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EXPONENTIALLY ASYMPTOTIC STABILITY FOR A CERTAIN CLASS OF SEMILINEAR VOLTERRA DIFFUSION EQUATIONS

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0. Introduction

In this paper we consider a semilinear Volterra diffusion equation in a bounded domain $\Omega(\subset \boldsymbol{R}^N)$ with smooth boundary $\partial\Omega$:

$$\frac{\partial u}{\partial t} = \Delta u+(a-bu-f*u)\,u\,, \quad t>0,\ x\in\Omega\,. \tag{0.1}$$

Here Δ is the usual Laplace operator with respect to x, a and b are positive constants, and $f*u$ denotes the convolution of a non-negative function $f(t)$ and $u(t, x)$:

$$f*u(t, x) = \int_0^t f(t-s)\,u(s, x)\,ds\,.$$

We impose no-flux condition at $\partial\Omega$ and non-negativity on the initial data:

$$\frac{\partial u}{\partial n} = 0\,, \qquad t>0,\ x\in\partial\Omega\,, \tag{0.2}$$

$$u(0, x) = u_0(x)\geq 0\,, \quad x\in\Omega\,, \tag{0.3}$$

where $\partial/\partial n$ denotes the exterior normal derivative to $\partial\Omega$ and $u_0(x)$ is a function of class $C^2(\bar{\Omega})$ satisfying $\partial u_0/\partial n=0$ on $\partial\Omega$.

Equation (0.1) comes from population dynamics; $u(t, x)$ denotes the population density at time t and position x of a single species which diffuses in Ω and grows obeying the *logistic law* (a is the difference of the ideal birth and death rate, $-bu-f*u$ expresses the crowding effect due not only to the present size but also to the past ones of the population). For instance, some kinds of bacteria which are cultured in a laboratory without replacing their solid medium are observed to die of the poisoning caused by the piled up stuff P that has come of decomposition and metabolism by themselves. The growth rate of their population density u seems to follow

$$\frac{\partial u}{\partial t} = \Delta u+(a-bu-cv)\,u\,, \quad t>0,\ x\in\Omega\,, \tag{0.4}$$

where c is a positive constant and $v(t, x)$ denotes the density of the poisonous substance P at time t and position x. Here we assume that P is in a solid state and does not move. We propose two models describing the growth dynamics of P: [model A] the poisonous substance P is directly produced by the bacteria and its density v satisfies

$$(0.5)\qquad \begin{cases} \dfrac{\partial v}{\partial t} = d_1 u - d_2 v\,, & t>0,\ x\in\Omega\,, \\ v(0, x) = 0\,, & x\in\Omega\,, \end{cases}$$

where the positive constants d_1 and d_2 denote the productivity of P by the bacteria and the decaying rate of P respectively; [model B] the poisonous substance P_2 is made, as a second product, from another first product P_1 which is directly produced by the bacteria and the densities v_1, v_2 of P_1, P_2 respectively satisfy

$$(0.6)\qquad \begin{cases} \dfrac{\partial v_1}{\partial t} = d_3 u - d_4 v_1\,, & t>0,\ x\in\Omega\,, \\ \dfrac{\partial v_2}{\partial t} = d_5 v_1 - d_6 v_2\,, & t>0,\ x\in\Omega\,, \\ v_1(0, x) = 0\,, & x\in\Omega\,, \\ v_2(0, x) = 0\,, & x\in\Omega\,, \end{cases}$$

where d_3, d_4, d_5 and d_6 are positive constants. The models A and B are biologically natural. For simplicity we assume that $d_4 = d_6$. For $t \geq 0$ and $x\in\Omega$, in the model A

$$(0.7)\qquad v(t, x) = d_1 \int_0^t \exp\{-d_2(t-s)\}\, u(s, x)\, ds\,,$$

and in the model B

$$v_1(t, x) = d_3 \int_0^t \exp\{-d_4(t-s)\}\, u(s, x)\, ds\,,$$

$$(0.8)\qquad v_2(t, x) = d_3 d_5 \int_0^t (t-s) \exp\{-d_4(t-s)\}\, u(s, x)\, ds$$

are derived from (0.5) and (0.6) respectively. After combining (0.4) with (0.7) ,or (0.4) with (0.8) (by replacing v with v_2), rewriting constants reduces the models to (0.1), where

$$(0.9)\qquad f(t) = \frac{\alpha}{T} \exp\left(-\frac{t}{T}\right) \quad (T>0,\ \alpha>0;\ \text{model A})\,,$$

$$(0.10)\qquad f(t) = \frac{\alpha t}{T^2} \exp\left(-\frac{t}{T}\right) \quad (T>0,\ 0<\alpha<8b;\ \text{model } B)\,.$$

In this paper we do not treat the case: $\alpha \geq 8b$ for the model B (as to that case, see, e.g., Tesei [15], Yamada [18], Yamada and Niikura [20]). The delay kernels given by (0.9) or (0.10) are popular in population dynamics (cf. Cushing [2]). Note that the kernel (0.9) represents the case when past densities have monotone decreasing influence but the kernel (0.10) represents the case when the maximum influence at any time t is due to the density at the previous time $t-T$.

In his book [16], Volterra has first introduced and studied the differential equations with time delay, in a spatially homogeneous situation, such as

$$\frac{du}{dt}=(a-bu-f*u)\,u\,,\quad t>0\,.$$

In this situation a useful survey of results is found in Cushing [2]. As for the spatially inhomogeneous solutions of (0.1)-(0.3), Schiaffino, Tesei, Yamada and Niikura have investigated existence, uniqueness, non-negativity, boundedness and asymptotic behavior (see [9, 15, 18, 19, 20]). In particular, Schiaffino has driven the global attractivity of the unique positive equilibrium $u_\infty=a/(b+\alpha)$ $(\alpha=\int_0^\infty f(t)\,dt)$ in [9] assuming monotonisity and, in some sense, smallness of f, and Yamada has loosen these conditions on f in [18] by giving a sufficient condition which is described in terms of the Laplace transform of f. Moreover, in [19], Yamada has stutied the (local) asymptotic stability of u_∞ as an application of the stability of a *fundamental solution* associated with the linearized problem. (For the results under Dirichlet boundary condition instead of (0.2), see Schiaffino and Tesei [10, 11, 12].)

On the other hand, it is well-known that the positive solution u of a semilinear diffusion equation (without time celay)

$$\frac{\partial u}{\partial t}=\Delta u+(a-bu)\,u\,,\quad t>0,\ x\in\Omega$$

under the initial boundary conditions (0.2) and (0.3) converges to $\tilde{u}^\infty=a/b$ as $t\to\infty$. More precisely, the convergence rate of u is given by

$$\|u(t,\cdot)-\tilde{u}_\infty\|_\infty\leq C\exp(-b\tilde{u}_\infty t)\,\|u_0-\tilde{u}_\infty\|_\infty\,,\quad t\geq 0\,, \tag{0.11}$$

where C is a positive constant depending on $\|u_0\|_\infty$. In this estimate we should note that the spectrum of the corresponding linearized operator $A+b\tilde{u}_\infty$ lies in $\{z\in \boldsymbol{C};\ \mathrm{Re}\,z\geq b\tilde{u}_\infty\}$. Here A denotes the operator $-\Delta$ with the homogeneous Neumann boundary condition (cf. Henry [4; Theorem 5.1.1]). For the solutions of a semilinear diffusion system with time delay, Yamada has estimated the decaying rate in terms of the fractional power of A in [19]. But his estimate depends on the spatial dimension N, so it is weaker than (0.11) (see Remark 2.6).

The main purpose of the present paper is to improve his estimate and to give a similar one to (0.11) (i.e., an estimate independent of N) for the positive solution of (0.1)–(0.3) under some conditions on the delay kernel f which allow the models A and B. In order to obtain such an estimate, we will study the ***retarded spectrum*** associated with (0.1) and (0.2), then construct a fundamental solution of a linear Volterra integrodifferential equation

$$\frac{dv}{dt}+Av+u_\infty(bv+f*v)=0\,, \quad t>0$$

by using a Dunford integral in the framework of L^p-theory (whereas Yamada has studied the stability of the fundamental solution within the framework of L^2-theory in [19]). This representation of the fundamental solution is analogous to that of analytic semigroups by Dunford integals and it is already used by Da Prato and Lunardi in the frame work of general Banach space theory (see, e.g., [3]). Its essential idea is based on the classical method of constructing solutions of differential equations by Laplace transforms. Compared with the theory of Da Prato and Lunardi, our results are new in point of accomplishing a uniform L^p-estimate of the fundamental solution with respect to p in order to apply it to the nonlinear problem (0.1)–(0.3). Although our main results will be stated for the models A and B because of biological importance, we will prove them in a generalized form. In fact, the same results hold true for any delay model which satisfies the conditions (H.1)–(H.4) (see Section 5.1).

The plan of this paper is as follows. In Section 1 we will give a summary of basic results on solutions of (0.1)–(0.3) which will be necessary later. In Section 2 we will state the main result (Theorem 2.5). The proof of the main result will be given in Section 6 as a corollary from the abstract theory: a reduction of a semilinear problem to a linearized one (Section 4), a representation and an estimate of the fundamental solution (Section 5). Section 3 is devoted to derive some essential properties of the delay kernels for the models A and B, as an introduction into the assumptions (A.1)-(A.4) in Section 4 and (H.1)-(H.4) in Section 5.

1. Preliminaries

In this section we will summarize some preliminary results on solutions of (0.1)-(0.3) where the delay kernel f need not be the type of (0.9) or (0.10).

For $1\leq p\leq\infty$, $\|\cdot\|_p$ denotes the usual $L_p(\Omega)$-norm. For $1<p<\infty$, we define a closed linear operator A_p in $L_p(\Omega)$ with domain $D(A_p)$ by

$$A_p u=-\Delta u\,, \quad D(A_p)=\{u\in W^{2,p}(\Omega);\, \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega\}\,,$$

where $W^{2,p}(\Omega)$ denotes the usual Sobolev space of measurable functions u on Ω such that u and its distributional derivatives up to order 2 belong to $L^p(\Omega)$. The domain $D(A_p)$ is a Banach space with the graph norm of A_p. We sometimes write A instead of A_p if there is no confusion.

Let X be any Banach space. $C([0, \infty); X)$ is the space of X-valued continuous functions on $[0, \infty)$. $C^1([0, \infty); X)$ is the space of X-valued continuously differentiable functions on $[0, \infty)$. For a measurable function f: $[0, \infty) \to X$, its Laplace transform $\hat{f}$ is defined by

$$\hat{f}(z) = \int_0^\infty \exp(-zt) f(t)\, dt\,, \tag{1.1}$$

whenever this integral exists.

Many authors have proved the existence and the uniqueness of solutions of (0.1)-(0.3) in the standard manner of parabolic differential equations.

Proposition 1.1. *If $f \in C^1([0, \infty))$ and $f(t) \geq 0$, then the initial boundary value problem (0.1)-(0.3) has a unique solution u such that*

$$u \in C^1([0, \infty); L^p(\Omega)) \cap C([0, \infty); D(A_p)) \quad \text{for} \quad 1 < p < \infty\,.$$

Moreover, u has the following properties:

(i) $0 \leq u(t, x) \leq \max\{\|u_0\|_\infty, a/b\}$ *for* $t \geq 0$, $x \in \bar{\Omega}$.

(ii) *If $u_0 (\geq 0)$ is not identically zero, then*

$$u(t, x) > 0 \quad \text{for} \quad t > 0,\ x \in \bar{\Omega}\,.$$

For the proof, see, e.g., Schiaffino [9], Yamada [18, 19].

REMARK 1.2. Since $D(A_p) \subset C^1(\bar{\Omega})$ for $p > N$, the solution u of (0.1)-(0.3) satisfies

$$u \in C([0, \infty); C^1(\bar{\Omega}))\,.$$

As for the asymptotic behavior of the solutions, we review some results on the stability of the *equilibrium* associated with (0.1)-(0.3). By formal calculation we can find that, if the solution $u(t, \cdot)$ tends to an equilibrium as $t \to \infty$, then the equilibrium should be a non-negative solution of the following boundary value problem:

$$\Delta u + (a - bu - \alpha u)\, u = 0\,, \quad x \in \Omega\,, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = 0\,, \quad x \in \partial\Omega\,, \tag{1.3}$$

where $\alpha = \int_0^\infty f(t)\, dt$.

Proposition 1.3. *Let $0 < \alpha < \infty$ and $0 < \theta < 1$. If the boundary value problem (1.2) and (1.3) has any non-negative solution $u(x) \in C^{2+\theta}(\bar{\Omega})$, then*

$$u(x) \equiv 0 \quad \text{or} \quad u(x) \equiv u_\infty \ \text{on}\ \Omega\,,$$

where $u_\infty = a/(b + \alpha)$.

In the proof of this proposition, the maximal (or minimal) positive solution and the concavity of the function $G(u)=(a-bu-\alpha u)\,u$ play a great role (see, e.g., [1]).

Clearly the constant functions 0 and u_∞ satisfy (1.2) and (1.3). It is a positive equilibrium in which we are interested. So, hereafter we call u_∞ the equilibrium associated with (0.1)-(0.3).

Schiaffino [9] and Yamada [18] have obtained some sufficient conditions for the global asymptotic stability of u_∞. The following result is due to Yamada [18; Theorem 3.2]:

Proposition 1.4. *Suppose* $f\in C^1([0,\infty))\cap L^1(0,\infty)$, $f(t)\geq 0$,

$$tf(t)\in L^1(0,\infty) \quad \text{and} \quad b+\inf_{\lambda\in R}\operatorname{Re}\hat{f}(i\lambda)>0\,. \tag{1.4}$$

Then, for each non-negative initial data u_0 *which is not identically zero in* Ω, *the solution* $u(t,x)$ *of* (0.1)-(0.3) *satisfies*

$$||u(t,\cdot)-u_\infty||_\infty\to 0 \quad \text{as } t\to\infty\,.$$

REMARK 1.5. In the model A, since (0.9) implies

$$\hat{f}(z)=\frac{\alpha}{Tz+1} \quad (\operatorname{Re} z>-\frac{1}{T})\,, \tag{1.5}$$

$f(t)$ fullfill the conditions in this proposition. On the other hand, in the case that we take $f(t)=(\alpha t/T^2)\exp(-t/T)$ with T and α positive, the condition (1.4) is equivalent to $\alpha<8b$ (i.e., the model B) because of

$$\hat{f}(z)=\frac{\alpha}{(Tz+1)^2} \quad (\operatorname{Re} z>-\frac{1}{T})\,. \tag{1.6}$$

For $\alpha\geq 8b$ in this case, bifurcation of non-constant periodic solutions can take place (see, e.g., Yamada [18, Section 5], Yamada and Niikura [20], Tesei [15]).

2. Main Result

We will state the asymptotic stability of u_∞ more precisely in terms of spectral analysis. Throughout this section, we assume that the kernel function f is given by (0.9) or (0.10), unless otherwise stated.

To this end we follow the usual linearization procedure. Put $w(t,x)=u(t,x)-u_\infty$ and substitute $u(t,x)=u_\infty+w(t,x)$ into (0.1), then we will get

$$\begin{aligned}\frac{\partial w}{\partial t}(t,x) &= \Delta w(t,x)-u_\infty\{bw(t,x)+\int_0^t f(t-s)\,w(s,x)\,ds\}\\ &\quad -w(t,x)\,\{bw(t,x)+\int_0^t f(t-s)\,w(s,x)\,ds\}\\ &\quad +u_\infty u(t,x)\int_t^\infty f(s)\,ds\,.\end{aligned} \tag{2.1}$$

Here we have used $a=bu_\infty+au_\infty$ and $\alpha=\int_0^\infty f(s)\,ds$. Neglecting second-order and residual terms in (2.1), we consider the following linear Volterra diffusion equation with a boundary condition to be a linear approximation of (0.1) and (0.2) about u_∞:

$$\frac{\partial v}{\partial t}=\Delta v-u_\infty(bv+f*v)\,,\quad t>0,\ x\in\Omega\,, \tag{2.2}$$

$$\frac{\partial v}{\partial n}=0\,,\qquad t>0,\ x\in\partial\Omega\,. \tag{2.3}$$

From a viewpoint of spectral analysis, we treat (2.2) and (2.3) as an abstract integrodifferential equation in the Banach space $L^p(\Omega)$:

$$\frac{dv}{dt}(t)+Av(t)+u_\infty\{bv(t)+\int_0^t f(t-s)\,v(s)\,ds\}=0\,, \tag{L}$$

and introduce the *characteristic problem* associated with (L):

$$\lambda v+Av+u_\infty(b+F(\lambda))\,v=0\,, \tag{2.4}$$

where $F(\lambda)$ is the analytic continuation of $\hat{f}(\lambda)$ out of the right half plane $\{\lambda\in\boldsymbol{C};\ \mathrm{Re}\,\lambda>-1/T\}$, namely,

$$\text{(model A)}\quad\begin{cases} f(t)=\dfrac{\alpha}{T}\exp\left(-\dfrac{t}{T}\right) & (t\geq 0)\,,\\[2ex] F(\lambda)=\dfrac{\alpha}{T\lambda+1} & \left(\lambda\in\boldsymbol{C},\ \lambda\neq-\dfrac{1}{T}\right), \end{cases}$$

$$\text{(model B)}\quad\begin{cases} f(t)=\dfrac{\alpha t}{T^2}\exp\left(-\dfrac{t}{T}\right) & (t\geq 0)\,,\\[2ex] F(\lambda)=\dfrac{\alpha}{(T\lambda+1)^2} & \left(\lambda\in\boldsymbol{C},\ \lambda\neq-\dfrac{1}{T}\right). \end{cases}$$

Lemma 2.1. *Suppose an analytic function $F(\lambda)$ can be defined for a complex number λ. For such λ, one of the following statements is true.*

(i) *For each $g\in L^2(\Omega)$, there exists a unique element $v\in D(A_2)$ such that*

$$\lambda v+A_2v+u_\infty(b+F(\lambda))\,v=g\,.$$

(ii) *There exists a non-trivial element $v\in D(A_2)$ such that*

$$\lambda v+A_2v+u_\infty(b+F(\lambda))\,v=0\,.$$

This lemma is shown in Yamada [19] (Lemma 3.2). It is easily derived from the compactness of the resolvents of the Laplace operator in a bounded domain by virtue of the Riesz-Schauder theorem.

DEFINITION 2.2. Let (ii) in Lemma 2.1 be true for a complex number λ ($\neq -1/T$). Such a λ is said to be a *characteristic value* of (2.4). Let σ(L) be the set of characteristic values of (2.4). We call σ(L) the ***retarded spectrum*** associated with (L) (see, e.g., Nakagiri [7], Yamada [19]).

REMARK 2.3. Actually we may consider σ(L) to be defined in the framework of L^p-theory for any p ($1<p<\infty$) in place of L^2-theory, because the eigenvalues of the Laplace operator are independent of p.

DEFINITION 2.4. $-\nu_f = \sup \{\mathrm{Re}\,\lambda;\ \lambda\in\sigma(\mathrm{L})\}$.

Our main result is the next theorem which says that ν_f determines the stability of u_∞.

Theorem 2.5. *In both the cases* (0.9) *and* (0.10), ν_f *in Definition* 2.4 *satisfies*

$$0<\nu_f\leqq\frac{1}{T}. \tag{2.5}$$

For any ε ($0<\varepsilon<\nu_f$) *and each non-negative initial data* u_0 ($\not\equiv 0$), *the solution* $u(t, x)$ *of* (0.1)-(0.3) *converges like*

$$\|u(t, \cdot)-u_\infty\|_\infty = O(\exp\{-(\nu_f-\varepsilon)\,t\}) \quad as\ t\to\infty\,. \tag{2.6}$$

REMARK 2.6. This theorem shows that positive solutions converges to the equilibrium with an exponential rate as time goes on. Theorem 4.3 in [19] combined with Sobolev's inequality leads to

$$\|u(t, \cdot)-u_\infty\|_\infty = O(\exp(-\zeta t)) \quad \text{as } t\to\infty\,,$$

where the constant ζ satisfies

$$0<\zeta<\frac{2}{N}\nu_f\,,$$

hence ζ depends on the spatial dimension N. On the other hand, in our Theorem 2.5 the convergence rate of $u(t, \cdot)$ is independent of N. Moreover, since ε is arbitrary, we may consider that (2.6) is almost the best estimate from a viewpoint of spectral analysis.

REMARK 2.7. We can also prove (2.6) in some situatinos other than (0.9) or (0.10) by making use of the results in Sections 4, 5. For example, let us consider the delay kernel f in the form of

$$f(t) = \frac{\alpha\rho(\omega^2+\rho^2)}{\omega^2+\rho^2+\omega\rho}\exp(-\rho t)\,(1+\sin\omega t) \quad (\alpha, \omega, \rho>0)\,. \tag{2.7}$$

This kernel has infinitely many (local) maxima. In this case the condition

$\rho > \omega$ is sufficient to get $\nu_f > 0$ and (2.6).

3. Lemmas

In this section we derive some properties of the delay kernel f in the model A or B, and then some estimates of the nonlinear terms in (2.1) are given. These will be essential when we use the results of Sections 4,5 in order to prove Theorem 2.5. In the following lemmas (except for Lemma 3.5 and 3.6), we assume that the delay kernel f is given by (0.9) or (0.10).

Lemma 3.1. (i) $\sup\{\lambda \in \boldsymbol{R};\ \exp(\lambda t) f(t) \in L^1(0, \infty)\} = 1/T$.

(ii) *The analytic continuation* $F(z)$ *of the Laplace transform* $\hat{f}(z)$ *is analytic everywhere except for the point* $z = -1/T$.

Proof. (i) and (ii) are clear by the forms (0.9) or (0.10) of f.
Note that

$$F(z) = \frac{\alpha}{Tz+1} : \quad \text{model A}, \tag{3.1}$$

$$F(z) = \frac{\alpha}{(Tz+1)^2} : \quad \text{model B}. \quad \blacksquare \tag{3.2}$$

DEFINITION 3.2. For $0 < \gamma < \pi/2$ and $R > 0$, we define a sector excluding a neighborhood of the origin by

$$S_{\gamma,R} = \{z \in \boldsymbol{C};\ |z| > R,\ |\arg z| < \frac{\pi}{2} + \gamma\}.$$

Lemma 3.3. (i) *Let* $0 < \gamma < \pi/2$ *and* $R > 0$ *be arbitrarily given. Then there exist* $0 < \gamma' < \pi/2$ *and* $R' > 0$ *such that* $z \in S_{\gamma',R'}$ *implies* $z + u_\infty(b + F(z)) \in S_{\gamma,R}$.

(ii) *For any* $0 < \gamma < \pi/2$ *there exists* $R > 0$ *such that*

$$\inf\left\{\left|1 + \frac{u_\infty}{z}(b + F(z))\right|;\ z \in S_{\gamma,R}\right\} > 0. \tag{3.3}$$

Proof. (i) is easily derived from the facts $bu_\infty > 0$ and $\lim_{|z| \to \infty} F(z) = 0$ (see (3.1) and (3.2)).

In order to obtain the inequality (3.3), we have only to choose R sufficiently large, for

$$\lim_{|z| \to \infty} \{1 + \frac{u_\infty}{z}(b + F(z))\} = 1. \quad \blacksquare$$

Lemma 3.4. $0 < \nu_f \leq \dfrac{1}{T}$.

Proof. We prove this inequality for the model A here.
Let λ be a characteristic value of (2.4). Then there exists a non-trivial ele-

ment $v \in D(A_2)$ such that

$$\lambda v + A_2 v + u_\infty (b + F(\lambda)) v = 0 .$$

In other words, $-\lambda - u_\infty(b+F(\lambda))$ is an eigenvalue of A_2. Therefore, by virtue of (3.1), λ satisfies the quadratic equation

$$\lambda^2 + (\mu_n + bu_\infty + \frac{1}{T}) \lambda + \frac{a+\mu_n}{T} = 0 \tag{3.4}$$

for some n, where $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots$ are the eigenvalues of A_2. Since the roots of (3.4) are

$$-\frac{1}{T} - \frac{1}{2}(\mu_n + bu_\infty - \frac{1}{T}) \pm \sqrt{\frac{1}{4}(\mu_n + bu_\infty - \frac{1}{T})^2 - \frac{\alpha u_\infty}{T}} ,$$

it is easy to verify that

$$\sup_{\lambda \in \sigma(L)} \operatorname{Re} \lambda = -\frac{1}{T}$$

$$\text{for} \quad bu_\infty > \frac{1}{T} ,$$

$$\max_{\lambda \in \sigma(L)} \operatorname{Re} \lambda = -\frac{1}{2}(bu_\infty + \frac{1}{T})$$

$$\text{for} \quad bu_\infty \leq \frac{1}{T} \quad \text{and} \quad a > (\sqrt{\alpha u_\infty} - \sqrt{\frac{1}{T}})^2 ,$$

$$\max_{\lambda \in \sigma(L)} \operatorname{Re} \lambda = -\frac{1}{2}(bu_\infty + \frac{1}{T}) + \sqrt{\frac{1}{4}(bu_\infty + \frac{1}{T})^2 - \frac{a}{T}}$$

$$\text{for} \quad bu_\infty < \frac{1}{T} \quad \text{and} \quad 0 < a \leq (\sqrt{\alpha u_\infty} - \sqrt{\frac{1}{T}})^2 .$$

Hence,

$$0 < \nu_f \leq \frac{1}{T} ,$$

where the equality holds for $bu_\infty \geq 1/T$.

The proof for the model B are given in Appendix. ■

By setting $w_\tau(t, x) = w(t+\tau, x)$ for $\tau > 0$, (2.1) is changed into

$$\begin{aligned} \frac{\partial w_\tau}{\partial t}(t, x) &= \Delta w_\tau(t, x) - u_\infty \{ b w_\tau(t, x) + \int_0^t f(t-s)\, w_\tau(s, x)\, ds \} \\ &\quad - w_\tau(t, x) \{ b w_\tau(t, x) + \int_0^t f(t-s)\, w_\tau(s, x)\, ds \} \\ &\quad + u(t+\tau, x) \{ u_\infty \int_{t+\tau}^\infty f(s)\, ds - \int_0^\tau f(t+\tau-s)\, w(s, x)\, ds \} . \end{aligned} \tag{3.5}$$

The next lemma naturally follows from Proposition 1.4.

Lemma 3.5. *Let the assumptions for f in Proposition 1.4 hold true and $u(t, x)$ be the solution of (0.1)-(0.3) with $u_0 \not\equiv 0$. Then, for any $\varepsilon > 0$, there eixsts a positive number τ such that*

$$|w_\tau(t, x)| \leq \varepsilon \qquad \text{for} \quad t \geq 0,\ x \in \bar{\Omega},$$

where $w_\tau(t, x) = u(t+\tau, x) - u_\infty$.

In an application of an estimate for the linearized problem (L) to the non-linear one (0.1)-(0.3), we will need some estimates for the nonlinear terms in (3.5).

Lemma 3.6. *Let $f \in L^1(0, \infty)$ and $f(t) \geq 0$ on $[0, \infty)$. Then, for each $v \in C([0, \infty); L^\infty(\Omega))$,*

$$\|bv(t) + \int_0^t f(t-s)\, v(s)\, ds\|_\infty \leq (b+\alpha) \max_{0 \leq s \leq t} \|v(s)\|_\infty, \quad t \geq 0$$

holds true with $\alpha = \int_0^\infty f(t)\, dt$.

Lemma 3.7. *Let u be the solution of (0.1)-(0.3) and $w = u - u_\infty$. Then, in the case of (0.9),*

$$\sup_{t,\tau>0} \exp\left(\frac{t}{T}\right) \|u(t+\tau, \cdot)\, u_\infty \int_{t+\tau}^\infty f(s)\, ds\|_\infty < \infty, \tag{3.6}$$

$$\sup_{t,\tau>0} \exp\left(\frac{t}{T}\right) \|u(t+\tau, \cdot) \int_0^\tau f(t+\tau-s)\, w(s, \cdot)\, ds\|_\infty < \infty \tag{3.7}$$

hold, and in the case of (0.10),

$$\sup_{t,\tau>0} (1+t)^{-1} \exp\left(\frac{t}{T}\right) \|u(t+\tau, \cdot)\, u_\infty \int_{t+\tau}^\infty f(s)\, ds\|_\infty < \infty, \tag{3.8}$$

$$\sup_{t,\tau>0} (1+t)^{-1} \exp\left(\frac{t}{T}\right) \|u(t+\tau, \cdot) \int_0^\tau f(t+\tau-s)\, w(s, \cdot)\, ds\|_\infty < \infty \tag{3.9}$$

hold.

Proof. Proposition 1.1 (i) leads us to the following inequalities:

$$\|u(t+\tau, \cdot)\, u_\infty \int_{t+\tau}^\infty f(s)\, ds\|_\infty \leq c \int_{t+\tau}^\infty f(s)\, ds, \tag{3.10}$$

$$\|u(t+\tau, \cdot) \int_0^\tau f(t+\tau-s)\, w(s, \cdot)\, ds\|_\infty \leq c \int_t^{t+\tau} f(s)\, ds, \tag{3.11}$$

where c is a positive constant independent of t and τ. In the case of (0.9), observing that

$$\int_{t+\tau}^{\infty} f(s)\,ds = \alpha \exp\left(-\frac{t+\tau}{T}\right) \leq \alpha \exp\left(-\frac{t}{T}\right),$$

$$\int_{t}^{t+\tau} f(s)\,ds = \alpha \exp\left(-\frac{t}{T}\right) \{1-\exp\left(-\frac{\tau}{T}\right)\} \leq \alpha \exp\left(-\frac{t}{T}\right),$$

(3.10) and (3.11) imply (3.6) and (3.7) respectively. Similarly, in the case of (0.10), we obtain (3.8) and (3.9) by virtue of

$$\begin{aligned}\int_{t+\tau}^{\infty} f(s)\,ds &= \frac{\alpha}{T}(T+t+\tau)\exp\left(-\frac{t+\tau}{T}\right)\\ &= \frac{\alpha}{T}\exp\left(-\frac{t}{T}\right)\{(T+t)\exp\left(-\frac{\tau}{T}\right)+\tau\exp\left(-\frac{\tau}{T}\right)\}\\ &\leq \frac{\alpha}{T}\exp\left(-\frac{t}{T}\right)(T+t+Te^{-1})\end{aligned}$$

and

$$\begin{aligned}\int_{t}^{t+\tau} f(s)\,ds &= \frac{\alpha}{T}\exp\left(-\frac{t}{T}\right)\{T+t-(T+t+\tau)\exp\left(-\frac{\tau}{T}\right)\}\\ &\leq \frac{\alpha}{T}\exp\left(-\frac{t}{T}\right)(T+t)\,. \quad \blacksquare\end{aligned}$$

4. Stability for a Semilinear Functional Differential Equation and Fundamental Solutions

In view of (3.5), we will investigate the asymptotic stability for a semilinear functional differential equation in $L^p(\Omega)$:

$$(4.1)\qquad \begin{cases} \dfrac{dv}{dt}(t)+\tilde{A}v(t) = \displaystyle\int_0^t g(t-s)\,v(s)\,ds+h_1[v](t)+h_2(t)\,, & t>0\,,\\ v(0) = v_0\,. \end{cases}$$

Here we impose the following conditions on $\tilde{A}$, g, $h_1[v]$ and h_2:

(A.1) Let $p\,(1<p<\infty)$ be arbitrarily fixed. $\tilde{A}$ is a closed linear operator densely defined in $L^p(\Omega)$ and $-\tilde{A}$ is the infinitesimal generator of a semigroup $\exp(-t\tilde{A})$ of class (C_0) on $L^p(\Omega)$.

(A.2) $g\in L^1(0, \infty)$.

(A.3) A nonlinear operator

$$h_1\colon C([0, \infty); C(\bar{\Omega}))\ni v \mapsto h_1[v]\in C([0, \infty); C(\bar{\Omega}))$$

satisfies, for $v\in C([0, \infty); C(\bar{\Omega}))$ and $t\geq 0$,

$$\frac{\|h_1[v](t)\|_\infty}{\|v(t)\|_\infty} \leq c_1 \max_{0\leq s\leq t} \|v(s)\|_\infty\,,$$

where $h_1[v](t)$ denotes the value of $h_1[v]\in C([0, \infty); C(\bar{\Omega}))$ at $t\in[0, \infty)$

and c_1 is a positive constant independent of t.

(A.4) A function $h_2 \in C([0, \infty); C(\bar{\Omega}))$ satisfies

$$\|h_2(t)\|_\infty \le c_2 \exp(-\rho_1 t) \qquad \text{for} \quad t \ge 0,$$

where ρ_1 and c_2 are positive constants independent of t.

We treat the nonlinear terms $h_1[v]+h_2$ in (4.1) as if they were previously given, and reduce (4.1) to a linear nonhomogeneous problem:

$$\text{(4.2)} \qquad \begin{cases} \dfrac{dv}{dt}(t)+\tilde{A}v(t) = \displaystyle\int_0^t g(t-s)\, v(s)\, ds + h(t), & t>0, \\ v(0) = v_0, \end{cases}$$

with $h \in C([0, \infty); L^p(\Omega))$ and $v_0 \in L^p(\Omega)$.

DEFINITION 4.1. Let $v \in C([0, \infty); L^p(\Omega))$ satisfy

$$\text{(4.3)} \qquad \begin{aligned} v(t) &= \exp(-t\tilde{A})\, v_0 \\ &\quad + \int_0^t \exp\{-(t-s)\tilde{A}\} \left\{\int_0^s g(s-r)\, v(r)\, dr + h(s)\right\} ds, \quad t \ge 0. \end{aligned}$$

We call such a function v a *mild solution* of (4.2) (cf. Yamada [19; p. 305]).

It is easy to see that $v \in C^1([0, \infty); L^p(\Omega)) \cap C([0, \infty); D(\tilde{A}))$ satisfying (4.2) is a mild solution.

The existence of mild solutions is proved by the method of successive approximation:

Lemma 4.2. *Let* (A.1) *and* (A.2) *be satisfied, and a function* $h \in C([0, \infty); L^p(\Omega))$ *be given. Then for each* $v_0 \in L^p(\Omega)$ *there exists a unique mild solution* v *of* (4.2).

Now we introduce a fundamental solution associated with (4.2).

DEFINITION 4.3. Let v be the mild solution of (4.2) with $h \equiv 0$ and $v_0 \in L^p(\Omega)$. Set

$$R(t; \tilde{A}, g)\, v_0 = v(t) \qquad \text{for} \quad t \ge 0.$$

We call the operator $R(t; \tilde{A}, g)$ a *fundamental solution* of

$$\text{(4.4)} \qquad \frac{dv}{dt}(t)+\tilde{A}v(t) = \int_0^t g(t-s)\, v(s)\, ds$$

(see, e.g., Yamada [19], Nakagiri [7]).

Proposition 4.4. *Under the assumptions* (A.1) *and* (A.2), *the fundamental solution* $R(t; \tilde{A}, g)$ *possesses the following properties.*

(i) $R(t; \tilde{A}, g)$ *is a bounded linear operator from* $L^p(\Omega)$ *into* $L^p(\Omega)$ *for each* $t \ge 0$.

(ii) *For each* $v_0 \in L^p(\Omega)$ *and* $h \in C([0, \infty); L^p(\Omega))$, *the mild solution* v *of* (4.2) *is represented by*

$$v(t) = R(t; \tilde{A}, g)\, v_0 + \int_0^t R(t-s; \tilde{A}, g)\, h(s)\, ds\,, \quad t \geq 0\,. \tag{4.5}$$

For the proof, see Yamada [19; pp. 305–307].

Suppose that

(A.5) The fundamental solution $R(t; \tilde{A}, g)$ satisfies, for all $t \geq 0$ and $v_0 \in L^\infty(\Omega)$,

$$||R(t; \tilde{A}, g)\, v_0||_\infty \leq c_3 \exp(-\rho_2 t)\, ||v_0||_\infty\,,$$

where c_3 and ρ_2 are positive constants independent of t and v_0.

We will derive exponentially asymptotic stability for (4.1) from the above estimate of the fundamental solution:

Theorem 4.5. *Let* (A.1), (A.2), (A.3), (A.4) *and* (A.5) *be fullfilled. Assume that* $v \in C^1([0, \infty); L^p(\Omega)) \cap C([0, \infty); D(\tilde{A}) \cap C(\bar{\Omega}))$ *satisfies* (4.1). *Then for each* $\varepsilon > 0$ *there exist positive numbers* C *and* δ *such that*

$$||v(t)||_\infty \leq \delta\,, \quad t \geq 0 \tag{4.6}$$

implies

$$||v(t)||_\infty \leq C \exp\{-(\rho-\varepsilon)\, t\}\,, \quad t \geq 0\,, \tag{4.7}$$

with $\rho = \min\{\rho_1, \rho_2\}$. *Here* C *and* δ *depend only on* ε, c_1, c_2 *and* c_3.

Proof. Let $\varepsilon > 0$ be arbitrarily fixed.

Since $v \in C([0, \infty); C(\bar{\Omega}))$, (A.3) and (A.4) imply $h_1[v] + h_2 \in C([0, \infty); L^p(\Omega))$. So we can consider v as a mild solution of (4.2) with $h = h_1[v] + h_2$, and by (4.5) v satisfies

$$v(t) = R(t; \tilde{A}, g)\, v_0 + \int_0^t R(t-s; \tilde{A}, g) \{h_1[v](s) + h_2(s)\}\, ds\,, \quad t \geq 0\,,$$

which leads us to the following inequality:

$$\begin{aligned} ||v(t)||_\infty \leq\ & c_3 \exp(-\rho_2 t)\, ||v_0||_\infty \\ & + c_1 c_3 \int_0^t \exp\{-\rho_2(t-s)\}\, ||v(s)||_\infty \max_{0 \leq r \leq s} ||v(r)||_\infty\, ds \\ & + c_2 c_3 \int_0^t \exp\{-\rho_2(t-s)\} \exp(-\rho_1 s)\, ds\,, \quad t \geq 0\,. \end{aligned} \tag{4.8}$$

Thus, supposing that (4.6) holds,

$$\begin{aligned} ||v(t)||_\infty \leq\ & c_3(\delta + c_2 t) \exp(-\rho t) \\ & + c_1 c_3\, \delta \int_0^t \exp\{-\rho(t-s)\}\, ||v(s)||_\infty\, ds\,, \quad t \geq 0\,. \end{aligned}$$

Consequently, an application of Gronwall's lemma yields

$$\begin{aligned}
&||v(t)||_\infty \\
&\leq \exp(-\rho t)\, [c_3(\delta+c_2 t)+c_1 c_3 \delta \int_0^t \exp\{c_1 c_3 \delta(t-s)\}\, c_3(\delta+c_2 s)\, ds] \\
&= \exp(-\rho t)\, \{(c_3 \delta+\frac{c_2}{c_1 \delta}) \exp(c_1 c_3 \delta t)-\frac{c_2}{c_1 \delta}\} \\
&\leq (c_3 \delta+\frac{c_2}{c_1 \delta}) \exp\{-(\rho-c_1 c_3 \delta)\, t\}\,, \quad t\geq 0\,.
\end{aligned}$$

Therefore, by choosing C and δ such that

$$C=\frac{\varepsilon}{c_1}+\frac{c_2 c_3}{\varepsilon},\ \delta=\frac{\varepsilon}{c_1 c_3},$$

we obtain (4.7). ■

5. Representation and Estimate of Fundamental Solution

In what follows, we take $A+bu_\infty$ and $-u_\infty f$ as $\tilde{A}$ and g respectively in (4.4) with $u_\infty=a/(b+\alpha)$ and $\alpha=\int_0^\infty |f(t)|\, dt$. But f need not be given by (0.9) or (0.10). Instead we allow general f which satisfies the properties deduced in Section 3: Lemma 3.1, 3.3 and 3.4 (see also (H.1)-(H.4) in Section 5.1). We simply write $R(t)$ in place of $R(t; A+bu_\infty, -u_\infty f)$. In other words, $R(t)$ is a fundamental solution of (L) in Section 2. For $1<p<\infty$, the operator norm of the bounded linear operators on $L^p(\Omega)$ is denoted by $|||\cdot|||_p$, or simply $|||\cdot|||$ if there is no confusion.

In this section, we will provide a representation of $R(t)$ by a Dunford integral under some assumptions for the delay kernel f. This representation will enable us to evaluate the decaying rate of $R(t)$.

First of all, we give a resolvent estimate which will play a crucial role in the proof of Theorem 5.4, 5.5 later.

Lemma 5.1. *There exist M_0, $R_0>0$ and $0<\gamma_0<\pi/2$ such that*

(5.1) $$|||(z+A)^{-1}|||_p\leq\frac{M_0}{|z|}, \qquad \text{for} \quad z\in S_{\gamma_0,R_0}\,.$$

where the sector S_{γ_0,R_0} is the one in Definition 3.2.

This lemma is derived from a well-known estimate of a Green's function associated with an elliptic boundary value problem. For the proof, see, e.g., Tanabe [13, 14; Theorem 17–8].

REMARK 5.2. In (5.1), the constants M_0, R_0 and γ_0 are independent of p $(1<p<\infty)$.

In the following subsections, let M_0, R_0 and γ_0 satisfy (5.1) and be fixed.

5.1. Notation and hypotheses on f

For the delay kernel f, we set

$$\lambda_f = \sup \{\lambda \in \boldsymbol{R};\ \exp(\lambda t) f(t) \in L^1(0, \infty)\}\ .$$

Clearly the Laplace transform $\hat{f}(z)$ of $f(t)$ can be defined by (1.1) at least in $\{z \in \boldsymbol{C};\ \mathrm{Re}\ z > -\lambda_f\}$.

We denote by $F(z)$ the analytic continuation of $\hat{f}(z)$ into the maximal region $D_f(\subset \boldsymbol{C})$ in which $F(z)$ is a single-valued and holomorphic function. If $F(z)$ could be multiple-valued, then D_f would be defined and fixed as any one of the greatest possible regiosn each of which corresponds to a branch of $F(z)$. (Most typical examples of $F(z)$ connected with bioligical models are single-valued functions.)

Putting

$$s(z) = z + u_\infty(b + F(z)) \qquad \text{for} \quad z \in D_f\ ,$$

we define the *retarded spectrum* associated with (L) by

$$\sigma(\mathrm{L}) = \{\lambda \in D_f;\ s(\lambda) \in \sigma(-A)\}\ ,$$

where $\sigma(-A)$ denotes the spectrum of $-A$ (cf. Remark 2.3).

Then

$$-\nu = \begin{cases} \sup\limits_{\lambda \in \sigma(L)} \mathrm{Re}\ \lambda & \text{if} \quad \sigma(\mathrm{L}) \neq \phi\ , \\ -\lambda_f & \text{if} \quad \sigma(\mathrm{L}) = \phi \end{cases}$$

is the *right edge* of the retarded spectrum $\sigma(\mathrm{L})$.

Impose the following conditions on f:

(H.1) $\lambda_f > 0$.

(H.2) For some $0 < \gamma < \pi/2$ and $R > 0$,

$$S_{\gamma,R} \cup \{z \in \boldsymbol{C};\ \mathrm{Re}\ z > -\lambda_f\} \subset D_f$$

holds true.

(H.3) $\nu > 0$.

(H.4) There exist $0 < \gamma < \pi/2$ and $R > 0$ such that

(i) $z \in S_{\gamma,R}$ implies $s(z) \in S_{\gamma_0,R_0}$,

(ii) $c_f = \inf \left\{ \left| \dfrac{s(z)}{z} \right|;\ z \in S_{\gamma,R} \right\} > 0$,

Hereafter γ and R are assumed to be fixed constants which satisfy (H.2) and (H.4).

Under the above assumptions, it is easy to see that the constant $\nu_f = \min\{\nu, \lambda_f\}$ has the following properties:

Lemma 5.3. (i) $\nu_f > 0$.

(ii) *If* $\mathrm{Re}\, z > -\nu_f$ *or* $z \in S_{\gamma,R}$, *then* $s(z) \in \rho(-A)$, *where* $\rho(-A)$ *is the resolvent set of* $-A$.

5.2. Submain theorems

Theorem 5.4. *If* (H.1), (H.2), (H.3) *and* (H.4) *are fullfilled, the fundamental solution* $R(t)$ *of* (L) *is explicitly given by*

$$R(t) = \frac{1}{2\pi i} \int_\Gamma \exp(tz)\,(s(z)+A)^{-1}\,dz\,, \quad t>0\,. \tag{5.2}$$

Here Γ *is a path running in the region* $S_{\gamma,R} \cup \{z \in \boldsymbol{C};\ \mathrm{Re}\, z > -\nu_f\}$ *from* $\infty \exp(-i\theta)$ *to* $\infty \exp(i\theta)$ *with* $\pi/2 < \theta < \pi/2 + \gamma$.

Proof. First we are going to see that $\exp(tz)\,(s(z)+A)^{-1}$ is integrable on Γ.

Since $(s(z)+A)^{-1}$ is holomorphic in $S_{\gamma,R} \cup \{\zeta;\ \mathrm{Re}\,\zeta > -\nu_f\}$ by virtue of Lemma 5.3 (ii), we may change Γ into $\Gamma_t = \Gamma_{t,+} \cup \Gamma_{t,0} \cup \Gamma_{t,-}$;

$$\Gamma_{t,\pm} = \{\rho \exp(\pm i\theta);\ \rho \geq \frac{1}{t}\}\,,$$

$$\Gamma_{t,0} = \{\frac{1}{t} \exp(i\sigma);\ -\theta \leq \sigma \leq \theta\}\,,$$

where $\theta\,(\pi/2 < \theta < \pi/2 + \gamma)$ is a constant such that the whole ray $\{\rho \exp(i\theta);\ \rho \geq 0\}$ is included in the region $S_{\gamma,R} \cup \{\zeta;\ \mathrm{Re}\,\zeta > -\nu_f\}$. It follows from (5.1) and (H.4) that

$$|||(s(z)+A)^{-1}||| \leq \frac{M_0}{c_f |z|} \qquad \text{for} \quad z \in S_{\gamma,R}\,. \tag{5.3}$$

Moreover, with the aid of the continuity of $(s(z)+A)^{-1}$ on a neighborhood of the origin, we get

$$|||(s(z)+A)^{-1}|||_p \leq \frac{M_p}{1+|z|} < \frac{M_p}{|z|} \quad \text{in} \quad S_{\gamma,R} \cup \{\zeta;\ \mathrm{Re}\,\zeta > -\nu_f\}\,, \tag{5.4}$$

where the positive constant M_p may possibly depend on p $(1<p<\infty)$. (5.4) implies

$$\begin{aligned} &\int_{\Gamma_{t,\pm}} |||\exp(tz)\,(s(z)+A)^{-1}|||_p\ |dz| \\ &\leq \int_{1/t}^{\infty} \exp(t\rho\cos\theta)\,\frac{M_p}{\rho}\,d\rho \\ &\leq \frac{M_p}{|\cos\theta|}\,, \quad t>0 \end{aligned} \tag{5.5}$$

and

$$(5.6)\qquad \begin{aligned}&\int_{\Gamma_{t,0}} |||\exp(tz)(s(z)+A)^{-1}|||_p\,|dz|\\ &\le \int_{-\theta}^{\theta} \exp(\cos\sigma)\,tM_p\,\frac{d\sigma}{t}\\ &\le 2\pi e\,M_p, \quad t>0,\end{aligned}$$

which assure the integrability of $\exp(tz)(s(z)+A)^{-1}$ on Γ_t (thus on Γ).

Namely the right-hand side of (5.2) is well defined. Denote it by $W(t)$ for simplicity.

Next we will show that the Laplace trasnforms of both sides of (5.2) coincide.

Let $E(t; x, y)$ be the heat kernel of $\exp(-tA)$, i.e.,

$$(5.7)\qquad \exp(-tA)\,u(x) = \int_\Omega E(t; x, y)\,u(y)\,dy\,.$$

It is easy to see that the well-known properties of $E(t; x, y)$:

$$(5.8)\qquad \begin{aligned}&E(t, x, y)>0\,, && t>0,\ x\in\Omega,\ y\in\Omega\,,\\ &\int_\Omega E(t; x, y)\,dy = 1\,, && t>0,\ x\in\Omega\end{aligned}$$

lead to

$$(5.9)\qquad |||\exp(-tA)|||_p \le 1\,, \quad t\ge 0,\ 1\le p\le\infty\,.$$

In particular, we have

$$|||\exp\{-t(A+bu_\infty)\}||| \le \exp(-bu^\infty t)\,(\le 1)\,, \quad t\ge 0\,,$$

so that an application of Gronwall's lemma to (4.3) with $h\equiv 0$ yields

$$|||R(t)||| \le \exp(\alpha u_\infty t)\,, \quad t\ge 0\,.$$

Then as it is shown in Yamada [19; p. 306], the Laplace transform $\hat{R}(z)$ of $R(t)$ is given by

$$\hat{R}(z) = (s(z)+A)^{-1} \qquad \text{for} \quad \operatorname{Re} z>\alpha u_\infty\,.$$

On the other hand, the Laplace transform $\hat{W}(z)$ of $W(t)$ can be defined at least for $\operatorname{Re} z>0$ because of the boundedness of $|||W(t)|||_p$ which follows from (5.5) and (5.6). Now let Γ be $\Gamma_+\cup\Gamma_-$;

$$\Gamma_\pm = \{\rho\exp(\pm i\theta);\ \rho\ge 0\}\,,$$

where θ is the same constant as one in the definition of Γ_t. Noting that $\operatorname{Re}(\zeta-z)\le -\operatorname{Re} z$ for $\zeta\in\Gamma$, by virtue of (5.4), we can see that

$$\begin{aligned}\hat{W}(z) &= \int_0^\infty \exp(-zt)\, W(t)\, dt \\ &= \int_0^\infty \exp(-zt)\, \{\frac{1}{2\pi i}\int_\Gamma \exp(\zeta t)\,(s(\zeta)+A)^{-1}\, d\zeta\}\, dt \\ &= \frac{1}{2\pi i}\int_\Gamma [\int_0^\infty \exp\{(\zeta-z)\,t\}\, dt]\,(s(\zeta)+A)^{-1}\, d\zeta \\ &= -\frac{1}{2\pi i}\int_\Gamma \frac{(s(\zeta)+A)^{-1}}{\zeta-z}\, d\zeta\end{aligned}$$

is valid for Re $z>0$. Hence, by Cauchy's integral formula and (5.4) again,

$$\hat{W}(z) = (s(z)+A)^{-1} \qquad \text{for} \quad \operatorname{Re} z>0\,,$$

Therefore

$$\hat{R}(z) = \hat{W}(z) \qquad \text{for} \quad \operatorname{Re} z>\alpha u_\infty\,,$$

from which the conclusion of the theorem is derived by the uniqueness theorem for Laplace transforms (see, e.g., Hille and Phillips [5; Theorem 6.3.2]). ■

Lemma 5.5. *Let $\kappa\subset D_f$ be a compact set and satisfy $s(\kappa)\cap\sigma(-A)=\phi$. Then*

$$\sup\{|||(s(z)+A)^{-1}|||_p;\ z\in\kappa,\ 2\le p<\infty\}<\infty\,. \tag{5.10}$$

Proof. First observe that the semigroup $\exp(-tA)$ satisfies

$$\|\exp(-tA)\,u\|_p\le c\,|\Omega|^{1/p}\exp(\omega t)\,\|u\|_2 \qquad \text{for} \quad \begin{cases} u\in L^2(\Omega)\,, \\ t\ge1,\ 2\le p<\infty\,, \end{cases} \tag{5.11}$$

where $|\Omega|$ denotes the volume of Ω and $c>0$, $\omega\in\boldsymbol{R}$ are constants depending on N and Ω but independent of u, t and p.

To see this inequality, we invoke the following estimate of the heat kernel which has appeared in the proof of Theorem 5.4:

$$|E(t;x,y)|\le\frac{c'}{t^{N/2}}\exp\left(\omega t-c''\frac{|x-y|^2}{t}\right) \qquad \text{for} \quad t>0,\ x,y\in\Omega\,. \tag{5.12}$$

Here c' and c'' are positive constants depending on N and Ω (see, e.g., Tanabe [13, 14; Theorem 17-8]). By (5.7), (5.8) and Schwarz' inequality,

$$\begin{aligned}|\exp(-tA)\,u(x)| &\le\int_\Omega E(t;x,y)\,|u(y)|\,dy \\ &\le\left\{\int_\Omega E(t;x,y)^2\,dy\right\}^{1/2}\|u\|_2\end{aligned}$$

holds for $u\in L^2(\Omega)$. Since

$$E(t;x,y) = E(t;y,x)$$

and

$$\int_\Omega E(t; x, y)\, E(s; y, x')\, dy = E(t+s; x, x'),$$

(5.12) implies

$$\|\exp(-tA)\, u\|_\infty \le \frac{c'^{1/2}}{(2t)^{N/4}} \exp(\omega t)\, \|u\|_2 \qquad \text{for} \quad u \in L^2(\Omega),\ t>0\,.$$

Hence (5.11) follows from the basic inequality:

$$\|u\|_p \le |\Omega|^{1/p}\, \|u\|_\infty \qquad \text{for} \quad u \in L^\infty(\Omega)\,. \tag{5.13}$$

As κ is compact and contained by D_f, there exists a real number $s_\kappa \equiv \min \{\mathrm{Re}\, s(z);\ z \in \kappa\}$. Let $\varphi_j (j=0, 1, 2, \cdots)$ be a complete orthonormal system in $L^2(\Omega)$ of the eigenfunctions of A corresponding to the eigenvalues μ_j (with $0=\mu_0<\mu_1\le\mu_2\le\cdots$). Define J by the smallest number such that

$$\mu_J > -s_\kappa\,. \tag{5.14}$$

Set

$$\delta = \min \{|s(z)+\mu_j|\,;\ 0\le j\le J-1,\ z\in\kappa\}\ (>0)\,,$$
$$\Phi = \max \{\|\varphi_j\|_\infty;\ 0\le j \le J-1\}\ (<\infty)\,.$$

By $(\cdot, \cdot)$ we denote the usual $L^2(\Omega)$-inner product.

Let $2\le p<\infty$ be fixed. Take $v\in L^p(\Omega)$ $(\subset L^2(\Omega))$ and put

$$v_\kappa = v - \sum_{j=0}^{J-1} (v, \varphi_j)\, \varphi_j = \sum_{j=J}^{\infty} (v, \varphi_j)\, \varphi_j\,.$$

Then we will show that

$$\|(s(z)+A)^{-1}\, v_\kappa\|_p \le c_\kappa\, \|v\|_p\,, \qquad z\in\kappa\,, \tag{5.15}$$

$$\Big\|\sum_{j=0}^{J-1} (s(z)+A)^{-1}\, (v, \varphi_j)\, \varphi_j\Big\|_p \le c_\kappa\, \|v\|_p\,, \quad z\in\kappa\,, \tag{5.16}$$

where c_κ is a positive constant depending on κ, N and Ω but independent of p, z and v. Clearly (5.15) and (5.16) imply (5.10).

Parseval's equality yields

$$\|\exp(-tA)\, v_\kappa\|_2 \le \exp(-\mu_J\, t)\, \|v_\kappa\|_2 \qquad \text{for} \quad t\ge 0\,,$$

so that, by (5.11),

$$\begin{aligned}
\|\exp(-tA)\, v_\kappa\|_p &= \|\exp(-A)\exp\{-(t-1)A\}\, v_\kappa\|_p \\
&\le c|\Omega|^{1/p} \exp(\omega)\, \|\exp\{-(t-1)A\}\, v_\kappa\|_2 \\
&\le c|\Omega|^{1/p} \exp\{\omega-\mu_J(t-1)\}\, \|v_\kappa\|_2 \\
&\le c|\Omega|^{1/2} \exp(\omega+\mu_J) \exp(-\mu_J\, t)\, \|v_\kappa\|_p\,, \quad t\ge 1\,.
\end{aligned}$$

Here we have used

$$\|u\|_2 \leq |\Omega|^{1/2-1/p} \|u\|_p \qquad \text{for} \quad n \in L^p(\Omega). \tag{5.17}$$

Moreover, with the aid of (5.9), we get

$$\|\exp(-tA)\, v_\kappa\|_p \leq c''' \exp(-\mu_J t) \|v_\kappa\|_p, \quad t \geq 0, \tag{5.18}$$

where $c''' = \exp(\mu_J) \max\{c|\Omega|^{1/2} e^{\omega}, 1\}$. Thus for $z \in \kappa$ we have

$$\begin{aligned} &\int_0^\infty |\exp\{-ts(z)\}| \, \|\exp(-tA)\, v_\kappa\|_p \, dt \\ &\leq \int_0^\infty \exp(-ts_\kappa)\, c''' \exp(-\mu_J t) \|v_\kappa\|_p \, dt. \end{aligned}$$

Hence (5.14) assures

$$(s(z)+A)^{-1} v_\kappa = \int_0^\infty \exp\{-ts(z)\} \exp(-tA)\, v_\kappa \, dt \qquad \text{for} \quad z \in \kappa,$$

and

$$\|(s(z)+A)^{-1} v_\kappa\|_p \leq \frac{c'''}{s_\kappa + \mu_J} \|v_\kappa\|_p \qquad \text{for} \quad z \in \kappa. \tag{5.19}$$

By (5.13) and (5.17),

$$\begin{aligned} \|v_\kappa\|_p &\leq \|v\|_p + \sum_{j=0}^{J-1} |(v, \varphi_j)| \, \|\varphi_j\|_p \\ &\leq \|v\|_p + |\Omega|^{1/p} \Phi \sum_{j=0}^{J-1} |(v, \varphi_j)| \\ &\leq \|v\|_p + |\Omega|^{1/p} \Phi J^{1/2} \{\sum_{j=0}^{J-1} |(v, \varphi_j)|^2\}^{1/2} \\ &\leq \|v\|_p + |\Omega|^{1/p} \Phi J^{1/2} \|v\|_2 \\ &\leq \{1 + (|\Omega| J)^{1/2} \Phi\} \|v\|_p. \end{aligned} \tag{5.20}$$

Therefore (5.15) is derived from (5.19) and (5.20).

On the other hand, since

$$\begin{aligned} \|(s(z)+A)^{-1} \varphi_j\|_p &= |s(z)+\mu_j|^{-1} \|\varphi_j\|_p \\ &\leq \delta^{-1} \|\varphi_j\|_p \end{aligned}$$

for $z \in \kappa$ and $0 \leq j \leq J-1$, (5.16) is deduced in the same way as (5.20). Thus we complete the proof. ■

Theorem 5.6. *Assume* (H.1), (H.2), (H.3) *and* (H.4). *For a given positive number* $\mathcal{E}(<\nu_f)$, *the fundamental solution* $R(t)$ *is evaluated like*

$$|||R(t)|||_p \leq C \exp\{-(\nu_f - \mathcal{E})\, t\}, \quad 2 \leq p < \infty, \; t \geq 0, \tag{5.21}$$

Here C *is a positive constant which is independent of* p *and* t.

Proof. Set

$$d = \inf \{|\mathrm{Im}\, z|\,;\, z \in S_{\gamma,R},\ \mathrm{Re}\, z = -\nu_f\}\ ,$$
$$\tilde{R} = \max\{\sqrt{d^2+\nu_f^2},\ R+\nu_f\}\ .$$

Let $\theta(\pi/2<\theta<\theta_\varepsilon)$ be arbitrarily fixed with $\tan\theta_\varepsilon = -d/\varepsilon\ (\pi/2<\theta_\varepsilon<\pi)$.

When $0<t<1/\tilde{R}$, we replace Γ with $\tilde{\Gamma}_t = \tilde{\Gamma}_{t,+} \cup \tilde{\Gamma}_{t,0} \cup \tilde{\Gamma}_{t,-}$ in (5.2):

$$\tilde{\Gamma}_{t,\pm} = \{-\nu_f+\varepsilon+\rho\exp(\pm i\theta);\ \rho \geq \frac{1}{t}\}\ ,$$
$$\tilde{\Gamma}_{t,0} = \{-\nu_f+\varepsilon+\frac{1}{t}\exp(i\sigma);\ -\theta\leq\sigma\leq\theta\}\ .$$

Since $\tilde{\Gamma}_t \subset S_{\gamma,R}$ for $0<t<1/\tilde{R}$, the following inequalities are deduced from (5.3):

$$|||(s(z)+A)^{-1}||| \leq \frac{M_0}{c_f\rho} \quad \text{for} \quad z\in\tilde{\Gamma}_{t,\pm}\ ,$$
$$|||(s(z)+A)^{-1}||| \leq \frac{M_0\tilde{R}t}{c_f R} \quad \text{for} \quad z\in\tilde{\Gamma}_{t,0}\ .$$

Hence we have

$$\begin{aligned}(5.22)\qquad &\int_{\tilde{\Gamma}_{t,\pm}} |\exp(tz)|\ |||(s(z)+A)^{-1}|||\ |dz| \\ &\leq \int_{1/t}^{\infty} \exp\{-(\nu_f-\varepsilon)t\}\exp(t\rho\cos\theta)\frac{M_0}{c_f\rho}\,d\rho \\ &\leq \frac{M_0}{c_f|\cos\theta|}\exp\{-(\nu_f-\varepsilon)t\}\ .\end{aligned}$$

and

$$\begin{aligned}(5.23)\qquad &\int_{\tilde{\Gamma}_{t,0}} |\exp(tz)|\ |||(s(z)+A)^{-1}|||\ |dz| \\ &\leq \int_{-\theta}^{\theta} \exp\{-(\nu_f-\varepsilon)t\}\exp(\cos\sigma)\frac{M_0\tilde{R}t}{c_f R}\frac{d\sigma}{t} \\ &\leq \frac{2\pi e M_0\tilde{R}}{c_f R}\exp\{-(\nu_f-\varepsilon)t\}\ .\end{aligned}$$

Now let $t\geq 1/\tilde{R}$ and we take $\tilde{\Gamma} = \tilde{\Gamma}_+ \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_-$ as Γ in (5.2) where

$$\tilde{\Gamma}_\pm = \{-\nu_f+\varepsilon+\rho\exp(\pm i\theta);\ \rho\geq\tilde{R}\}\ ,$$
$$\tilde{\Gamma}_0 = \{-\nu_f+\varepsilon+\rho\exp(-i\theta);\ 0\leq\rho\leq\tilde{R}\}$$
$$\cup\{-\nu_f+\varepsilon+\rho\exp(i\theta);\ 0\leq\rho\leq\tilde{R}\}\ .$$

Since $\tilde{\Gamma}_\pm \subset S_{\gamma,R}$ and $\tilde{R}\geq 1/t$, a similar calculation to (5.22) yields

$$(5.24)\quad \int_{\tilde{\Gamma}_\pm} |\exp(tz)|\ |||(s(z)+A)^{-1}|||\ |dz| \leq \frac{M_0}{c_f|\cos\theta|}\exp\{-(\nu_f-\varepsilon)t\}\ .$$

The path $\tilde{\Gamma}_0$ is compact and $s(\tilde{\Gamma}_0)\cap\sigma(-A)=\phi$ by virtue of Lemma 5.3 (ii). Thus, by choosing $\tilde{\Gamma}_0$ as κ in Lemma 5.5, we get

$$|||(s(z)+A)^{-1}|||_p\leq c_0 \qquad \text{for} \quad 2\leq p<\infty, z\in\tilde{\Gamma}_0,$$

where c_0 is a positive constant independent of p and z. Therefore,

$$\begin{aligned}(5.25)\qquad &\int_{\tilde{\Gamma}_0} |\exp(tz)|\ |||(s(z)+A)^{-1}|||\ |dz| \\ &\leq 2\int_0^{\tilde{R}} \exp\{-(\nu_f-\varepsilon)t\} \exp(t\rho\cos\theta)\, c_0\, d\rho \\ &\leq 2c_0\tilde{R}\exp\{-(\nu_f-\varepsilon)t\}\ .\end{aligned}$$

Consequently (5.21) follows from (5.2) combined with (5.22) and (5.23), or (5.24) and (5.25). ■

REMARK 5.7. From the above proof, the constant C in (5.21) is considered to be

$$C=\frac{1}{2\pi}\Big[\frac{M_0}{c_f|\cos\theta|}+2\tilde{R}\max\{c_0, \frac{\pi e M_0}{c_f R}\}\Big].$$

Here $\lim_{\varepsilon\to0}|\mathrm{con}\,\theta|=0$, so that we do not know whether ε is negligible or not in (5.21) under the conditions (H.1)-(H.4) only.

But in some cases we can drop ε in (5.21). For instance, we will consider the model A: (0.9).

In this case the retarded spectrum $\sigma(\mathrm{L})$ has at most one accumulation point $-1/T$. To see this, we have only to note that there exist some subsequences $\{\lambda_k\}\subset\sigma(\mathrm{L})$ and $\{j_k\}\subset \boldsymbol{N}$ such that

$$\begin{cases} j_k\nearrow\infty & \text{as } k\to\infty\,, \\ \lambda_k\to\lambda^* & \text{as } k\to\infty\,, \\ \lambda_k+bu_\infty+\dfrac{\alpha u_\infty}{T\lambda_k+1}=-\mu_{j_k} & (k=1,2,3,\cdots) \end{cases}$$

for each accumulation point λ^* of $\sigma(\mathrm{L})$, where $0=\mu_0<\mu_1\leq\mu_2\leq\cdots$ are the eigenvalues of A (cf. (3.4)). Moreover, we will find that the set $\{\mathrm{Re}\,\lambda;\ \lambda\in\sigma(\mathrm{L})\}$ has at most one accumulation point $-1/T$, because the set $\sigma(\mathrm{L})\cap\{z;\ x-1<\mathrm{Re}\, z<x\}$ is bounded for each negative number x (see Lemma 5.3 (ii)).

If $bu_\infty<1/T$ holds true, we can get the following results:

case 1 $\left(bu_\infty<\frac{1}{T},\ a>(\sqrt{\alpha u_\infty}-\sqrt{\frac{1}{T}})^2\right)$;

$$-\nu_f=-\frac{1}{2}(bu_\infty+\frac{1}{T})>-\frac{1}{T},$$

$$\sigma(\mathrm{L})\cap\{z;\ \mathrm{Re}\,z=-\nu_f\}=\{-\nu_f\pm i\sqrt{\frac{a}{T}-\nu_f^2}\}:\ \text{simple roots},$$

case 2 $(bu_\infty<\frac{1}{T}, a=(\sqrt{\alpha u_\infty}-\sqrt{\frac{1}{T}})^2)$;

$-\nu_f=-\frac{1}{2}(bu_\infty+\frac{1}{T})>-\frac{1}{T}$,

$\sigma(\mathrm{L})\cap\{z;\ \mathrm{Re}\,z=-\nu_f\}=\{-\nu_f\}$: double root,

case 3 $(bu_\infty<\frac{1}{T}, 0<a<(\sqrt{\alpha u_\infty}-\sqrt{\frac{1}{T}})^2)$;

$-\nu_f=-\frac{1}{2}(bu_\infty+\frac{1}{T})+\sqrt{\frac{1}{4}(bu_\infty+\frac{1}{T})^2-\frac{a}{T}}>-\frac{1}{T}$,

$\sigma(\mathrm{L})\cap\{z;\ \mathrm{Re}\,z=-\nu_f\}=\{-\nu_f\}$: simple root

(see the proof of Lemma 3.4). Thus, by replacing Γ properly, we can prove that there exists a positive constant C (independent of p and t) such that

$$|||R(t)|||_p\leq\begin{cases}C\exp(-\nu_f t), & :\ \text{case } 1, 3,\\ C(1+t)\exp(-\nu_f t), & :\ \text{case } 2,\end{cases}\qquad 2\leq p<\infty,\ t\geq 0 \tag{5.26}$$

(cf. [3; Proposition 1.4]). On the other hand, the condition $bu_\infty\geq 1/T$ implies that $-\nu_f$ coincides the accumulation point $-1/T$ of $\sigma(\mathrm{L})$ as mentioned in the proof of Lemma 3.4.

6. Proof of Theorem 2.5

As we have proved (2.5) (i.e., Lemma 3.4), we will show (2.6) for a given positive number ε.

The hypotheses (H.1), (H.2), (H.3) and (H.4) are verified by Lemmas 3.1 (i), 3.1 (ii), 3.4, 3.3 respectively. In particular, $\lambda_f=1/T$ and ν in Section 5 is ν_f in Definition 2.4. Hence, by Theorem 5.6, we have

$$||R(t)\,v_0||_p\leq C\exp\{-(\nu_f-\varepsilon)\,t\}\ ||v_0||_p\,,\quad 2\leq p<\infty, t\geq 0\,,$$

for each element $v_0\in L^\infty(\Omega)$. Since the constant C is independent of p, letting $p\to\infty$ we get

$$||R(t)\,v_0||_\infty\leq C\exp\{-(\nu_f-\varepsilon)\,t\}\ ||v_0||_\infty\,,\quad t\geq 0\,. \tag{6.1}$$

Now we apply the results of Section 4 by taking $A+bu_\infty$ and $-u_\infty f$ as $\tilde{A}$ and g respectively.

The inequality (6.1), in other words, asserts that the assumption (A.5) is valid for $R(t)$ and $\rho_2=\nu_f-\varepsilon$. Needless to say, the assumptions (A.1) and (A.2) hold true (note that $\exp(-tA)$ is an analytic semgiroup on $L^p(\Omega)$ for $1<p<\infty$). In view of Lemma 3.6, we define an operator h_1 by

$$h_1[v](t) = -v(t)\,\{bv(t)+\int_0^t f(t-s)\,v(s)\,ds\}$$

for $v\in C([0,\infty);\,C(\bar{\Omega}))$ and $t\geq 0$ in order that h_1 may satisfy (A.3). Let h_2 be the last two terms in (3.5):

$$h_2(t) = u(t+\tau,\,\cdot)\,\{u_\infty\int_{t+\tau}^{\infty} f(s)\,ds-\int_0^{\tau} f(t+\tau-s)\,w(s,\,\cdot)\,ds\}\;,\quad t\geq 0\,.$$

Remark 1.2 and Lemma 3.7 assure that h_2 satisfies (A.4) where $\rho_1=1/T$ in the case of (0.9) and $\rho_1=1/T-\varepsilon'$ for any small $\varepsilon'>0$ in the case of (0.10). Set $v(t)=w_\tau(t,\,\cdot)$ and $v_0=w_\tau(0,\,\cdot)\,(=u(\tau,\,\cdot)-u_\infty)$ in (3.5), and we will find that (4.1) holds for v. Moreover, for each $\delta>0$, (4.6) follows from Lemma 3.5 by choosing $\tau>0$ sufficiently large. Thus, by virtue of Theorem 4.5, we can see that

$$\|w_\tau(t,\,\cdot)\|_\infty\leq C\exp\{-(\nu_f-2\varepsilon)\,t\}\;,\quad t\geq 0\,, \tag{6.2}$$

for some $\tau>0$. Here C is a positive constant independent of τ and t.

To complete the proof, we have only to rewrite (6.2) as an estimate for $u-u_\infty$ and replace t by $t-\tau$. ■

7. Some Improvements of the Main Result

7.1. As we have mentioned in Remark 2.7, we can show (2.6) for the delay kernel (2.7) with $\rho>\omega$ in the following way.

The facts

$$\hat{f}(z) = c_0\left\{\frac{1}{z+\rho}+\frac{\omega}{(z+\rho)^2+\omega^2}\right\}, \qquad c_0 = \frac{(\omega^2+\rho^2)\,\rho\alpha}{\omega^2+\rho^2+\omega\rho} \tag{7.1}$$

and $\rho>\omega$ lead us to

$$\inf_{\lambda\in\mathbf{R}} \operatorname{Re}\hat{f}(i\lambda) = 0\,,$$

which assures that the delay kernel f satisfies the assumptions of Proposition 1.4.

By virtue of (7.1), we see that λ is a characteristic value of (2.4) if and only if λ satisfies

$$\begin{aligned}&\lambda^4+(3\rho+bu_\infty+\mu_j)\,\lambda^3+\{c_0\,u_\infty+\omega^2+3\rho^2+3\rho(bu_\infty+\mu_j)\}\,\lambda^2\\&+\{\omega\,c_0\,u_\infty+2\rho\,c_0\,u_\infty+\omega^2\,\rho+\rho^3+(\omega^2+3\rho^2)\,(bu_\infty+\mu_j)\}\,\lambda\\&+\omega^2\,c_0\,u_\infty+\rho^2\,c_0\,u_\infty+\omega\rho\,c_0\,u_\infty+(\omega^2\,\rho+\rho^3)\,(bu_\infty+\mu_j)=0\end{aligned} \tag{7.2}$$

for some j, where $0=\mu_0<\mu_1\leq\mu_2\leq\cdots$ are the eigenvalues of A_2. After some tedious calculation which use the similar technique to the one in Appendix, we can see that the ıoots of the algebraic equation (7.2) belong to $\{z\in \boldsymbol{C};\ \mathrm{Re}\,\lambda\leq -\nu\}$ for some $\nu>0$. (Note that $\rho>\omega$.) Hence (H.3) is valid.

Moreover we can verify the similar results for the delay kernel (2.7) to Lemma 3.1, 3.3, 3.7.

Therefore we can apply the results of Sections 4, 5 to the case of (2.7) in order to prove (2.6).

It seems to the author that the condition $\rho>\omega$ indicates that the more rapidly the delay kernel f decays or the more slowly f oscillates, the more stable u_∞ is.

7.2. In Section 4, instead of (A.3) and (A.5), we will assume the following conditions respectively:

(A.3)′ A nonlinear operator

$$h_1\colon C([0,\infty);\, C(\bar\Omega))\ni v\mapsto h_1[v]\in C([0,\infty);\, C(\bar\Omega))$$

satisfies, for $v\in C([0,\infty);\, C(\bar\Omega))$ and $t\geq 0$,

$$\|h_1[v](t)\|_\infty\leq c_1\|v(t)\|_\infty\{\|v(t)\|_\infty+\int_0^t \exp\{-\rho_3(t-s)\}\,\|v(s)\|_\infty\,ds\}\,,$$

where c_1 and ρ_3 are positive constants independent of t.

(A.5)′ The fundamental solution $R(t;\tilde A, g)$ satisfies, for all $t\geq 0$ and $v_0\in L^\infty(\Omega)$,

$$\|R(t;\tilde A, g)\,v_0\|_\infty\leq c_3\, q(t)\,\|v_0\|_\infty\,,$$

where $q(t)=(1+t^m)\exp(-\rho_2 t)$ and c_3, $\rho_2(>0)$, $m(\geq 0)$ are constants independent of t and v_0.

Lemma 7.1. *Let* (A.1), (A.2), (A.3)′, (A.4) *and* (A.5)′ *be fullifilled, and* $\rho_1>\rho_2$ *hold. Assume that* $v\in C^1([0,\infty);\, L^p(\Omega))\cap C([0,\infty);\, D(\tilde A)\cap C(\bar\Omega))$ *satisfies* (4.1). *Then there exist positive numbers* C *and* δ *such that* (4.6) *implies*

$$\|v(t)\|_\infty\leq Cq(t)\,,\quad t\geq 0\,, \tag{4.7$'$}$$

where C *and* δ *depend only on* c_1, c_2, c_3, ρ_1, ρ_2, ρ_3 *nad* m.

For the proof of this lemma, after deriving

$$\begin{aligned}\|v(t)\|_\infty &\\ \leq c_3\, q(t)\,\|v_0\|_\infty &\\ +\int_0^t c_3\, q(t-s)\, c_1\,\|v(s)\|_\infty^2\,ds &\\ +\int_0^t c_3\, q(t-s)\, c_1\,\|v(s)\|_\infty \int_0^s \exp\{-\rho_3(s-r)\}\,\|v(r)\|_\infty\,dr\,ds &\\ +\int_0^t c_3\, q(t-s)\, c_2\exp(-\rho_1 s)\,ds\,,\quad t\geq 0 &\end{aligned} \tag{7.3}$$

in place of (4.8), we have only to apply (4.7) to the right-hand side of (7.3).

In the model A with $bu_\infty<1/T$, the assumptions (A.3)′ and (A.5)′ hold true for $\rho_5=1/T$, $\rho_4=\nu_f$ and $m=0$ or $m=1$ instead of (A.3) and (A.5) in Section 6 (see Remark 5.7). Thus, by using Lemma 7.1 in place of Theorem 4.5, we can prove

$$\|u(t,\cdot)-u_\infty\|_\infty = \begin{cases} O(\exp(-\nu_f t)) & : bu_\infty<\frac{1}{T},\ a\neq(\sqrt{\alpha u_\infty}-\sqrt{\frac{1}{T}})^2, \\ O((1+t)\exp(-\nu_f t)) & : bu_\infty<\frac{1}{T},\ a=(\sqrt{\alpha u_\infty}-\sqrt{\frac{1}{T}})^2 \end{cases} \quad \text{as } t\to\infty \tag{7.4}$$

for the solution u of (0.1)-(0.3) with non-negative initial data $u_0(\not\equiv 0)$. Therefore ε in (2.6) can be dropped and (7.4) is the best estimate in this case. We will discuss this result in detail elsewhere.

Appendix

We will prove Lemma 3.4 for the model B.

Before proceeding to the proof, we review a lemma on the distribution of zero of polynomials:

Lemma. *A cubic equation*

$$z^3+a_1 z^2+a_2 z+a_3=0$$

with real coefficients has no roots with non-negative real parts if and only if

$$\begin{cases} a_1>0, \\ a_3>0, \\ a_1 a_2-a_3>0. \end{cases}$$

This lemma is a special case of Hurwitz' criterion (see, e.g., Wall [17; Section 6], Marden [8; p. 141]).

Proof of Lemma 3.4 for the model B. In the same way as the proof for the model A, it follows from (3.2) that λ is a characteristic value of (2.4) if and only if

$$T^2\lambda^3+\{T(bu_\infty+\mu_j)+2\}\,T\lambda^2+\{2T(bu_\infty+\mu_j)+1\}\,\lambda+a+\mu_j=0 \tag{1}$$

for some j, where $0=\mu_0<\mu_1\leq\mu_2\leq\cdots$ are the eigenvalues of A_2.

By setting $\Lambda=T(\lambda+\nu)$, $N=T\nu$, $B=Tbu_\infty$ and $M_j=T\mu_j(j=0,1,2,\cdots)$ for a fixed real number ν, (1) is rewritten as

$$\text{(2)}\qquad \begin{aligned}&\Lambda^3+(M_j+B+2-3N)\,\Lambda^2\\&+[2(1-N)\,M_j+\{3N^2-2(B+2)\,N+2B+1\}]\,\Lambda\\&+(1-N)^2\,M_j-\{N^3-(B+2)\,N^2+(2B+1)\,N-aT\}=0\,.\end{aligned}$$

By the preceding lemma, (2) has no roots with Re $\Lambda\geq 0$ for every j $(=0, 1, 2, \cdots)$ if and only if

$$\text{(3)}\qquad M_j>3N-B-2\,,$$

$$\text{(4)}\qquad (1-N)^2\,M_j>P_1(N)\,,$$

$$\text{(5)}\qquad 2(1-N)\,M_j^2+4(1-N)\,(B+1-2N)\,M_j>P_2(N)$$

for any j. Here $P_1(N)$ and $P_2(N)$ are defined by

$$\begin{aligned}P_1(N)&=N^3-(B+2)\,N^2+(2B+1)\,N-aT\,,\\P_2(N)&=8N^3-8(B+2)\,N^2+2\,(B^2+6B+5)\,N\\&\quad-2B^2-5B-2+aT\,.\end{aligned}$$

If $N<\min\,\{1, (B+1)/2\}$, $P_1(N)<0$ and $P_2(N)<0$, then the inequalities (3), (4) and (5) hold true for every j because of $M_j\geq 0$. Since $\alpha<8b$ and $a=(b+\alpha)u_\infty$,

$$P_2(0)=-2(B-1)^2-(9B-aT)<0\,.$$

This inequality and $P_1(0)=-aT<0$ imply that $P_1(N)<0$ and $P_2(N)<0$ hold true on some neighborhood of $N=0$. Hence there exists a positive number N_0 such that any root Λ of (2) satisfies Re $\Lambda<0$ for each $j\geq 0$ whenever $0<N<N_0$.

Therefore the inequality Re $\lambda<-\nu$ holds for each characteristic value λ if $0<\nu<N_0/T$. Namely,

$$\nu_f=-\sup_{\lambda\in\sigma(L)}\operatorname{Re}\lambda>0\,.$$

On the other hand, in order to prove $\nu_f<1/T$, it suffices to show that there exists a positive number ε and a root λ of (1) for $j=0$ such that Re $\lambda\geq -1/T+\varepsilon$.

To see this, set $\tilde{\Lambda}=T(\lambda-\varepsilon)+1$, $\delta=-1+T\varepsilon$ and $B=Tbu_\infty$ for a fixed real number ε. Then the equation (1) with $j=0$ is reduced to

$$\text{(6)}\qquad \begin{aligned}&\tilde{\Lambda}^3+(3\delta+B+2)\,\tilde{\Lambda}^2+\{3\delta^2+2(B+2)\,\delta+2B+1\}\,\tilde{\Lambda}\\&+\delta^3+(B+2)\,\delta^2+(2B+1)\,\delta+aT=0\,.\end{aligned}$$

Taking account of the preceding lemma, we put

$$\begin{aligned}a_1&=3\delta+B+2\,,\\a_2&=3\delta^2+2(B+2)\,\delta+2B+1\,,\\a_3&=\delta^3+(B+2)\,\delta^2+(2B+1)\,\delta+aT\,,\end{aligned}$$

and

$$P_3(\delta)=a_1 a_2-a_3\,.$$

It is easy to see

$$P_3(-1)=B-aT=-\alpha u_\infty T<0\,,$$

which imply that $P_3(\delta)<0$ holds in a neighborhood of $\delta=-1$. Since the inequality $P_3(\delta)<0$ means that (6) has a root $\tilde{\Lambda}$ with Re $\tilde{\Lambda}\geq 0$, by choosing a positive number ε sufficiently small, we can find a root λ of (1) with Re $\lambda\geq -1/T+\varepsilon$. Thus we complete the proof. ■

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