# HOLOMORPHIC EQUIVALENCE PROBLEM FOR A CERTAIN CLASS OF UNBOUNDED REINHARDT DOMAINS IN $C^2$

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## Introduction

An answer to the holomorphic equivalence problem for arbitrary bounded Reinhardt domains was given in [1]. It seems interesting to investigate whether a similar result on unbounded Reinhardt domains holds or not.

Now, for a pair $(a, b)$ of non-negative real constants with $(a, b) \neq (0, 0)$ and a positive real constant $r$, let us consider an unbounded Reinhardt domain $D_{a,b}(r)$ in $C^2$ given by

$$D_{a,b}(r) = \{(z, w) \in C^2 \,|\, |z|^a |w|^b < r\}.$$

Here, when $ab=0$, for example, when $b=0$, the domain $D_{a,0}(r)$ is understood as

$$D_{a,0}(r) = \{(z, w) \in C^2 \,|\, |z|^a < r\}.$$

We are concerned with the holomorphic automorphisms and the equivalence of the domains $D_{a,b}(r)$. In the present paper, we confine ourselves to the case where $a$ and $b$ are integers. The case where $a$ and $b$ are arbitary non-negative real constants will be treated in the subsequent paper [3].

Our main result of this paper can be stated as the following theorem.

**Theorem.** *If a domain $D_{a,b}(r)$ with $(a, b) \in (Z_{\geq 0})^2$ is biholomorphic to a domain $D_{u,v}(s)$ with $(u, v) \in (Z_{\geq 0})^2$, then there exists a transformation $\varphi$ given by*

$$\varphi : C^2 \ni (z, w) \mapsto (\alpha z, \beta w) \in C^2$$

*or*

$$\varphi : C^2 \ni (z, w) \mapsto (\gamma w, \delta z) \in C^2$$

*such that $\varphi(D_{a,b}(r)) = D_{u,v}(s)$, where $Z_{\geq 0}$ denotes the set of non-negative intergers and $\alpha, \beta, \gamma, \delta$ are non-zero complex constants.*

This paper is organized as follows. In Section 1, we recall basic concepts and results on Reinhardt domains. In particular, we give a general formulation of the holomorphic equivalence problem for Reinhardt domains as well as the

motivation of the consideration of domains $D_{a,b}(r)$.   In Section 2, we introduce the notion of a Liouville foliation, which plays a key role in our investigation. Section 3 is devoted to the study of a certain class of unbounded Reinhardt domains in $(C^*)^2$.   The results are used in Section 4 for discussing the holomorphic automorphisms and the equivalence of domains $D_{a,b}(r)$ with $(a, b) \in (Z_{\geq 0})^2$.

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## 1.   Basic concepts on Reinhardt domains

We first collect some notations and terminology.   The set of non-zero complex numbers is denoted by $C^*$.   The multiplicative group of complex numbers of absolute value 1 is denoted by $U(1)$.   An *automorphism* of a complex manifold $M$ means a biholomorphic mapping of $M$ onto itself.   The group of all automorphisms of $M$ is denoted by Aut($M$).   Two complex spaces are said to be *holomorphiclaly equivalent* if there is a biholomorphic mapping between them.

We now recall some basic concepts and results on Reinhardt domains (cf. [2, Section 2]).   Write $T = (U(1))^n$.   The group $T$ acts as a group of automorphisms on $C^n$ by

$$(\alpha_1, \cdots, \alpha_n) \cdot (z_1, \cdots, z_n) = (\alpha_1 z_1, \cdots, \alpha_n z_n)$$
$$\text{for} \quad (\alpha_1, \cdots, \alpha_n) \in T \quad \text{and} \quad (z_1, \cdots, z_n) \in C^n.$$

By definition, a *Reinhardt domain* $D$ in $C^n$ is a domain in $C^n$ which is stable under $T$, that is, such that $\alpha \cdot D \subset D$ for every $\alpha \in T$.   The group $T$ then acts as a group of automorphisms on $D$.   The subgroup of Aut($D$) induced by $T$ is denoted by $T(D)$.

An automorphism $\varphi$ of $(C^*)^n$ is called an *algebraic automorphism* of $(C^*)^n$ if the components of $\varphi$ are given by Laurent monomials, that is, $\varphi$ is of the form

$$\varphi: (C^*)^n \ni (z_1, \cdots, z_n) \mapsto (w_1, \cdots, w_n) \in (C^*)^n,$$
$$w_i = \alpha_i z_1^{a_{1i}} \cdots z_n^{a_{ni}}, \quad i = 1, \cdots, n,$$

where $(a_{ij}) \in GL(n, Z)$ and $(\alpha_i) \in (C^*)^n$.   The set $\text{Aut}_{alg}((C^*)^n)$ of all algebraic automorphisms of $(C^*)^n$ forms a subgroup of Aut $((C^*)^n)$.

Let $\varphi$ be an algebraic automorphism of $(C^*)^n$ and write $\varphi(z) = (\varphi_1(z), \cdots, \varphi_n(z))$.   In general, the components $\varphi_1, \cdots, \varphi_n$ have zeroes or poles along each coordinate hyperplane.   If, for two domains $D$ and $D'$ in $C^n$ not necessarily contained in $(C^*)^n$, they have no poles on $D$ and $\varphi: D \to C^n$ maps $D$ biholomorphically onto $D'$, then we say that $\varphi$ induces a biholomorphic mapping of $D$ onto $D'$.

Consider a biholomorphic mapping $\varphi: D \to D'$ between two Reinhardt do-

mains $D$ and $D'$ in $C^n$. Then $\varphi$ is induced by an algebraic automorphism of $(C^*)^n$ if and only if it is equivariant with respect to the $T$-actions, or equivalently if and only if it has the property that $\varphi T(D)\varphi^{-1} = T(D')$. Biholomorphic mappings between Reinhardt domains equivariant with respect to the $T$-actions may be considered as natural isomorphisms in the category of Reinhardt domians. In view of this observation, we say that two Reinhardt domians in $C^n$ are *algebraically equivaelnt* if there is a biholomorphic mapping between them induced by an algebraic automorphism of $(C^*)^n$.

In terms of the notion of algebraic equivalence, the holomorphic equivalnce problem for Reinhardt domains may be formulated as the problem of studying the relationship between the holomorphic equivalence of Reinhardt domains and the algebraic equivalence of them. It is clear that if two Reinhardt domains in $C^n$ are algebraically equivalent, then they are holomorphically equivalent. What we have to ask is whether the converse assertion holds or not:

**Problem.** *If two Reinhardt domains $D$ and $D'$ in $C^n$ are holomorpically equivalent, then are they algebraically equivalent?*

To specify this problem to the case where both $D$ and $D'$ contain the origin, we need the following lemma, whose proof is straightforward, and is omitted.

**Lemma 1.1.** (cf. [1, Section 4]). *Let $\varphi$ be a biholomorphic mapping between two domains in $C^n$ both containing the origin. If the components of $\varphi$ are given by Laurent monomials, then $\varphi$ is induced by an algebraic automorphism of $(C^*)^n$ of the form*

$$(C^*)^n \ni (z_1, \cdots, z_n) \mapsto (w_1, \cdots, w_n) \in (C^*)^n,$$
$$w_i = \alpha_i z_{\sigma(i)}, \quad i = 1, \cdots, n,$$

*where $\sigma$ is a permutation of $\{1, \cdots, n\}$ and $(\alpha_1, \cdots, \alpha_n) \in (C^*)^n$.*

The concept of an algebraic automorphism of a Reinhardt domain will be needed later. An automorphism of a Reinhardt domain $D$ in $C^n$ is called an *algebraic automorphism* of $D$ if it is induced by an algebraic automorphism of $(C^*)^n$. The set $\mathrm{Aut}_{\mathrm{alg}}(D)$ of all algebraic automorphisms of $D$ forms a subgroup of $\mathrm{Aut}(D)$.

We conclude this section with observations which motivates the consideration of the unbounded domains $D_{a,b}(r)$ given in the introduction.

In order to express a Reinhardt domain $D$ in $C^n$ geometrically, it is convenient to consider the image of $D^* := D \cap (C^*)^n$ under the mapping ord: $(C^*)^n \to R^n$ given by

$$\mathrm{ord}\,(z_1, \cdots, z_n) = (-(2\pi)^{-1} \log |z_1|, \cdots, -(2\pi)^{-1} \log |z_n|).$$

The subset ord$(D^*)$ of $R^n$ is called the *logarithmic image* of $D$. Clearly, ord

$(D^*)$ is a domain in $\boldsymbol{R}^n$.

Here are some observations about the relationships between Reinhardt domains and their logarithmic images (cf. [1, Section 2]). Firstly, if $D$ is a Reinhardt domain $\boldsymbol{C}^n$, then $D$ is bounded if and only if

$$\operatorname{ord}(D^*) \subset \{(\xi_1, \cdots, \xi_n) \in \boldsymbol{R}_u \mid \xi_1 > c_1, \cdots, \xi_n > c_n\}$$

for some constants $c_1, \cdots, c_n$. Secondly, if $D$ is a Reinhardt domain in $(\boldsymbol{C}^*)^n$, then $D$ is algebraically equivalent to a bounded Reinhardt domain if and only if the logarithmic image $\operatorname{ord}(D)$ of $D$ has the convex hull containing no complete straight lines. Thirdly, if a Reinhardt domain $D$ in $\boldsymbol{C}^n$ is pseudoconvex, then the logarithmic image $\operatorname{ord}(D^*)$ of $D$ is a convex domain in $\boldsymbol{R}^n$.

Let $D$ be a Reinhardt domain in $\boldsymbol{C}^n$. In order to discuss the holomorphic equivalence problem, we may assume without loss of generality that $D$ is pseudoconvex. The observations above give a consequence that $D$ is essentially unbounedd when a convex domain in $\boldsymbol{R}^n$ given as the logarithmic image ord $(D^*)$ of $D$ contains complete straight lines. In case $n = 2$, the condition that ord $(D^*)$ contains complete straight lines implies that $\operatorname{ord}(D^*)$ is of the form

$$\operatorname{ord}(D^*) = \{(\xi_1, \xi_2) \in \boldsymbol{R}^2 \mid c < a_1\xi_1 + a_2\xi_2 < c'\},$$

that is, $D^*$ is of the form

$$D^* = \{(z_1, z_2) \in \boldsymbol{C}^2 \mid e^c < |z_1|^{-a_1/2\pi} |z_2|^{-a_2/2\pi} < e^{c'}\}$$

for some real constants $a_1, a_2, c, c'$ with $(a_1, a_2) \neq (0, 0)$ and $-\infty \leq c < c' \leq +\infty$. The domains $D_{a,b}(r)$ with which we are concerned are basic objects among those Reinhardt domains in $\boldsymbol{C}^2$ whose logarithmic images are half-planes.

## 2. Liouville foliation

Let $M$ be a complex manifold. A collection $\{\Sigma_\alpha\}_{\alpha \in A}$ of subsets $\Sigma_\alpha$, $\alpha \in A$ of $M$ is called a *Liouville foliation* on $M$ if the following four conditions are satisfied:

(L1)   If $\alpha_1, \alpha_2 \in A$ and $\alpha_1 \neq \alpha_2$, then $\Sigma_{\alpha_1} \cap \Sigma_{\alpha_2} = \emptyset$;

(L2)   $\bigcup_{\alpha \in A} \Sigma_\alpha = M$;

(L3)   For each subset $\Sigma_\alpha$, any bounded holomorphic function on $M$ takes a constant value on $\Sigma_\alpha$;

(L4)   For every $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \neq \alpha_2$, there exists a bounded holomorphic function $h$ on $M$ such that the constant values of $h$ on $\Sigma_{\alpha_1}$ and $\Sigma_{\alpha_2}$ are different.

If there exists a Liouville foliation on $M$, then we say that $M$ has a Liouville foliation. The following lemma shows that $M$ has at most one Liouville foliation.

**Lemma 2.1.** *If $\{\Sigma_\alpha\}_{\alpha \in A}$ and $\{\Sigma'_{\alpha'}\}_{\alpha' \in A'}$, are two Liouville foliations on a complex manifold $M$, then they coincide, that is, there exists a bijective correspondence $\tau: A \to A'$ between the index sets $A$ and $A'$ such that $\Sigma_\alpha = \Sigma'_{\tau(\alpha)}$ for every $\alpha \in A$.*

Proof. We first show that if $\Sigma_\alpha \cap \Sigma'_{\alpha'} \neq \emptyset$, say $p \in \Sigma_\alpha \cap \Sigma'_{\alpha'}$, then $\Sigma_\alpha = \Sigma'_{\alpha'}$. Suppose contrarily that $\Sigma_\alpha \neq \Sigma'_{\alpha'}$. Then there exists a point $q \in M$ such that $q \in \Sigma_\alpha \backslash \Sigma'_{\alpha'}$, or $q \in \Sigma'_{\alpha'} \backslash \Sigma_\alpha$, where, for example, $\Sigma_\alpha \backslash \Sigma'_{\alpha'}$ denotes the intersection of $\Sigma_\alpha$ and the complee complement of $\Sigma'_{\alpha'}$ in $M$. We may assume without loss of generality that $q \in \Sigma'_{\alpha'} \backslash \Sigma_\alpha$. Since $p \in \Sigma_\alpha$ and $q \notin \Sigma_\alpha$, it follows from (L4) that there exists a bounded holomorphic function $h$ on $M$ such that $h(p) \neq h(q)$. But, since $p \in \Sigma'_{\alpha'}$ and $q \in \Sigma'_{\alpha'}$, this contradicts (L3).

Now, it follows from (L1), (L2) and what we have shown above that, to each element $\alpha \in A$, there is associated a unique element $\tau(\alpha) \in A'$ for which $\Sigma_\alpha = \Sigma'_{\tau(\alpha)}$. The desired correspondence is given by $A \ni \alpha \mapsto \tau(\alpha) \in A'$. q.e.d.

The next proposition plays a key role in our investigation.

**Proposition 2.1.** *If $\varphi: M \to M'$ is a biholomorphic mapping between two complex manifolds $M$ and $M'$, and if $M$ and $M'$ have Liouville foliations $\{\Sigma_\alpha\}_{\alpha \in A}$ and $\{\Sigma'_{\alpha'}\}_{\alpha' \in A'}$, respectively, then there exists a bijective correspondence $\tau: A \to A'$ between the index sets $A$ and $A'$ such that $\varphi(\Sigma_\alpha) = \Sigma'_{\tau(\alpha)}$ for every $\alpha \in A$.*

Proof. It is readily verified that $\{\varphi(\Sigma_\alpha)\}_{\alpha \in A}$ is a Liouville foliation on $M'$. We have only to apply the above lemma to the Liouville foliations $\{\varphi(\Sigma_\alpha)\}_{\alpha \in A}$ and $\{\Sigma'_{\alpha'}\}_{\alpha' \in A'}$ on $M'$. q.e.d.

### 3. Domains $D^*_{a,b}$

For an element $(a, b)$ of $\mathbf{Z}^2$ with $(a, b) \neq (0, 0)$, we define an unbounded Reinhardt domain $D^*_{a,b}$ in $(\mathbf{C}^*)^2$ by

$$D^*_{a,b} = \{(z, w) \in (\mathbf{C}^*)^2 \mid |z|^a |w|^b < 1\}.$$

Note that $D^*_{1,0} = \Delta^* \times \mathbf{C}^*$, where $\Delta^* = \{\zeta \in \mathbf{C} \mid 0 < |\zeta| < 1\}$.

In this section, we discuss the automorphisms and the equivalence of the domains $D^*_{a,b}$. The results will be used in the next section.

We begin with a remark that $D^*_{a,b} = D^*_{u,v}$ if $(a, b) = k(u, v)$ for a positive integer $k$. Consequently, in the study of domains $D^*_{a,b}$, we may assume without loss of generality that $(a, b)$ is a primitive element of the free module $\mathbf{Z}^2$, that is, $(a, b)$ is not a positive integral multiple of any element of the free module $\mathbf{Z}^2$ except itself. Throughout this section, we assume that $(a, b)$ is a primitive element of $\mathbf{Z}^2$.

We now discuss the equivalence of domains $D^*_{a,b}$.

**Lemma 3.1.** *Every domain $D_{a,b}^*$ is algebraically equivalent to the domain* $D_{1,0}^*$.

Proof.   Since $(a, b)$ is a primitive element of $\mathbf{Z}^2$, we can find integers $c$, $d$ for which $ad - bc = 1$. Using these integers $c$, $d$, we define an algebraic automorphism $\varphi$ of $(\mathbf{C}^*)^2$ by

$$\varphi \colon (\mathbf{C}^*)^2 \ni (z, w) \mapsto (z^a w^b, z^c w^d) \in (\mathbf{C}^*)^2 .$$

Then it is readily verified that $\varphi(D_{a,b}^*) = D_{1,0}^*$.                                   q.e.d.

As an immediate consequence of this lemma, we obtain the following proposition.

**Proposition 3.1.** *Every two domains $D_{a,b}^*$ and $D_{u,v}^*$ are algebraically equivalent.*

REMARK.   In the above proposition, $D_{u,v}^*$ is algebraically equivalent to $D_{a,b}^*$ under a transformation of the form

$$\psi \colon D_{u,v}^* \ni (z, w) \mapsto (z^p w^q, z^r w^s) \in D_{a,b}^* ,$$

where

$$A \colon = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in GL(2, \mathbf{Z})$$

satisfies

(3.1)                                   $(a, b)A = (u, v) .$

By Lemma 3.1, in order to describe the automorphisms of domains $D_{a,b}^*$, we need to investigate the automorphisms of the domain $D_{1,0}^*$.

**Lemma 3.2.** *If $G(D_{1,0}^*)$ denotes the subgroup of $\mathrm{Aut}(D_{1,0}^*)$ consisting of all transformations of the form*

$$D_{1,0}^* \ni (z, w) \mapsto (z, \lambda(z)w) \in D_{1,0}^* ,$$

*where $\lambda$ is a nowhere-vanishing holomorphic function on the punctured unit disk $\Delta^*$, then*

$$\mathrm{Aut}\,(D_{1,0}^*) = G(D_{1,0}^*) \cdot \mathrm{Aut}_{\mathrm{alg}}\,(D_{1,0}^*) .$$

*Furthermore, $\mathrm{Aut}_{\mathrm{alg}}(D_{1,0}^*)$ consists of all transformations of the form*

$$D_{1,0}^* \ni (z, w) \mapsto (\alpha z, \beta z^k w^\varepsilon) \in D_{1,0}^* ,$$

*where $k \in \mathbf{Z}$, $\varepsilon = \pm 1$, $\alpha \in U(1)$, and $\beta \in \mathbf{C}^*$.*

Proof.   Note first that $D_{1,0}^*$ has a Liouville foliation.   Indeed, for each $\zeta \in$

$\Delta^*$, set $\Sigma_\zeta = \{(z, w) \in D^*_{1,0} \mid z = \zeta\}$. Clearly, the collection $\{\Sigma_\zeta\}_{\zeta \in \Delta^*}$ of the subsets $\Sigma_\zeta$, $\zeta \in \Delta^*$, of $D^*_{1,0}$ satisfies (L1) and (L2). Since, for each $\zeta \in \Delta^*$, the subset $\Sigma_\zeta$ is an analytic subset of $D^*_{1,0}$ which is holomorphically equivalent to $C^*$, (L3) follows by Liouville's theorem. To see (L4), consider the bounded holomorphic function $h$ on $D^*_{1,0}$ given by $h(z, w) = z$. Then, for every $\zeta \in \Delta^*$, we have $\Sigma_\zeta = \{(z, w) \in D^*_{1,0} \mid h(z, w) = \zeta\}$. This implies that if $\zeta, \zeta' \in \Delta^*$ and $\zeta \neq \zeta'$, then the constant values of $h$ on $\Sigma_\zeta$ and $\Sigma_{\zeta'}$ are different, and (L4) is verified.

Now, let $f$ be any element of $\mathrm{Aut}(D^*_{1,0})$. By Proposition 2.1, there exists a bijective correspondence $\tau: \Delta^* \to \Delta^*$ between $\Delta^*$ and itself such that $f(\Sigma_\zeta) = \Sigma_{\tau(\zeta)}$ for every $\zeta \in \Delta^*$. This implies that $f$ can be written in the form

$$f: D^*_{1,0} \ni (z, w) \mapsto (\tau(z), \theta(z, w)) \in D^*_{1,0},$$

where $\theta(z, w)$ is a holomorphic function on $D^*_{1,0}$. As a consequence, the mapping $\tau: \Delta^* \to \Delta^*$ is holomorphic. Since $\tau$ is bijective, it follows that $\tau(z) = \alpha z$ for a constant $\alpha \in U(1)$.

To determine $\theta(z, w)$, let $\theta(z, w) = \sum_{n=-\infty}^{\infty} \gamma_n(z) w^n$ be the Laurent expansion of $\theta(z, w)$ with respect to $w$, where $\gamma_n(z)$, $n \in Z$, are holomorphic functions on $\Delta^*$. Fix any point $\zeta$ of $\Delta^*$. It follows from the relation $f(\Sigma_\zeta) = \Sigma_{\tau(\zeta)}$ that the mapping $C^* \ni w \mapsto \theta(\zeta, w) \in C^*$ gives a biholomorphic mapping of $C^*$ onto itself, and hence that $\theta(\zeta, w) = \gamma w^\varepsilon$ for a constant $\gamma \in C^*$, where $\varepsilon = 1$ or $\varepsilon = -1$. By the uniqueness of the Laurent expansion, we have $\gamma_n(\zeta) = 0$ for all $n \neq \pm 1$, while we have either $\gamma_1(\zeta) = \gamma \neq 0$ and $\gamma_{-1}(\zeta) = 0$ or $\gamma_1(\zeta) = 0$ and $\gamma_{-1}(\zeta) = \gamma \neq 0$. Since this holds for every $\zeta \in \Delta^*$, we see that $\theta(z, w) = \lambda(z) w^\varepsilon$ for a nowhere-vanishing holomorphic function $\lambda$ on $\Delta^*$, where $\varepsilon = 1$ or $\varepsilon = -1$.

The results of the preceding paragraphs show that every automorphism $f$ of $D^*_{1,0}$ is given by

$$f: D^*_{1,0} \ni (z, w) \mapsto (\alpha z, \lambda(z) w^\varepsilon) \in D^*_{1,0},$$

where $\varepsilon = \pm 1$, $\alpha \in U(1)$, and $\lambda$ is a nowhere-vanishing holomorphic function on $\Delta^*$. If we set $f' \in G(D^*_{1,0})$ and $f'' \in \mathrm{Aut}_{\mathrm{alg}}(D^*_{1,0})$ as

$$f': D^*_{1,0} \ni (z, w) \mapsto (z, (\lambda \circ \pi)(z) w) \in D^*_{1,0}$$

and

$$f'': D^*_{1,0} \ni (z, w) \mapsto (\alpha z, w^\varepsilon) \in D^*_{1,0},$$

where $\pi$ denotes the automorphism of $\Delta^*$ given by $\pi(z) = \alpha^{-1} z$, then $f = f' \circ f''$, which proves the first assertion. To prove the second assertion, it is enough to observe that $f \in \mathrm{Aut}_{\mathrm{alg}}(D^*_{1,0})$ if and only if $\lambda(z) = \beta z^k$, where $k \in Z$ and $\beta \in C^*$.

**Proposition 3.2.** *If $G(D^*_{a,b})$ denotes the subgroup of $\mathrm{Aut}(D^*_{a,b})$ consisting of all transformations of the form*

(3.2)         $D_{a,b}^* \ni (z, w) \mapsto (\lambda(z^a w^b)^{-b} z, \ \lambda(z^a w^b)^a w) \in D_{a,b}^*$ ,

*where $\lambda$ is a nowhere-vanishing holomorphic function on the punctured unit disk $\Delta^*$, then*

$$\mathrm{Aut}\,(D_{a,b}^*) = G(D_{a,b}^*) \cdot \mathrm{Aut}_{\mathrm{alg}}\,(D_{a,b}^*)\,.$$

*Furthermore, $\mathrm{Aut}_{\mathrm{alg}}(D_{a,b}^*)$ consists of all transformations of the form*

(3.3)         $D_{a,b}^* \ni (z, w) \mapsto (\alpha z^p w^q, \ \beta z^r w^s) \in D_{a,b}^*$ ,

*where $\alpha, \beta \in C^*$ satisfy*

(3.4)                              $|\alpha|^a |\beta|^b = 1$

*and*

$$A := \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in GL(2, \mathbf{Z})$$

*satisfies*

(3.5)                         $(a, b)A = (a, b)\,.$

Proof. Using the biholomorphic mapping $\varphi \colon D_{a,b}^* \to D_{1,0}^*$ given in Lemma 3.1, we have $\mathrm{Aut}(D_{a,b}^*) = \varphi^{-1} \mathrm{Aut}(D_{1,0}^*)\varphi$, and hence, by Lemma 3.2, $\mathrm{Aut}(D_{a,b}^*)$ $= \varphi^{-1} G(D_{1,0}^*)\varphi \cdot \varphi^{-1} \mathrm{Aut}_{\mathrm{alg}}(D_{1,0}^*)\varphi$. Since $\varphi \in \mathrm{Aut}((C^*)^2)$, it follows that $\mathrm{Aut}_{\mathrm{alg}}$ $(D_{a,b}^*) = \varphi^{-1} \mathrm{Aut}_{\mathrm{alg}}(D_{1,0}^*)\varphi$. On the other hand, a straightforward computation yields that $G(D_{a,b}^*) = \varphi^{-1} G(D_{1,0}^*)\varphi$. Thus we conclude the first assertion. The second assertion follows immediately from the relation $\mathrm{Aut}_{\mathrm{alg}}(D_{a,b}^*) = \varphi^{-1} \mathrm{Aut}_{\mathrm{alg}}$ $(D_{1,0}^*)\varphi$ and the second assertion of Lemma 3.2.                     q.e.d.

## 4. Automorphisms and equivalence of domains $D_{a,b}$ with $(a, b) \in (\mathbf{Z}_{\geq 0})^2$

In what follows, we always deal with domains $D_{a,b}(r)$ for which $(a, b) \in (\mathbf{Z}_{\geq 0})^2$.

We begin with preliminary observations. Firstly, for every positive constant $r$, the domain $D_{a,b}(r)$ is algebraically equivalent to the domain $D_{a,b}(1)$ under a suitable transformation the form

$$C^2 \ni (z, w) \mapsto (\alpha z, \beta w) \in C^2,$$

where $(\alpha, \beta) \in (C^*)^2$. Hence, in order to discuss the automorphisms and the equivalence of domains $D_{a,b}(r)$, it is enough to deal with domains $D_{a,b}(1)$. For brevity, we set $D_{a,b} = D_{a,b}(1)$. Secondly, in the study of the domains $D_{a,b}$, we may assume without loss of generality that $(a, b)$ is a primitive element of $\mathbf{Z}^2$. In fact, as remarked at the beginning of the preceding section, we have $D_{p,q} = D_{u,v}$

if $(p, q) = k(u, v)$ for a positive integer $k$. Throughout this section, we assume that $(a, b)$ is a primitive element of $\mathbf{Z}^2$. In particular, in case $ab = 0$, we have $(a, b) = (1, 0)$ or $(a, b) = (0, 1)$. Note that $D_{1,0} = \Delta \times \mathbf{C}$ and $D_{0,1} = \mathbf{C} \times \Delta$, where $\Delta = \{\zeta \in \mathbf{C} \mid |\zeta| < 1\}$.

We now present a basic lemma.

**Lemma 4.1.** *Every domain $D_{a,b}$ has a Liouville foliation.*

Proof. Consider first the case where $ab \neq 0$. For each $\zeta \in \Delta$, set $\Sigma_\zeta = \{(z, w) \in D_{a,b} \mid z^a w^b = \zeta\}$. Clearly, the collection $\{\Sigma_\zeta\}_{\zeta \in \Delta}$ of the subsets $\Sigma_\zeta$, $\zeta \in \Delta$, of $D_{a,b}$ satisfies (L1) and (L2). Since, for each $\zeta \in \Delta^*$, the subset $\Sigma_\zeta$ is an analytic subset of $D_{a,b}$ which is holomorphically equivalent to $\mathbf{C}^*$ under the transformation $\varphi$ given in the proof of Lemma 3.1, and since $\Sigma_0$ is the analytic subset of $D_{a,b}$ given by $\Sigma_0 = \{(z, w) \in \mathbf{C}^2 \mid zw = 0\}$, (L3) follows by Liouville's theorem. To see (L4), consider the bounded holomorphic function $h$ on $D_{a,b}$ given by $h(z, w) = z^a w^b$. Then, for every $\zeta \in \Delta$, we have $\Sigma_\zeta = \{(z, w) \in D_{a,b} \mid h(z, w) = \zeta\}$. This implies that if $\zeta, \zeta' \in \Delta$ and $\zeta \neq \zeta'$, then the constant values of $h$ on $\Sigma_\zeta$ and $\Sigma_{\zeta'}$ are different, and (L4) is verified.

Consider next the case where $ab = 0$. Suppose $(a, b) = (1, 0)$. Then, for each $\zeta \in \Delta$, set $\Sigma_\zeta = \{(z, w) \in D_{1,0} \mid z = \zeta\}$. As in the proof of Lemma 3.2, we can show that $\{\Sigma_{\zeta \in \Delta}\}$ is a Liouville foliation on $D_{1,0}$. When $(a, b) = (0, 1)$, a similar construction gives a Liouville foliation on $D_{0,1}$ as well. q.e.d

An application of this lemma gives the following result on automorphisms.

**Theorem 4.1.** *According as the cases* (i) $ab = 0$ *and* (ii) $ab \neq 0$, *the automorphisms of $D_{a,b}$ are described as follows.*

(i) Aut $(D_{1,0})$ *consists of all transformations of the form*

$$D_{1,0} \ni (z, w) \mapsto (\tau(z), \lambda(z)w + \mu(z)) \in D_{1,0},$$

*where $\tau \in \mathrm{Aut}(\Delta)$, $\lambda$ is a nowhere-vanishing holomorphic function on $\Delta$, and $\mu$ is a holomorphic function on $\Delta$. Also, $\mathrm{Aut}(D_{0,1})$ is given by $\mathrm{Aut}(D_{0,1}) = \sigma \mathrm{Aut}(D_{1,0})\sigma^{-1}$, where*

$$\sigma : D_{1,0} \ni (z, w) \mapsto (w, z) \in D_{0,1}.$$

(ii) *If $G(D_{a,b})$ denotes the subgroup of $\mathrm{Aut}(D_{a,b})$ consisting of all transformations of the form*

$$D_{a,b} \ni (z, w) \mapsto (\lambda(z^a w^b)^{-b} z, \lambda(z^a w^b)^a w) \in D_{a,b},$$

*where $\lambda$ is a nowhere-vanishing holomorphic function on $\Delta$, then*

$$\mathrm{Aut}(D_{a,b}) = G(D_{a,b}) \cdot \mathrm{Aut}_{\mathrm{alg}}(D_{a,b}).$$

Proof. Consider first the case (i). To prove the first assertion, let $\{\Sigma_\zeta\}_{\zeta \in \Delta}$

be the Liouville foliation on $D_{1,0}$ given in Lemma 4.1. If $f$ is any element of $\mathrm{Aut}(D_{1,0})$, then, by Proposition 2.1, there eixsts a bijective correspondence $\tau$: $\Delta \to \Delta$ between $\Delta$ and itself such that $f(\Sigma_\zeta) = \Sigma_{\tau(\zeta)}$ for every $\zeta \in \Delta$. This implies that $f$ can be written in the form

$$f: D_{1,0} \ni (z, w) \mapsto (\tau(z), \theta(z, w)) \in D_{1,0},$$

where $\theta(z, w)$ is a holomorphic function on $D_{1,0}$. As a consequence, the mapping $\tau: \Delta \to \Delta$ is holomorphic. Since $\tau$ is bijective, $\tau$ is an automorphism of $\Delta$.

To determine $\theta(z, w)$, let $\theta(z, w) = \sum\limits_{n=0}^{\infty} \gamma_n(z) w^n$ be the Taylor expansion of $\theta(z, w)$ with respect to $w$, where $\gamma_n(z)$, $n = 0, 1, 2, \cdots$, are holomorphic functions on $\Delta$. Fix any point $\zeta$ of $\Delta$. It follows from the relation $f(\Sigma_\zeta) = \Sigma_{\tau(\zeta)}$ that the mapping $\boldsymbol{C} \ni w \mapsto \theta(\zeta, w) \in \boldsymbol{C}$ gives a biholomorphic mapping of $\boldsymbol{C}$ onto itself, and hence that $\theta(\zeta, w) = \gamma w + \delta$ for constants $\gamma \in \boldsymbol{C}^*$ and $\delta \in \boldsymbol{C}$. By the uniqueness of the Taylor expansion, we have $\gamma_n(\zeta) = 0$ for all $n \neq 0, 1$, while we have $\gamma_1(\zeta) \neq 0$. Since this holds for every $\zeta \in \Delta$, we see that $\theta(z, w) = \lambda(z) w + \mu(z)$ for a nowhere-vanishing holomorphic function $\lambda$ on $\Delta$ and a holomorphic function $\mu$ on $\Delta$.

The results of the preceding paragraphs conclude the first assertion of (i). For the second assertion of (i), it is enough to observe that $\sigma$ gives a biholomorphic mapping between $D_{1,0}$ and $D_{0,1}$.

Consider next the case (ii). Let $\{\Sigma_\zeta\}_{\zeta \in \Delta}$ be the Liouville foliation on $D_{a,b}$ given in Lemma 4.1. If $f$ is any element of $\mathrm{Aut}(D_{a,b})$, then, by Proposition 2.1, there exists a bijective correspondence $\tau: \Delta \to \Delta$ between $\Delta$ and itself such that $f(\Sigma_\zeta) = \Sigma_{\tau(\zeta)}$ for every $\zeta \in \Delta$. As a consequence, $\Sigma_0$ and $\Sigma_{\tau(0)}$ must be holomorphically equivalent. As we saw in the proof of Lemma 4.1, for every $\zeta \in \Delta^*$, the analytic subset $\Sigma_\zeta$ is non-singular, while the analytic subset $\Sigma_0 = \{(z, w) \in \boldsymbol{C}^2 \mid zw = 0\}$ is singular. Therefore we must have $\Sigma_{\tau(0)} = \Sigma_0$, so that $f(\Sigma_0) = \Sigma_0$. Since $D_{a,b}$ is the disjoint union of $D_{a,b}^*$ and $\Sigma_0$, this implies that $f(D_{a,b}^*) = D_{a,b}^*$, and hence that the restriction $f^*$ of $f$ to $D_{a,b}^*$ gives an automorphism of $D_{a,b}^*$.

By Proposition 3.2, there exist an element $f'$ of $G(D_{a,b}^*)$ written in the form (3.2) and an element $f''$ of $\mathrm{Aut}_{\mathrm{alg}}(D_{a,b}^*)$ written in the form (3.3) such that $f^* = f' \circ f''$. Using (3.4) and (3.5), we see that $f^*$ is written as

(4.1)
$$\begin{aligned} f^*: D_{a,b}^* \ni (z, w) \mapsto \\ (\alpha(\lambda \circ \pi)(z^a w^b)^{-b} z^p w^q, \ \beta(\lambda \circ \pi)(z^a w^b)^a z^r w^s) \in D_{a,b}^*, \end{aligned}$$

where $\pi$ denotes the automorphism of $\Delta^*$ given by $\pi(\zeta) = \alpha^a \beta^b \zeta$. Note that $\lambda \circ \pi$ is a nowhere-vanishing holomorphic function on $\Delta^*$. Since $f^*$ has the holomorphic extension $f$ from $D_{a,b}^*$ to $D_{a,b}$, it follows that $(\lambda \circ \pi)(\zeta)$ has at most pole at $\zeta = 0$, and hence that $(\lambda \circ \pi)(\zeta) = \zeta^k \lambda^*(\zeta)$ for an integer $k$ and a nowhere-

vanishing holomorphic function $\lambda^*$ on $\Delta$. Substituting this into (4.1), we have

$$(4.2) \qquad f^*(z, w) = (\alpha\lambda^*(z^a w^b)^{-b} z^{p'} w^{q'}, \ \beta\lambda^*(z^a w^b)^a z^{r'} w^{s'}),$$

where $p' = -kab+p$, $q' = -kb^2+q$, $r' = ka^2+r$, and $s' = kab+s$. If we define an element $g$ of $G(D_{a,b})$ by

$$g(z, w) = ((\lambda^* \circ \pi^{-1})(z^a w^b)^b z, \ (\lambda^* \circ \pi^{-1})(z^a w^b)^{-a} w),$$

where $\pi$ is regarded as an element of $\mathrm{Aut}(\Delta)$, then, by (4.2) and (3.5), for $(z, w) \in D^*_{a,b}$,

$$(g \circ f)(z, w) = (g \circ f^*)(z, w)$$
$$= (\alpha z^{p'} w^{q'}, \ \beta z^{r'} w^{s'}).$$

Since $g \circ f \in \mathrm{Aut}(D_{a,b})$ and since $D_{a,b}$ contains the origin, it follows from Lemma 1.1 that $g \circ f \in \mathrm{Aut}_{\mathrm{alg}}(D_{a,b})$. If we write $g' = g \circ f$, then $f = g^{-1} \circ g'$, where $g^{-1} \in G(D_{a,b})$ and $g' \in \mathrm{Aut}_{\mathrm{alg}}(D_{a,b})$. This proves (ii). q.e.d.

In view of the observations made at the beginning of this section, our theorem stated in the introduction is an immediate consequence of Theorem 4.2 below.

**Theorem 4.2.** *If $D_{a,b}$ and $D_{u,v}$ are holomorphically equivalent, then they are algebraically equivalent under the identity transformation or the transformation of the form*

$$C^2 \ni (z, w) \mapsto (w, z) \in C^2 .$$

Proof. Let $\varphi: D_{u,v} \to D_{a,b}$ be a biholomorphic mapping of $D_{u,v}$ onto $D_{a,b}$, and let $\{\Sigma_\zeta\}_{\zeta \in \Delta}$ and $\{\Sigma'_{\zeta'}\}_{\zeta' \in \Delta}$ be the Liouville foliations on $D_{a,b}$ and $D_{u,v}$ given in Lemma 4.1, respectively. By Proposition 2.1, there exists a bijective correspondence $\tau: \Delta \to \Delta$ between $\Delta$ and itself such that $\varphi(\Sigma'_{\zeta'}) = \Sigma_{\tau(\zeta')}$ for every $\zeta' \in \Delta$.

Suppose first $ab = 0$. If $uv \neq 0$, then the analytic subset $\Sigma'_0$ is singular. Since, for every $\zeta \in \Delta$, the analytic subset $\Sigma_\zeta$ is non-singular, this contradicts the relation $\varphi(\Sigma'_0) = \Sigma_{\tau(0)}$. We thus conclude that $uv = 0$, and our assertion follows immediately.

Suppose next $ab \neq 0$. Then, arguing as in the preceding paragraph, we see that $uv \neq 0$. Since the analytic subset $\Sigma'_0$ is singular, the analytic subset $\Sigma_{\tau(0)} = \varphi(\Sigma'_0)$ is also singular. Note that, for every $\zeta \in \Delta^*$, the analytic subset $\Sigma_\zeta$ is non-singular. Therefore we must have $\Sigma_{\tau(0)} = \Sigma_0$, so that $\varphi(\Sigma'_0) = \Sigma_0$. Since $D_{u,v}$ is the disjoint union of $D^*_{u,v}$ and $\Sigma'_0$, while $D_{a,b}$ is the disjoint union of $D^*_{a,b}$ and $\Sigma_0$, this implies that $\varphi(D^*_{u,v}) = D^*_{a,b}$, and hence that the restriction $\varphi^*$ of $\varphi$ to $D^*_{u,v}$ gives a biholomorphic mapping of $D^*_{u,v}$ onto $D^*_{a,b}$.

By Proposition **3.1**, there exists a biholomorphic mapping $\psi': D^*_{u,v} \to D^*_{a,b}$ of $D^*_{u,v}$ onto $D^*_{a,b}$ induced by an algebraic automorphism of $(\boldsymbol{C}^*)^2$. Hence $\varphi^*$ can be written in the form $\varphi^* = f \circ \psi'$, where $f \in \mathrm{Aut}(D^*_{a,b})$. According to Proposition 3.2, we write $f = f' \circ f''$, where $f' \in G(D^*_{a,b})$ and $f'' \in \mathrm{Aut}_{\mathrm{alg}}(D^*_{a,b})$. Then we have $\varphi^* = f' \circ f'' \circ \psi'$, and the biholomorphic mapping $f'' \circ \psi'$ of $D^*_{u,v}$ onto $D^*_{a,b}$ is induced by an algebraic automorphism of $(\boldsymbol{C}^*)^2$.

Note that if, for constants $\alpha$, $\beta \in \boldsymbol{C}^*$ with $|\alpha|^a |\beta|^b = 1$, we set the transformation $\pi$ as

$$\pi : \boldsymbol{C}^2 \ni (z, w) \mapsto (\alpha z, \beta w) \in \boldsymbol{C}^2,$$

which is an automorphism of $D^*_{a,b}$ as well as an automorphism of $D_{a,b}$, and if $h$ is any elemet element of $G(D^*_{a,b})$, then $\pi \circ h = h' \circ \pi$ for some $h' \in G(D^*_{a,b})$. Therefore, replacing $\varphi$ by $\pi \circ \varphi$ for a suitable transformation $\pi$ of this form, we may assume that $\psi := f'' \circ \psi'$ is given as in the remark after Proposition 3.1.

Write $f'$ in the form (3.2). Using (3.1), we see that $\varphi^*$ is written as

(4.3)
$$\begin{aligned} \varphi^* : D^*_{u,v} \ni (z, w) \mapsto \\ (\lambda(z^a w^b)^{-b} z^p w^q, \ \lambda(z^a w^b)^a z^r w^s) \in D^*_{a,b} . \end{aligned}$$

Note that $\lambda$ is a nowhere-vanishing holomorphic function on $\Delta^*$. Since $\varphi^*$ has the holomorphic extension $\varphi$ from $D^*_{u,v}$ to $D_{u,v}$, it follows that $\lambda(\zeta)$ has at most pole at $\zeta = 0$, and hence that $\lambda(\zeta) = \zeta^k \lambda^*(\zeta)$ for rn integer $k$ and a nowhere-vanishing holomorphic function $\lambda^*$ on $\Delta$. Substituting this into (4.3), we have

(4.4)
$$\varphi^*(z, w) = (\lambda^*(z^a w^b)^{-b} z^{p'} w^{q'}, \ \lambda^*(z^a w^b)^a z^{r'} w^{s'}) ,$$

where $p' = -kab + p$, $q' = -kb^2 + q$, $r' = ka^2 + r$, and $s' = kab + s$. If we define an element $g$ of $G(D_{a,b})$ by

$$g(z, w) = (\lambda^*(z^a w^b)^b z, \ \lambda^*(z^a w^b)^{-a} w) ,$$

then, by (4.4) and (3.1), for $(z, w) \in D^*_{u,v}$,

$$\begin{aligned} (g \circ \varphi)(z, w) &= (g \circ \varphi^*)(z, w) \\ &= (z^{p'} w^{q'}, z^{r'} w^{s'}) . \end{aligned}$$

Since $g \circ \varphi$ is a biholomorphic mapping of $D_{u,v}$ onto $D_{a,b}$ and since both $D_{u,v}$ and $D_{a,b}$ contain the origin, it follows from Lemma 1.1 that $g \circ \varphi$ is either the identity transformation or the transformation of the form

$$\boldsymbol{C}^2 \ni (z, w) \mapsto (w, z) \in \boldsymbol{C}^2.$$

This completes the proof of Theorem 4.2.

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