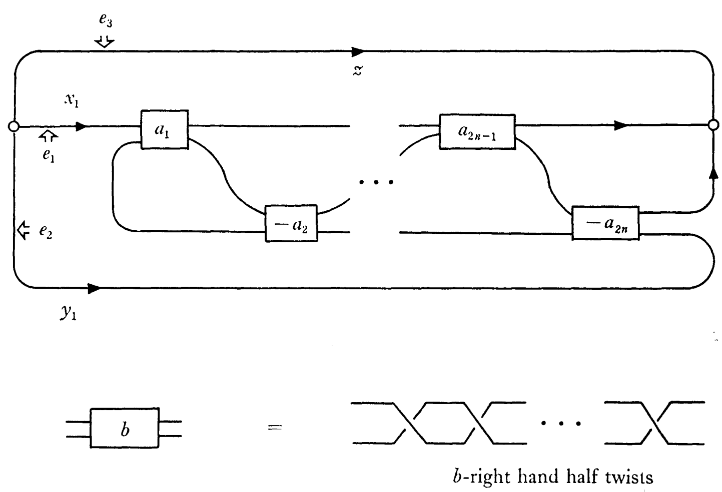

Fig. 2.4

Case 1. Suppose that $\mu(z)=b^\gamma$, where γ is equal to 0, 1 or 2. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is an S^3.

Case 2. Suppose that $\mu(x_i)=b^{\alpha_i}$, where α_i is equal to 1 or 2, $i=1, \cdots, l$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Further, we can see that $\tilde{M}_\mu(L)$ is a lens space.

(3) Let L be a pseudo-rational θ-curve $\theta(p_1, q_1; p_2, q_2)$ illustrated in Figure 2.5, where

$$\frac{p_1}{q_1} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{2}}}} \quad \text{and} \quad \frac{p_2}{q_2} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1}}}}$$

(see [2]).

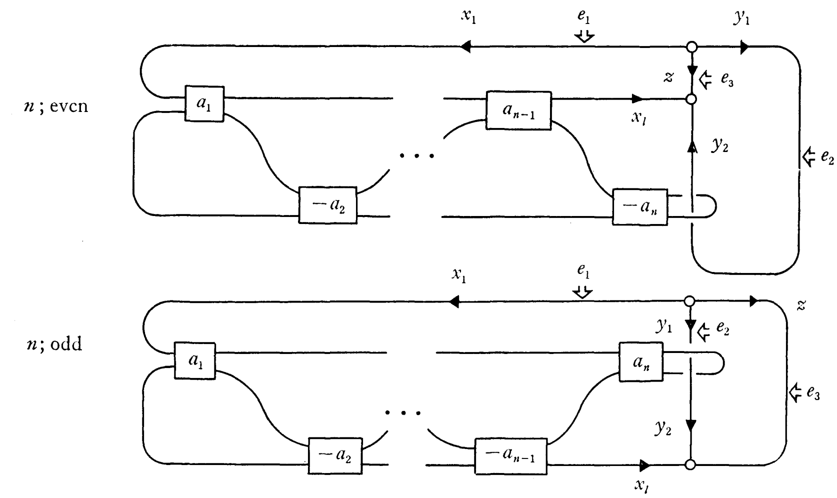

Fig. 2.5

Case 1. Suppose that $\mu(z)=b^{\gamma}$, where γ is equal to 0,1 or 2. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, if $p_2\equiv 0$ (mod 3), then $\tilde{M}_\mu(L)$ is an S^3. If $p_2\not\equiv 0$ (mod 3), then $\tilde{M}_\mu(L)$ is a real projective 3-space P^3.

Case 2. Suppose that $\mu(y_j)=b^{\beta_j}$, where β_j is equal to 0, 1 or 2, $i=1, 2$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, if $p_1\equiv 0$ (mod 3), then $\tilde{M}_\mu(L)$ is an S^3. If $p_1\not\equiv 0$ (mod 3), then $\tilde{M}_\mu(L)$ is a P^3.

Case 3. Suppose that $\mu(x_i)=b^{\alpha_i}$, where α_i is equal to 1 or 2, $i=1, \cdots, l$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one.

(4) Let L be the Kinoshita's θ-curve illustrated in Figure 2.6 (see [5]). Note that L has the symmetry for three edges. We assume that $\mu(z_k)=b^{\gamma_k}$, where γ_k is equal to 1 or 2 for $k=1, 2, 3$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is a lens space $L(5, 2)$.

(5) Let L be a θ-curve illustrated in Figure 2.7. Note that L has the symmetry for e_2 and e_3.

Case 1. Suppose that $\mu(z_k)=b^{\gamma_k}$, where γ_k is equal to 1 or 2 for $k=1, 2, 3, 4$. Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is $L(4, 1)$.

Case 2. Suppose that $\mu(x_i)=b^{\alpha_i}$, where α_i is equal to 0, 1 or 2 for $i=1, 2$.

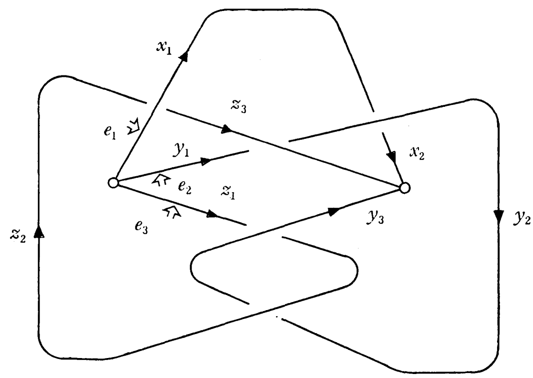

Fig. 2.6

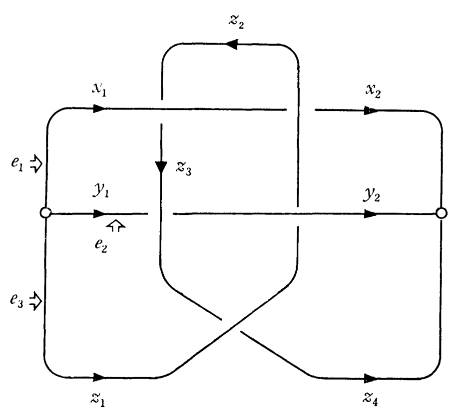

Fig. 2.7

Then we have $\nu=3$. Hence, the number of conjugacy classes of μ is equal to four. Actually, the set of $\tilde{M}_\mu(L)$ consists of S^3, $S^2\times S^1$, $L(3, 1)$ and $L(3, 1)$.

3. Three-fold branched coverings of spatial handcuff graphs

In this section, let L denote a spatial handcuff graph which consists of three edges e_1, e_2 and e_3, where e_3 has distinct endpoints A and B, and e_1 and e_2 are loops based at A and B, respectively. Suppose that e_3 is oriented from A to B. We shall use the same notations as Section 2. Then $G=\pi_1(S^3-L)$ is generated by $x_1, \cdots, x_l; y_1, \cdots, y_m; z_1, \cdots, z_n$, where each of x_i, y_j and z_k corresponds to a meridian of each of e_1, e_2 and e_3, respectively. Let $r_1=x_1x_l^{-1}z_1=1$ be the relation corresponding to A. By applying $ba=ab^{-1}$ to $r_1=1$, we have $\alpha_{11}-\alpha_{1l}+\gamma_{11}\equiv 0$ (mod 2). Further we obtain $\alpha_{11}=\alpha_{1l}$ by using the argument in Section 2. Hence we have $\gamma_{11}=0$, which leads $\gamma_{1k}=0$. Suppose that $\alpha_{1i}=\beta_{1j}=1$, then $\tilde{M}_\mu(L)$ is an orientable 3-manifold. Thus we have equations (1) in Section 2. If we define ν as similar to Section 2, then we have

Theorem 3.1. *The number of conjugacy classes of S_3-representations of L, each of which satisfies* (1), *is equal to* $(3^{\nu}-3)/3!$.

Proof. Using the similar argument to the proof of Theorem 2.1, we can prove the statement of the theorem.

Since one of the relations of G is a consequence of the others, the deficiency of G is equal to two. Hence $\nu \geq 2$. Therefore we have

Collorary 3.2. *There exists at least one S_3-representation of L which satisfies* (1).

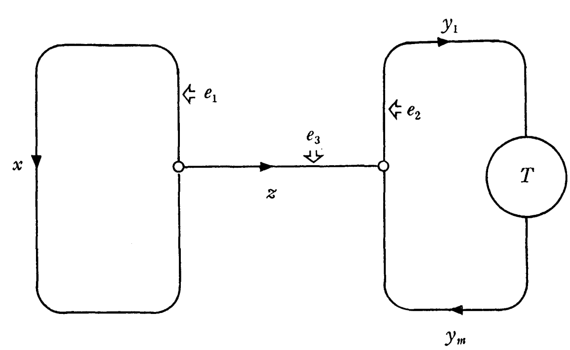

Fig. 3.1

EXAMPLES. (1) Let L be a handcuff graph illustrated in Figure 3.1, where T is a 1-string tangle. Let K be a constituent knot e_2 of L. Let $\tilde{M}_2(K)$ be the two-fold branched covering of K and $\tilde{M}_3(K)$ the three-fold irregular branched covering of K. If we denote the Betti number of $H_1(\tilde{M}_2(K); Z_3)$ by λ, then $\nu=\lambda+2$. Note that the number of conjugacy classes of S_3-representations of K is equal to $(3^{\lambda+1}-3)/3!$. Suppose that μ satisfies (1). Then by Theorem 3.1, the number of conjugacy classes of μ is equal to $(3^{\lambda+2}-3)/3!$. Actually, the set of $\tilde{M}_\mu(L)$ consists of one $\tilde{M}_2(K)$ and $3(3^{\lambda+1}-3)/3!\ \tilde{M}_3(K)\#(S^2\times S^1)$'s.

(2) Let L be a rational handcuff graph $\phi(p, q)$ illustrated in Figure 3.2, where

$$\frac{p}{q}=a_1+\cfrac{1}{a_2+\cfrac{1}{\ddots+\cfrac{1}{a_{2n+1}}}}$$

(see Harikae [3]). Suppose that μ satisfies (1). Then we have $\nu=2$. Hence, the number of conjugacy classes of μ is equal to one. Actually, $\tilde{M}_\mu(L)$ is an S^3.

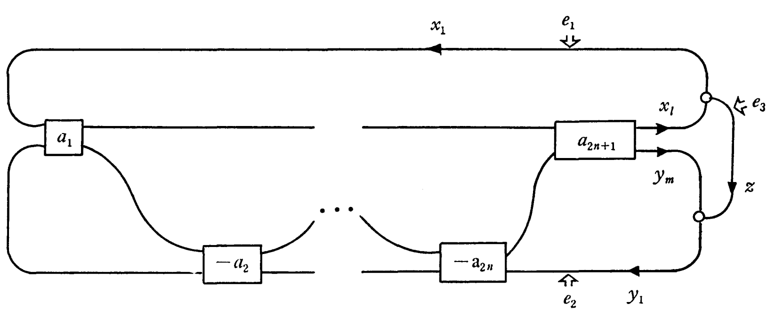

Fig. 3.2

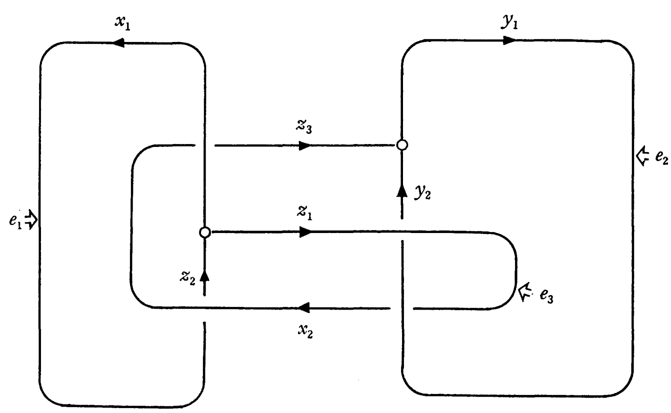

Fig. 3.3

(3) Let L be a handcuff graph illustrated in Figure 3.2 (see [5]). Suppose that μ satisfies (1). Then we have $\nu=3$. Hence, the number of conjugacy classes of μ is equal to four. Actually, the set of $\tilde{M}_\mu(L)$ consists of S^3, $S^2\times S^1$, $L(3, 1)$ and $L(3, 2)$.

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