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THE REVERSIBLE MEASURES OF MULTI-DIMENSIONAL GINZBURG-LANDAU TYPE CONTINUUM MODEL

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1. Introduction

In this paper we investigate a stochastic dynamics for continuum fields on $\boldsymbol{R}^d$ with interactions prescribed by Ginzburg-Landau type Hamiltonian. The main problems discussed here are to clarify the structure of the family of reversible measures (r.m.'s) of this dynamics, especially, we are interested in (1) the characterization, (2) the construction and (3) showing the uniqueness of r.m.'s. For the characterization problem the classical notion of Gibbs states (e.g., for the lattice systems) is extended to the continuum fields. In our present situation Gibbs states are Markovian random fields over $\boldsymbol{R}^d$ and they are given as local perturbations from Gaussian fields, which is determined by the so-called DLR equation. Then the answer to the first problem will be given by establishing the equivalence between reversibility and Gibbs property. The r.m.'s and therefore the Gibbs states will be constructed for a wide class of potentials, while for the uniqueness we require the strict-convexity for the self-potential appearing in the Hamiltonian. In this uniqueness domain, we also verify the strong mixing property of the Gibbs states. This is one of examples which show stochastic dynamics is useful for the study of properties of Gibbs states.

Now let us explain the dynamics we shall discuss in this paper more explicitly. It is described by the so-called time-dependent Ginzburg-Landau equation (TDGL eq.) of non-conservative type:

$$(1.1)\qquad dS_t(x) = -\frac{1}{2} D\mathcal{H}(x, S_t)\, dt + dw_t(x)\,, \quad t>0,\ x\in \boldsymbol{R}^d\,,$$

where w_t is a cylindrical Brownian motion (c.B.m.) on $L^2(\boldsymbol{R}^d)$, see [7, 8]. The solution S_t determines a random time evolution of real-valued continuum field on $\boldsymbol{R}^d$. The Hamiltonian $\mathcal{H}$ is formally given as the sum of two terms, local-interaction and self-interaction:

$$(1.2)\qquad \mathcal{H}(S) = \int_{\boldsymbol{R}^d} \{\frac{1}{2} \mathcal{A}S(x)\cdot S(x) + V(x, S(x))\}\, dx, \quad S\colon \boldsymbol{R}^d \to \boldsymbol{R}\,.$$

Here $\mathcal{A}$ is a symmetric differential operator of order $2m$ having the form

$$\mathcal{A}f(x) = \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^{\alpha}\{a_{\alpha,\beta} D^{\beta} f\}(x), \tag{1.3}$$

with coefficients $a_{\alpha,\beta}=a_{\beta,\alpha}\in C_b^{\infty}(\boldsymbol{R}^d)$ and $V\in \boldsymbol{V}$; the class of all measurable functions $V=V(x, s)$ on $\boldsymbol{R}^d\times\boldsymbol{R}$ such that $V(x, \cdot)\in C^1(\boldsymbol{R})$ for a.e. $x\in\boldsymbol{R}^d$ and their derivatives $V'=\frac{\partial V}{\partial s}$ in s are bounded and Lipschitz continuous (i.e., $\operatorname{esssup}_{x,s} |V'(x, s)|<\infty$ and $|V'(x, s)-V'(x, \bar{s})|\leq \text{const}\, |s-\bar{s}|$, $x\in\boldsymbol{R}^d$, $s, \bar{s}\in\boldsymbol{R}$). We adopt the usual notation: $|\alpha|=\sum_{i=1}^d \alpha_i$, $D^{\alpha}=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_d}\right)^{\alpha_d}$ and $\xi^{\alpha}=\xi_1^{\alpha_1}\cdots \xi_d^{\alpha_d}$ for $\alpha=(\alpha_1, \cdots, \alpha_d)\in\boldsymbol{Z}_+^d=\{0, 1, 2, \cdots\}^d$ and $\xi\in\boldsymbol{R}^d$. In (1.1), $D\mathcal{H}(x, S)$ is the (formal) functional derivative of $\mathcal{H}(S)$ and therefore, in more mathematical terminology, we consider instead of (1.1) the following stochastic partial differential equation (SPDE):

$$dS_t(x) = -\frac{1}{2}\mathcal{A}S_t(x)\,dt-\frac{1}{2}V'(x, S_t(x))\,dt+dw_t(x). \tag{1.4}$$

We assume that $m>d/2$ and $\mathcal{A}$ is uniformly strongly elliptic, i.e.

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}(x)\,\xi^{\alpha}\xi^{\beta}\geq c|\xi|^{2m},\ x, \xi\in\boldsymbol{R}^d,\ c\}>0, \tag{1.5}$$

and strictly positive, i.e.

$$\gamma=\inf\{\int_{\boldsymbol{R}^d} f\mathcal{A}f\,dx;\ \int_{\boldsymbol{R}^d} f^2\,dx = 1, f\in C_0^{\infty}(\boldsymbol{R}^d)\}>0. \tag{1.6}$$

Then it is more convenient to regard the following function $U\equiv U_V$ as the substantial self-potential function for the dynamics instead of V:

$$U(x, s) = \frac{\gamma}{2}s^2+V(x, s),\quad x\in\boldsymbol{R}^d, s\in\boldsymbol{R}. \tag{1.7}$$

The results in [8] give the existence, uniqueness and some regularity properties of solutions S_t of the SPDE (1.4). Indeed, fix a positive symmetric function $\chi\in C^{\infty}(\boldsymbol{R}^d)$ satisfying $\chi(x)=|x|$ for x; $|x|\geq 1$ and introduce Hilbert spaces $\boldsymbol{L}_r^2=L^2(\boldsymbol{R}^d, e^{-2r\chi(x)}\,dx)$, $r\in\boldsymbol{R}$, having inner products defined by

$$\langle S, S'\rangle_r = \int_{\boldsymbol{R}^d} S(x)\cdot S'(x)\,e^{-2r\chi(x)}\,dx,\ S, S'\in\boldsymbol{L}_r^2. \tag{1.8}$$

The corresponding norms are denoted by $|\cdot|_r$. The space $C(\boldsymbol{R}^d)$ with the usual topology determined from uniform convergence on all bounded sets is denoted by $\boldsymbol{C}$. Then, it is one of the consequences of Theorem 2.1 in [8] that the SPDE (1.4) has a unique solution S_t satisfying

$$S_t\in C([0, \infty), \boldsymbol{L}_r^2)\cap C((0, \infty), \boldsymbol{C})\ \text{a.s.} \tag{1.9}$$

if the initial data $S_0 \in \boldsymbol{L}_r^2$, $r>0$. We set $\boldsymbol{L}_e^2 = \bigcap_{r>0} \boldsymbol{L}_r^2$, a countably Hilbertian space.

The first part (Sect.'s 2,3,4) of this paper is devoted to the characterization problem of r.m.'s. The form of the TDGL eq. (1.1) suggests that its r.m. might be given by

$$(1.10) \qquad e^{-\mathcal{H}(S)} \text{“}dS\text{”}/\text{normalization},$$

where “dS”$=\prod_{x\in \boldsymbol{R}^d} dS(x)$ is the Feynman's measure (“Lebesgue measure” on the space of functions on $\boldsymbol{R}^d$). The measure like (1.10) would be understood as the local perturbation from the Gaussian measure on $\boldsymbol{C}$ with covariance operator $\mathcal{A}^{-1}$. However, in order to localize the problem, it turns out to be necessary to know the boundary data of the random fields. For this purpose important roles will be played by the weak C^{m-1} property for the random fields, which determines m generalized random fields $Y=\{Y_i\}_{i=0}^{m-1}$ on the smooth boundary ∂G of bounded region G. Then the regularity condition $(RC)_{\partial G}^s$, $s<m$, for Y enables us to reconstruct the fields inside G (see Sect. 2).

The second part consists of Sect.'s 5, 6, 7 and 8. The uniqueness of r.m.'s of the SPDE (1.4) is shown by assuming the strict convexity (in s) of the potential $U=U_V$:

$$(1.11) \quad U(x,\cdot)\in C^2(\boldsymbol{R}) \quad \text{for a.e. } x\in\boldsymbol{R}^d \text{ and } \operatorname*{essinf}_{x\in\boldsymbol{R}^d, s\in\boldsymbol{R}} \frac{\partial^2 U}{\partial s^2}(x,s)\geq\gamma_0>0\,.$$

An effective tool is the energy estimate for (1.4), cf. [4]. Moreover, by using this estimate, it is possible to construct the r.m.'s for sufficiently wide class of potentials V. We shall also discuss the TDGL eq. of conservative type:

$$(1.12) \qquad dS_t(x) = \frac{1}{2}\Delta D\mathcal{H}(x, S_t)\,dt + (-\Delta)^{1/2}\,dw_t(x)\,,$$

or, more mathematically saying, the SPDE:

$$(1.13) \qquad dS_t(x) = \frac{1}{2}\Delta\{\mathcal{A}S_t(x)+V'(x,S_t(x))\}\,dt + d\{\operatorname{div}\boldsymbol{w}_t(x)\}\,,$$

where $\Delta=\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}$ is the Laplacian on $\boldsymbol{R}^d$ and $\operatorname{div}\boldsymbol{w}_t(x)=\sum_{j=1}^d \frac{\partial}{\partial x_j} w_t^j(x)$, $\boldsymbol{w}_t(x) = \{w_t^j(x)\}_{j=1}^d$ being a c.B.m. on the space $L^2(\boldsymbol{R}^d, \boldsymbol{R}^d)$. Note that the process $(-\Delta)^{1/2}\,w_t(x)$ is equivalent in law to $\operatorname{div}\boldsymbol{w}_t(x)$. See [8] for the existence and uniqueness of solutions S_t of (1.13) satisfying (1.9). Since the eq. (1.12) is unchanged (at least formally) under the replacement of $\mathcal{H}$ by another Hamiltonian $\mathcal{H}_\lambda$:

$$(1.14) \qquad \mathcal{H}_\lambda(S) = \mathcal{H}(S) - \int_{\boldsymbol{R}^d} \lambda(x)\,S(x)\,dx$$

with $\lambda=\lambda(x)$ satisfying $\Delta\lambda=0$, the SPDE (1.13) might have a family of reversible measures (i.e., the Gibbs states corresponding to $\mathcal{H}_\lambda$), see Sect. 7 for details. Finally in Sect. 8 the uniform mixing property of the Gibbs states is analyzed by using the dynamics (1.4). The result will be applied in another paper [9] to discuss the hydrodynamic limit for the TDGL eq. (1.13) of conservative type.

2. Gibbs states on $\boldsymbol{R}^d$

In this section we shall define the Gibbs states associated with the Hamiltonian $\mathcal{H}$ determined by (1.2) and then formulate the result which gives the equivalence between reversibility and Gibbs property.

2.1. Stochastic Dirichlet problem

Let $\mathcal{V}$ be the family of all bounded open sets in $\boldsymbol{R}^d$ with boundaries being $(d-1)$-dimensional C^∞-manifolds. Assume a family of m generalized random fields (g.r.f.'s) $Y=\{Y_i(\psi); \psi\in L^2(\Gamma)\}_{i=0}^{m-1}$ on $\Gamma=\partial G$, $G\in\mathcal{V}$, is given and satisfies the regularity condition $(RC)_\Gamma^s$ with some s: $\frac{d}{2}\vee(m-\frac{1}{2})<s<m$; namely, Y is supposed to fulfill the following bounds

$$E[Y_i(\psi)^2]\leq \text{const}\ \|\psi\|^2_{H^{-s+i+1/2}(\Gamma)},\ \psi\in L^2(\Gamma),\ 0\leq i\leq m-1\,,$$

where $H^s(\Gamma)$ is the Sobolev space of order s on Γ [14], see also Definition 3.1 in [8]. (When $d=1$, we interpret $\|\psi\|^2_{H^s(\Gamma)}=\sum_{a\in\Gamma}\psi(a)^2$, $s\in\boldsymbol{R}$, as usual.) Let us consider the following stochastic Dirichlet (SD) problem:

$$\text{(2.1)}\qquad \begin{cases} \mathcal{A}X=0 & \text{in } G \\ \partial_i^+X=Y_i & \text{on } \Gamma\,,\quad 0\leq i\leq m-1\,, \end{cases}$$

where $\partial_i^+=\dfrac{\partial^i}{\partial n(\underline{x})^i}$; $n(\underline{x})$ is the inner normal unit vector at $\underline{x}\in\Gamma$.

The mathematical meaning to this equation is given in the following manner: Let $f=f^\psi\in H^{2m}(G)\cap H_0^m(G)$, $\psi\in L^2(G)$, be the solution of the *dual* problem to (2.1):

$$\text{(2.2)}\qquad \begin{cases} \mathcal{A}f=\psi & \text{in } G \\ \partial_i^+f=0 & \text{on } \Gamma\,,\quad 0\leq i\leq m-1\,, \end{cases}$$

in other words, $f^\psi=\mathcal{A}_G^{-1}\psi$. Here $\mathcal{A}_G$ is the Friedrichs extension on the space $L^2(G)$ of the operator $\mathcal{A}$ with domain $C_0^\infty(G)$, the space of C^∞-functions on G vanishing near Γ. We shall use in Sect. 3 the fact that $H^{2m}(G)\cap H_0^m(G)=H^{2m}(G)\cap H_0^{m+1/2}(G)$, [14, Theorem 11.5, p 62]. Let $\{\delta_i\}_{i=0}^{2m-1}$ be a system of boundary differential operators of order $2m-i-1$, respectively, determined by the Green's formula [2, 14], i.e.,

$$\langle f, \mathcal{A} f' \rangle_G = \langle \mathcal{A} f, f' \rangle_G + \sum_{i=0}^{2m-1} \langle \partial_i^\dagger f, \delta_i f' \rangle_\Gamma \tag{2.3}$$

for every $f, f' \in C^\infty(\bar{G})$, where $\langle f, f' \rangle_G = \int_G f f' \, dx$, $\langle g, g' \rangle_\Gamma = \int_\Gamma g g' \, d\sigma(\underline{x})$ and $d\sigma = d\sigma_\Gamma$ is the volume element on Γ. By the solution of SD problem (2.1) we mean the g.r.f. X on G defined by

$$\langle X, \psi \rangle_G = \sum_{i=0}^{m-1} Y_i(\delta_i f^\psi), \quad \psi \in L^2(G). \tag{2.4}$$

Note that, if the functions $\{Y_i\}_{i=0}^{m-1}$ on Γ are smooth enough, then X determined by this relation actually solves the Dirichlet problem (2.1). We have extended the notion of solutions through this relation to the case with random data $Y = \{Y_i\}$ which take values in the space of generalized functions (a.s.).

Lemma 2.1. *It holds* $X(\cdot) \in C^\infty(G)$ (a.s.) *for the solution* X *of* (2.1).

Proof: First we note that the map

$$\Xi^{-s}(G) \ni \psi \mapsto \delta_i f^\psi \in H^{-s+i+1/2}(\Gamma)$$

is continuous, where the space $\Xi^{-s}(G)$ is introduced as follows: Let $\rho \in C(\bar{G})$ be a function which is positive on G, $=0$ on Γ and satisfies $\rho(x) \sim \mathrm{dis}\,(x, \partial G)$ as $x \sim \partial G$ and let $\Xi^s(G)$, $s \in \boldsymbol{Z}_+$, be a Hilbert space consisting of all u such that $\rho(x)^{|\alpha|} D^\alpha u \in L^2(G)$ for every α; $|\alpha| \le s$. This space is equipped with the natural inner product. Then $\Xi^s(G)$ is defined for $s \ge 0$ by using interpolation technique and finally $\Xi^{-s}(G)$ is introduced as its dual space, see [14, p 188] for detail. Now the regularity condition $(RC)_\Gamma^s$ on Y implies a bound:

$$E[\langle X, \psi \rangle_G^2] \le \mathrm{const}\, \|\psi\|_{\Xi^{-s}(G)}^2 .$$

This proves that $X|_{G'} \in H^{s-(d/2)-\delta}(G')$ (a.s.) for every $\delta > 0$ and $G' \in \mathcal{V}$ such that $\bar{G}' \subset G$ and therefore $X \in L^2_{loc}(G)$ (a.s.); use Minlos-Gross-Sazonov's theorem. However, it is easy to see $\langle X, \mathcal{A}\varphi \rangle_G = 0$ (a.s.) for every $\varphi \in C_0^\infty(G)$ and consequently $\langle X, \mathcal{A}\varphi \rangle_G = 0$ for every $\varphi \in C_0^\infty(G)$ with probability one. This shows $X \in C^\infty(G)$ (a.s.), see e.g. [1, p 66]. □

Let ν_0^G be a centered Gaussian measure on $L^2(G)$ with covariance operator $\bar{\mathcal{A}}_G^{-1}$, i.e.,

$$E^{\nu_0^G}[\langle X, \varphi \rangle_G \langle X, \varphi' \rangle_G] = \langle \varphi, \bar{\mathcal{A}}_G^{-1} \varphi' \rangle_G, \quad \varphi, \varphi' \in L^2(G).$$

Note that $\nu_0^G(L^2(G)) = 1$ since $\bar{\mathcal{A}}_G^{-1}$ is a trace-class operator on $L^2(G)$. In fact, it is well-known that $\bar{\mathcal{A}}_G$ has eigenfunctions $\{\bar{\varphi}_n\}_{n=1}^\infty$ being complete in $L^2(G)$ and corresponding eigenvalues $\{0 < \lambda_n \nearrow\}_{n=1}^\infty$ such that $\lambda_n \sim c n^{2m/d}$, $c > 0$, as $n \to \infty$, see [1]. We can regard ν_0^G as a probability measure on the space $C(\bar{G})$ by the following lemma.

Lemma 2.2. *For ν_0^G-a.e. $X \in L^2(G)$, there exists a version $\tilde{X} \in C(\bar{G})$ of X in the sense that $X(x) = \tilde{X}(x)$ holds for* a.e. $x \in G$.

Proof. Consider a stochastic process X_t on $L^2(G)$ defined by

$$X_t = T_t X_0 + \int_0^t T_{t-u}\, dw_u\,, \quad X_0 \in L^2(G)\,,$$

where T_t is the semigroup on $L^2(G)$ generated by $-\frac{1}{2}\bar{\mathcal{A}}_G$ and w_t is a c.B.m. on $L^2(G)$. Clearly $T_t X_0 \in \mathcal{D}(\bar{\mathcal{A}}_G) \subset C(\bar{G})$ for $X_0 \in L^2(G)$ and $t > 0$, while the second term of the RHS has a version also belonging to $C(\bar{G})$ (a.s.); see [7]. We therefore obtain the conclusion since if the distribution of X_0 is ν_0^G then it is also true for X_t (to see this, e.g., use the Fourier series expansion of X_t based on $\{\bar{\varphi}_n\}$). □

2.2. Two definitions of the Gibbs states

Set $\Omega = \boldsymbol{C}$ and denote $X(x, \omega) = \omega(x)$ for $x \in \boldsymbol{R}^d$ and $\omega \in \Omega$. Let $\mathcal{B}$ be the Borel field of Ω, $\mathcal{B}(G) = \sigma\{X(x);\, x \in G\}$ if G is open in $\boldsymbol{R}^d$ and $\mathcal{B}(C) = \cap\, \{\mathcal{B}(G);\, G$ is open, $G \supset C\}$ if C is closed. For $\Gamma = \partial G$, $G \in \mathcal{V}$, we also introduce $\mathcal{B}_-(\Gamma) = \cap\, \{\mathcal{B}(O \cap (G^c)^\circ);\, O$ is open, $O \supset \partial G\}$.

We give the notion of weak C^{m-1} property of r.f.'s, see [8] and also [15]. For a real-valued r.f. $\tilde{X} = \{\tilde{X}(x);\, x \in \boldsymbol{R}^d\}$, set

$$\begin{aligned} F_{\tilde{X}}(h, \psi) &\equiv F_{\tilde{X}}(h, \psi; \Gamma) \\ &= \int_\Gamma \psi(\underline{x})\, \tilde{X}(\underline{x} + h \cdot n(\underline{x}))\, d\sigma(\underline{x})\,, \end{aligned}$$

for $|h| < h_0$, $h_0 > 0$, and $\psi \in L^2(\Gamma) \equiv L^2(\Gamma, d\sigma)$.

DEFINITION 2.1. The r.f. $\tilde{X}$ is called weakly C^{m-1} at Γ if there exists $h_0 > 0$ such that $F_{\tilde{X}}(\cdot, \psi; \Gamma) \in C^{m-1}((-h_0, h_0))$ a.s. for every $\psi \in L^2(\Gamma)$.

Now we introduce the definition of Gibbs states. We denote by $\mathcal{B}_b(\boldsymbol{R}^d \times \boldsymbol{R})$ the class of all bounded and measurable functions on $\boldsymbol{R}^d \times \boldsymbol{R}$ and assume the following condition on the potential $V = V(x, s)$:

$$V \in \mathcal{B}_b(\boldsymbol{R}^d \times \boldsymbol{R})\,. \tag{2.5}$$

DEFINITION 2.2. We call $\mu \in \mathcal{P}(\Omega)$, the family of probability measures on $(\Omega, \mathcal{B})$, a V-Gibbs state if it satisfies the following two conditions for every $G \in \mathcal{V}$:

(1) The r.f. X distributed by μ is weakly C^{m-1} at $\Gamma = \partial G$ and

$$Y_i(\psi; \Gamma) = \frac{d^i}{dh^i} F_X(h, \psi; \Gamma)\,|_{h=0}\,, \quad 0 \le i \le m-1\,,$$

satisfies the regularity condition $(RC)_\Gamma^s$ for every $s < m$.

(2) For every $\mathcal{B}(G)$-measurable bounded function Ψ on Ω and $\mathcal{B}((G^c)^\circ)$-measurable bounded function Φ on Ω, μ satisfies the so-called DLR equation:

$$E^\mu[\Phi\cdot\Psi] = E^\mu[\Phi(\omega)\, E^{\mu_\omega^G}[\Psi]]\,.$$

In this definition the finite-volume Gibbs distribution μ_ω^G is defined as follows: Consider the solution $X'(\omega)=\{X'(x,\omega), x\in G\}$ of SD problem (2.1) with $Y_i=Y_i(\cdot\,;\Gamma)$ given in (1). Note that X' is $\mathcal{B}(\Gamma)$-measurable and even $\mathcal{B}_-(\Gamma)$-measurable. Define $\nu_\omega^G(\cdot)=\nu_0^G(\cdot-X'(\omega))$ and then

$$d\mu_\omega^G(X(\cdot)) = Z_{\omega,G}^{-1}\exp\{-\int_G V(x,X(x))\,dx\}\, d\nu_\omega^G(X(\cdot))\,, \tag{2.6}$$

where $Z_{\omega,G}$ is a normalization constant.

We introduce another definition of Gibbs states. Let $\nu\in\mathcal{P}(\Omega)$ be a centered Gaussian measure with covariance operator $\bar{\mathcal{A}}^{-1}$, where $\bar{\mathcal{A}}$ is the Friedrichs extension of $(\mathcal{A}, C_0^\infty(\boldsymbol{R}^d))$ on the space $L^2(\boldsymbol{R}^d)$, see [2]. Similar notion of Gibbs states was introduced and studied by Fröhlich [5].

DEFINITION 2.3. We call $\mu\in\mathcal{P}(\Omega)$ a ν-l.a.c. (locally absolutely continuous with respect to ν) V-Gibbs state if, for every $G\in\mathcal{V}$, μ is absolutely continuous *w.r.t.* ν on $\mathcal{B}(G)$ and the condition (2) holds with μ_ω^G constructed by the formula (2.6) with ν_ω^G replaced by the regular conditional probability distribution $\nu(\cdot\,|\,\mathcal{B}((G^c)^\circ))$ of ν *w.r.t.* $\mathcal{B}((G^c)^\circ)$. Note that μ_ω^G is defined for μ—a.s.ω.

We fix $r>0$ and denote by $\mathcal{G}(V)$ and $\mathcal{G}'(V)$ the families of all V-Gibbs states and ν-l.a.c. V-Gibbs states, respectively, satisfying $E^\mu[|S|_r^2]<\infty$.

2.3. Formulation of the result

Let $L_b(\boldsymbol{L}_r^2)$ be the class of all $b=b(x,S)$, $x\in\boldsymbol{R}^d$, $S\in\boldsymbol{L}_r^2$, being bounded (i.e., $|b|_{r,(\infty)}=\sup_S|b(\cdot,S)|_r<\infty$) and Lipschitz continuous as functions of $\boldsymbol{L}_r^2\to\boldsymbol{L}_r^2$. For $V=V(x,s)$ we set $b_V(x,S)=-\dfrac{1}{2}V'(x,S(x))$, $x\in\boldsymbol{R}^d$, $S\in\boldsymbol{L}_r^2$, where $V'=\dfrac{\partial V}{\partial s}$, and assume the following condition:

$$V(x,\cdot)\in C^1(\boldsymbol{R}) \text{ in } s \text{ for a.e. } x\in\boldsymbol{R}^d \quad\text{and}\quad b_V\in\bigcap_{r>0}L_b(\boldsymbol{L}_r^2)\,. \tag{2.7}$$

Then, under this slightly milder condition than "$V\in\boldsymbol{V}$", the TDGL eq. (1.4) has a unique solution satisfying (1.9), see [8]. Let $\mathcal{D}$ be the class of all functions Ψ on $\boldsymbol{L}_r^2$ having the form:

$$\Psi(S) = \psi(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle),\ S\in\boldsymbol{L}_r^2\,, \tag{2.8}$$

with $k=1,2,\cdots$, $\psi=\psi(\alpha_1,\cdots,\alpha_k)\in C_b^2(\boldsymbol{R}^k)$ and $\varphi_1,\cdots,\varphi_k\in C_0^\infty(\boldsymbol{R}^d)$, where $\langle S,\varphi\rangle=\int_{\boldsymbol{R}^d}S(x)\,\varphi(x)\,dx$.

A probability measure $\mu\in\mathcal{P}(\boldsymbol{C}\cap\boldsymbol{L}_r^2)$ is called a reversible measure of the SPDE (1.4), if it satisfies

$$E^{\mu}[\Phi(S_0)\,\Psi(S_t)] = E^{\mu}[\Psi(S_0)\,\Phi(S_t)] \tag{2.9}$$

for every $t>0$ and Φ, $\Psi\in\mathcal{D}$, where the superscript μ in both sides of (2.9) means that μ is the initial distribution of the solution S_t. Note that $\mathcal{D}$ is a determining class of measures on the space $\boldsymbol{C}\cap\boldsymbol{L}_r^2$. We denote by $\mathcal{R}(V)$ the family of all reversible probability measures of (1.4). We can now formulate the theorem which is one of our main results in this paper.

Theorem 2.1. *Assume the conditions* (2.5) *and* (2.7) *on the potential* V. *Then, we have* $\mathcal{G}(V)=\mathcal{G}'(V)=\mathcal{R}(V)$.

The proof for $\mathcal{G}(V)=\mathcal{R}(V)$ will be given in the next two sections. We notice finally in this section that $\mathcal{G}(V)=\mathcal{G}'(V)$ follows easily by assuming $\mathcal{G}(0)=\mathcal{R}(0)$ and $\nu\in\mathcal{R}(0)$ (see Proposition 6.1). In fact, $\nu\in\mathcal{G}(0)$ implies $\nu(\cdot\,|\,\mathcal{B}((G^c)^\circ))\,(\omega)=\bar{\nu}_\omega^G$, ν—a.s.ω, where $\bar{\nu}_\omega^G$ is the shifted measure of ν_0^G by the solution $\bar{X}'_\omega$ of the SD problem (2.1) with boundary fields $\{\bar{Y}_i\}$ which are determined based on the measure ν(i.e., $\bar{\nu}_\omega^G$ denotes the measure ν_ω^G defined especially based on ν instead of μ). Take an arbitrary $G'\in\mathcal{V}$ such that $\bar{G}'\subset G$. Then, since $\bar{X}'_\omega\in C^\infty(G)$, ν—a.s.ω, we have $\langle\mathcal{A}(\bar{X}'_\omega\cdot\varphi),\bar{X}'_\omega\cdot\varphi\rangle_G<\infty$, ν—a.s.ω, for every $\varphi\in C_0^\infty(G)$ satisfying $\varphi\equiv1$ on G'. This verifies $\bar{\nu}_\omega^G\sim\nu_0^G$ on $\mathcal{B}(G')$, ν—a.s.ω, (see [12, p 118]) and therefore $\nu\sim\nu_0^G$ on $\mathcal{B}(G')$, where $\sim$ means the equivalence of two measures.

Now suppose $\mu\in\mathcal{G}(V)$. Then, the same argument as above implies $\nu_\omega^G\sim\nu_0^G$ on $\mathcal{B}(G')$, μ—a.s.ω, where ν_ω^G is the measure defined from μ as described just after Definition 2.2. This shows $\mu\sim\nu_0^G$ on $\mathcal{B}(G')$. Therefore we obtain $\mu\sim\nu$ on $\mathcal{B}(G')$ and consequently μ is ν-l.a.c. Especially, we see that $\bar{\nu}_\omega^G$ is defined for μ—a.s.ω and $\bar{\nu}_\omega^G=\nu_\omega^G$, μ—a.s.ω. Hence we get $\mu\in\mathcal{G}'(V)$. The converse assertion is easy; notice that, since $\nu\in\mathcal{G}(0)$, the condition (1) of Definition 2.2 holds for ν and therefore for every $\mu\in\mathcal{G}'(V)$.

3. The proof of Theorem 2.1; reversible $\Rightarrow$ Gibbs

In this section $\mu\in\mathcal{R}(V)$ is given and fixed. The potential $V=V(x,s)$ satisfies the conditions (2.5) and (2.7). First we notice the following integrability property of μ, whose validity will be shown in Sect. 5 using coupling method, see Proposition 5.1 and note $|S|_r\le|S|_{r'}$ if $0<r<r'$.

Lemma 3.1. $E^{\mu}[e^{\beta|S|_r^2}]<\infty$ *with some* $\beta=\beta(r)>0$ *for every* $r>0$.

Let us introduce the (formal) generator $\mathcal{L}=\mathcal{L}_V$ of the process S_t determined by the TDGL eq. (1.4): For $\Psi\in\mathcal{D}$ having the form (2.8),

$$\mathcal{L}\Psi(S) = -\frac{1}{2}\sum_{i=1}^{k}\frac{\partial\psi}{\partial\alpha_i}(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle) \times\{\langle S,\mathcal{A}\varphi_i\rangle+\langle V'(\cdot,S(\cdot)),\varphi_i\rangle\} + \frac{1}{2}\sum_{i,i'=1}^{k}\frac{\partial^2\psi}{\partial\alpha_i\partial\alpha_{i'}}(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle)\langle\varphi_i,\varphi_{i'}\rangle. \tag{3.1}$$

We denote for $\Psi\in\mathcal{D}$

$$D\Psi(x,S) = \sum_{i=1}^{k}\frac{\partial\psi}{\partial\alpha_i}(\langle S,\varphi_1\rangle,\cdots,\langle S,\varphi_k\rangle)\,\varphi_i(x). \tag{3.2}$$

Lemma 3.2. *For every $\Phi,\Psi\in\mathcal{D}$,*

$$E^{\mu}[\Phi\cdot\mathcal{L}\Psi] = -\frac{1}{2}E^{\mu}[\langle D\Phi(\cdot,S),D\Psi(\cdot,S)\rangle] \tag{3.3}$$

Proof. Noting the integrability of μ (Lemma 3.1), we have

$$\frac{d}{dt}E^{\mu}[\Phi(S_0)\,\Psi(S_t)] = E^{\mu}[\Phi(S_0)\,\mathcal{L}\Psi(S_t)] \to E^{\mu}[\Phi(S_0)\,\mathcal{L}\Psi(S_0)], \quad \text{as } t\downarrow 0,$$

and therefore the reversibility condition (2.9) on μ implies

$$E^{\mu}[\Phi\cdot\mathcal{L}\Psi] = E^{\mu}[\Psi\cdot\mathcal{L}\Phi], \quad \Phi,\Psi\in\mathcal{D}. \tag{3.4}$$

However this verifies (3.3) with the help of a trivial equality

$$\mathcal{L}(\Phi\cdot\Psi)-\Phi\mathcal{L}\Psi-\Psi\mathcal{L}\Phi = \langle D\Phi(\cdot,S),D\Psi(\cdot,S)\rangle. \tag{3.5}$$

□

Let $\bar{\mathcal{D}}$ be the completion of the space $\mathcal{D}$ with respect to the norm $|||\cdot|||$ defined by

$$|||\Phi|||^2 = ||\Phi||^2_{L^2(d\mu)}+||D\Phi||^2_{L^2(dx\times d\mu)}, \quad \Phi\in\mathcal{D}. \tag{3.6}$$

Notice that $D\Phi\in L^2(dx\times d\mu)$, $\Phi\in\bar{\mathcal{D}}$, is determined uniquely by the procedure of this completion. The formula (3.3) still holds for $\Phi\in\bar{\mathcal{D}}$ and $\Psi\in\mathcal{D}$ or even for Ψ of the form (2.8) with ψ which may be unbounded but satisfies $|\psi(\alpha)|\le$ const $\{1+|\alpha|\}$; note Lemma 3.1.

DEFINITION 3.1. Let K be a subspace of $\boldsymbol{C}$ and Λ a real-valued measurable function on $K\times\boldsymbol{C}$. A probability measure $m\in\mathcal{P}(\boldsymbol{C})$ is called K-quasi-invariant with cocycle $\Lambda(\varphi,S)$ iff $m(\varphi+\cdot)$ and m are mutually equivalent and it holds $m(\varphi+dS)=e^{\Lambda(\varphi,S)}\,m(dS)$ for every $\varphi\in K$.

Lemma 3.3. *μ is $C_0^\infty(\boldsymbol{R}^d)$-quasi-invariant with cocycle defined by*

$$\Lambda(\varphi, S) = \int_{R^d} [V(x, S(x)) - V(x, \varphi(x)+S(x)) \tag{3.7}$$
$$- \{S(x) + \frac{1}{2} \varphi(x)\} \mathcal{A}\varphi(x)]\, dx\,, \quad \varphi \in C_0^\infty(\boldsymbol{R}^d),\ S \in \boldsymbol{C}\,.$$

Proof. Fix $\varphi \in C_0^\infty(\boldsymbol{R}^d)$ and put $F_t(S) = \Lambda(t\varphi, S - t\varphi)$ for $t \in \boldsymbol{R}$ and $S \in \boldsymbol{C}$. Then we see $F_t \in \bar{\mathcal{D}}$ and moreover

$$\frac{\partial}{\partial t} F_t(S) = -\langle DF_t(\cdot, S), \varphi\rangle + 2\mathcal{L}\Psi_1(S)\,,$$

where $\Psi_1 = \langle S, \varphi\rangle$. Hence, for every $\Phi \in \mathcal{D}$

$$\frac{\partial}{\partial t} \int \Phi(S)\, e^{-\Lambda(t\varphi, S)}\, \mu(t\varphi + dS) \tag{3.8}$$
$$= \frac{\partial}{\partial t} \int \Phi(S - t\varphi)\, e^{-F_t(S)}\, \mu(dS) = 0\,.$$

Here the formula (3.3) has been used by taking $\Phi = \Phi_1$, which is defined by $\Phi_1(S) = \Phi(S - t\varphi)\, e^{-F_t(S)}$, and $\Psi = \Psi_1$; note that $\Phi_1 \in \bar{\mathcal{D}}$. The conclusion follows immediately from (3.8). □

The following lemma is a consequence of Theorems 3.1 and 3.2 in [8]. Indeed we apply these results by considering the TDGL eq. (1.4) with initial distribution μ; note that its solution S_t, $t > 0$, is always μ-distributed and also that Lemma 3.1 guarantees the integrability $E^\mu[|S|_r^2] < \infty$, $r > 0$, which is required for using these results. We say a r.f. $S = \{S(x);\ x \in \boldsymbol{R}^d\}$ satisfies the regularity condition $(RC)_G^s$, $s > 0$, $G \in \mathcal{V}$, if

$$E[\langle S, \psi\rangle_G^2] \leq \text{const}\, \|\psi\|_{H^{-s}(G)}^2,\ \psi \in L^2(G)\,.$$

Lemma 3.4. *Take $G \in \mathcal{V}$. Then a μ-distributed r.f. $S = \{S(x);\ x \in \boldsymbol{R}^d\}$ is weakly C^{m-1} at $\Gamma = \partial G$. Moreover the family of g.r.f.'s $Y = \{Y_i(\psi) \equiv \frac{d^i}{dh^i} F_S(h, \psi; \Gamma)|_{h=0}\}_{i=0}^{m-1}$ on Γ and the r.f. S satisfy the regularity conditions $(RC)_\Gamma^s$ and $(RC)_G^s$, respectively, for every s: $0 < s < m$.*

We shall sometimes denote the Friedrichs extension $\bar{\mathcal{A}}_G$ introduced in Sect. 2 simply by $\mathcal{A}$ when there is no afraid of confusion. The domain of this operator is given by $\mathcal{D}(\mathcal{A}) = H^{2m}(G) \cap H_0^{m+1/2}(G)$. We define a function $\Lambda_G(\varphi, S; \omega)$ by

$$\Lambda_G(\varphi, S; \omega) = \sum_{i=0}^{m-1} Y_i(\delta_i\varphi)(\omega) - \frac{1}{2} \|\sqrt{\mathcal{A}}\varphi\|_{L^2(G)}^2 \tag{3.9}$$
$$+ \int_G [V(x, S(x)) - V(x, \varphi(x) + S(x)) - S(x) \cdot \mathcal{A}\varphi(x)]\, dx\,,$$

for $\varphi \in \mathcal{D}(\mathcal{A})$, $S \in \boldsymbol{C}$, $\omega \in \Omega(=\boldsymbol{C})$, where $\{Y_i(\psi) = Y_i(\psi)(\omega)\}_{i=0}^{m-1}$ is a family of g.r.f.'s on Γ appearing in Lemma 3.4. Note that $\Lambda_G(\varphi, S; \omega)$ is $\mathscr{B}(G)$-measurable in S and $\mathscr{B}_-(\Gamma)$-measurable in ω. The regular conditional probability distribution $\mu(\cdot\,|\,\mathscr{B}((G^c)^\circ))(\omega)$ of μ given $\mathscr{B}((G^c)^\circ)$ will be occasionally regarded as a probability measure on $C(\bar{G})$ instead of $\boldsymbol{C}$ for μ-a.s. ω.

Proposition 3.1. *$\mu(\cdot\,|\,\mathscr{B}((G^c)^\circ))(\omega)$ is $\mathscr{D}(\mathscr{A})$-quasi-invariant with cocycle $\Lambda_G(\varphi, S; \omega)$ for μ-a.s. ω.*

Proof. Lemma 3.3 shows with the help of [16, Proposition 3] that $\mu(\cdot\,|\,\mathscr{B}((G^c)^\circ))(\omega)$ is $C_0^\infty(G)$-quasi-invariant with cocycle $\Lambda_G(\varphi, S)$ μ-a.s. ω, where

$$\begin{aligned}\Lambda_G(\varphi, S) = \int_G [V(x, S(x)) - V(x, \varphi(x)+S(x)) - S(x)\cdot\mathscr{A}\varphi(x)]\,dx \\ -\frac{1}{2}\|\sqrt{\mathscr{A}}\varphi\|_{L^2(G)}^2, \quad \varphi\in C_0^\infty(G),\ S\in\boldsymbol{C}\,.\end{aligned} \tag{3.10}$$

This means

$$\begin{aligned}&\int \Phi(S-\varphi)\,\mu(dS\,|\,\mathscr{B}((G^c)^\circ))(\omega) \\ &= \int \Phi(S)\,e^{\Lambda_G(\varphi,S)}\,\mu(dS\,|\,\mathscr{B}((G^c)^\circ))(\omega),\ \mu\text{—a.s. } \omega\,,\end{aligned} \tag{3.11}$$

for every $\Phi\in C_b(\Omega)$ and $\varphi\in C_0^\infty(G)$. The goal is, however, to prove the quasi-invariance for the space $\mathscr{D}(\mathscr{A})$. To this end, for $\varphi\in\mathscr{D}(\mathscr{A})$, we take an approximating sequence $\{\varphi_n\in C_0^\infty(G)\}_{n=1}^\infty$ in such a manner that $\|\varphi_n-\varphi\|_{H^{m+1/2}(G)}\to 0$ as $n\to\infty$. This is certainly possible because $\mathscr{D}(\mathscr{A})\subset H_0^{m+1/2}(G)$. We shall verify that $\Lambda_G(\varphi_n, S)$ converges to $\Lambda_G(\varphi, S; S)$ as $n\to\infty$ in a proper sense. It is easy to see

$$\int_G V(x, \varphi_n(x)+S(x))\,dx \to \int_G V(x, \varphi(x)+S(x))\,dx\,, \quad n\to\infty\,, \tag{3.12}$$

for every $S\in\boldsymbol{C}$ and, using Gårding's inequality [1],

$$\|\sqrt{\mathscr{A}}\varphi_n\|_{L^2(G)} \to \|\sqrt{\mathscr{A}}\varphi\|_{L^2(G)}\,, \quad n\to\infty\,. \tag{3.13}$$

The limit of the remainder term in $\Lambda_G(\varphi_n, S)$ is given by the next lemma.

Lemma 3.5. *As $n\to\infty$, $\int_G S(x)\mathscr{A}\varphi_n(x)dx$ converges to $\int_G S(x)\,\mathscr{A}\varphi(x)\,dx - \sum_{i=0}^{m-1} Y_i(\delta_i\varphi)(S)$ in the space $L^2(d\mu)$.*

Proof. Take a non-negative function $\eta\in C_0^\infty(\boldsymbol{R}^d)$ satisfying $\eta(-x)=\eta(x)$, $\int_{\boldsymbol{R}^d}\eta(x)\,dx=1$ and $\eta\equiv 0$ on $\{|x|\geq 1\}$. We set $\eta^\varepsilon(x)=\varepsilon^{-d}\,\eta(x/\varepsilon)$ and $S^\varepsilon(x)=S*\eta^\varepsilon(x)$ for $S\in\boldsymbol{C}$. Then, since $\varphi\in\mathscr{D}(\mathscr{A})$ implies $\delta_i\varphi=0$ for $i\geq m$, Green's formula (2.3) shows

$$\langle S^\varepsilon, \mathscr{A}\varphi\rangle_G = \langle\mathscr{A}S^\varepsilon, \varphi\rangle_G + \sum_{i=0}^{m-1}\langle\partial_i^+ S^\varepsilon, \delta_i\varphi\rangle_\Gamma\,,$$

while it is clear that

$$\langle S^\varepsilon, \mathcal{A}\varphi_n\rangle_G = \langle \mathcal{A}S^\varepsilon, \varphi_n\rangle_G$$

for $\varphi_n \in C_0^\infty(G)$. Therefore the proof can be completed by combining the following four assertions:

$$\lim_{n\to\infty} E^\mu[\langle \mathcal{A}S^\varepsilon, \varphi_n-\varphi\rangle_G^2] = 0\,, \quad 0<\varepsilon<1\,, \tag{3.14}$$

$$\lim_{\varepsilon\downarrow 0} \sup_n E^\mu[\langle S-S^\varepsilon, \mathcal{A}\varphi_n\rangle_G^2] = 0\,, \tag{3.15}$$

$$\lim_{\varepsilon\downarrow 0} E^\mu[\langle S-S^\varepsilon, \mathcal{A}\varphi\rangle_G^2] = 0\,, \tag{3.16}$$

$$\lim_{\varepsilon\downarrow 0} E^\mu[\{\langle \partial_i^+ S^\varepsilon, \delta_i\varphi\rangle_\Gamma - Y_i(\delta_i\varphi)\}^2] = 0\,, \quad 0\le i\le m-1\,. \tag{3.17}$$

The first (3.14) is easily shown. Indeed, since $\lim_{n\to\infty} ||\varphi_n-\varphi||_{L^2(G)}=0$, we may only notice $E^\mu[||\mathcal{A}S^\varepsilon||^2_{L^2(G)}]<\infty$, which follows from the integrability condition of μ and an equality $\mathcal{A}S^\varepsilon(x)=\{(\mathcal{A}\eta^\varepsilon)*S\}(x)$.

In order to prove (3.15) and (3.16), we notice

$$\begin{aligned} E[\langle S-S^\varepsilon, \mathcal{A}\psi\rangle_G^2] &= E[\langle S, \mathcal{A}(\psi-\eta^\varepsilon*\psi)\rangle_{G_1}^2] \\ &\le \text{const}\, ||\mathcal{A}(\psi-\eta^\varepsilon*\psi)||^2_{H^{-s}(G_1)} \\ &\le \text{const}\, ||\psi-\eta^\varepsilon*\psi||^2_{H^{2m-s}(G_1)}\,, \quad \psi\in\mathcal{D}(\mathcal{A}),\ 0<\varepsilon<1\,, \end{aligned} \tag{3.18}$$

for every s: $0<s<m$, where $G_1=\{y\in\boldsymbol{R}^d;\ \text{dis}\,(y, G)<1\}$ and we regard $\psi\equiv 0$ on G^c. We have used $(RC)^s_{G_1}$, $0<s<m$, for the first inequality in (3.18) and the result of [14, p 195, Th. 8.3] by noting $\partial_i^+\{\psi-\eta^\varepsilon*\psi\}\,|_{\partial G_1}=0$, $0\le i\le m-1$, for the second. Therefore (3.15) and (3.16) can be established by showing

$$\lim_{\varepsilon\downarrow 0} \sup_n ||\varphi_n-\eta^\varepsilon*\varphi_n||_{H^{2m-s}(G)} = 0 \tag{3.19}$$

and

$$\lim_{\varepsilon\downarrow 0} ||\varphi-\eta^\varepsilon*\varphi||_{H^{2m-s}(G_1)} = 0\,, \quad \varphi\in\mathcal{D}(\mathcal{A})\,, \tag{3.20}$$

respectively, for every s: $m-1/2<s<m$; the details are omitted since the argument is standard.

Finally, to prove (3.17), we put $\psi=\delta_i\varphi\in H^{i+1/2}(\Gamma)\subset L^2(\Gamma)$ and notice that

$$Y_i(\psi) = \frac{d^i}{dh^i} F_S(h, \psi; \Gamma)\,|_{h=0}$$

and

$$\langle \partial_i^+ S^\varepsilon, \psi\rangle_\Gamma = \int_{\boldsymbol{R}^d} \eta^\varepsilon(y) \frac{d^i}{dh^i} F_{\tau_y S}(h, \psi; \Gamma)\,|_{h=0}\, dy\,,$$

where τ_y, $y\in\boldsymbol{R}^d$, is the shift operator on $\boldsymbol{C}$ defined by $(\tau_y S)(x)=S(x-y)$, $S\in\boldsymbol{C}$. However, Theorem 3.1-(iii) in [8] proves

$$\lim_{\varepsilon\downarrow 0}\sup_{y;|y|\leq\varepsilon} E^{\mu}[|\frac{d^i}{dh^i}F_S(h,\psi;\Gamma)|_{h=0}-\frac{d^i}{dh^i}F_{\tau_y S}(h,\psi;\Gamma)|_{h=0}|^2]=0\,,$$

and this implies (3.17). □

Now we continue the proof of Proposition 3.1. We can find a subsequence $\{n'\}$ of $\{n\}$ such that $\Lambda_G(\varphi_{n'}, S)\to\Lambda_G(\varphi, S; S)$ as $n'\to\infty$ μ-a.s. S and therefore $\Lambda_G(\varphi_{n'}, S)\to\Lambda_G(\varphi, S;\omega)$ as $n'\to\infty$ $\mu(\cdot\,|\,\mathscr{B}((G^c)^\circ))(\omega)$-a.s. S, μ-a.s.ω; combine Lemma 3.5 with (3.12) and (3.13). Noting that the equality (3.11) holds with φ replaced by φ_n, we take the limit $n\to\infty$ of both sides. Since Sobolev's imbedding theorem guarantees that $\varphi_n\to\varphi$ in $\boldsymbol{C}$, we see that the LHS of (3.11) with $\varphi=\varphi_n$ converges as $n\to\infty$ to $\int\Phi(S-\varphi)\,\mu(dS\,|\,\mathscr{B}((G^c)^\circ))(\omega)$. On the other hand, assuming $\Phi\geq 0$, we can use Fatou's lemma to see

$$\varliminf_{n'\to\infty}\{\text{the RHS of (3.11) with }\varphi=\varphi_{n'}\}\geq\int\Phi(S)\,e^{\Lambda_G(\varphi,S;\omega)}\,\mu(dS\,|\,\mathscr{B}((G^c)^\circ))(\omega)\,.$$

Therefore we obtain

$$\int\Phi(S-\varphi)\,\mu(dS\,|\,\mathscr{B}((G^c)^\circ))(\omega)\geq\int\Phi(S)\,e^{\Lambda_G(\varphi,S;\omega)}\,\mu(dS\,|\,\mathscr{B}((G^c)^\circ))(\omega),\ \mu\text{—a.s.}\omega\,,$$

for every non-negative $\Phi\in C_b(\Omega)$ and $\varphi\in\mathscr{D}(\mathscr{A})$. This proves the conclusion, namely (3.11) holds for all $\Phi\in C_b(\Omega)$ and $\varphi\in\mathscr{D}(\mathscr{A})$; see [11, Lemma (3.5)]. □

Let $X'=X'(x,\omega)$, $x\in G$, $\omega\in\Omega$, be the solution of SD problem (2.1) with the boundary data $Y=\{Y_i(\psi)\}_{i=0}^{m-1}$, which is determined by Lemma 3.4. We define a function $\Lambda'_G(\varphi, S;\omega)$ by

$$\Lambda'_G(\varphi,S;\omega)=-\frac{1}{2}\|\sqrt{\mathscr{A}}\varphi\|^2_{L^2(G)}-\langle S,\mathscr{A}\varphi\rangle_G$$
$$+\int_G[V(x,S(x)+X'(x,\omega))-V(x,\varphi(x)+S(x)+X'(x,\omega))]\,dx\,,$$

for $\varphi\in\mathscr{D}(\mathscr{A})$, $S\in C_\omega(G)$, $\omega\in\Omega(=\boldsymbol{C})$. Here $C_\omega(G)=\{S\in C(G);\ S+X'(\omega)\in C(\bar{G})\}$ and we set $\langle S,\mathscr{A}\varphi\rangle_G=\langle S+X'(\omega),\mathscr{A}\varphi\rangle_G-\langle X'(\omega),\mathscr{A}\varphi\rangle_G$, where the first term is defined by the usual integral and the second by (2.4).

Lemma 3.6. *$\mu(\cdot+X'(\omega)\,|\,\mathscr{B}((G^c)^\circ))(\omega)$ is $\mathscr{D}(\mathscr{A})$-quasi-invariant with cocycle $\Lambda'_G(\varphi, S;\omega)$ for μ-a.s. ω.*

Proof. From Proposition 3.1, we have

$$\log\frac{\mu(\varphi+dS+X'(\omega)\,|\,\mathscr{B}((G^c)^\circ))(\omega)}{\mu(dS+X'(\omega)\,|\,\mathscr{B}((G^c)^\circ))(\omega)}$$
$$=\Lambda_G(\varphi, S+X'(\omega);\omega),\quad\varphi\in\mathscr{D}(\mathscr{A}),\ \mu\text{—a.s. }\omega\,.$$

However the RHS coincides with $\Lambda'_G(\varphi,S;\omega)$. Indeed $\varphi\in\mathscr{D}(\mathscr{A})$ implies $\mathscr{A}\varphi\in L^2(G)$ and $f^{\mathscr{A}\varphi}=\varphi$, so that $\langle X',\mathscr{A}\varphi\rangle_G=\sum_{i=0}^{m-1}Y_i(\delta_i\varphi)$ from (2.4). □

Define $\nu_{0,\omega}^{G}\in\mathcal{P}(C_\omega(G))$, for μ-a.s. ω, by

$$\nu_{0,\omega}^{G}(dS)=Z_\omega^{-1}\exp\{\int_G V(x, S(x)+X'(x,\omega))\,dx\}\,\mu(dS+X'(\omega)\,|\,\mathcal{B}((G^c)^\circ))(\omega),$$

where Z_ω is the normalization constant. Now the proof of Theorem 2.1; reversible $\Rightarrow$ Gibbs can be completed by showing the next proposition. We recall Sect. 2.1 for the definition of $\nu_0^G\in\mathcal{P}(C(\bar{G}))$.

Proposition 3.2. $\nu_{0,\omega}^{G}=\nu_0^G$, μ-a.s. ω.

Proof. It is easily seen from Lemma 3.6 that $\nu_{0,\omega}^{G}$ is $\mathcal{D}(\mathcal{A})$-quasi-invariant with cocycle

$$\text{(3.21)}\quad \Lambda_G^0(\varphi, S)=-\frac{1}{2}\|\sqrt{\mathcal{A}}\varphi\|_{L^2(G)}^2-\langle S, \mathcal{A}\varphi\rangle_G,\ \varphi\in\mathcal{D}(\mathcal{A}),\ S\in C_\omega(G).$$

Let $\{\bar{\varphi}_n\}_{n=1}^\infty$ and $\{\lambda_n>0\}_{n=1}^\infty$ be the sets of all eigenfunctions and the corresponding eigenvalues of the operator $\mathcal{A}$, respectively. We denote by π: $C(G)\cap L^2(G)\ni S\mapsto\{w_n=\langle S, \bar{\varphi}_n\rangle\}_{n=1}^\infty\in l^2$, the map giving the coefficients of Fourier series expansion of S based on the CONS (complete orthonormal system) $\{\bar{\varphi}_n\}_{n=1}^\infty$ of $L^2(G)$. We also denote by $\boldsymbol{R}^{(N)}$ the class of all $c=(c_n)\in\boldsymbol{R}^N$ such that $c_n=0$ for all but finitely many n's. Then the quasi-invariance of $\nu_{0,\omega}^{G}$ proves that $\nu_{0,\omega}^{G}(L^2(G))=1$, μ-a.s. ω, (use similar argument to [16, Lemma 8]) and the image measure $\nu_{0,\omega}^{G}\circ\pi^{-1}\in\mathcal{P}(l^2)$ of $\nu_{0,\omega}^{G}$ is $\boldsymbol{R}^{(N)}$-quasi-invariant with cocycle

$$\begin{aligned}\Lambda^*(c, w)&=\Lambda_G^0\Big(\sum_{n=1}^\infty c_n\bar{\varphi}_n, \sum_{n=1}^\infty w_n\bar{\varphi}_n\Big)\\&=-\sum_{n=1}^\infty\lambda_n\Big(w_n+\frac{1}{2}c_n\Big)c_n\end{aligned}$$

for $c=(c_n)\in\boldsymbol{R}^{(N)}$ and $w=(w_n)\in l^2$. However, this proves $\nu_{0,\omega}^{G}\circ\pi^{-1}=\nu_0^G\circ\pi^{-1}$ (see [16]) and consequently we obtain $\nu_{0,\omega}^{G}=\nu_0^G$. □

4. The proof of Theorem 2.1; Gibbs $\Rightarrow$ reversible

In this section we always assume $\mu\in\mathcal{G}(V)$. The argument in the previous section can be essentially followed in the converse direction.

Lemma 4.1. *The Gibbs state* μ *is* $C_0^\infty(\boldsymbol{R}^d)$*-quasi-invariant with cocycle* $\Lambda(\varphi, S)$ *defined by* (3.7).

Proof. We begin with observing that the Gaussian measure $\nu_0^G\in\mathcal{P}(C(\bar{G}))$ is $\mathcal{D}(\mathcal{A})$-quasi-invariant with cocycle $\Lambda_G^0(\varphi, S)$ defined by (3.21). This is actually shown by transforming ν_0^G into a measure on the space l^2 by using the map π introduced in the proof of Proposition 3.2. We especially find as a consequence that ν_ω^G is also $C_0^\infty(G)$-quasi-invariant with cocycle $\Lambda_G^0(\varphi, S)$ for μ-a.s. ω, since $\langle X'(\omega), \mathcal{A}\varphi\rangle_G=0$ μ-a.s. ω if $\varphi\in C_0^\infty(G)$; see Sect. 2.2. for the definition of ν_ω^G. It is then easy to show that the finite-volume Gibbs state $\mu_\omega^G\in\mathcal{P}(C(G))$ is $C_0^\infty(G)$-quasi-invariant with cocycle $\Lambda_G(\varphi, S)$ defined by (3.10) for μ-a.s. ω. Now take an arbitrary $\varphi\in C_0^\infty(\boldsymbol{R}^d)$ and a non-negative $\Psi\in\mathcal{D}$ of the form (2.8). Let $G\in\mathcal{V}$ be an open set such that it includes supp φ and supp φ_i, φ_i appearing in (2.8), for every $1\leq i\leq k$. Then, by using the DLR equation, we obtain

$$\begin{aligned}E^{\mu(\varphi+\cdot)}[\Psi] &= E^\mu[\Psi(S-\varphi)]\\ &= E^\mu[E^{\mu_\omega^G}[\Psi(S-\varphi)]]\\ &= E^\mu[E^{\mu_\omega^G}[\Psi(S)\,e^{\Lambda_G(\varphi,S)}]]\\ &= E^\mu[\Psi(S)\,e^{\Lambda(\varphi,S)}]\,,\end{aligned}$$

which concludes the proof of lemma. □

This lemma gives particularly an information on the integrability of μ:

Corollary 4.1. *For every $\varphi\in C_0^\infty(\boldsymbol{R}^d)$, $E^\mu[e^{\Lambda(\varphi,S)}]<\infty$ and $E^\mu[e^{|\langle S,\mathcal{A}\varphi\rangle|}]<\infty$.*

Lemma 4.2. *The Gibbs state μ satisfies the equality* (3.3) *for every Φ, $\Psi\in\mathcal{D}$.*

Proof. Notice that the LHS of (3.3) is integrable because of Corollary 4.1. First we prove (3.3) for every $\Phi\in\mathcal{D}$ and $\Psi=\Psi_0\in\mathcal{D}$ having the form $\Psi_0(S)=\psi(\langle S, \varphi\rangle)$ with $\psi\in C_b^2(\boldsymbol{R})$, $\varphi\in C_0^\infty(\boldsymbol{R}^d)$. Indeed this can be done by differentiating the both sides of the following equality in t and then setting $t=0$:

$$E^\mu[\Phi(S)\,\psi'(\langle S, \varphi\rangle)] = E^\mu[\Phi(S-t\varphi)\,\psi'(\langle S-t\varphi, \varphi\rangle)\,e^{-\Lambda(t\varphi, S-t\varphi)}]\,,\quad t\in\boldsymbol{R}\,,$$

which follows from Lemma 4.1. The next remark is that if (3.3) is true for every $\Phi\in\mathcal{D}$ and some $\Psi=\Psi_1\in\mathcal{D}$ then this equality still holds for every $\Phi\in\mathcal{D}$ and $\Psi=\Psi_1\cdot\Psi_0$, where Ψ_0 is an arbitrary function of the above form. Actually we may just use (3.5). Therefore the recursive application of this fact verifies (3.3) for every $\Phi\in\mathcal{D}$ and $\Psi\in\mathcal{D}$ of the form $\Psi(S)=\prod_{i=1}^k \psi_i(\langle S, \varphi_i\rangle)$ with $k=1, 2, \cdots$, $\psi_i\in C_b^2(\boldsymbol{R})$ and $\varphi_i\in C_0^\infty(\boldsymbol{R}^d)$. This completes the proof. □

Proof of Theorem 2.1: Gibbs ⇒ reversible: Take an arbitrary separable Hilbert space $\boldsymbol{B}$ in such a way that $\boldsymbol{L}_r^2\subset\boldsymbol{B}\subset\{C_0^\infty(\boldsymbol{R}^d)\}'=$ {Schwartz's space of generalized functions} with the inclusion map of $\boldsymbol{L}_r^2\to\boldsymbol{B}$ being compact, e.g., take the dual space of $H^m(\boldsymbol{R}^d)$ with respect to the space $\boldsymbol{L}_r^2$. Then, $\mu\in\mathcal{P}(\boldsymbol{L}_r^2)$ is extended naturally on $\boldsymbol{B}$ and also every $\Phi\in\mathcal{D}$ can be regarded as a function on $\boldsymbol{B}$. Let us consider a symmetric form on $L^2(\boldsymbol{B}, d\mu)$ defined by

$$\tilde{\mathcal{E}}(\Phi, \Psi) = \frac{1}{2}\int_B \langle D\Phi, D\Psi\rangle\,\mu(dz),\ \Phi, \Psi\in\mathcal{D}\,,$$

(we regard $\langle\cdot,\cdot\rangle$ as an inner product of the Hilbert space $L^2(\boldsymbol{R}^d, dx)$). Note that $\mathcal{D}$ is dense in $L^2(\boldsymbol{B}, d\mu)$ and Lemma 4.2 implies $\tilde{\mathcal{E}}(\Phi, \Psi)=-(\Phi, \mathcal{L}\Psi)_{L^2(\boldsymbol{B}, d\mu)}$ for $\Phi, \Psi\in\mathcal{D}$. We see easily that $(\tilde{\mathcal{E}}, \mathcal{D})$ is a closable Markovian symmetric form on $L^2(\boldsymbol{B}, d\mu)$. Moreover, its minimal closed extension determines a diffusion process z_t on $\boldsymbol{B}$, i.e., there exists a process z_t with generator being the Friedrichs extension A of $(\mathcal{L}, \mathcal{D})$ on the space $L^2(\boldsymbol{B}, d\mu)$. To this end, we rely on the paper of Kusuoka [13]. Indeed we have only to check the condition (C.2) of [13] and this follows from the assumption $E^\mu[|S|_r^2]<\infty$ since it implies $E^\mu[\|z\|_{\boldsymbol{B}}^2]<\infty$. Then it is possible to show that z_t is in fact an $\boldsymbol{L}_r^2$-valued diffusion process. Actually this follows by applying the same argument used for the proof of Propositions 3.7 and 3.8 in [13] by replacing M, B and B_0 there with $\boldsymbol{B}$, $\boldsymbol{L}_r^2$ and $\boldsymbol{B}_0$, respectively, in our situation. Here $\boldsymbol{B}_0$ is a Hilbert space constructed as follows: Fix an arbitrary CONS $\{e_n\}_{n=1}^\infty$ of $\boldsymbol{L}_r^2$ and take $1\leq\lambda_n\nearrow\infty$, $n\to\infty$, such that

$$\sum_{n=1}^\infty \lambda_n E^\mu[\langle S, e_n\rangle_{\boldsymbol{L}_r^2}^2]<\infty . \tag{4.1}$$

The space $\boldsymbol{B}_0$ is the completion of $C_0^\infty(\boldsymbol{R}^d)$ with respect to the norm

$$\|S\|_{\boldsymbol{B}_0}^2=\sum_{n=1}^\infty \sqrt{\lambda_n}\langle S, e_n\rangle_{\boldsymbol{L}_r^2}^2 .$$

From the construction the imbedding of $\boldsymbol{B}_0\to\boldsymbol{L}_r^2$ is compact and, using (4.1), we see that the Choquet capacity $\widetilde{\mathrm{Cap}}\,(\boldsymbol{B}-\boldsymbol{B}_0)$ on $\boldsymbol{B}$ vanishes. Now, since $A\Psi=\mathcal{L}\Psi$ μ-a.e. for $\Psi\in\mathcal{D}$, the similar argument given in Fukushima and Stroock [6, Theorem (2.9)] verifies that the distribution on $C([0,\infty), \boldsymbol{L}_r^2)$ of the process z_t coincides with that of our process S_t (i.e., the solution of (1.4)) if the initial distributions of z_0 and S_0 are common and absolutely continuous with respect to μ; we should notice that the well-posedness of $(\mathcal{L}, \mathcal{D})$-martingale problem is established by Theorem 4.1 in [8]. The proof is therefore completed since μ is a reversible measure of z_t. □

REMARK 4.1. For a uniformly positive function $c=c(S)\in\mathcal{D}$, the solution of an SPDE:

$$\begin{aligned} dS_t(x) &= -\frac{c(S_t)}{2}\mathcal{A}S_t(x)\,dt-\frac{c(S_t)}{2}V'(x, S_t(x))\,dt \\ &\quad +\frac{1}{2}Dc(x, S_t)\,dt+\sqrt{c(S_t)}\,dw_t(x) \end{aligned} \tag{4.2}$$

can be constructed by means of the time change method. Small changes in the proof of Theorem 2.1 show that the family of all reversible probability measures of this equation coincides with $\mathcal{G}(V)$.

5. Energy inequality and stationary measures

As a slight generalization of the eq. (1.4) we consider the following SPDE for a given $b=b(x, S)\in \cap_{r>0} L_b(\boldsymbol{L}_r^2)$:

$$(5.1;b) \qquad dS_t(x) = -\frac{1}{2}\mathcal{A}S_t(x)\,dt+b(x, S_t)\,dt+dw_t(x)\,,$$

which has a unique solution satisfying (1.9), see [8]. The purpose of the present section is to investigate the class $\mathcal{S}(b)$ of all stationary probability measures $\mu\in\mathcal{P}(\boldsymbol{C}\cap\boldsymbol{L}_r^2)$ of this equation. We introduce a norm

$$|\sigma|_{r,m} = \{\sum_{|\alpha|\le m}|D^\alpha\sigma|_r^2\}^{1/2},\ \sigma\in C_0^\infty(\boldsymbol{R}^d),\ r\in\boldsymbol{R}\,.$$

Lemma 5.1. *For every $\delta>0$, there exist $r_1=r_1(\delta)$, $C=C(\delta)>0$ such that*

$$(5.2) \qquad \langle\sigma, \mathcal{A}\sigma\rangle_r\ge(\gamma-\delta)|\sigma|_r^2+C|\sigma|_{r,m}^2\,, \quad \sigma\in C_0^\infty(\boldsymbol{R}^d),\ |r|\le r_1\,.$$

Proof. Consider an operator $\mathcal{A}'=\mathcal{A}-\gamma+\delta$, $\delta>0$. Then $\mathcal{A}'$ is uniformly strongly elliptic and strictly positive. Therefore, by Gårding's inequality [1], the norm $||f||_{\mathcal{A}'}=\langle f, \mathcal{A}'f\rangle^{1/2}$ is equivalent to the Sobolev norm $||f||_m=\{\sum_{|\alpha|\le m}||D^\alpha f||_{L^2(\boldsymbol{R}^d)}^2\}^{1/2}$ on the space $C_0^\infty(\boldsymbol{R}^d)$. To complete the proof, we see

$$\begin{aligned}\langle\sigma, \mathcal{A}\sigma\rangle_r &= \sum_{|\alpha|,|\beta|\le m}\int_{\boldsymbol{R}^d} a_{\alpha,\beta}\,D^\alpha\sigma\cdot D^\beta\{\sigma e^{-2r\chi}\}\,dx\\ &= ||\sigma e^{-r\chi}||_{\mathcal{A}'}^2+(\gamma-\delta)|\sigma|_r^2+R\,,\end{aligned}$$

where

$$R = \sum_{|\alpha|,|\beta|\le m}\int_{\boldsymbol{R}^d} a_{\alpha,\beta}[D^\alpha\sigma\cdot D^\beta\{\sigma e^{-2r\chi}\}-D^\alpha\{\sigma e^{-r\chi}\}\,D^\beta\{\sigma e^{-r\chi}\}]\,dx\,.$$

However, since $|e^{-r\chi(x)}D^\alpha e^{r\chi(x)}|\le\text{const}\,|r|$ holds for $x\in\boldsymbol{R}^d$, $|r|\le 1$ and $1\le|\alpha|\le m$, we obtain by using Leibniz's rule

$$(5.3) \qquad |e^{-r\chi(x)}D^\alpha\{e^{r\chi(x)}f(x)\}-D^\alpha f(x)|\le\text{const}\,|r|\sum_{\alpha':\,\alpha'<\alpha}|D^{\alpha'}f(x)|\,,$$

for all $|r|\le 1$, $|\alpha|\le m$ and $f\in C_0^\infty(\boldsymbol{R}^d)$, where $\alpha'=(\alpha_1', \cdots, \alpha_d')<\alpha=(\alpha_1, \cdots, \alpha_d)$ means that $\alpha'\le\alpha$ (i.e. $\alpha_i'\le\alpha_i$ for $1\le i\le d$) and $|\alpha'|<|\alpha|$. This, by taking $f=\sigma e^{-r\chi}$, verifies a bound:

$$|R|\le\text{const}\,|r|\,||\sigma e^{-r\chi}||_m^2\times\sup_{x\in\boldsymbol{R}^d;\,|\alpha|,|\beta|\le m}|a_{\alpha,\beta}(x)|\,,\ |r|\le 1\,.$$

Hence, the conclusion follows since (5.3) also implies that $|\sigma|_{r,m}\le\text{const}\,||\sigma e^{-r\chi}||_m$ with *const* independent of r; $|r|\le 1$. □

REMARK 5.1. The two constants $r_1=r_1(\delta)$ and $C=C(\delta)$ in Lemma 5.1 depend on $\mathcal{A}$ only through the following four quantities: the constant c appearing in the condition (1.5), γ, $\sup\{|a_{\alpha,\beta}(x)|\,;x\in\boldsymbol{R}^d,\ |\alpha|,\ |\beta|\le m\}$ and the modulus of continuity of $\{a_{\alpha,\beta}\}_{|\alpha|=|\beta|=m}$; c.f. [1].

Corollary 5.1. *(Energy inequalities) Suppose that two functions* $\sigma_t(x)$ *and* $v_t(x)$ *are given and satisfy*

$$(5.4)\qquad \begin{cases} \dfrac{\partial}{\partial t}\sigma_t(x) = -\dfrac{1}{2}\mathcal{A}\sigma_t(x)+v_t(x)\,, & t>0, x\in \boldsymbol{R}^d \\ \langle\sigma_t, v_t\rangle_r \leq \dfrac{c_1}{2}|\sigma_t|_r^2+c_2 \end{cases}$$

with some $c_1<\gamma$ *and* $c_2=c_2(r)>0$ *for every* $r>0$. *Then, for each* c *and* r *such that* $0<c<\gamma-c_1$, $0<r<r_1(\gamma-c_1-c)$, *we have*

$$(5.5)\qquad |\sigma_t|_r^2\leq e^{-ct}|\sigma_0|_r^2+2c_2/c,\ t>0$$

and

$$(5.6)\qquad \int_0^t |\sigma_u|_{r,m}^2\, du\leq (C(\gamma-c_1-c))^{-1}\{|\sigma_0|_r^2+2c_2t\},\ t>0\,.$$

Proof. From Lemma 5.1 and condition (5.4), we have

$$\begin{aligned}\frac{d}{dt}|\sigma_t|_r^2 &= -\langle\sigma_t, \mathcal{A}\sigma_t\rangle_r+2\langle\sigma_t, v_t\rangle_r \\ &\leq -(\gamma-\delta-c_1)|\sigma_t|_r^2-C(\delta)|\sigma_t|_{r,m}^2+2c_2\end{aligned}$$

for every $\delta>0$ and $0<r<r_1(\delta)$. This implies the conclusion; take $\delta=\gamma-c_1-c$. □

We show as the first application of this corollary the existence of stationary measures of (5.1; b) and their uniform integrability.

Proposition 5.1. $\mathcal{S}(b)\neq\emptyset$ *for every* $b\in\bigcap_{r>0}L_b(\boldsymbol{L}_r^2)$. *Moreover, for every sufficiently small* $r>0$, *there exists* $\beta=\beta(r)>0$ *such that*

$$(5.7)\qquad \sup\{E^\mu[e^{\beta|S|_r^2}];\ \mu\in\mathcal{S}(b),\ |b|_{r,(\infty)}\leq K\}<\infty,\ K>0\,.$$

Proof. Consider the following coupling: Let S_t and $\bar{S}_t$ be the solutions of the SPDE's (5.1; b) and (5.1; 0), respectively, with common c.B.m.'s $w_t(x)$. As for the initial data, we choose an arbitrary point $S\in\boldsymbol{L}_e^2$ for the process S_t, while we assume $\bar{S}_0$ is a ν-distributed random variable (r.v.) which is independent of $\{w_t\}$. We shall denote this coupling simply by $\{S_t, \bar{S}_t\}\sim\{(b, \delta_S), (0, \nu)\}$. Note that ν is stationary for $\bar{S}_t$, see Proposition 6.1 below. The difference $\sigma_t=S_t-\bar{S}_t$ of these two processes satisfies the condition (5.4) with $v_t(x)=b(x, S_t)$ and arbitrary $0<c_1<\gamma$. We denote the distribution of S_t on $\boldsymbol{L}_r^2$ by μ_t and its Cesàro mean by $\bar{\mu}_T$, i.e. $\bar{\mu}_T=\dfrac{1}{T}\displaystyle\int_0^T\mu_t\,dt$. Then, $\{\bar{\mu}_T\}_{T\geq 1}$ is tight on $\boldsymbol{L}_r^2$ for sufficiently small r. In fact, the $\boldsymbol{L}_r^2$-valued r.v. $S(t,\omega)=S_t(\omega)$ realized on the probability space $(\tilde{\Omega}_T, \tilde{P}_T)=(\Omega\times[0, T], P\times\frac{dt}{T})$ has the distribution $\bar{\mu}_T$, where (Ω, P) is the probability space on which the c.B.m. w_t is defined. Note that $S(t, \omega)$ has a decomposition $S(t, \omega)=\bar{S}(t, \omega)+\sigma(t, \omega)$ and the r.v. $\bar{S}(t, \omega)=\bar{S}_t(\omega)$ defined on $(\tilde{\Omega}_T, \tilde{P}_T)$ is always ν-distributed. Therefore, since $\nu(\boldsymbol{L}_r^2)=1$, the tightness of $\{\bar{\mu}_T\}$ follows if we can show the tightness of the $\boldsymbol{L}_r^2$-valued r.v.'s $\{\sigma(t, \omega)=\sigma_t(\omega)\}$. However, this is verified by observing that (5.6) implies

$$
\begin{aligned}
E^{\tilde{P}_T}[|\sigma(t, \omega)|_{r',m}^2] &= \frac{1}{T}\int_0^T E[|\sigma_t|_{r',m}^2]\, dt \\
&\leq \text{const}\ \{1+E[|\sigma_0|_{r'}^2]\}<\infty
\end{aligned} \tag{5.8}
$$

with *const* independent of $T\geq 1$ for every sufficiently small $r'>0$ and the imbedding map of $H_{r'}^m(\boldsymbol{R}^d)\rightarrow\boldsymbol{L}_r^2$ is compact if $0<r'<r$ (see Remark 2.1 in [8] for the definition of $H_{r'}^m(\boldsymbol{R}^d)$ and the compactness of imbedding map). Now, find a sequence $\{T_n\nearrow\infty\}$ and $\mu\in\mathcal{P}(\boldsymbol{L}_r^2)$ such that $\bar{\mu}_{T_n}\Rightarrow\mu$ weakly on $\boldsymbol{L}_r^2$. Then, it is easy to prove $\mu\in\mathcal{S}(b)$ by noting that the family of distributions $\{P_S\}_{S\in\boldsymbol{L}_r^2}$ of the solutions of (5.1; b) starting from $S\in\boldsymbol{L}_r^2$ has the Feller property, or, what amounts to the same, the map $S\in\boldsymbol{L}_r^2\mapsto P_S\in\mathcal{P}(C([0, \infty), \boldsymbol{L}_r^2))$ is continuous (this property is shown from the well-posedness of the corresponding martingale problem, Theorem 4.1 in [8], by employing the usual compactness argument).

To show the uniform integrability (5.7), we use the coupling $(S_t, \bar{S}_t)$ again, but this time we suppose the initial data S_0 of the process S_t is distributed by $\mu\in\mathcal{S}(b)$, i.e. we consider the coupling $\{S_t, \bar{S}_t\}\sim\{(b, \mu), (0, \nu)\}$. Then, we see from (5.5) that

$$
|S_t-\bar{S}_t|_r^2\leq e^{-ct}|S_0-\bar{S}_0|_r^2+C,\ t>0\,,
$$

for arbitrary $0<c<\gamma$ with some $C>0$ if $r>0$ is sufficiently small. Therefore we obtain

$$
\begin{aligned}
E^{\mu}[e^{\beta(|S|_r^2\wedge N)}] &= E[e^{\beta(|S_t|_r^2\wedge N)}] \\
&\leq E[\exp\{\beta a(C_1+|\bar{S}_t|_r^2)+\beta(N\wedge\frac{a}{a-1}e^{-C_2t}|S_0-\bar{S}_0|_r^2)\}] \\
&\leq E^{\nu}[\exp\{\beta ap(C_1+|\bar{S}|_r^2)\}]^{1/p} \\
&\quad\times E[\exp\{\beta q(N\wedge\frac{a}{a-1}e^{-C_2t}|S_0-\bar{S}_0|_r^2)\}]^{1/q}
\end{aligned} \tag{5.9}
$$

for every $t>0$, $a>1$ and $p, q>1$ such that $1/p+1/q=1$. Now use Lebesgue's dominated convergence theorem for letting $t\rightarrow\infty$ in the RHS of (5.9) and then apply Fatou's lemma to take the limit $N\rightarrow\infty$ in the LHS of (5.9). The conclusion for small r follows by noticing that ν is a Gaussian measure on $\boldsymbol{L}_r^2$, $r>0$, so that it holds $E^{\nu}[e^{\beta'|S|_r^2}]<\infty$ with some $\beta'>0$. □

We introduce the so-called Vasershtein metric on $\mathcal{P}(\boldsymbol{L}_r^2)$: Let $\mathcal{P}_2(\boldsymbol{L}_r^2)$, $r>0$, be the class of all $\mu\in\mathcal{P}(\boldsymbol{L}_r^2)$ satisfying $E^\mu[|S|_r^2]<\infty$. For $\mu_1,\mu_2\in\mathcal{P}_2(\boldsymbol{L}_r^2)$, we set

$$d_r(\mu_1,\mu_2)=\inf_m \sqrt{E^m[|S_1-S_2|_r^2]}\,,$$

where the infimum is taken over all $m\in\mathcal{P}(\boldsymbol{L}_r^2\times\boldsymbol{L}_r^2)$ satisfying $m\circ\pi_i^{-1}=\mu_i$, $i=1,2$; the maps π_i are projections defined by π_i: $\boldsymbol{L}_r^2\times\boldsymbol{L}_r^2\ni(S_1,S_2)\mapsto S_i\in\boldsymbol{L}_r^2$. We denote by $||\Phi||_{L,r}$ the Lipschitz constant of the functions Φ on $\boldsymbol{L}_r^2$ with respect to the norm $|\cdot|_r$, $r>0$, i.e.

$$|\Phi(S)-\Phi(\bar{S})|\le||\Phi||_{L,r}|S-\bar{S}|_r\,.$$

The second application of the energy inequalities is to give the uniqueness of stationary measures and show an ergodic property of the process under a proper condition on $b(x,S)$.

Proposition 5.2. *(i) Let b and $\bar{b}\in\bigcap_{r>0}L_b(\boldsymbol{L}_r^2)$ be given and satisfy*

$$\langle b(\cdot,S)-\bar{b}(\cdot,\bar{S}),S-\bar{S}\rangle_r\le\frac{c_1}{2}|S-\bar{S}|_r^2+c_2,\ S,\bar{S}\in\boldsymbol{L}_r^2\,, \tag{5.10}$$

with some $c_1<\gamma$ and $c_2=c_2(r)>0$ for every $r>0$. Then, for each $0<c<\gamma-c_1$ and $0<r<r_1(\gamma-c_1-c)$, we have

$$d_r(\mu,\bar{\mu})\le\sqrt{2c_2/c}\,,\quad \mu\in\mathcal{S}(b),\bar{\mu}\in\mathcal{S}(\bar{b})\,. \tag{5.11}$$

(ii) If $b\in\bigcap_{r>0}L_b(\boldsymbol{L}_r^2)$ satisfies

$$\langle b(\cdot,S)-b(\cdot,\bar{S}),S-\bar{S}\rangle_r\le\frac{c_1}{2}|S-\bar{S}|_r^2,\ S,\bar{S}\in\boldsymbol{L}_r^2\,, \tag{5.12}$$

with some $c_1<\gamma$ for every $r>0$, then $\sharp\mathcal{S}(b)=1$. Moreover the process S_t has an ergodic property in the following sense:

$$|E_S[\Phi(S_t)]-E^\mu[\Phi]|\le\sqrt{2\{|S|_r^2+E^\mu[|S|_r^2]\}}||\Phi||_{L,r}e^{-ct/2},\ \mu\in\mathcal{S}(b)\,, \tag{5.13}$$

for every c; $0<c<\gamma-c_1$, $S\in\boldsymbol{L}_r^2$, Lipschitz continuous Φ on $\boldsymbol{L}_r^2$ and $0<r<r_1(\gamma-c_1-c)$.

Proof. The assertion (i) follows easily by applying (5.5) for the difference $\sigma_t(x)=S_t(x)-\bar{S}_t(x)$ of the coupling $\{S_t,\bar{S}_t\}\sim\{(b,\mu),(\bar{b},\bar{\mu})\}$, where $\mu\in\mathcal{S}(b)$ and $\bar{\mu}\in\mathcal{S}(\bar{b})$. The uniqueness statement $\sharp\mathcal{S}(b)\le1$ in (ii) is an immediate consequence of (i). To show (5.13), use (5.5) and the coupling $\{S_t,\bar{S}_t\}\sim\{(b,\delta_S),(b,\mu)\}$. □

Now we apply this result to the equation (1.4). The potential functions $V=V(x,s)$ and $\bar{V}=\bar{V}(x,s)$ appearing in the rest of this section are assumed to satisfy the conditions (2.7) and (1.11). Notice that these two conditions imply $\gamma\ge\gamma_0$.

(I) The function $b=b_V(x, S)$ satisfies (5.12) with $c_1=\gamma-\gamma_0$ and this, in particular, shows $\mathcal{S}(b_V)=\{\mu_V\}$ with single $\mu_V\in\mathcal{P}(\boldsymbol{C}\cap\boldsymbol{L}_r^2)$.

(II) The estimate (5.10) holds for the pair of two functions $b=b_V$ and $\bar{b}=b_{\bar{V}}$ by taking $c_1=\gamma-\gamma_0+\varepsilon$ and $c_2=\frac{1}{2\varepsilon}|b_V-b_{\bar{V}}|^2_{r,(\infty)}$ for arbitrary ε: $0<\varepsilon<\gamma_0$. In fact, to see this, decompose the LHS of (5.10) into the sum of the LHS of (5.12) and $\langle b(\cdot, \bar{S})-\bar{b}(\cdot, \bar{S}), S-\bar{S}\rangle_r$. Then, the latter is bounded from above by $\frac{\varepsilon}{2}|S-\bar{S}|_r^2+\frac{1}{2\varepsilon}|b-\bar{b}|^2_{r,(\infty)}$. Therefore, we have

$$d_r(\mu_V, \mu_{\bar{V}})\leq \text{const}\ |b_V-b_{\bar{V}}|_{r,(\infty)}, \tag{5.14}$$

for every sufficiently small $r>0$. The *const* can be taken independently of r.

(III) For $g\in\mathcal{B}_1(\boldsymbol{R}^d)$ and $\lambda\in\boldsymbol{L}_e^2$, set

$$V_{g,\lambda}(x, s) = g(x)\ V(x, s)-\lambda(x)\ s\ . \tag{5.15}$$

Here $\mathcal{B}_1(\boldsymbol{R}^d)$ is the class of all measurable functions g on $\boldsymbol{R}^d$ such that $0\leq g\leq 1$ (a.e.). Then, since $V_{g,\lambda}$ still satisfies (2.7) and (1.11), we have $\#\mathcal{S}(b_{V_{g,\lambda}})=1$. Moreover, for given two pairs of functions (g, λ) and $(\bar{g}, \bar{\lambda})\in\mathcal{B}_1(\boldsymbol{R}^d)\times\boldsymbol{L}_e^2$, the following estimate holds:

$$|b_{V_{g,\lambda}}-b_{V_{\bar{g},\bar{\lambda}}}|_{r,(\infty)}\leq |\{g(\cdot)-\bar{g}(\cdot)\}\ b_V(\cdot, S)|_{r,(\infty)}+\frac{1}{2}|\lambda-\bar{\lambda}|_r\ . \tag{5.16}$$

The first term in the RHS of (5.16) is bounded further by $\frac{1}{2}|g-\bar{g}|_r\times \operatorname{esssup}_{x,s}|V'(x, s)|$ if $V=V(x, s)\in\boldsymbol{V}$; the class introduced in Sect. 1. These remarks will be useful in [9].

6. Construction of reversible measures

Here we shall show the set $\mathcal{R}(V)$ of all reversible measures of the TDGL eq. (1.4) is nonempty for every $V=V(x, s)$ satisfying the condition (2.7). Let us begin with the simplest case $V=0$. The centered Gaussian measure on $\boldsymbol{C}\cap\boldsymbol{L}_e^2$ with covariance operator $\bar{\mathcal{A}}^{-1}$ is denoted by ν; see Sect. 2.2.

Proposition 6.1. $\mathcal{R}(0)=\{\nu\}$

Proof. Proposition 5.2-(ii) shows $\#\mathcal{R}(0)\leq 1$ so that the conclusion follows by proving $\nu\in\mathcal{R}(0)$. To this end, we use Lemma 4.3 in [8] to see

$$E^{P_S}[e^{\sqrt{-1}\langle S_t,\varphi\rangle}] = \exp\left\{\sqrt{-1}\langle S, \varphi_t\rangle-\frac{1}{2}\int_0^t \|\varphi_u\|^2_{L^2}\,du\right\}, \tag{6.1}$$

for $t\geq 0$, $\varphi\in C_0^\infty(\boldsymbol{R}^d)$, where $\varphi_t=e^{-t\bar{\mathcal{A}}/2}\varphi$ and P_S denotes the distribution of the solution of the SPDE (5.1; 0) starting from $S\in\boldsymbol{L}_r^2$. Noting that

(6.2)
$$E^{\nu}[e^{\sqrt{-1}\langle S,\varphi\rangle}] = \exp\{-\frac{1}{2}\langle \mathcal{A}^{-1}\varphi, \varphi\rangle\},$$

we obtain

(6.3)
$$\begin{aligned} &E^{\nu}[e^{\sqrt{-1}\langle S_t,\varphi\rangle} e^{\sqrt{-1}\langle S_0,\psi\rangle}] \\ &= \exp\left\{-\frac{1}{2}\|\mathcal{A}^{-1/2}(\varphi_t+\psi)\|_{L^2}^2 - \frac{1}{2}\int_0^t \|\varphi_u\|_{L^2}^2\, du\right\} \\ &= \exp\left\{-\frac{1}{2}\langle \mathcal{A}^{-1}\varphi, \varphi\rangle - \frac{1}{2}\langle \mathcal{A}^{-1}\psi, \psi\rangle - \langle \mathcal{A}^{-1}\varphi_t, \psi\rangle\right\}\end{aligned}$$

for every $\varphi, \psi \in C_0^\infty(\boldsymbol{R}^d)$. However the RHS of (6.3) is symmetric in φ and ψ and this proves the equality (2.9) for functions Φ and Ψ of the forms $\Phi(S)=e^{\sqrt{-1}\langle S,\varphi\rangle}$ and $\Psi(S)=e^{\sqrt{-1}\langle S,\psi\rangle}$. Therefore the standard approximation argument concludes the proof. □

Let $\mathcal{B}_{b,0}(\boldsymbol{R}^d\times\boldsymbol{R})$ be the family of all functions $V\in\mathcal{B}_b(\boldsymbol{R}^d\times\boldsymbol{R})$ such that $V(x, s)=0$ a.e. on $\{|x|\geq K\}\times\boldsymbol{R}$ with some $K>0$.

Lemma 6.1. *Suppose* $V\in\mathcal{B}_{b,0}(\boldsymbol{R}^d\times\boldsymbol{R})$ *is given. Define* $\mu_V\in\mathcal{P}(\boldsymbol{C}\cap\boldsymbol{L}_r^2)$ *by*

(6.4)
$$d\mu_V(X(\cdot)) = Z_V^{-1}\exp\{-\int_{\boldsymbol{R}^d} V(x, X(x))\,dx\}\, d\nu(X(\cdot)),$$

where Z_V *is a normalization constant. Then,* $\mathcal{G}(V)=\{\mu_V\}$.

Proof. It is easy to show that $\mu_V\in\mathcal{G}'(V)$. Conversely, suppose $\mu\in\mathcal{G}(V)$ is given. Then, the probability measure $\tilde{\mu}$ defined by

$$d\tilde{\mu} = Z^{-1}\exp\{\int_{\boldsymbol{R}^d} V(x, X(x))\,dx\}\, d\mu\,, \quad Z = \text{normalization}$$

belongs to the class $\mathcal{G}'(0)$. This verifies $\tilde{\mu}=\nu$ and consequently $\mu=\mu_V$. □

Proposition 6.2. $\mathcal{R}(V)\neq\emptyset$ *for every* $V=V(x, s)$ *satisfying* (2.7).

Proof. We construct an approximating sequence $\{V_n(x, s)\}_{n=1}^\infty$ of the function $V(x, s)$ in the following manner: For a.e. $x\in\boldsymbol{R}^d$, let $\sigma=\sigma_x(v)\in[0, \infty]$, $v\geq 0$, be the right continuous inverse function of $v_x(\sigma)=\sup_{|s|\leq\sigma}\{|V(x, s)|+|V'(x, s)|\}$, $\sigma\geq 0$. We also prepare, for each $\sigma\geq 0$, a function $\varphi_\sigma\in C^\infty(\boldsymbol{R})$ satisfying that $\varphi_\sigma(s)=s$, $|s|\leq\sigma-1$; $\varphi_\sigma(s)=\sigma$, $|s|\geq\sigma$ and $|\varphi_\sigma'(s)|\leq 2$, $|\varphi_\sigma''(s)|\leq 2$, $s\in\boldsymbol{R}$. Set $\varphi_\infty(s)=s$. Then, we define $V_n(x, s)=1_{\{|x|\leq n\}}\cdot V(x, \varphi_{\sigma_x(n)}(s))$, $(x, s)\in\boldsymbol{R}^d\times\boldsymbol{R}$. The sequence $\{V_n\}$ constructed as above has the following properties: (i) $V_n\in\mathcal{B}_{b,0}(\boldsymbol{R}^d\times\boldsymbol{R})$, (ii) V_n satisfies (2.7) (iii) For a.e. $x\in\boldsymbol{R}^d$, $V_n(x, s_n)\to V(x, s)$ and $V_n'(x, s_n)\to V'(x, s)$ if $s_n\to s$ in $\boldsymbol{R}$ (iv) $\sup_n |b_{V_n}|_{r,(\infty)}\leq 2|b_V|_{r,(\infty)}<\infty$. Let μ_n be the unique element of $\mathcal{G}(V_n)$. Then, $\{\mu_n\}_n$ is tight on $\boldsymbol{L}_r^2$. In fact, we use the coupling $\{S_t, \bar{S}_t\}\sim\{(b_{V_n}, \mu_n), (0, \nu)\}$. Since the distribution $\mu_t^{(n)}$ of S_t is μ_n for every $t\geq 0$, we have $\mu_n=\int_0^1 \mu_t^{(n)}\, dt$. Therefore, the similar argument used in the proof of Proposition 5.1 shows the tightness of $\{\mu_n\}_n$; note that the property (iv) of $\{V_n\}$ with the help of (5.7) implies that c_2 in (5.4) can be taken independently of n and also $\sup_n E^{\mu_n}[|S|_r^2]<\infty$ which is used to derive an estimate like (5.8). Let $\{P_S\}_{S\in \boldsymbol{L}_r^2}$ and $\{P_S^n\}_{S\in \boldsymbol{L}_r^2}$ be the distributions on $C([0, \infty), \boldsymbol{L}_r^2)$ of the solutions of the SPDE (1.4) and the same SPDE with V replaced by V_n, respectively, which start from S. The next remark is that $P_{S_n}^n \Rightarrow P_S$ weakly on $C([0, \infty), \boldsymbol{L}_r^2)$ if $S_n \to S$ in $\boldsymbol{L}_r^2$. To this end, we first prove the tightness of $\{P_{S_n}^n\}_n$ on $C([0, \infty), \boldsymbol{L}_r^2)$. This, however, follows from the property (iv) of $\{V_n\}$ by using Remark 2.1-(ii) in [8]. Then, noting the property (iii), it is shown that every limit point of $\{P_{S_n}^n\}$ solves the martingale problem (m.p.) associated with the SPDE (1.4). Since this m.p. is well-posed (see Theorem 4.1 in [8]), we obtain the convergence $P_{S_n}^n \Rightarrow P_S$. Now it is easy to show that every limit point of $\{\mu_n\}$ belongs to the class $\mathcal{R}(V)$. □

We obtain the following by combining this proposition with the final remark in the previous section.

Corollary 6.1. *If* $V=V(x, s)$ *satisfies* (2.7) *and* (1.11), *then* $\#\mathcal{R}(V)=1$.

We finally summarize the result for the TDGL eq. (1.4) by assuming that $V\in \boldsymbol{V}$.

Theorem 6.1. (*i*) $\mathcal{R}(V)\neq\emptyset$.
(*ii*) *If* V *satisfies* (1.11), *then* $\#\mathcal{R}(V)=1$. *Moreover, the solution* S_t *of* (1.4) *has an ergodic property in the sense of* (5.13) *for every* $0<c<\gamma_0$ *and* $0<r<r_1(\gamma_0-c)$. *The unique reversible probability measure* μ *is given by the* (*thermodynamic*) *limit of* $\mu_n\in\mathcal{R}(1_{\{|x|\leq n\}}\cdot V)$, *i.e.* $d_r(\mu_n, \mu)\to 0$ *as* $n\to\infty$.

7. Reversible measures of the TDGL eq. of conservative type

Here we discuss the reversible measures of the TDGL eq. (1.13) of conservative type under the assumption that V satisfies (2.7). For this purpose, we use the cut-off method so that we consider the SPDE (1.13) with $V_{g,\lambda}$ in place of V for each $g\in\mathcal{B}_1(\boldsymbol{R}^d)$ and $\lambda\in\boldsymbol{L}_e^2$:

$$(7.1)\qquad dS_t(x)=\frac{1}{2}\Delta\{\mathcal{A}S_t(x)+V'_{g,\lambda}(x, S_t(x))\}\, dt+d\,\{\operatorname{div} \boldsymbol{w}_t(x)\}\ ,$$

where $V_{g,\lambda}$ is the function defined by (5.15) and $V'_{g,\lambda}=\frac{\partial}{\partial s}V_{g,\lambda}$, see [8] for the existence and uniqueness of solutions satisfying (1.9). Let us denote by $\mathcal{R}_C(V; g, \lambda)$ and $\mathcal{R}_C(V)$ the classes of all reversible measures of the SPDE's (7.1) and (1.13), respectively.

Lemma 7.1. $\nu \in \mathcal{R}_C(V; 0, 0)$

This lemma is shown similarly to Proposition 6.1. Indeed, the RHS of (6.1) is now replaced by $\exp\{\sqrt{-1}\langle S, \varphi_t\rangle - \frac{1}{2}\int_0^t \|\nabla\varphi_u\|_{L^2}^2 du\}$, where $\varphi_t = e^{-t\mathcal{A}'}\varphi$, $\mathcal{A}' = -\Delta\mathcal{A}/2$.

Denote by $\mathcal{B}_0(\boldsymbol{R}^d)$ the class of all $g \in \mathcal{B}(\boldsymbol{R}^d)$ (i.e., measurable functions on $\boldsymbol{R}^d$) having compact supports, namely, $g(x)=0$ a.e. on $\{|x| \geq K\}$ with some $K>0$. We suppose $g \in \mathcal{B}_1(\boldsymbol{R}^d) \cap \mathcal{B}_0(\boldsymbol{R}^d)$ and $\lambda \in \boldsymbol{L}_e^2 \cap \mathcal{B}_0(\boldsymbol{R}^d)$. Let $\eta \in C_0^\infty(\boldsymbol{R}^d)$ be a symmetric and non-negative function satisfying $\eta \equiv 0$ on $\{|x| \geq 1\}$ and $\int_{\boldsymbol{R}^d} \eta(x)\, dx = 1$. We introduce probability measures $\mu_{g,\lambda}$ and $\mu_{g,\lambda;\eta}$ on $\boldsymbol{C} \cap \boldsymbol{L}_e^2$ by

$$(7.2)\quad d\mu_{g,\lambda}(S) = Z_{g,\lambda}^{-1} \exp\{-\Psi_{g,\lambda}(S)\}\, d\nu(S), \quad \Psi_{g,\lambda}(S) = \int_{\boldsymbol{R}^d} V_{g,\lambda}(x, S(x))\, dx,$$

and

$$(7.3)\quad d\mu_{g,\lambda;\eta}(S) = Z_{g,\lambda;\eta}^{-1} \exp\{-\Psi_{g,\lambda;\eta}(S)\}\, d\nu(S), \quad \Psi_{g,\lambda;\eta}(S) = \Psi_{g,\lambda}(S*\eta),$$

where $Z_{g,\lambda}$ and $Z_{g,\lambda;\eta}$ are normalization constants and $*$ means the convolution. Note that $\mu_{g,\lambda}$ is the unique element of $\mathcal{G}(V_{g,\lambda})$; see Lemma 6.1. Let us consider the SPDE:

$$(7.4)\qquad dS_t(x) = \Delta\{\frac{1}{2}\mathcal{A}S_t(x) + b_{g,\lambda;\eta}(x, S_t)\}\, dt + d\{\operatorname{div} \boldsymbol{w}_t(x)\},$$

where

$$(7.5)\qquad b_{g,\lambda;\eta}(x, S) = \frac{1}{2}\int_{\boldsymbol{R}^d} \eta(x-y)\, V'_{g,\lambda}(y, S*\eta(y))\, dy.$$

Lemma 7.2. *$\mu_{g,\lambda;\eta}$ is reversible for the eq.* (7.4).

Proof. Let us denote by $\{Q_S\}_{S\in \boldsymbol{L}_r^2}$ and $\{P_S\}_{S\in \boldsymbol{L}_r^2}$ the distributions on the space $\tilde{\Omega} = C([0, \infty), \boldsymbol{L}_r^2)$ of the solutions of the SPDE's (7.4) and (7.4) with $g \equiv \lambda \equiv 0$, respectively, starting from $S \in \boldsymbol{L}_r^2$. Then, the Cameron-Martin-Girsanov's formula (infinite-dimensional version) shows

$$\frac{dQ_S}{dP_S} = R_t \quad \text{on} \quad \mathcal{F}_t.$$

Here $\mathcal{F}_t = \sigma\{S_u; u \leq t\}$, $t \geq 0$, is a usual family of σ-fields on $\tilde{\Omega}$, R_t is a martingale defined on $(\tilde{\Omega}, P_S, \{\mathcal{F}_t\})$ by

$$R_t = \exp\left\{-\int_0^t \langle b_{g,\lambda;\eta}(\cdot, S_u), dm_u\rangle - \frac{1}{2}\int_0^t \|\nabla b_{g,\lambda;\eta}(\cdot, S_u)\|_{L^2(\boldsymbol{R}^d)}^2 du\right\}$$

and $m_t = S_t - S_0 - \frac{1}{2}\int_0^t \Delta\mathcal{A}S_u\, du$ is a $\{C_0^\infty(\boldsymbol{R}^d)\}'$-valued process. The process m_t has a representation $m_t = \operatorname{div} \boldsymbol{w}_t$ with a c.B.m. $\boldsymbol{w}_t$ on $L^2(\boldsymbol{R}^d, \boldsymbol{R}^d)$, which is realized on the probability space $(\tilde{\Omega}, P_S, \{\mathcal{F}_t\})$, cf. Lemma 4.2 in [8]. However, by using Itô's formula, we have

$$\int_0^t \langle b_{g,\lambda;\eta}(\cdot, S_u), dm_u\rangle = \frac{1}{2}\Psi_{g,\lambda;\eta}(S_t) - \frac{1}{2}\Psi_{g,\lambda;\eta}(S_0) + \int_0^t F(S_u)\, du,$$

where

$$F(S) = \frac{1}{2}\langle S, \mathcal{A}\Delta b_{g,\lambda;\eta}(\cdot, S)\rangle + \frac{1}{4}\|\nabla\eta\|_{L^2}^2 \int_{R^d} V''_{g,\lambda}(x, S*\eta(x))\, dx,$$

and this verifies the conclusion with the help of Lemma 7.1. □

Proposition 7.1. *If* $g\in\mathcal{B}_1(\boldsymbol{R}^d)\cap\mathcal{B}_0(\boldsymbol{R}^d)$ *and* $\lambda\in\boldsymbol{L}_e^2\cap\mathcal{B}_0(\boldsymbol{R}^d)$, *then* $\mu_{g,\lambda}\in\mathcal{R}_C(V; g, \lambda)$.

Proof. Set $\mu_n=\mu_{g,\lambda;\eta_n}$, where $\eta_n(x)=n^d\eta(nx)$, $n=1, 2, \cdots$, $x\in\boldsymbol{R}^d$. We denote by $\{P_S\}_{S\in\boldsymbol{L}_r^2}$ and $\{P_S^n\}_{S\in\boldsymbol{L}_r^2}$ the distributions on $\tilde{\Omega}=C([0,\infty), \boldsymbol{L}_r^2)$ of the solutions of the SPDE's (7.1) and (7.4) with η replaced by η_n, respectively. Then, noting that $\sup_n|b_{g,\lambda;\eta_n}(\cdot, S)|_{r,(\infty)}<\infty$, the tightness of $\{\mu_n\}_n$ on $\boldsymbol{L}_r^2$ is shown; see the proof of Proposition 6.2. This estimate also gives the tightness of $\{P_{S_n}^n\}$ for $\{S_n\}$ such that $S_n\to S$ in $\boldsymbol{L}_r^2$. It is therefore verified that $P_{S_n}^n\Rightarrow P_S$ weakly on $\tilde{\Omega}$, see the proof of Proposition 6.2 again and the results of [8] as well. This limiting procedure completes the proof since μ_n converges weakly on $\boldsymbol{L}_r^2$ to $\mu_{g,\lambda}$ as $n\to\infty$. □

Set $\mu_\lambda^{(n)}=\mu_{1_{\{|x|\le n\}},1_{\{|x|\le n\}}\cdot\lambda}$ for $\lambda\in\boldsymbol{L}_e^2$. Then, the family $\{\mu_\lambda^{(n)}\}_n$ is tight, because $\sup_n|b_n(x,\cdot)|_{r,(\infty)}<\infty$ holds for $b_n(x, S)=\frac{1}{2}1_{\{|x|\le n\}}V'_{1,\lambda}(x, S(x))$. The following theorem can be verified similarly to the proof of Proposition 7.1 (or Proposition 6.2). Note that the eq. (7.1) with $g\equiv 1$ and λ satisfying $\Delta\lambda=0$ is just the same equation as (1.13). We use Theorem 6.1-(ii) (note that the conclusion of this theorem is true even if V is replaced by $V_{1,\lambda}$) to show the assertion (ii) below.

Theorem 7.1. (*i*) *Suppose* $\lambda\in\boldsymbol{L}_e^2\cap C^2(\boldsymbol{R}^d)$ *satisfies* $\Delta\lambda=0$. *Then, every limit point of* $\{\mu_\lambda^{(n)}\}_n$ *belongs to the class* $\mathcal{R}_C(V)$.
(*ii*) *Suppose* $V\in\boldsymbol{V}$ *and the strict-convexity condition* (1.11). *Then, the convex-hull of* $\{\mathcal{R}(V_{1,\lambda}); \lambda\in\boldsymbol{L}_e^2\cap C^2(\boldsymbol{R}^d), \Delta\lambda=0\}$ *is in the set* $\mathcal{R}_C(V)$.

8. Uniform mixing property of Gibbs states

Under the strict-convexity condition (1.11) of U, there exists a unique stationary (and reversible) probability measure $\mu_{g,\lambda}$, $g\in\mathcal{B}_1(\boldsymbol{R}^d)$, $\lambda\in\boldsymbol{L}_e^2$, of the TDGL eq. (1.4) with $V=V_{g,\lambda}$; the function defined by (5.15). We assume in this section that the Eidel'man-type estimate holds for the fundamental solution $q(t, x, y)$ of the parabolic operator $\frac{\partial}{\partial t}+\frac{1}{2}\mathcal{A}$ globally in time:

$$(8.1)\qquad |q(t, x, y)|\leq K_1 t^{-d/2m} e^{-\tilde{\gamma}t}\bar{q}(t^{-1/2m}\cdot|x-y|)\,,\quad t>0, x, y\in \boldsymbol{R}^d\,,$$

where $\tilde{\gamma}, K_1>0$ and $\bar{q}(r)=\exp\{-K_2 r^{2m/(2m-1)}\}$, $K_2>0$; see Appendix for this condition. For $G\subset \boldsymbol{R}^d$ and $\mathcal{B}(G)$-measurable $\Phi\in C(\boldsymbol{C})$, we set

$$||\Phi||_G = \sup\{|\Phi(S)-\Phi(\bar{S})|/||S-\bar{S}||_{L^2(G,dx)};\ S|_G\neq\bar{S}|_G\}\,.$$

The goal is to prove the following theorem which gives the decay, valid uniformly in g and λ, of the correlations of functionals with respect to $\mu_{g,\lambda}$.

Theorem 8.1. *Assume* (1.11) *on the potential U and* (8.1) *on the operator $\mathcal{A}$. Then, if $r>0$ is sufficiently small, we can take for every $M>1$ positive constants c and C in such a way that*

$$\begin{aligned}&|E^{\mu_{g,\lambda}}[\Phi_1\Phi_2]-E^{\mu_{g,\lambda}}[\Phi_1]E^{\mu_{g,\lambda}}[\Phi_2]|\\ &\leq C\prod_{i=1}^{2}\{||\Phi_i||_{B_i}+\sqrt{E^{\mu_{g,\lambda}}[\Phi_i^2]}\}\exp\{-c|x_1-x_2|\}\,,\end{aligned}$$

for every $\mathcal{B}(B_i)$-measurable Φ_i, $i=1, 2$, $B_i=B(x_i, a)$ being balls with centers $x_i\in\boldsymbol{R}^d$ and radius a, whenever $|\lambda|_r\leq M$ and $0\leq a\leq M$. The constants c and C may depend on r, M but not on x_1, x_2, g, λ, Φ_1 and Φ_2.

We denote by $S_t\sim\{g, \lambda, w_t\}$ if S_t is the solution of the SPDE (1.4) with $V=V_{g,\lambda}$ and c.B.m. w_t. Before starting the proof of the theorem, we prepare an estimate on the difference between two solutions of (1.4) with different c.B.m.'s. Namely, we assume that a bounded open set $\tilde{G}$ in $\boldsymbol{R}^d$ and c.B.m.'s w_t, $\bar{w}_t$ are given and satisfy the following two conditions:

(8.2) $w_t(x)=\bar{w}_t(x)$ on $\tilde{G}$, i.e. $\langle w_t, \varphi\rangle=\langle\bar{w}_t, \varphi\rangle$ for $\varphi\in C_0^\infty(\tilde{G})$,

(8.3) $\{w_t(x); x\in\tilde{G}^c\}$ and $\{\bar{w}_t(x); x\in\tilde{G}^c\}$ are independent of each other, i.e. two systems $\{\langle w_t, \varphi\rangle; \varphi\in C_0^\infty(\tilde{G}^c)\}$ and $\{\langle\bar{w}_t, \varphi\rangle; \varphi\in C_0^\infty(\tilde{G}^c)\}$ are mutually independent.

Consider two solutions $S_t\sim\{g, \lambda, w_t\}$ and $\bar{S}_t\sim\{g, \lambda, \bar{w}_t\}$ starting from the same point; $S_0=\bar{S}_0\in\boldsymbol{L}_e^2$. In the following lemma, G is always an open set satisfying $G\subset\tilde{G}$. Similar method of coupling was used by [10].

Lemma 8.1. *There exist positive constants C_1 and C_2 which are independent of $G, \tilde{G}, g, \lambda, S_0$ such that*

$$(8.4)\qquad \sup_{x\in G}E[|S_t(x)-\bar{S}_t(x)|^2]\leq C_1\exp\left\{C_2 t-K_2\left(\frac{R^{2m}}{t}\right)^{1/(2m-1)}\right\},\ t>0\,,$$

whenever supp $g\subset G$ *and* $R\equiv\text{dis}\,(G, \tilde{G}^c)\geq 1$.

Proof. Recalling that the solutions of the SPDE are defined through the stochastic integral equation [8], we have

$$S_t(x)-\bar{S}_t(x) = I_t(x)+II_t(x)\,, \tag{8.5}$$

where

$$I_t(x) = \int_0^t \int_{R^d} q(t-u, x, y)\, d\{w_u(y)-\bar{w}_u(y)\}\, dy\,,$$

$$II_t(x) = -\frac{1}{2}\int_0^t \int_{R^d} q(t-u, x, y)\, g(y)\{V'(S_u(y))-V'(\bar{S}_u(y))\}\, dudy\,.$$

Set $f(t)$=the LHS of (8.4). Then, by using Schwarz's inequality and (8.1), we obtain

$$E[II_t(x)^2]\leq\frac{1}{4}\,||V''||_\infty^2\int_0^t du\int_{R^d}|q(t-u, x, y)|\,dy$$
$$\times\int_0^t du\int_G |q(t-u, x, y)|\cdot E[|S_u(y)-\bar{S}_u(y)|^2]\,dy$$
$$\leq C\int_0^t f(u)\,du,\ t>0\,,$$

with C independent of (t, x) and hence, applying (8.5) with the help of Gronwall's inequality

$$f(t)\leq 2e^{2Ct}\sup_{0\leq u\leq t}\sup_{x\in G} E[I_u(x)^2]\,.$$

Therefore (8.4) is verified, since we have for $0\leq u\leq t$ and $x\in G$:

$$E[I_u(x)^2] = 2\int_0^u dv\int_{\tilde{G}^c} q^2(v, x, y)\,dy$$
$$\leq \text{const}\ \bar{q}(t^{-1/2m} R)\int_0^\infty v^{-d/2m}\, e^{-\tilde{\gamma} v}\, dv\,. \quad \square$$

Proof of Theorem 8.1: The asserted estimate is trivial when $|x_1-x_2|\leq 8M$, so that we assume $|x_1-x_2|>8M$ in the sequel. Take four balls $G_i=B(x_i, b)$ and $\tilde{G}_i=B(x_i, \tilde{b})$, $i=1, 2$, in such a way that $a<b=\frac{1}{8}|x_1-x_2|<\tilde{b}=\frac{1}{4}|x_1-x_2|$. We construct two independent c.B.m.'s $\bar{w}_t^{(1)}$ and $\bar{w}_t^{(2)}$ from arbitrarily chosen three independent c.B.m.'s w_t, $w_t^{(1)}$, $w_t^{(2)}$ on $L^2(\boldsymbol{R}^d)$ in the following manner:

$$\langle\bar{w}_t^{(i)}, \varphi\rangle = \langle w_t, 1_{\tilde{G}_i}\cdot\varphi\rangle+\langle w_t^{(i)}, 1_{\tilde{G}_i^c}\cdot\varphi\rangle,\ \varphi\in C_0^\infty(\boldsymbol{R}^d),\ i=1, 2\,.$$

Then, by introducing three stochastic processes $S_t\sim\{g, \lambda, w_t\}$ and $S_t^{(i)}\sim\{g, \lambda, \bar{w}_t^{(i)}\}$, $i=1, 2$, we have an identity for every $t\geq 0$:

$$E^{\mu_{g,\lambda}}[\Phi_1\,\Phi_2]-E^{\mu_{g,\lambda}}[\Phi_1]\,E^{\mu_{g,\lambda}}[\Phi_2] = E^{\mu_{g,\lambda}}[I(t, S)]+E^{\mu_{g,\lambda}}[II(t, S)]\,,$$

where

$$I \equiv I(t, S) = E_S[\Phi_1(S_t)\,\Phi_2(S_t)] - E_S[\Phi_1(S_t^{(1)})\,\Phi_2(S_t^{(2)})]$$
$$II \equiv II(t, S) = E_S[\Phi_1(S_t)]\,E_S[\Phi_2(S_t)] - E^{\mu_g,\lambda}[\Phi_1]\,E^{\mu_g,\lambda}[\Phi_2]\,.$$

The subscript S means the starting point of the processes. Notice the facts that two processes $S_t^{(1)}$ and $S_t^{(2)}$ both starting from $S \in \boldsymbol{L}_e^2$ are mutually independent and also all the laws of three processes $S.$, $S.^{(1)}$ and $S.^{(2)}$ are the same. The first task is to give estimates on I, so that we decompose it into

$$\begin{aligned} I &= E_S[\Phi_2(S_t^{(2)})\,\{\Phi_1(S_t) - \Phi_1(S_t^{(1)})\}] + E_S[\Phi_1(S_t)\,\{\Phi_2(S_t) - \Phi_2(S_t^{(2)})\}] \\ &\equiv I_1 + I_2\,. \end{aligned}$$

In order to give further bounds on I_1, we set

$$\begin{aligned} I_{1,1} &= E_S[\{\Phi_1(S_t) - \Phi_1(\tilde{S}_t)\}^2]\,, \\ I_{1,2} &= E_S[\{\Phi_1(\tilde{S}_t) - \Phi_1(\bar{S}_t)\}^2]\,, \\ I_{1,3} &= E_S[\{\Phi_1(\bar{S}_t) - \Phi_1(S_t^{(1)})\}^2]\,, \end{aligned}$$

where $\tilde{S}_t \sim \{1_{G_1}\cdot g, \lambda, w_t\}$ and $\bar{S}_t \sim \{1_{G_1}\cdot g, \lambda, \bar{w}_t^{(1)}\}$. The first term $I_{1,1}$ is bounded as follows:

$$\begin{aligned} I_{1,1} &\leq \|\Phi_1\|_{B_1}^2\, E_S[\|S_t - \tilde{S}_t\|_{L^2(B_1,dx)}^2] \\ &\leq \text{const}\, \|\Phi_1\|_{B_1}^2\, e^{2ra}\, E_S[\,|\tau_{x_1} S_t - \tau_{x_1} \tilde{S}_t|_r^2] \\ &\leq \text{const}\, \|\Phi_1\|_{B_1}^2 \int_{\{|x|\geq b\}} e^{-2r\chi(x)}\, dx\,, \end{aligned}$$

where $\tau_x S$ is defined by $\tau_x S(y) = S(y+x)$ for $y \in \boldsymbol{R}^d$. We have used (5.5) for $\sigma_t = \tau_{x_1} S_t - \tau_{x_1} \tilde{S}_t$ making similar calculations to those in (II) and (III) of Sect. 5 to derive the last inequality; $r>0$ is sufficiently small. The same estimate can be derived for $I_{1,3}$. On the other hand, using Lemma 8.1, we get

$$\begin{aligned} I_{1,2} &\leq \|\Phi_1\|_{B_1}^2\, E_S[\|\tilde{S}_t - \bar{S}_t\|_{L^2(B_1,dx)}^2] \\ &\leq \|\Phi_1\|_{B_1}^2\, |B_1| \times C_1 \exp\left\{C_2\, t - K_2\left(\frac{R^{2m}}{t}\right)^{1/(2m-1)}\right\} \end{aligned}$$

where $R = \text{dis}\,(G_1, \tilde{G}_1^c) = \tilde{b} - b = \frac{1}{8}|x_1 - x_2|$. These three estimates on $I_{1,1} - I_{1,3}$ can be summarized into an estimate on I_1 by using

$$|I_1| \leq \sqrt{E_S[\Phi_2^2(S_t^{(2)})]}\, \sum_{i=1}^{3} \sqrt{I_{1,i}}\,.$$

The other term I_2 can be bounded similarly. We therefore obtain, by choosing t such that $t = t_0 \cdot R$ with fixed t_0: $0 < t_0 < (K_2/C_2)^{(2m-1)/2m}$,

$$\begin{aligned} (8.6) \qquad E^{\mu_g,\lambda}[\,|I(t, S)|\,] \leq \text{const}\, &\{\|\Phi_1\|_{B_1} \sqrt{E^{\mu_g,\lambda}[\Phi_2^2]} \\ &+ \|\Phi_2\|_{B_2} \sqrt{E^{\mu_g,\lambda}[\Phi_1^2]}\}\, \exp\{-\text{const}\, |x_1 - x_2|\}\,. \end{aligned}$$

Now we move to the estimate on the term II:

$$E^{\mu_g,\lambda}[|II(t, S)|] \leq \sqrt{E^{\mu_g,\lambda}[\Phi_2^2]\cdot I_{2,1}(t)} + \sqrt{E^{\mu_g,\lambda}[\Phi_1^2]\cdot I_{2,2}(t)}$$

where

$$I_{2,i}(t) = E^{\mu_g,\lambda}[|E_S[\Phi_i(S_t)] - E^{\mu_g,\lambda}[\Phi_i]|^2],\ i = 1, 2\,.$$

However, Proposition 5.2-(ii) can be applied to give a bound on $I_{2,i}(t)$. Indeed, take $\Phi=\Phi_i\circ\tau_{x_i}^{-1}$ which is defined by $\Phi\circ\tau_x^{-1}(S)=\Phi(\tau_x^{-1} S)$, then we obtain

$$I_{2,i}(t) \leq \text{const}\, ||\Phi_i||_{B_i}^2\, e^{-ct}$$

by noting (5.7) (remember $|\lambda|_r \leq M$) and $||\Phi\circ\tau_{x_i}^{-1}||_{L,r} \leq ||\Phi_i||_{B_i}\, e^{ra}$. We therefore get the similar estimate to (8.6) also for $E^{\mu_g,\lambda}[|II(t, S)|]$ by taking t: $t=t_0\cdot R$ with the same t_0 as before and this completes the proof. □

REMARK 8.1. Two constants c, C and possible region of $r>0$ in Theorem 8.1 depend on $\mathcal{A}$ and V only through the following quantities: three constants $\tilde{\gamma}$, K_1, K_2 in (8.1), four quantities listed in Remark 5.1, γ_0, $||V'||_\infty$ and $||V''||_\infty$.

Appendix

Here we prove the following global estimate on the fundamental solution $q(t, x, y)$ of the parabolic operator $\dfrac{\partial}{\partial t}+\dfrac{1}{2}\mathcal{A}$.

Proposition A.1. *The function* $q(t, x, y)$ *has a bound*:

$$\text{(A.1)}\quad |q(t, x, y)| \leq K_1\, t^{-d/2m}\, e^{Ct} \exp\left\{-K_2\left(\frac{|x-y|^{2m}}{t}\right)^{1/(2m-1)}\right\},\ t>0,\ x, y\in \boldsymbol{R}^d\,,$$

with positive constants K_1, K_2 *and* C *which depend only on the following two quantities*: c *appearing in the condition* (1.5) *and* $\sup\{|D^{\alpha'}a_{\alpha,\beta}(x)|;\ x\in\boldsymbol{R}^d,\ |\alpha|, |\beta|\leq m, \alpha'\leq\alpha \text{ or } |\alpha|=|\beta|=m, |\alpha'|=1\}$. *In particular, if we consider an operator* $\tilde{\mathcal{A}}=\mathcal{A}+2(C+\tilde{\gamma})$ *instead of* $\mathcal{A}$, *then* $\tilde{\mathcal{A}}$ *satisfies the estimate* (8.1).

Proof. It is verified in [3, Theorem 2.1, p71] that (A.1) holds locally in time, i.e., (A.1) holds for $0<t\leq 1$ with $C=0$ and $K_1, K_2>0$ which depend on those two quantities listed above (notice that the latter quantity controls especially the Hölder constants of $\{a_{\alpha,\beta}\}_{|\alpha|=|\beta|=m}$). We denote the constants K_1 and K_2 appearing in this local estimate by K_1' and K_2', respectively, for discrimination. Let us consider the operators $\{T_{t,s};\ 0\leq s\leq t<\infty\}$ defined by

$$\text{(A.2)}\qquad T_{t,s}f(x) = \int_{\boldsymbol{R}^d} \psi_t^{-1}(x)\, q(t-s, x, y)\, \psi_s(y) f(y)\, dy\,,$$

where $\psi_t(x)=\exp\left\{-K_2\left(\dfrac{|x|^{2m}}{t}\right)^{1/2(m-1)}\right\}$, $t>0$, $x\in\boldsymbol{R}^d$ and $K_2=\dfrac{1}{2}K_2'$. Then,

$\{T_{t,s}\}$ has the semigroup property: $T_{t,\tau}\, T_{\tau,s}=T_{t,s}$, $0\leq s\leq\tau\leq t<\infty$. Moreover, we have

$$\text{(A.3)}\qquad T_{t,s}\colon L^\infty\to L^\infty\,,\quad \|T_{t,s}\|_{L^\infty\to L^\infty}\leq K_3'\equiv K_1'\int_{\boldsymbol{R}^d}\psi_1(y)\,dy\,,$$

$$\text{(A.4)}\qquad T_{t,s}\colon L^1\to L^\infty\,,\quad \|T_{t,s}\|_{L^1\to L^\infty}\leq K_1'(t-s)^{-d/2m}\,,$$

for $0\leq s<t<\infty$ if $t-s\leq 1$, where $L^1=L^1(\boldsymbol{R}^d)$, $L^\infty=L^\infty(\boldsymbol{R}^d)$ and $\|T\|_{E\to E'}$ denotes the operator norm of $T\colon E\to E'$ for two normed spaces E and E'. Indeed, these two estimates are consequences of

$$|\psi_t^{-1}(x)\,q(t-s,x,y)\,\psi_s(y)|\leq K_1'(t-s)^{-d/2m}\,\psi_{t-s}(x-y),\ 0<t-s\leq 1\,,$$

which is shown from the local estimate on q by noting

$$\left(\frac{|x-y|^{2m}}{t-s}\right)^{1/(2m-1)}+\left(\frac{|y|^{2m}}{s}\right)^{1/(2m-1)}\geq\left(\frac{|x|^{2m}}{t}\right)^{1/(2m-1)},\ 0<s<t<\infty,\ x,y\in\boldsymbol{R}^d\,,$$

see [3, p36]. Now, by employing a similar argument to [17, p232], the semigroup property of $\{T_{t,s}\}$ combined with (A.3) verifies

$$\text{(A.5)}\qquad T_{t,s}\colon L^\infty\to L^\infty\,,\quad \|T_{t,s}\|_{L^\infty\to L^\infty}\leq K_3'\,e^{C'(t-s)}$$

for all $0\leq s\leq t<\infty$, where $C'=\max\{0,\log K_3'\}$. Therefore, we obtain

$$\text{(A.6)}\qquad \|T_{t,s}\|_{L^1\to L^\infty}\leq K_1(t-s)^{-d/2m}\,e^{C(t-s)},\ 0\leq s<t<\infty\,,$$

with $K_1=\max\{K_1', K_1'K_3'e^{-C'}\}$ and $C=C'+\dfrac{d}{2m}$. In fact, (A.6) follows from (A.4) when $0<t-s\leq 1$. On the other hand, when $t-s>1$,

$$\begin{aligned}\|T_{t,s}\|_{L^1\to L^\infty}&\leq\|T_{t,s+1}\|_{L^\infty\to L^\infty}\|T_{s+1,s}\|_{L^1\to L^\infty}\\&\leq K_3'\,e^{C'(t-s-1)}\times K_1'\\&\leq K_1'K_3'\,e^{-C'}\,e^{(C'+(d/2m))(t-s)}(t-s)^{-d/2m}\,,\end{aligned}$$

where we have used a simple inequality $1\leq e^{(d/2m)t}\,t^{-d/2m}$, $t>0$. The estimate (A.6) gives a bound on the kernel of $T_{t,s}$:

$$|\psi_t^{-1}(x)\,q(t-s,x,y)\,\psi_s(y)|\leq K_1(t-s)^{-d/2m}\,e^{C(t-s)},\ 0\leq s<t<\infty,\ x,y\in\boldsymbol{R}^d\,.$$

This implies the estimate (A.1) for $y=0$ and then for general $y\in\boldsymbol{R}^d$ by considering the operator $\mathcal{A}^{(y)}$ with coefficients $\{a_{\alpha,\beta}^{(y)}\}$ obtained by sifting the original $\{a_{\alpha,\beta}\}$ by y. □

REMARK A.1. (1) It is possible to derive (A.1) by looking at the arguments in [3] carefully. Nevertheless, we have exposed a simple proof based on the local estimate for the sake of completeness.
(2) When $\mathcal{A}=\sum_{|\alpha|=|\beta|=m}(-1)^{|\alpha|}D^\alpha\{a_{\alpha,\beta}D^\beta\cdot\}$ with constant coefficients $a_{\alpha,\beta}$,

we can take $C=0$ in (A.1). In fact, this is an easy consequence of the local estimate combined with the scaling law of q: $c^d q(c^{2m}t, cx, cy)=q(t, x, y)$ for every $c>0$.

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