Morioka, T.
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HYPOELLIPTICITY FOR SEMI-ELLIPTIC OPERATORS WHICH DEGENERATE ON HYPERSURFACE

TATSUSHI MORIOKA

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1. Introduction and results

Let us denote a coordinate of $T^*(\boldsymbol{R}^n)$ by the following notation:

$$T^*(\boldsymbol{R}^n) = \{(x, y; \xi, \eta) \colon x, \xi \in \boldsymbol{R}^{n_1} \quad \text{and} \quad y, \eta \in \boldsymbol{R}^{n_2}\} .$$

Here $n=n_1+n_2$. In this paper, we shall study the hypoellipticity of semi-elliptic operators in $\boldsymbol{R}^n$ which degenerate at $x=0$. It is well known that non-degenerate semi-elliptic operators are hypoelliptic. For the definition of semi-elliptic operators, see Kumano-go [5, p.85]. We consider a differential operator of the form

$$L = a(x, y, D_x) + g(x)\, b(x, y, D_y) \quad \text{in } \boldsymbol{R}^n = \boldsymbol{R}^{n_1}_x \times \boldsymbol{R}^{n_2}_y , \tag{1.1}$$

satisfying the following conditions. (Throughout this paper, the coefficients of differential operators are assumed to be functions of the class C^∞.)

(A.1) $g(0)=0$ and $g(x)>0$ for $x \neq 0$.
(A.2) $a(x, y, D_x)$ is a differential operator of order $2l$ and

$$\operatorname{Re} a(x, y, \xi) \geqq C_1 |\xi|^{2l}$$

holds for sufficiently large $|\xi|$.
(A.3) $b(x, y, D_y)$ is a differential operator of order $2m$ and

$$\operatorname{Re} b(x, y, \eta) \geqq C_2 |\eta|^{2m}$$

holds for sufficiently large $|\eta|$. Here C_1 and C_2 are positive constants and l, m are positive integers.

Our main result is the following:

Theorem 1. *Let L be an operator of the form* (1.1) *satisfying* (*A*.1)–(*A*.3). *Then L is hypoelliptic, i.e.,*

$$\text{sing supp } Lu = \text{sing supp } u \quad \text{for} \quad u \in \mathcal{D}' .$$

Taniguchi [12] showed that L is hypoelliptic if $g(x)$ is non-negative and for any x^0 there exists a multi-index α suth that $D_x^\alpha g(x^0) \neq 0$, under the assumptions (A.2) and (A.3). Morimoto [7] showed that L is hypoelliptic if L satisfies (A.1)—(A.3) and the following condition (G).

(G) For any multi-index β there exists a constant C such that

$$|D_x^\beta g(x)| \leqq C g(x)^{1-\sigma|\beta|} \quad \text{near} \quad x = 0\,,$$

where σ satisfies $0<\sigma<1/(2lm+2m-2l)$.

Theorem 1 implies that (G) can be eliminated for L to be hypoelliptic.

Next we study the hypoellipticity of the following operator in $\boldsymbol{R}^3$

$$L = D_x^{2l} + g(x)\, D_y^{2m} + D_t^{2k} \tag{1.2}$$

satisfying (A.1), where l, m and k are positive integers. Note that the following operator in $\boldsymbol{R}^2$

$$L_0 = D_x^{2l} + g(x)\, D_y^{2m}$$

satisfying (A.1) is hypoelliptic, in view of Theorem 1 (Cf. Theorem 5 in Fediĭ [1]). But it is known that for L to be hypoelliptic, we have to restrict the vanishing order of $g(x)$ at $x=0$. The following theorem is about a sufficient condition for the hypoellipticity.

Theorem 2. *Let L be an operator of the form* (1.2) *satisfying* (*A*.1). *Assume moreover that*

$$\lim_{x\to 0} |x|^{l/k} |\log g(x)| = 0\,. \tag{A.4}$$

Then L is hypoelliptic.

In the case where $l=m=k$, the hypoellipticity of L was studied in Morimoto [9]. Hoshiro [4] studied more general case. Morimoto [9] showed that L is hypoelliptic under the assumption $l=m=k$ if $g(x)$ satisfies (A.1), (A.4) and the following condition (G)'.

(G)' For any j there exists a constant C such that

$$|g^{(j)}(x)| \leqq C g(x)^{1-\sigma j} \quad \text{near} \quad x = 0\,,$$

where $g^{(j)}(x) = D_x^j g(x)$ and σ satisfies $0<\sigma<1/(2l^2)$.

Theorem 2 implies that (G)' can be eliminated for L to be hypoelliptic.

We also remark that a certain condition which is almost complementary to (A.4) is also sufficient for the non-hypoellipticity of L. Indeed, Morimoto [8] has shown that L is not hypoelliptic if $g(x) \geqq 0$ and

$$\liminf_{x\to 0} |x|^{l/k} |\log g(x)| \neq 0\,. \tag{A.5}$$

EXAMPLE 1. Let σ be a positive constant. Theorem 2 and the condition (A.5) show that the operator

$$L = D_x^{2l} + e^{-|x|^{-\sigma}} D_y^{2m} + D_t^{2k}$$

is hypoelliptic if and only if $\sigma < l/k$ (Cf. Proposition 3.1 in [9]).

The plan of this paper is as follows. In Section 2, we introduce Sobolev spaces which are necessary to prove Theorem 1. In Section 3, we explain our microlocal energy method and complete the proof of Theorem 1. The proof of Theorem 2 will be given in Sections 4 and 5. Finally in Section 6, we prove the lemma in Section 3.

2. Preliminaries

We begin this section by preparing the following Sobolev spaces which are necessary for the proof of Theorem 1.

DEFINITION. We denote by $H^{d,j}(-\infty<d,j<\infty)$ the space of all distributions $u \in \mathcal{S}'(\boldsymbol{R}^n)$ satisfying

$$\iint |\hat{u}(\xi, \eta)|^2 \langle\xi\rangle^{2d} \langle\eta\rangle^{2j} \, d\xi \, d\eta < \infty$$

($\xi \in \boldsymbol{R}^{n_1}$, $\eta \in \boldsymbol{R}^{n_2}$), where $\hat{u}$ is the Fourier transform of u and $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. $u \in H^{0,\infty}$ means $u \in H^{0,k}$ for any k.

Furthermore we say that a distribution u is locally of the class $H^{d,j}$ at (x^0, y^0) ($x^0 \in \boldsymbol{R}^{n_1}_x$, $y^0 \in \boldsymbol{R}^{n_2}_y$) if there exists a function $\phi \in C_0^\infty(\boldsymbol{R}^n)$ with $\phi = 1$ in a neighborhood of (x^0, y^0) such that $\phi u \in H^{d,j}$.

Now let L be an operator of the form (1.1) satisfying (A.1)–(A.3), then we have the following proposition.

Proposition 2.1. *If $Lu \in C^\infty$ and $u \in H^{0,j}$ at $(0, y^0)$, then it follows that $u \in H^{0,\infty}$ at $(0, y^0)$.*

Proposition 2.2. *In the case where $Lu \in C_0^\infty$ at $(0, y^0)$, there exists a function $\phi(x, y) \in C_0^\infty(\boldsymbol{R}^n)$ with $\phi = 1$ near $x=0$ such that for any M*

$$\iint_\Omega |\widehat{\phi u}(\xi, \eta)|^2 \langle\xi, \eta\rangle^{2M} \, d\xi \, d\eta < \infty \,, \tag{2.1}$$

where $\Omega = \{(\xi, \eta) \in \boldsymbol{R}^n;\ |\xi|^\rho \geqq |\eta|\}$ with $\rho = \min(1, l/(m+1))$.

Theorem 1 follows from Proposition 2.1 and 2.2. Indeed, it follows from (2.1) that $u \in H^{0,j}$ at $(0, y^0)$ for some j. Note that $2|\eta|^{2/\rho} \geqq |\xi|^2 + |\eta|^2$ for $(\xi, \eta) \notin \Omega$, $|\eta| \geqq 1$. Then from Proposition 2.1 we see that $u \in H^{0,\infty}$ at $(0, y^0)$. Therefore we have

$$\iint_{\boldsymbol{R}^n\setminus\Omega} |\widehat{\phi u}(\xi, \eta)|^2 \langle \xi, \eta \rangle^{2M}\, d\xi\, d\eta < \infty \tag{2.2}$$

for any M. From (2.1) and (2.2) we see that $u \in C^\infty$ at $(0, y^0)$.

The remaining part of this section is devoted to the proof of Proposition 2.2. We say that $r(x, y, \xi, \eta) \in C^\infty(\boldsymbol{R}^n_{x,y} \times \boldsymbol{R}^n_{\xi,\eta})$ belongs to $S^j_{\rho,\delta}$ if for any multi-indices α and β there exists a constant $C_{\alpha\beta}$ suth that

$$|r^{(\alpha)}_{(\beta)}(x, y, \xi, \eta)| \leqq C_{\alpha\beta} \langle \xi, \eta \rangle^{j-\rho|\alpha|+\delta|\beta|},$$

where $r^{(\alpha)}_{(\beta)} = \partial^\alpha_{\xi,\eta} D^\beta_{x,y} r$. Furthermore we say that the pseudo-differential operator R belongs to $OPS^j_{\rho,\delta}$ if the symbol of R belongs to $S^j_{\rho,\delta}$.

Now take four non-negative functions $\varphi_k(t) \in C^\infty_0(\boldsymbol{R})$, $k=1, 2, 3, 4$ with $\varphi_k = 1$ in $\{|t| \leqq 9-2k\}$ and $\varphi_k = 0$ in $\{|t| \geqq 10-2k\}$. Take moreover four non-negative functions $\lambda_k \in C^\infty(\boldsymbol{R})$, $k=1, 2, 3, 4$ with $\lambda_k = 1$ in $\{|t| \geqq 2k+1\}$ and $\lambda_k = 0$ in $\{|t| \leqq 2k\}$. We define $\psi_k(\xi, \eta)$ by

$$\psi_k(\xi, \eta) = \varphi_k(|\xi|^{-\rho}|\eta|)\, \lambda_k(|\xi|^2 + |\eta|^2),$$

where $\rho = \min(1, l/(m+1))$. It is easy to check that $\psi_{k+1} \Subset \psi_k$ $(k=1, 2, 3)$, i.e., the support of ψ_{k+1} is contained in a neighborhood of the closed set where $\psi_k = 1$. We define the pseudo-differential operator Ψ_k by $\Psi_k = \psi_k(D_y, D_y)$. Take $q = \psi_1/p$, where p denotes the symbol of L. Then the pseudo-differential operator R is defined as follows:

$$R = QL - \Psi_1, \tag{2.3}$$

where $Q = q(x, y, D_x, D_y)$. Now we see that $R \in OPS^{-\rho}_{\rho,0}$. Indeed, the asymptotic expansion gives

$$\sigma_{QL} - \psi_1 \sim \sum_{|\gamma| \geqq 1} (\gamma!)^{-1} q^{(\gamma)} p_{(\gamma)},$$

where σ_{QL} denotes the symbol of QL. Observing that $\psi_k \in S^{-\rho}_{\rho,0}$ and

$$|p^{(\alpha)}_{(\beta)}(x, y, \xi, \eta)| \leqq C_{\alpha\beta} \langle \xi, \eta \rangle^{2l-\rho|\alpha|}$$

in the support of ψ_1 together with the assumption (A.2), we have $q^{(\gamma)} p_{(\gamma)} \in S^{-\rho|\gamma|}_{\rho,0}$. Therefore it follows that $R \in OPS^{-\rho}_{\rho,0}$. From (2.3) we have

$$\Psi_2\, QL = \Psi_2(I+R). \tag{2.4}$$

Since $R \in OPS^{-\rho}_{\rho,0}$, we can define $E \in OPS^0_{\rho,0}$ to have the following asymptotic expansion:

$$E \sim I + \sum_{j=1}^\infty (-1)^j R^j \tag{2.5}$$

and then,

$$E(I+R)-I\in OPS^{-\infty}\,. \tag{2.6}$$

Now from (2.4) we see that

$$\Psi_3 E\Psi_2\, QL = \Psi_3 E(I+R)-\Psi_3 E(1-\Psi_2)\,(I+R)\,. \tag{2.7}$$

Since $\psi_3 \Subset \psi_2$, it follows that $\Psi_3 E(1-\Psi_2)\in OPS^{-\infty}$. Then in view of (2.6) and (2.7), we have

$$BL = \Psi_3 + K \tag{2.8}$$

with $B=\Psi_3 E\Psi_2 Q$ and $K\in OPS^{-\infty}$.

REMARK. Q belongs to $OPS^{-2l}_{\rho,0}$ and so does B.

Now we consider the case where $Lu\in C^\infty$ at $(0, y^0)$. We may assume $u\in\mathcal{E}'$. Let $\phi(x, y)\in C_0^\infty$ be a function with $\phi=1$ near $(0, y^0)$ such that $Lu\in C^\infty$ in a neighborhood of the support of ϕ. Then from (2.8) we have

$$\phi BLu = \phi\Psi_3 u+\phi Ku\,. \tag{2.9}$$

Since B has the pseudo-local property, we see that $\phi BLu\in C_0^\infty$. So we obtain $\phi\Psi_3 u\in C_0^\infty$. Moreover we have

$$\Psi_4(\phi u) = \Psi_4\phi\Psi_3 u+\Psi_4\phi(1-\Psi_3)\, u\,. \tag{2.10}$$

Since $\psi_4\Subset\psi_3$, we see that $\Psi_4(\phi u)\in H^\infty$. This implies (2.1) holds and the proof is completed.

The proof of Proposition 2.1 will be given in Section 3.

3. Microlocal energy method

In this section we give the proof of Proposition 2.1. Here we use the microlocal energy method of Mizohata [6].

First we define the microlocal smoothness of distributions as follows.

DEFINITION. Let $(x^0, y^0)\in \boldsymbol{R}^{n_1}_x\times\boldsymbol{R}^{n_2}_y$ and $\eta^0\in\boldsymbol{R}^{n_2}$ with $|\eta^0|=1$. For $u\in\mathcal{D}'$ we say that u is microlocally of the class $H^{0,\infty}$ at $(x^0, y^0; \eta^0)$ if there exists a function $\phi\in C_0^\infty(\boldsymbol{R}^n)$ with $\phi=1$ in a neighborhood of (x^0, y^0) and a conic neighborhood $\Gamma_0(\subset\boldsymbol{R}^{n_2})$ of η^0 such that

$$\iint_\Delta |\widehat{\phi u}(\xi, \eta)|^2\langle\eta\rangle^{2s}\, d\xi\, d\eta<\infty$$

for any positive number s, where

$$\Delta = \boldsymbol{R}^{n_1}\times\Gamma_0 = \{(\xi, \eta)\colon \xi\in\boldsymbol{R}^{n_1}, \eta\in\Gamma_0\}\,.$$

REMARK. $u\in H^{0,\infty}$ at (x^0, y^0) if and only if u is microlocally of the class $H^{0,\infty}$

at $(x^0, y^0; \eta^0)$ for all $\eta \in \boldsymbol{R}^{n_2}$ with $|\eta^0|=1$.

Now let $\Phi \in C_0^\infty(\boldsymbol{R}^{n_2})$ be a function satisfying

$$\Phi = 1 \quad \text{in} \quad \{z: |z| \leqq r'\}$$

and

$$\Phi = 0 \quad \text{in} \quad \{z: |z| \geqq r\} .$$

Here we assume that $0<r'<r<1$.

Our microlocalizers $\{\alpha_n(\eta), \beta(y)\}$ are defined in such a way that

$$\alpha_n(\eta) = \Phi(\frac{\eta}{n}-\eta^0), \quad \beta(y) = \Phi(y-y^0),$$

where n is a positive integer.

Our microlocal energy is

$$S_{N,n}\, u = \sum_{|p+q|\leqq N} \|c_{pq}^n\, \alpha_n^{(p)}(D_y)\,(\beta_{(q)}\, u)\|_{L^2}^2, \quad u \in \mathcal{S}'(\boldsymbol{R}^n)$$

with $c_{pq}^n = n^{(1/2)(|p|-|q|)}$.

REMARK. $\alpha_n^{(p)}$ and $\beta_{(q)}$ denote $\partial_\eta^p\, \alpha_n$ and $D_y^q\, \beta$, respectively.

Now we have the following lemma whose proof will be given in Section 6.

Lemma 3.1. *Let $u \in H^{0,j}$ at $(0, y^0)$ for some j. Then $u \in H^{0,\infty}$ at $(0, y^0; \eta^0)$ if and only if there exists a function $\chi(x) \in C_0^\infty(\boldsymbol{R}_x^{n_1})$ with $\chi=1$ in a neighborhood of $x=0$ such that*

$$S_{N,n}(\chi u) = O(n^{-2s})$$

when $n \to \infty$, for any fixed N and s.

REMARK. In general, $c_n = O(n^{-k})$ means that there exists a constant B such that $|c_n| \leqq Bn^{-k}$ when n is large.

Let us now begin the proof of Proposition 2.1. The semi-ellipticity of L except at $x=0$ enables us to know that the right hand side of the equation

$$\psi L(\chi u) = \psi\,[a(x, y, D_x), \chi(x)]\, u + \chi \psi L u$$

is of the class C_0^∞ if Lu is of the class C^∞ at $(0, y^0)$. Here we supposed that $\chi(x) \in C_0^\infty(\boldsymbol{R}^{n_1})$ and $\psi(y) \in C_0^\infty(\boldsymbol{R}^{n_2})$ have their supports in small neighborhoods of $x=0$ and $y=y^0$, respectively. So we can write

$$\psi L v = h, \tag{3.1}$$

where $v=\chi u$ and $h \in C_0^\infty$. In view of Lemma 3.1, our aim is to show that

$S_{N,n}\, v=O(n^{-2s})$ for any N and s. We choose $r>0$ sufficiently small such that $\beta\subset\psi$. Let us operate $\alpha_n^{(p)}\,\beta_{(q)}$ to the both sides of (3.1), namely,

$$\alpha_n^{(p)}\,\beta_{(q)}\,Lv=\alpha_n^{(p)}\,\beta_{(q)}\,h\,. \tag{3.2}$$

The asymptotic expansion gives

$$\begin{aligned}\operatorname{Re}(Lv_{n,p,q}, v_{n,p,q}) &= \operatorname{Re}(-[\alpha_n^{(p)}\,\beta_{(q)}, L]\,v+h_{n,p\,q}, v_{n,p,q})\\ &= \operatorname{Re}\sum_{k=1}^{5} I_k\end{aligned} \tag{3.3}$$

with

$$\begin{aligned}
&v_{n,p,q}=\alpha_n^{(p)}\,\beta_{(q)}\,v\,,\quad h_{n,p,q}=\alpha_n^{(p)}\,\beta_{(q)}\,h\,,\\
&I_1=-\sum_{1\leq|\mu+\nu|\leq N-|p+q|}(-1)^{|\nu|}(\nu!)^{-1}(\mu!)^{-1}(g(x)\,b_{(\mu)}^{(\nu)}\,v_{n,p+\mu,q+\nu}, v_{n,p,q})\,,\\
&I_2=-(g(x)\,r_{n,p,q}^{N}\,v, v_{n,p,q})\,,\\
&I_3=-\sum_{1\leq|\mu|\leq N-|p+q|}(\mu!)^{-1}(a_{(\mu)}\,v_{n,p+\mu,q}, v_{n,p,q})\,,\\
&I_4=-(\tau_{n,p,q}^{N}\,v, v_{n,p,q})\\
&I_5=(h_{n,p,q}, v_{n,p,q})\,,
\end{aligned}$$

where N is a large integer whose definition will be given later.

REMARK. $([\alpha_n^{(p)}\,\beta_{(q)}, a]\,v, v_{n,p,q})=I_3+I_4$ and $([\alpha_n^{(p)}\,\beta_{(q)}, g(x)\,b]\,v, v_{n,p,q})=I_1+I_2$.

Now we are going to estimate each $|I_k|$. From Gårding's inequality, it follows that

$$||\Lambda_y^m\,w||^2\leq C_1\operatorname{Re}(bw, w)+C_2\,||w||^2\quad\text{for}\quad w\in\mathcal{S}(\boldsymbol{R}^n)\,,$$

where C_1 and C_2 are positive constants.

Taking $w=g(x)^{1/2}\,v_{n,p,q}$ and noticing that $c^{-1}\,n\leq|\eta|\leq cn$ for $\eta\in\operatorname{supp}\alpha_n$, we have the following estimate:

$$\begin{aligned}&n^{2m}\,||g(x)^{1/2}\,v_{n,p,q}||^2\\ &\qquad\leq C_3\operatorname{Re}(g(x)\,bv_{n,p,q}, v_{n,p\,q})+C_4\,||v_{n,p,q}||^2\,.\end{aligned} \tag{3.4}$$

On the other hand, we see that for any $K>0$

$$\begin{aligned}&|(b_{(\mu)}^{(\nu)}\,g(x)^{1/2}\,v_{n,p+\mu,q+\nu}, g(x)^{1/2}\,v_{n,p,q})|\\ &\qquad\leq Kn^{-2m}\,||b_{(\mu)}^{(\nu)}\,g(x)^{1/2}\,v_{n,p+\mu,q+\nu}||^2+K^{-1}\,n^{2m}\,||g(x)^{1/2}\,v_{n,p,q}||^2\,.\end{aligned} \tag{3.5}$$

Noticing that the order of $b_{(\mu)}^{(\nu)}$ is $2m-|\nu|$, we obtain from (3.4) and (3.5)

$$\begin{aligned}|I_1|\leq&\sum_{|1\leq\mu+\nu|\leq N-|p+q|}\{K^{-1}(C_5\operatorname{Re}(g(x)\,bv_{n,p,q}, v_{n,p,q})\\ &+C_6\,||v_{n,p,q}||^2)+Kn^{-2|\nu|}(C_7\operatorname{Re}(g(x)\,bv_{n,p+\mu,q+\nu}, v_{n,p+\mu,q+\nu})\\ &+C_8\,||v_{n,p+\mu,q+\nu}||^2)\}\,.\end{aligned} \tag{3.6}$$

Since Gårding's inequality is also applicable to $a(x, y, D_x)$, we obtain

$$
\begin{aligned}
|I_3| \leqq \sum_{1 \leqq |\mu| \leqq N-|p+q|} \{ & K^{-1}(C_9 \operatorname{Re}(av_{n,p,q}, v_{n,p,q}) \\
& + C_{10} \|v_{n,p,q}\|^2) + K(C_{11} \operatorname{Re}(av_{n,p+\mu,q}, v_{n,p+\mu,q}) \\
& + C_{12} \|v_{n,p+\mu,q}\|^2)\} .
\end{aligned} \tag{3.7}
$$

We estimate $|I_2|$, $|I_4|$ and $|I_5|$ in the following way:

$$|I_2| \leqq K \|r^N_{n,p,q}\, v\|^2 + K^{-1} \|v_{n,p,q}\|^2 , \tag{3.8}$$

$$|I_4| \leqq K \|\tau^N_{n,p,q}\, v\|^2 + K^{-1} \|v_{n,p,q}\|^2 , \tag{3.9}$$

$$|I_5| \leqq K \|h_{n,p,q}\|^2 + K^{-1} \|v_{n,p,q}\|^2 . \tag{3.10}$$

Writing the symbol of $r^N_{n,p,q}$ by an oscillatory integral together with the fact that $c^{-1} n \leqq |\eta| \leqq cn$ for $\eta \in \operatorname{supp} \alpha_n$, and introducing the partition of unity in the η-space, we see that

$$\|r^N_{n,p,q}\|_{H^{0,j} \to L^2} \leqq \text{const.}\ |r^N_{n,p,q}|^{(j)}_k \leqq \text{const.}\ n^{-N+|q|+2m-j-1} ,$$

where $|\omega|^{(j)}_k$ denotes a seminorm in $S^j_{1,0}(\boldsymbol{R}^{n_2})$, i.e.,

$$|\omega|^{(j)}_k = \max_{|\mu+\nu| \leqq k} \sup |\omega^{(\nu)}_{(\mu)}(y, \eta)| \langle \eta \rangle^{-j+|\nu|} .$$

For detail, [6, p.58]. Therefore we have

$$\|c^n_{pq}\, r^N_{n,p,q}\, v\| \leqq c^n_{pq} \|r^N_{n,p,q}\|_{H^{0,j} \to L^2} \|\psi v\|_{H^{0,j}} \leqq \text{const.}\ n^{-(1/2)N+2m-j-1}$$

for $|p+q| \leqq N$.

Then if we choose N sufficiently large such that $-\frac{1}{2} N + 2m - j - 1 < -s$, we obtain

$$\|c^n_{pq}\, r^N_{n,p,q}\, v\| = O(n^{-s}) \tag{3.11}$$

for $|p+q| \leqq N$. By the parallel way, we also obtain

$$\|c^n_{pq}\, \tau^N_{n,p,q}\, v\| = O(n^{-s}) \tag{3.12}$$

for $|p+q| \leqq N$ if we choose N sufficiently large.

Let us now observe that $c^n_{pq} = n^{-(1/2)(|\mu|-|\nu|)}\, c^n_{p+\mu,q+\nu}$, then from (3.3) and (3.6)–(3.12) we have for any $\varepsilon > 0$

$$
\begin{aligned}
& \frac{1}{2} \operatorname{Re}(aw_{n,p,q}, w_{n,p,q}) + \frac{1}{2} \operatorname{Re}(g(x)\, bw_{n,p,q}, w_{n,p,q}) \\
& \leqq K \sum_{1 \leqq |\mu+\nu| \leqq N-|p+q|} n^{-|\mu+\nu|} (C_{13} \operatorname{Re}(g(x)\, bw_{n,p+\mu,q+\nu}, w_{n,p+\mu,q+\nu}) \\
& \quad + C_{14} \|w_{n,p+\mu,q+\nu}\|^2)
\end{aligned} \tag{3.13}
$$

$$+K \sum_{1 \leq |\mu| \leq N-|p+q|} n^{-|\mu|}(C_{15} \operatorname{Re}(aw_{n,p+\mu,q}, w_{n,p+\mu,q})$$
$$+C_{16} \|w_{n,p+\mu,q}\|^2) + \varepsilon \|w_{n,q,q}\|^2$$
$$+K\|c^n_{pq} h_{n,p,q}\|^2 + O(n^{-2s}),$$

by taking K sufficiently large. Here $w_{n,p,q} = c^n_{pq} v_{n,p,q}$.

Summing up the both sides of (3.10) with respect to (p, q) satisfying $|p+q| \leq N$, then we see that

$$\frac{1}{2} A_{N,n} + \frac{1}{2} B_{N,n} \leq KC_{17} n^{-1} A_{N,n} + KC_{18} n^{-1} B_{N,n} + 2\varepsilon S_{N,n} v + KS_{N,n} h + O(n^{-2s}), \tag{3.14}$$

where

$$A_{N,n} = \sum_{|p+q| \leq N} \operatorname{Re}(aw_{n,p,q}, w_{n,p,q}),$$
$$B_{N,n} = \sum_{|p+q| \leq N} \operatorname{Re}(g(x)\, bw_{n,p,q}, w_{n,p,q}).$$

By Gårding's inequality, $B_{N,n} \geq 0$ holds for sufficiently large n. $S_{N,n} h = O(n^{-2s})$ since $h \in C_0^\infty$. So from (3.14) we have

$$\frac{1}{4} A_{N,n} \leq \varepsilon S_{N,n} v + O(n^{-2s}). \tag{3.15}$$

Recall that ε can be any positive number.

Lemma 3.2. *If we choose the supports of $\chi(x)$ sufficiently small, then we have*

$$A_{N,n} \geq \delta S_{N,n} v$$

for some $\delta > 0$.

Proof of Lemma 3.2. By Gårding's inequality, we have

$$\operatorname{Re}(aw, w) \geq K_1 \|\Lambda_x^l w\|^2 - K_2 \|w\|^2 \quad \text{for} \quad w \in \mathcal{S}(\boldsymbol{R}^n).$$

Taking $w = v_{n,p,q}$, we see that

$$\operatorname{Re}(av_{n,p,q}, v_{n,p,q}) \geq \delta \|v_{n,p,q}\|^2, \tag{3.16}$$

since we can write

$$v_{n,p,q}(x, y) = \chi(x)\, \alpha_n^{(p)}(D_y)\, (\beta_{(q)} \chi_1 u)(x, y),$$

where $\chi_1(x) \in C_0^\infty(\boldsymbol{R}^{n_1})$ and $\chi \Subset \chi_1$.

Multiplying the both sides of (3.16) by $(c^n_{pq})^2$ and summing up with respect to (p, q) satisfying $|p+q| \leq N$, we obtain the desired estimate. Q.E.D.

REMARK. From (2.1) and the definition of $\alpha_n(\eta)$, we see that $v_{n,p,q} \in H^\infty$.

Since the support of $v_{n,p,q}$ is compact with respect to the x variable, Poincaré's inequality is applicable.

From Lemma 3.2 and (3.15), it follows that

$$S_{N,n}\, v = O(n^{-2s})\,.$$

In view of Lemma 3.1, the proof of Proposition 2.1 is completed.

4. The uncertainty principle

The proof of Theorem 2 is devided into the following two propositions.

Proposition 4.1. *Let $g(x)$ be a smooth function satisfying (A.1) and (A.4). Then for any $\varepsilon>0$ there exists a constant $N>0$ such that*

$$\begin{aligned}(4.1)\qquad &(\log|\eta|)^{2k}\int|\varphi(x)|^2\,dx\\ &\leqq\varepsilon\left(\int|\varphi^{(l)}(x)|^2\,dx+\eta^{2m}\int g(x)|\varphi(x)|^2\,dx\right)\end{aligned}$$

for $\varphi\in C_0^\infty(-1,1)$ and $|\eta|\geqq N$. Here $\varphi^{(l)}(x)=\left(\frac{d}{dx}\right)^l\varphi(x)$.

Proposition 4.2. *Let L be an operator of the form (1.2). If $g(x)$ satisfies (4.1), then L is hypoelliptic.*

The proof of Proposition 4.2 will be given in the next Section. We shall devote the remaining part of this section to the proof of Proposition 4.1. We prepare the uncertainty principle of Morimoto [10]. Consider a symbol of the form

$$(4.2)\qquad p(x,\xi)=\xi^{2l}+V(x)\,,$$

where $V(x)\geqq 0$ belongs to $C^\infty(\boldsymbol{R})$. For $\delta>0$ and $x_0\in\boldsymbol{R}$, we denote by $B_\delta(x_0)$ a box

$$\{(x,\xi)\colon |x-x_0|\leqq\frac{1}{2}\delta,\ |\xi|\leqq\frac{1}{2}\delta^{-1}\}\,.$$

Clearly the volume of $B_\delta(x_0)$ is equal to 1.

Theorem A. *Let $p(x,\xi)$ be the above symbol and let M be a positive constant. Assume that there exists a constant $c>1-2^{-l+1}$ such that for any $\delta>0$ and any $x_0\in\boldsymbol{R}$,*

$$(4.3)\qquad \mu(\{(x,\xi)\in B_\delta(x_0)\colon p(x,\xi)\geqq M\})\geqq c\,,$$

where μ is Lebesgue measure on $\boldsymbol{R}^2$. Then there exists a constant $K>0$ which is *independent of M and $V(x)$ such that*

$$(4.4) \qquad (p(x, D)u, u) \geqq KM\|u\|^2 \quad for \quad u \in C_0^\infty(\boldsymbol{R}).$$

Theorem A is a paticular case of Theorem 1 in [10]. For the proof of Theorem A, see Section 1 and 2 of [10].

Now Proposition 4.1 follows from Theorem A. Indeed, taking $V(x)=\eta^{2m} g(x)$, i.e.,

$$p(x, \xi) = \xi^{2l}+\eta^{2m} g(x)$$

and $M=\varepsilon^{-1}(\log|\eta|)^{2k}$, by the assumption (A.4) we see that $p(x, \xi)$ and M satisfy (4.3), when $|\eta|$ is sufficiently large. It follows from Theorem A that (4.4) holds. Clearly, (4.4) implies (4.1). Thus the proof of Proposition 4.1 is completed.

5. Proof of Proposition 4.2

The proof of Proposition 4.2 is quite analogous to the one of Theorem 2 in [4]. In this section $H^{d,j}$ denotes a space of distributions in $\mathcal{S}'(\boldsymbol{R}^3)$ satisfying

$$\iiint |\hat{u}(\xi, \eta, \tau)|^2 \langle\xi\rangle^{2d} \langle\eta, \tau\rangle^{2j} \, d\xi \, d\eta \, d\tau < \infty,$$

where we denote a coordinate of $T^*(\boldsymbol{R}^3)$ by $(x, y, t; \xi, \eta, \tau)$. The local and microlocal smoothness of distributions are defined in the same way as in Sections 2 and 3. Now for the proof of Proposition 4.2, it suffices to show that u is microlocally of the class $H^{0,\infty}$ at $(0, y_0, t_0; \eta_0, \tau_0)$ for all $(\eta_0, \tau_0) \in \boldsymbol{R}^2$ with $\eta_0^2+\tau_0^2=1$, when $Lu \in C^\infty$ and $u \in H^{0,j}$ at $(0, y_0, t_0)$.

We end the preparation of the proof of Proposition 4.2 by recalling the microlocal energy method which Hoshiro has used in [3]. Choose first a sequnce $\Psi_N \in C_0^\infty(\boldsymbol{R}^2)$, $N=1, 2\ldots$ with

$$\Psi_N = 1 \quad \text{in} \quad \{(y, t): y^2+t^2 \leqq r'^2\}$$

and

$$\Psi_N = 0 \quad \text{in} \quad \{(y, t): y^2+t^2 \geqq r^2\}$$

satisfying

$$|D^{p+\mu}\Psi_N| \leqq C_\mu (CN)^{|p|} \quad \text{for} \quad |p| \leqq N,$$

where C_μ and C are constants which are independent of N. We supposed that $0<r'<r<1$. We define our microlocalizers as follows:

$$\alpha_n(\eta, \tau) = \Psi_{N_n}\left(\frac{\eta}{n}-\eta_0, \frac{\tau}{n}-\tau_0\right),$$

$$\beta_n(y, t) = \Psi_{N_n}(y-y_0, t-t_0)$$

with $N_n=[\log n]+1$.

Our microlocal energy of $v\in\mathcal{S}'(\boldsymbol{R}^3)$ is defined in such a way that

$$S_n^M v=\sum_{|p+q|\leqq N_n}\|c_{pq}^n\,\alpha_n^{(p)}(D_y, D_t)\,(\beta_{n(q)}\,v)\|_{L^2}^2$$

with $c_{pq}^n=n^{|p|}(M\log n)^{-|p+q|}$. Here $\alpha_n^{(p)}=\partial_\eta^{p_1}\,\partial_\tau^{p_2}\,\alpha_n$ and $\beta_{n(q)}=D_y^{q_1}\,D_t^{q_2}\,\beta_n$ with $p=(p_1, p_2)$ and $q=(q_1, q_2)$, respectively.

Then we have the following lemma whose proof was given in Section 6 of [3].

Lemma 5.1. *Let $u\in H^{0,j}$ at $(0, y_0, t_0)$ for some j. Then u is microlocally of the class $H^{0,\infty}$ at $(0, y_0, t_0; \eta_0, \tau_0)$ if and only if there exists a function $\chi(x)\in C_0^\infty(\boldsymbol{R})$ with $\chi=1$ in a neighborhood of $x=0$ such that for any positive number s, there exists a constant M such that*

$$S_n^M(\chi u)=O(n^{-2s}),$$

when n is large.

Proof of Proposition 4.2. Let $\chi(x)$ and $\psi(y, t)$ be smooth functions whose supports are contained in small neighborhoods of $x=0$ and $(y, t)=(y_0, t_0)$, respectively. The semi-ellipticity of L except at $x=0$ enables us to know that the right hand side of the equation

$$\psi L(\chi u)=\psi[D_x^{2k}, \chi(x)]\,u+\chi\psi Lu$$

is of the class C_0^∞ if Lu is of the class C^∞ at $(0, y_0, t_0)$. So we can write

$$\psi Lv=h, \tag{5.1}$$

where $v=\chi u$ and $h\in C_0^\infty$.

Assume that $|p+q|\leqq N_n$ and $r>0$ is chosen sufficiently small so that $\beta_n\Subset\psi$. Let us operate $\alpha_n^{(p)}\,\beta_{n(q)}$ to the both sides of (5.1), namely,

$$\alpha_n^{(p)}\,\beta_{n(q)}\,Lv=\alpha_n^{(p)}\,\beta_{n(q)}\,h. \tag{5.2}$$

Furthermore the asymptotic expansion gives

$$\begin{aligned}(Lv_{n,p,q}, v_{n,p,q})=&-\sum_{|\nu|\neq 0}(-1)^{|\nu|}(\nu!)^{-1}(L^{(\nu)}\,v_{n,p,q+\nu}, v_{n,p,q})\\&+(h_{n,p,q}, v_{n,p,q}),\end{aligned} \tag{5.3}$$

where $v_{n,p,q}=\alpha_n^{(p)}\,\beta_{n(q)}\,v$, $h_{n,p,q}=\alpha_n^{(p)}\,\beta_{n(q)}\,h$ and $L^{(\nu)}$ is the differential operator whose symbol is $\partial_{\eta,\tau}^\nu\,L(x; \xi, \eta, \tau)$.

Thus we have the following estimate:

$$\begin{aligned}(Lv_{n,p,q}, v_{n,p,q})\leqq&\sum_{|\nu|\neq 0}|(L^{(\nu)}\,v_{n,p,q+\nu}, v_{n,p,q})|\\&+\varepsilon\|v_{n,p,q}\|^2+\varepsilon^{-1}\|h_{n,p,q}\|^2,\end{aligned} \tag{5.4}$$

where $\varepsilon>0$ is an arbitrary constant. Moreover, multiplying the both sides of (5.4) by $(c^n_{pq})^2$, we obtain

$$(5.5)\qquad \begin{aligned}(Lw_{n,p,q}, w_{n,p,q}) \leqq \sum_{|\nu|\neq 0} (M\log n)^{|\nu|}\, |(L^{(\nu)} w_{n,p,q+\nu}, w_{n,p,q})| \\ +\varepsilon\, \|w_{n,p,q}\|^2 + \varepsilon^{-1} \|c^n_{pq}\, h_{n,p,q}\|^2 ,\end{aligned}$$

where $w_{n,p,q}=c^n_{pq}\, v_{n,p,q}$. Note that $c^n_{pq}=(M\log n)^{|\nu|}\, c^n_{p,q+\nu}$.

Now we are going to estimate the first term on the right hand side of (5.5) under the assumptions that $|\eta_0|\geqq|\tau_0|$ and $|\eta_0|\leqq|\tau_0|$, respectively.

A. *In the case where* $|\eta_0|\geqq|\tau_0|$

(1) For $\nu=(d, 0)$ (i.e., $L^{(\nu)}=\text{const}, g(x)\, D_y^{2m-d}$), we see the following. Since $c^{-1} n\leqq|\eta|\leqq cn$ for $(\eta, \tau)\in \text{supp}\ \alpha_n$, we have

$$\begin{aligned}&|(L^{(\nu)} w_{n,p,q+\nu}, w_{n,p,q})| \\ &= \text{const.}\, \left|\iiint g(x)\, \eta^{2m-d}\, w^{\wedge}_{n,p,q+\nu}(x; \eta, \tau)\, \overline{w^{\wedge}_{n,p,q}(x; \eta, \tau)}\, dx\, d\eta\, d\tau\right| \\ &\leqq \text{const.}\, n^{-d} \iiint g(x)\, \eta^{2m}(|w^{\wedge}_{n,p,q+\nu}|^2+|w^{\wedge}_{n,p,q}|^2)\, dx\, d\eta\, d\tau ,\end{aligned}$$

where $w^{\wedge}_{n,p,q}(x; \eta, \tau)$ denotes the partial Fourier transform of $w_{n,p,q}(x, y, t)$ with respect to (y, t).

Therefore it follows that

$$(5.6)\qquad \begin{aligned}&(M\log n)^{|\nu|}\, |(L^{(\nu)} w_{n,p,q+\nu}, w_{n,p,q})| \\ &\leqq \varepsilon\, \{(Lw_{n,p,q+\nu}, w_{n,p,q+\nu})+(Lw_{n,p,q}, w_{n,p,q})\} ,\end{aligned}$$

when n is large.

(2). For $\nu=(0, d)$ (i,e., $L^{(\nu)}=\text{const.}\ D_t^{2k-d}$), we have

$$\begin{aligned}&(M\log n)^{|\nu|}\, |(L^{(\nu)} w_{n,p,q+\nu}, w_{n,p,q})| \\ &= \text{const.}\, (M\log n)^d \left|\iiint \tau^{2k-d}\, w^{\wedge}_{n,p,q+\nu}(x; \eta, \tau)\, \overline{w^{\wedge}_{n,p,q}(x; \eta, \tau)}\, dx\, d\eta\, d\tau\right| .\end{aligned}$$

Observing that for any $\varepsilon>0$ there exists a constant $K>0$ such that

$$(M\log n)^d\, |\tau|^{2k-d} \leqq \varepsilon\tau^{2k}+K(M\log n)^{2k} ,$$

then from (4.1) we have

$$\begin{aligned}&(M\log n)^{|\nu|}\, |(L^{(\nu)} w_{n,p,q+\nu}, w_{n,p,q})| \\ &\leqq \text{const.} \iiint \{\varepsilon\tau^{2k}+K(M\log n)^{2k}\} \\ &\quad\times \{|w^{\wedge}_{n,p,q+\nu}|^2+|w^{\wedge}_{n,p,q}|^2\}\, dx\, d\eta\, d\tau \\ &\leqq \text{const.}\ \varepsilon\, \{(Lw_{n,p,q+\nu}, w_{n,p,q+\nu})+(Lw_{n,p,q}, w_{n,p,q})\} ,\end{aligned}$$

when n is large. Note that $\log n \sim \log|\eta|$ in the support of $w^{\wedge}_{n,p,q}(x; \eta, \tau)$.

Combining (1) and (2), we obtain for any $\varepsilon>0$

(5.7) $$(Lw_{n,p,q}, w_{n,p,q}) \leqq \text{const. } \varepsilon \sum_{|\nu| \leqq 2m+2k} (Lw_{n,p,q+\nu}, w_{n,p,q+\nu}) + \varepsilon^{-1} \|c^n_{pq} h_{n,q,q}\|^2 + \varepsilon \|w_{n,p,q}\|^2 ,$$

when n is large.

B. *In the case where* $|\eta_0| \leqq |\tau_0|$

(3). For $\nu=(d, 0)$, we have

$$(M \log n)^{|\nu|} |(L^{(\nu)} w_{n,p,q+\nu}, w_{n,p,q})|$$
$$= \text{const.} (M \log n)^d \left| \iiint g(x) \eta^{2m-d} \hat{w_{n,p,q+\nu}}(x; \eta, \tau) \overline{\hat{w_{n,p,q}}(x; \eta, \tau)} \, dx \, d\eta \, d\tau \right| .$$

Observing that for any $\varepsilon>0$ there exists a constant $K>0$ such that

$$(M \log n)^d |\eta|^{2m-d} \leqq \varepsilon \eta^{2m} + KM^{2m} n ,$$

we have

$$(M \log n)^{|\nu|} |(L^{(\nu)} w_{n,p,q+\nu}, w_{n,p,q})|$$
$$\leqq \text{const.} \iiint g(x) \{\varepsilon \eta^{2m} + KM^{2m} n\}$$
$$\times \{|\hat{w_{n,p,q+\nu}}|^2 + |\hat{w_{n,p,q}}|^2\} \, dx \, d\eta \, d\tau$$
$$\leqq \text{const. } \varepsilon \{(Lw_{n,p,q+\nu}, w_{n,p,q+\nu}) + (Lw_{n,p,q}, w_{n,p,q})\} ,$$

when n is large.

(4). For $\nu=(0, d)$, we have

$$|(L^{(\nu)} w_{n,p,q+\nu}, w_{n,p,q})|$$
$$= \text{const.} \left| \iiint \tau^{2k-d} \hat{w_{n,p,q+\nu}}(x; \eta, \tau) \overline{\hat{w_{n,p,q}}(x; \eta, \tau)} \, dx \, d\eta \, d\tau \right|$$
$$\leqq \text{const. } n^{-d} \iiint \tau^{2k} (|\hat{w_{n,p,q+\nu}}|^2 + |\hat{w_{n,p,q}}|^2) \, dx \, d\eta \, d\tau$$
$$\leqq \varepsilon \{(Lw_{n,p,q+\nu}, w_{n,p,q+\nu}) + (Lw_{n,p,q}, w_{n,p,q})\} ,$$

when n is large.

Combining (3) and (4), we also obtain (5.7) for any $\varepsilon>0$.

Now, let us sum up (5.7) with respect to (p, q) satisfying $|p+q| \leqq N_n - 2(m+k)$. Then the first term of the right hand side of (5.5) will be absorbed into the left hand side (by taking ε sufficiently small). Thus we have

(5.8) $$\sum_{|p+q| \leqq N_n} (Lw_{n,p,q}, w_{n,p,q}) \leqq \varepsilon S^M_n v + O(n^{-2s}) ,$$

since the microlocal energy of h is rapidly decreasing as $n \to \infty$. To establish (5.8), we used

$$\sum_{N_n - 2(m+k) \leqq |p+q| \leqq N_n} (Lw_{n,p,q}, w_{n,p,q}) = O(n^{-2s}) ,$$

for sufficiently large M. Cf. Lemma 1 in [2].

By Poincaré's inequality,

$$(Lw_{n,p,q}, w_{n,p,q}) \geqq \|D_x^l w_{n,p,q}\|^2 \geqq \delta \|w_{n,p,q}\|^2$$

holds for some $\delta > 0$. Therefore have

$$\sum_{|p+q| \leqq N_n} (Lw_{n,p,q}, w_{n,p,q}) \geqq \delta S_n^M v .$$

So from (5.8), we see that for any number s there exists a constant M such that

$$S_n^M(\chi u) = O(n^{-2s}) . \tag{5.9}$$

In view of Lemma 5.1, it follows that $u \in H^{0,\infty}$ at $(0, y_0, t_0; \eta_0, \tau_0)$ for any (η_0, τ_0) with $\eta_0^2 + \tau_0^2 = 1$. Now the proof is completed.

6. Proof of Lemma 3.1

Here we give the proof of Lemma 3.1.

Necessity. First we suppose $\phi = \chi(x)\psi(y)$ is a cut-off function which satisfies $\phi u \in H^{0,\infty}$ at η^0-direction, i.e.,

$$\iint_\Delta |\widehat{\phi u}(\xi, \eta)|^2 \langle \eta \rangle^{2j} \, d\xi \, d\eta < \infty \quad \text{for any } j$$

with

$$\Delta = \boldsymbol{R}^{n_1} \times \Gamma_0 = \{(\xi, \eta) : \xi \in \boldsymbol{R}^{n_1}, \eta \in \Gamma_0\} ,$$

where Γ_0 is a small cone which contains η^0. Furthermore, choose $r > 0$ sufficiently small so that $\beta \Subset \psi$ and supp $\alpha_n \Subset \Gamma_0$. Now let us take $v = \chi u$ and consider the following estimate:

$$\begin{aligned} n^{2j} \|\alpha_n^{(p)} \beta_{(q)} v\|^2 &= \|n^j \alpha_n^{(p)}(\eta) (\widehat{\beta_{(q)} v})(\xi, \eta)\|^2 \\ &\leqq \text{const.} \|\alpha_n^{(p)}(\eta) (\widehat{\beta_{(q)} v})(\xi, \eta) \langle \eta \rangle^j\|^2 \\ &\leqq \text{const.} \iint_\Delta |\widehat{\phi u}(\xi, \eta)|^2 \langle \eta \rangle^{2j} \, d\xi \, d\eta . \end{aligned}$$

Recall that $c^{-1} n \leqq |\eta| \leqq cn$ for $\eta \in$ supp α_n. Then we see that

$$\|\alpha_n^{(p)} \beta_{(q)} v\| = O(n^{-j}) \quad \text{for any } j .$$

So it follows that

$$S_{N,n}(\chi u) = O(n^{-2s})$$

for any N and s. Q.E.D.

Sufficiency. If $S_{N,n}(\chi u) = O(n^{-2s})$, we have

$$\|\alpha_n \beta v\| = O(n^{-s}) ,$$

where $v=\chi u$. Let us observe that

$$\sum_{n=1}^{\infty} \alpha_n(\eta)^2 n^{2s-1} \geqq \text{const.} \langle\eta\rangle^{2s}$$

for η contained in some conic neighborhood Γ_0 of η^0 with $|\eta|\geqq 1$. Then we see that

$$\iint_{\Delta'} |\widehat{\beta v}(\xi, \eta)|^2 \langle\eta\rangle^{2s'} d\xi\, d\eta \leqq \sum_{n=1}^{\infty} \|\alpha_n \beta v\|^2 n^{2s'-1} < \infty$$

for any $s'<s$, where $\Delta'=\Delta \cap \{(\xi, \eta)\colon |\eta|\geqq 1\}$. Now it follows that u is microlocally of the class $H^{0,\infty}$ at $(0, y^0; \eta^0)$ if $S_{N,n}\, v$ is rapidly decreasing as $n \to \infty$.
Q.E.D.

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Tatsushi Morioka
Department of Mathematics
Faculty of Sciences
Osaka University