Jeong, J.M.
Osaka J. Math.
28 (1991), 347–365

STABILIZABILITY OF RETARDED FUNCTIONAL DIFFERENTIAL EQUATION IN HILBERT SPACE

JIN-MUN JEONG

(Received March 29, 1990)

1. Introduction

This paper is concerned with the stabilizability on the retarded functional differential equation

$$(1.1)\qquad \frac{d}{dt}u(t)=A_0u(t)+\int_{-h}^{0}a(s)\,A_1u(t+s)\,ds+\Phi_0f(t)\,,$$

$$(1.2)\qquad u(0)=g^0\,,\quad u(s)=g^1(s),\ s\in[-h,0)$$

in a Hilbert space H. Here, A_0 is the operator associated with a sesquilinear form $a(u, v)$ which is defined in $V\times V$ and satisfies Gårding's inequality

$$\mathrm{Re}\ a(u,u)\geq c_0\,||u||^2-c_1\,|u|^2\,,\quad c_0>0\,,\quad c_1\geq 0$$

where V is another Hilbert space such that $V\subset H\subset V^*$. The notations $|\cdot|$, $||\cdot||$ denote the norms of H, V respectively as usual. A_1 is a bounded linear operator from V to V^* such that it maps $D(A_0)$ into H, and $a(\cdot)$ is a real valued Hölder continuous function in $[-h, 0]$. Φ_0 is a bounded linear operator from some Banach space U to H.

We will establish a necessary and sufficient condition in order that the initial value problem (1.1), (1.2) is stabilizable in the sense that for any $g\in Z=H\times L^2(-h,0;V)$ there exists $f\in L^2(0,\infty;U)$ such that for the solution u of (1.1), (1.2) we have

$$\int_0^\infty \{|u(t)|^2+\int_{-h}^0 ||u(t+s)||^2\,ds\}\,dt<\infty\,.$$

Our result is analogous to a recent result by G. Da Prato and A. Lunardi [1] for an integrodifferential parabolic equation of Volterra type

$$(1.3)\qquad \frac{d}{dt}u(t)=Au(t)+\int_0^t K(t-s)\,u(s)\,ds+\Phi f(s),\ t\geq 0,\ u(0)=u_0$$

in a Banach space X, where A is a not necessarily densely defined closed linear operator generating an analytic semigroup, K is a measurable function with vaules in $L(D(A), X)$ and Φ is a bounded linear operator from some Banach space

Y to X. Under the assumption that $F(\lambda)$ which is the analytic continuation of $(\lambda-A-\hat{K}(\lambda))^{-1}$ where $\hat{K}$ is the Laplace transform of K has only a finite number of singularities in the half plane Re $\lambda\geq-\omega$, $\omega\geq0$, and all these singularities are poles of $F(\lambda)$ such that the coefficients of the negative powers in the Laurent expansions of $F(\lambda)$ around them are all operators of finite rank, they established a necessary and sufficient condition in order that for every $u_0\in\overline{D(A)}$ there exists f satisfying sup $||e^{\omega t}f(t)||_Y<\infty$ such that the solution u of (1.3) satisfies sup $||e^{\omega t}\,u(t)||_X<\infty$.

When we investigate the equation (1.1), it is natural and usual to consider the equivalent enalrged system for the unknown functions $u(t)$ and u_t, where $u_t(s)=u(t+s)$ for $s\in[-h, 0)$, since it enables us to express the solution with the aid of the solution semigroup (cf. [2], [3], [7]). Since we necessarily consider the adjoint equation in the study of a stabilization problem, it is convenient to consider the original equation in V^* as in [11] so that the enalrged system is an equation in the space $Z=H\times L^2(-h, 0; V)$:

$$\frac{d}{dt}\,x(t) = Ax(t)+\Phi f(t),\ x(0) = g = (g^0, g^1)$$

where A is the infinitesimal generator of the associated solution semigroup $S(t)$ and Φ is the operator defined by $\Phi f=(\Phi_0 f, 0)$. Thus, we are led to studying the stabilizability in the sense stated above.

We assume that

$$\sigma_+ = \sigma(A)\cap\{\lambda\colon \mathrm{Re}\,\lambda>0\}$$

consists entirely of a finite number of eigenvalues of A with generalized eigenspaces of finite dimension and

$$\sup\,\{\mathrm{Re}\,\lambda\colon \lambda\in\sigma(A)\backslash\sigma_+\}<0\,.$$

Let P be the spectral projection corresponding to σ_+:

$$P=\frac{1}{2\pi i}\int_\Gamma(\lambda-A)^{-1}\,d\lambda\,,$$

where Γ is a rectifiable Jordan curve surrounding σ_+ inside but no other point of $\sigma(A)$. In our case the exponential decay of $||S(t)(I-P)||$ is not so evident as the corresponding fact for Volterra equations of [1]. We will show that owing to the absense of a discrete delay term in (1.1) $S(t)$ is Hölder continuous in $(3h, \infty)$ in the operator norm, and so eventually norm continuous. Hence the aforesaid exponential decay of $||S(t)(I-P)||$ follows from Theorem 1.20 of [6], and we can proceed as in [1] to establish the desired result.

We note here that it is shown in G. Di Blasio, K. Kunisch and E. Sinestrari [3] that $S(t)$ is differentiable in (h, ∞), hence eventually norm continuous, if

$a(\cdot)\in W^{1,2}(-h, 0)$ for a general equation of the form (1.1) in a Hilbert space. However, if there exists a discrete delay term, this is not the case as the following counter example shows. For the equation

$$\frac{d}{dt}u(t) = A_0 u(t)+A_0 u(t-1) \tag{1.4}$$

we have

$$S(t)(g^0, 0) = (W(t)g^0, W(t+\cdot)g^0)$$

where

$$W(t) = \sum_{j=0}^{n} \frac{1}{j!} A_0^j e^{(t-j)A_0}(t-j)^j,\ t\in[n, n+1],\ n = 1, 2, \cdots$$

is the fundamental solution of (1.4) which is not norm continuous at $t=1, 2, \cdots$.

2. Assumptions and main theorem

Let H and V be complex Hilbert spaces such that V is a dense subspace of H and the imbedding of V into H is continuous. The norms of H and V are denoted by $|\cdot|$ and $||\cdot||$ respectively. Identifying the antidual of H with H we may consider $V\subset H\subset V^*$. Let $a(u, v)$ be a bounded sesquilinear form defined in $V\times V$ satisfying Gårding's inequality

$$\operatorname{Re} a(u, u)\geq c_0 ||u||^2-c_1|u|^2 \tag{2.1}$$

where c_0 and c_1 are constants such that $c_0>0$ and $c_1\geq 0$. Let A_0 be the operator associated with this sesquilinear form:

$$(A_0u, v) = -a(u, v), \quad u, v\in V. \tag{2.2}$$

The operator A_0 is a bounded linear form V to V^*. The realization of A_0 in H which is the restriction of A_0 to

$$D(A_0) = \{u\in V: A_0u\in H\}$$

is also denoted by A_0. It is known that A_0 generates an analytic semigroup in both of H and V^*. Let A_1 be a bounded linear operator from V to V^* such that A_1 maps $D(A_0)$ endowed with the graph norm of A_0 to H continuously. Let $a(s)$ be a real valued Hölder continuous function on the interval $[-h, 0]$, where h is a fixed positive number. Let U be a complex Banach space and Φ_0 be a bounded linear operator from U to H. We are interested in the stabilization of the initial value problem of the retarded functional differential equation

$$\frac{d}{dt}u(t) = A_0u(t)+\int_{-h}^{0} a(s)\, A_1u(t+s)\, ds+\Phi_0 f(t), \tag{2.3}$$

$$u(0) = g^0, \quad u(s) = g^1(s) \quad a.e.\ s\in[-h, 0), \tag{2.4}$$

where $g=(g^0,g^1)\in Z=H\times L^2(-h,0;V)$, $f\in L^2(0,T;U)$. Applying Theorem 4.1 of [2] to (2.3) considered as an equation in V^* we see that there exists a solution semigroup $S(t)$ associated with (2.3), (2.4):

$$S(t)g=\begin{pmatrix}u(t;g)\\ u_t(\cdot\,;g)\end{pmatrix},\ t\geq 0,\ g=(g^0,g^1)\in Z\,,$$

where $u(t;g)$ is the solution of (2.3), (2.4) with $f(t)\equiv 0$, and $u_t(s;g)=u(t+s;g)$, $s\in[-h,0)$. $S(t)$ is a C_0-semigroup in Z whose infinitesimal generator is denoted by A.

In view of Theorem 4.2 of [2] A is characterized as

$$\begin{aligned}D(A)=\{&(\phi^0,\phi^1)\colon \phi^1\in W^{1,2}(-h,0;V),\ \phi^0=\phi^1(0)\,,\\ &A_0\phi^0+\int_{-h}^0 a(s)\,A_1\phi^1(s)\,ds\in H\}\,,\\ A(\phi^0,\phi^1)&=(A_0\phi^0+\int_{-h}^0 a(s)\,A_1\phi^1(s)\,ds,\ \dot\phi^1)\,.\end{aligned}$$

We assume

$$\sigma(A)\cap\{\lambda\colon \mathrm{Re}\,\lambda=0\}=\phi\,. \tag{2.5}$$

Set

$$\sigma_+=\sigma(A)\cap\{\lambda\colon \mathrm{Re}\,\lambda>0\},\ \sigma_-=\sigma(A)\cap\{\lambda\colon \mathrm{Re}\,\lambda<0\}\,. \tag{2.6}$$

We assume also that σ_+ is a finite set and sup $\{\mathrm{Re}\,\lambda\colon \lambda\in\sigma_-\}<0$, that is,

$$\sigma_+=\{\lambda_1,\cdots,\lambda_N\}\,, \tag{2.7}$$

$$-\omega_0=\sup\{\mathrm{Re}\,\lambda\colon \lambda\in\sigma_-\}<0\,, \tag{2.8}$$

and for each $j=1,\cdots,N$, the spectral projection

$$P_{\lambda_j}=\frac{1}{2\pi i}\int_{\Gamma_{\lambda_j}}(\lambda-A)^{-1}\,d\lambda \tag{2.9}$$

is an operator of finite rank, where Γ_{λ_j} is a small circle centered at λ_j such that it surrounds no point of $\sigma(A)$ except λ_j. As is well known λ_j is an eigenvalue of A and $Z_{\lambda_j}=\mathrm{Im}\,P_{\lambda_j}$ is the generalized eigenspace for λ_j. It is also well known that λ_j is a pole of $(\lambda-A)^{-1}$ whose order we denote by m_j.

We consider also the adjoint problem

$$\frac{d}{dt}v(t)=A_0^*v(t)+\int_{-h}^0 a(s)\,A_1^*\,v(t+s)\,ds\,, \tag{2.10}$$

$$v(0)=\varphi^0,\ v(s)=\varphi^1(s)\quad s\in[-h,0)\,, \tag{2.11}$$

where A_0^*, $A_1^*\in B(V;V^*)$ are adjoint operators of A_0, A_1 and $\varphi^0\in H$, $\varphi^1\in L^2(-h,0;V)$. The solution semigroup associated with this problem is denoted by $S_T(t)=e^{tA_T}$. Just as in [7] it can be shown that $\overline{\lambda_1}, \cdots, \overline{\lambda_N}$ are eigenvalues of A_T. The spectral projection and the generalized eigenspace for $\overline{\lambda_j}$ are denoted by $P^T_{\overline{\lambda_j}}$ and $Z^T_{\overline{\lambda_j}}$ respectively.

The structural operator F is defined by

$$Fg = ([Fg]^0, [Fg]^1),\ g = (g^0, g^1) \in Z\,,$$
$$[Fg]^0 = g^0,\ [Fg]^1(s) = \int_{-h}^{s} a(\tau)\, A_1 g^1(\tau - s)\, d\tau\,.$$

Here and in what follows we denote the first and second components of the element φ of Z by $[\varphi]^0$ and $[\varphi]^1$ respectively. F is a bounded linear operator from Z to its adjoint $Z^*=H\times L^2(-h, 0; V^*)$. Putting $x(t)=(u(t), u_t)$ where $u_t(s)=u(t+s)$, $s\in[-h, 0)$, and $g=(g^0, g^1)$ the problem (2.3), (2.4) is transformed to the problem

$$\frac{d}{dt}x(t) = Ax(t)+\Phi f(t)\,, \tag{2.12}$$

$$x(0) = g \tag{2.13}$$

in Z, where Φ is the operator defined by $\Phi f=(\Phi_0 f, 0)$. The mild solution of (2.12), (2.13) is defined by

$$x(t) = S(t)\,g+\int_0^t S(t-s)\,\Phi f(s)\,ds\,. \tag{2.14}$$

We call the first component of the right hand side of (2.14) the mild solution of (2.3), (2.4). Following [3], [7] we set

$$\Delta(\lambda) = \lambda - A_0 - \int_{-h}^{0} e^{\lambda s}\, a(s)\, A_1\, ds\,,$$
$$\Delta_T(\lambda) = \lambda - A_0^* - \int_{-h}^{0} e^{\lambda s}\, a(s)\, A_1^*\, ds\,,$$
$$\Delta_T^{(i)}(\lambda) = (d/d\lambda)^i\, \Delta_T(\lambda) = \delta_{i1} - \int_{-h}^{0} s^i\, e^{\lambda s}\, a(s)\, A_1^*\, ds,\ i = 1, 2, \cdots,$$

where δ_{i1} is the Kronecker symbol, i.e., $\delta_{11}=1$, $\delta_{i1}=0$ if $i\neq 1$.

The main theorem of this paper is

Theorem. *The following statements are equivalent*:

(*i*) *For any* $g\in Z$ *there exists an* $f\in L^2(0, \infty; U)$ *such that the mild solution* u *of* (2.3), (2.4) *satisfies*

$$\int_0^\infty \{|u(t)|^2+\int_{-h}^{0} \|u(t+s)\|^2\, ds\}\, dt<\infty\,.$$

(*ii*) *For each* $j=1, \cdots, N$

$$\Phi_0^* \phi_n^0 = 0,\ n = 0, \cdots, m_j - 1$$

implies

$$\phi_n^0 = 0,\ n = 0, \cdots, m_j - 1$$

for any elements $\phi_0^0, \cdots, \phi_{m_j-1}^0 \in V$ *such that*

$$\sum_{i=n}^{m_j-1} (-1)^{i-n} \Delta_T^{(i-n)}(\overline{\lambda_j})\, \phi_i^0/(i-n)! = 0,\ n = 0, \cdots, m_j - 1\,.$$

REMARK 1. If $m_j = 1$, the statement (ii) of the theorem reduces to

$$\Phi_0^* \phi_0^0 = 0,\ \Delta_T(\overline{\lambda_j})\, \phi_0^0 = 0 \quad \text{implies} \quad \phi_0^0 = 0\,.$$

REMARK 2. Set

$$\rho(\Delta) = \{\lambda : \Delta(\lambda) \text{ is an isomorphism from } V \text{ to } V^*\}\,,$$
$$\sigma(\Delta) = \boldsymbol{C} \backslash \rho(\Delta)\,.$$

According to Riemann-Lebesgue's lemma $\int_{-h}^0 e^{\lambda s} a(s) ds$ tends to 0 as $|\mathrm{Im}\,\lambda| \to \infty$ uniformly in $\{\lambda : \mathrm{Re}\,\lambda \geq c\}$ for any real number c. Hence

$$\Delta(\lambda) = \{I - \int_{-h}^0 e^{\lambda s} a(s)\, ds\, A_1 (\lambda - A_0)^{-1}\}\, (\lambda - A_0)$$

has a bounded inverse if $\mathrm{Re}\,\lambda \geq c$ and $|\mathrm{Im}\,\lambda|$ is sufficiently large. Consequently $\sigma(\Delta) \cap \{\lambda : \mathrm{Re}\,\lambda \geq c\}$ is bounded.

Suppose that the imbedding of V to H is compact and $A_1 = A_0$. Then in view of Theorem 1 of [4]

$$\sigma(\Delta) = \sigma(A) = \{\lambda : m(\lambda) = 0\} \cup \{\lambda : m(\lambda) \neq 0,\ \lambda/m(\lambda) \in \sigma(A_0)\}\,,$$

where

$$m(\lambda) = 1 + \int_{-h}^0 e^{\lambda s} a(s)\, ds\,.$$

If $\sigma(\Delta) \cap \{\lambda : \mathrm{Re}\,\lambda = 0\}$ is empty and $m(\lambda) \neq 0$ for $\mathrm{Re}\,\lambda > 0$, then the assumptions of the theorem are satisfied.

3. Stabilizability of functional differential equations

In this section we consider the stabilizability of the equation

$$\frac{d}{dt} x(t) = Ax(t) + \Phi f(t)\,, \tag{3.1}$$

$$x(0) = g \tag{3.2}$$

in a general Banach space X, where A is the infinitesimal generator of a C_0-

semigroup $S(t)$ and Φ_0 is a bounded linear operator from some Banach space U to X. We assume that A satisfies (2.5), (2.7), (2.8) and the spectral projection P_{λ_j} defined by (2.9) is of finite rank for each $j=1, \cdots, N$. Hence λ_j is a pole of $(\lambda-A)^{-1}$ whose order is denoted by m_j. We assume also that $S(t)$ is eventually norm continuous, i.e. $S(t)$ is continuous in the operator norm in the interval (t', ∞) for some $t' \geq 0$. The Laurent expansion of $(\lambda-A)^{-1}$ around λ_j is

$$(\lambda-A)^{-1}=\sum_{n=0}^{m_j-1} \frac{Q_{\lambda_j}^n}{(\lambda-\lambda_j)^{n+1}}+R_0(\lambda), \tag{3.3}$$

where $Q_{\lambda_j}^0=P_{\lambda_j}$, $Q_{\lambda_j}=(A-\lambda_j) P_{\lambda_j}$, and $R_0(\lambda)$ is the holomorphic part of $(\lambda-A)^{-1}$ at $\lambda=\lambda_j$. It is known that $Q_{\lambda_j}^n=(A-\lambda_j)^n P_{\lambda_j}$, and by assumption $Q_{\lambda_j}^{m_j}=0$. Put

$$P=\sum_{j=1}^{N} P_{\lambda_j}, j=1, \cdots, N \tag{3.4}$$

$$X_+=\operatorname{Im} P,\ X_-=\operatorname{Im}(I-P), \tag{3.5}$$

$$S_+(t)=S(t)|_{X_+},\ S_-(t)=S(t)|_{X_-} \tag{3.6}$$

where $S(t)|_{X_+}$, $S(t)|_{X_-}$ are the restrictions of $S(t)$ to X_+, X_-, respectively. Since both Im P and Im $(I-P)$ are closed and invariant under $S(t)$, we can see that

$$A_+=A|_{X_+},\quad A_-=A|_{D(A)\cap X_-}, \tag{3.7}$$

$$S_+(t)=e^{tA_+},\quad S_-(t)=e^{tA_-}, \tag{3.8}$$

$$\sigma(A_+)=\sigma_+,\quad \sigma(A_-)=\sigma_-. \tag{3.9}$$

$S(t)P$ is extended to the whole real line so that

$$S(t)P=\sum_{j=1}^{N}\sum_{n=0}^{m_j-1}\frac{1}{n!}e^{\lambda_j t}t^n Q_{\lambda_j}^n,\ -\infty<t<\infty. \tag{3.10}$$

Lemma 3.1. *For any $\omega\in(0, \omega_0)$ there exists a constant M such that*

$$||S(t)(I-P)||\leq M||I-P||e^{-\omega t},\ t\geq 0.$$

Proof. For each eventually norm continuous semigroup $s(A)=\omega(A)$ is valid where $s(A)=\sup\{\operatorname{Re}\lambda: \lambda\in\sigma(A)\}$, $\omega(A)=\inf\{\omega: ||S(t)||\leq Me^{\omega t}$ for some M and any $t\geq 0\}$ (see e.g. [6: p. 109]). Therefore for any $\omega>s(A)$, there exists M such that $||S(t)||\leq Me^{\omega t}$. By assumption (2.8) $-\omega_0=\sup\{\operatorname{Re}\lambda: \lambda\in\sigma_-\}<0$. If $S(t)$ is eventually norm continuous, then so is $S_-(t)$. Hence for $0<\omega<\omega_0$ there exists a constant M such that $||S_-(t)||\leq Me^{-\omega t}$. Thus,

$$||S(t)(I-P)||=||S_-(t)(I-P)||\leq M||I-P||\,e^{-\omega t}.$$

Now we consider the following initial value problem:

$$
\begin{cases} \dfrac{d}{dt} u(t) = Au(t) + \phi(t), \ t \geq 0 \\ u(0) = u_0 , \end{cases} \tag{3.11}
$$

where $\phi \in L^2(0, \infty; X)$. Let $u(t)$ be the mild solution of the equation (3.11), i.e.,

$$
u(t) = S(t) u_0 + \int_0^t S(t-s) \phi(s) \, ds . \tag{3.12}
$$

The following Lemma is related to Proposition 1.1 of [1], and we mimic its proof.

Lemma 3.2. *For the equation* (3.11), *the mild solution* $u(t)$ *belongs to* $L^2(0, \infty; X)$ *if and only if*

$$
S(t) P u_0 + \int_0^\infty S(t-s) P \phi(s) = 0, \ t \geq 0 . \tag{3.13}
$$

Proof. We set $u(t) = v(t) + z(t)$, with

$$
v(t) = S(t)(I-P) u_0 + \int_0^t S(t-s)(I-P) \phi(s) \, ds - \int_t^\infty S(t-s) P \phi(s) \, ds ,
$$

$$
z(t) = S(t) P u_0 + \int_0^\infty S(t-s) P \phi(s) \, ds, \ t \geq 0 .
$$

With the aid of Lemma 3.1, (3.10) and Hausdorff-Young's inequality it is easily seen that $v \in L^2(0, \infty; X)$. In view of (3.10) and Lemma 3.1 $z(t)$ is of the form

$$
z(t) = \sum_{j=1}^{N} \sum_{n=0}^{m_j - 1} e^{\lambda_j t} t^n y_{j,n}, \ y_{j,n} \in D(A) .
$$

Since $\mathrm{Re}\, \lambda_j > 0$ for each $j = 1, \cdots, N$ and the function $t \mapsto e^{\lambda_j t} t^n$ ($j = 1, \cdots, N$, $n = 0, \cdots, m_j - 1$) are linearly independent, we see that $z \in L^2(0, \infty; X)$ iff $z(t) \equiv 0$.

The following Proposition is concerned with the stabilizability of (3.1), (3.2).

Proposition 3.1. *The following statements are equivalent*:
(i) *For any* $g \in X$, *there exists* $f \in L^2(0, \infty; U)$ *such that the mild solution of* (3.1), (3.2) *belongs to* $L^2(0, \infty; X)$.
(ii) *For each* $j = 1, \cdots, N$

$$
\{x^* \in Z^*_{\bar{\lambda}_j} : \Phi^* (A^* - \bar{\lambda}_j)^k x^* = 0, \ k = 0, \cdots, m_j - 1\} = \{0\} ,
$$

where $Z^*_{\bar{\lambda}_j}$ *is the generalized eigenspace for* $\bar{\lambda}_j$ *which is an eigenvalue of* A^*.

Proof. In view of Lemma 3.2 the solution of (3.1), (3.2) belongs to $L^2(0, \infty; X)$ iff

$$
S(t) P g + \int_0^\infty S(t-s) P \Phi f(s) \, ds \equiv 0 .
$$

By virtue of (3.10) and $\sum_{k=0}^{m_j-1}\sum_{n=k}^{m_j-1}=\sum_{n=0}^{m_j-1}\sum_{k=0}^{n}$ this is equivalent to

$$Q_{\lambda_j}^n g+\int_0^\infty e^{-\lambda_j s}\sum_{k=n}^{m_j-1}\frac{(-s)^{k-n}}{(k-n)!}Q_{\lambda_j}^k\Phi f(s)\,ds=0\,,$$

$j=1,\cdots,N$, $n=0,\cdots,m_j-1$. Following the proof of Theorem 2.1 or 2.3 of [1] we see (i) holds iff

(iii) For $1\leq j\leq N$, if $x_n^*\in X^*(n=0,\cdots,m_j-1)$ satisfies

$$\Phi^*\sum_{n=0}^{m_j-1-k}(Q_{\lambda_j}^{k+n})^*x_n^*=0\,,\quad k=0,\cdots,m_j-1\,, \tag{3.14}$$

then

$$\sum_{n=0}^{m_j-1}(Q_{\lambda_j}^n)^*x_n^*=0\,.$$

Suppose that (iii) is true, and $x^*\in Z_{\bar{\lambda}_j}^*$, $\Phi^*(A^*-\bar{\lambda}_j)^k x^*=0$, $k=0,\cdots,m_j-1$. Put

$$x_n^*=\begin{cases}x^* & \text{if } n=0\,,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\Phi^*\sum_{n=0}^{m_j-1-k}(Q_{\lambda_j}^{k+n})^*x_n^* &=\Phi^*(Q_{\lambda_j}^k)^*x^*\\ &=\Phi^*(A^*-\bar{\lambda}_j)^k(P_{\lambda_j})^*x^*\\ &=\Phi^*(A^*-\bar{\lambda}_j)^k x^*\\ &=0\,,\end{aligned} \tag{3.15}$$

$k=0,\cdots,m_j-1$. By (iii)

$$x^*=(P_{\lambda_j})^*x^*=\sum_{n=0}^{m_j-1}(Q_{\lambda_j}^n)^*x_n^*=0\,.$$

This shows that (iii) implies (ii).

Conversely, suppose (ii) is true and (3.14) holds. Set

$$x^*=\sum_{n=0}^{m_j-1}(Q_{\lambda_j}^n)^*x_n^*\,.$$

Then $x^*\in Z_{\bar{\lambda}_j}^*$ and

$$\begin{aligned}\Phi^*(A^*-\bar{\lambda}_j)^k x^* &=\Phi^*\sum_{n=0}^{m_j-1}(A^*-\bar{\lambda}_j)^{k+n}(P_{\lambda_j})^*x_n^*\\ &=\Phi^*\sum_{n=0}^{m_j-1-k}(Q_{\lambda_j}^{k+n})^*x_n^*\\ &=0\end{aligned}$$

for $k=0,\cdots,m_j-1$. By (ii) $x^*=0$. Hence (iii) is true.

4. Some inequalities on the fundamental solution of (2.3), (2.4)

In this section we establish the Hölder continuity results concerning the fundamental solution of the equation (2.3), (2.4) in a Banach space X, where A_0 is the infinitesimal generator of an analytic semigroup $T(t)$ and A_1 is a closed linear operator in X with domain containing that of A_0. We may assume without loss of generality that A_0 has an everywhere defined bounded inverse.

By definition the fundamental solution $W(t)$ is a bounded linear operator valued function satisfying

$$\frac{d}{dt}W(t)=A_0W(t)+\int_{-h}^{0}a(s)A_1W(t+s)\,ds\,,$$
$$W(0)=I\,,\quad W(s)=0\quad s\in[-h,0)\,.$$

The main object of this section is to prove the following

Proposition 4.1. *For $h<t<t'\le nh$, $n>1$, and $0<\kappa<\rho$, we have*

$$\|W(t')-W(t)\|\le C_n\log\left(\frac{t'}{t}\right), \tag{4.1}$$

$$\|A_0(W(t')-W(t))\|\le C_{n,\kappa}(t'-t)^{\kappa}(t-h)^{-\kappa}, \tag{4.2}$$

$$\|A_0(W(t')-W(t))A_0^{-1}\|\le C_{n,\kappa}(t'-t)^{\kappa} \tag{4.3}$$

where C_n and $C_{n,\kappa}$ are constants dependent on n and n, κ, respectively but not on t and t'.

For the sake of simplicity we assume that $T(t)$ is uniformly bounded: Then

$$\|T(t)\|\le K(t\ge0),\ \|A_0T(t)\|\le\frac{K}{t}\,(t>0),\ \|A_0^2T(t)\|\le\frac{K}{t^2}\,(t>0) \tag{4.4}$$

for some constant K(e.g. [9]). Since $a(\cdot)$ is Hölder continuous of order ρ, we can set

$$|a(s)|\le H_0,\ |a(s)-a(\tau)|\le H_1(s-\tau)^{\rho} \tag{4.5}$$

for some constants H_0, H_1. From (4.4) for $0<s<t$

$$\|A_0T(t)-A_0T(s)\|=\left\|\int_s^tA_0^2T(\tau)\,d\tau\right\|\le K\frac{t-s}{ts}\,, \tag{4.6}$$

$$\|T(t)-T(s)\|=\left\|\int_s^tA_0T(\tau)\,d\tau\right\|\le K\log\frac{t}{s}\,. \tag{4.7}$$

As is easily seen for any $t>0$ and $0<\alpha<1$

$$\log(1+t)\le\frac{t^{\alpha}}{\alpha}\,. \tag{4.8}$$

Combining this with (4.4) we get

$$\|T(t)-T(s)\| \leq \frac{K}{\alpha}\left(\frac{t-s}{s}\right)^{\alpha} \tag{4.9}$$

for $0<s<t$ and $0<\alpha<1$. Set

$$B(t)=A_0\int_0^t T(t-s)\,a(-s)\,ds\,,\quad 0\leq t\leq h\,.$$

Lemma 4.1. *$B(t)$ is strongly continuous in $[0, h]$, and hence uniformly bounded:*

$$\|B\|_\infty=\sup_{0\leq t\leq h}\|B(t)\|<\infty\,.$$

Furthermore, $B(t)$ is Hölder continuous in $(0, h]$, and for each $\kappa\in(0, \rho)$ there exists a constant C_κ such that

$$\|B(t')-B(t)\|\leq C_\kappa(t'-t)^\kappa\, t^{-\kappa}$$

for $0<t<t'\leq h$.

Proof. Since

$$\begin{aligned}B(t)&=A_0\int_0^t T(t-s)\,a(-s)\,ds\\&=\int_0^t A_0T(t-s)\,(a(-s)-a(-t))\,ds-(I-T(t))\,a(-t)\,,\end{aligned}$$

it follows without difficulty from (4.4), (4.5) that $B(t)$ is strongly continuous in $[0, h]$.

For $0<t<t'\leq h$

$$\begin{aligned}B(t')-B(t)&=A_0\int_0^{t'} T(t'-s)\,a(-s)\,ds-A_0\int_0^t T(t-s)\,a(-s)\,ds\\&=\int_t^{t'} A_0\,T(t'-s)\,(a(-s)-a(-t'))\,ds\\&\quad+\int_0^t A_0\,T(t'-s)-A_0\,T(t-s))\,(a(-s)-a(-t))\,ds\\&\quad+(T(t')-T(t'-t))\,(a(-t)-a(-t'))\\&\quad+(T(t')-T(t))\,a(-t')\\&\quad+(T(t)-I)\,(a(-t')-a(-t))\,.\end{aligned}$$

It follows from (4.4), (4.5), (4.6), (4.7), and (4.8) that

$$\left\|\int_t^{t'} A_0\,T(t'-s)\,(a(-s)-a(-t'))\,ds\right\|\leq\int_t^{t'} KH_1(t'-s)^{\rho-1}\,ds=\frac{KH_1}{\rho}\,(t'-t)^\rho\,,$$

$$\begin{aligned}\left\|\int_0^t (A_0\,T(t'-s)-A_0\,T(t-s))\,(a(-s)-a(-t))\,ds\right\|&\leq\int_0^t KH_1\frac{t'-t}{(t'-s)\,(t-s)}\,(t-s)^\rho\,ds\\&\leq KH_1(t'-t)^\kappa\int_0^t (t-s)^{\kappa-\rho-1}\,ds\\&\leq\frac{KH_1}{\rho-\kappa}\,(t'-t)^\kappa\, t^{\rho-\kappa}\end{aligned}$$

for $0<\kappa<\rho$,

$$\|(T(t')-T(t'-t))(a(-t)-a(-t'))\|\le 2KH_1(t'-t)^\rho ,$$
$$\|(T(t')-T(t))\, a(-t)\|\le KH_0 \log\frac{t'}{t}\le KH_0\frac{1}{\kappa}(t'-t)^\kappa t^{-\kappa} ,$$
$$\|(T(t)-I)(a(-t')-a(-t))\|\le (K+1)\, H_1(t'-t)^\rho .$$

Combining these we obtain the desired inequality.

Set

$$V(t)=\begin{cases} A_0(W(t)-T(t))\,, & t\in(0,h] \\ A_0\, W(t)\,, & t\in(nh,(n+1)h],\ n=1,2,\cdots. \end{cases} \tag{4.10}$$

Then, the integral equation to be satisfied by $V(t)$ in $[nh,(n+1)h]$ is

$$V(t)=V_0(t)+\int_{nh}^{t} B(t-\tau)\, A_1\, A_0^{-1}\, V(\tau)\, d\tau\,, \tag{4.11}$$

where

$$V_0(t)=\int_0^t B(t-\tau)\, A_1 T(\tau)\, d\tau$$

in $[0,h]$, and

$$V_0(t)=A_0\, T(t)+\int_0^{t-h} T(t-h-\tau)\, B(h)\, A_1\, W(\tau)\, d\tau$$
$$+\int_{t-h}^{nh} B(t-\tau)\, A_1\, W(\tau)\, d\tau$$

in $[nh,(n+1)h]$, $n=1,2,\cdots$. As is stated in [10] $V(t)$ is bounded in each of the interval $[nh,(n+1)h]$, $n=1,2,\cdots$. Using this and by (4.10) we get

Proposition 4.2. *Let $W(t)$ be the fundamental solution of equation* (2.3), (2.4) *then for any natural number n there exists a constant C_n such that*

$$\|W(t)\|\le C_n,\ \|A_0\, W(t)\|\le\frac{C_n}{t},\ \|A_1\, W(t)\|\le\frac{C_n}{t}$$
$$\Big\|\int_s^t A_0\, W(\tau)\, d\tau\Big\|\le C_n,\ \Big\|\int_s^t A_1\, W(\tau)\, d\tau\Big\|\le C_n$$

for $0\le s<t\le nh$.

Proof of Proposition 4.1. For $nh\le t<t'\le(n+1)h$

$$\begin{aligned} V_0(t')-V_0(t) &= (A_0\, T(t')-A_0\, T(t)) \\ &+\int_{t-h}^{t'-h} T(t'-h-\tau)\, B(h)\, A_1\, W(\tau)\, d\tau \\ &+\int_0^{t-h} (T(t'-h-\tau)-T(t-h-\tau)-T(t'-h)+T(t-h))\, B(h)\, A_1\, W(\tau)\, d\tau \end{aligned}$$

$$
\begin{aligned}
&+(T(t'-h)-T(t-h))\,B(h)\int_0^{t-h} A_1\,W(\tau)\,d\tau \\
&+\int_{t'-h}^{nh}(B(t'-\tau)-B(t-\tau))\,A_1W(\tau)\,d\tau \\
&-\int_{t-h}^{t'-h} B(t-\tau)\,A_1\,W(\tau)\,d\tau \\
&=I_1+I_2+I_3+I_4+I_5+I_6\,.
\end{aligned}
$$

From (4.6)

$$||I_1||\leq K\frac{t'-t}{t't}\leq K\Big(\frac{t'-t}{t'}\Big)^{\kappa}t\,.$$

In view of (4.4) and Proposition 4.2

$$
\begin{aligned}
||I_2||&\leq\int_{t-h}^{t'-h}K||B(h)||\,C_n\tau^{-1}\,d\tau=KC_n\,||B(h)||\log\frac{t'-h}{t-h}\\
&\leq\kappa^{-1}KC_n\,||B(h)||(t'-t)^{\kappa}(t-h)^{-\kappa}\,.
\end{aligned}
$$

In what follows throughout the proof let μ be any number such that $\kappa<\mu<\rho$. In view of (4.9)

$$
\begin{aligned}
&||T(t'-h-\tau)-T(t-h-\tau)||\leq\frac{K}{\mu}\Big(\frac{t'-t}{t-h-\tau}\Big)^{\mu},\\
&||T(t'-h)-T(t-h)||\leq\frac{K}{\mu}\Big(\frac{t'-t}{t-h}\Big)^{\mu}\leq\frac{K}{\mu}\Big(\frac{t'-t}{t-h-\tau}\Big)^{\mu}.
\end{aligned}
$$

Hence

$$(4.12)\quad ||T(t'-h-\tau)-T(t-h-\tau)-T(t'-h)+T(t-h)||\leq\frac{2K}{\mu}\Big(\frac{t'-t}{t-h-\tau}\Big)^{\mu}.$$

Similarly

$$
\begin{aligned}
&||T(t'-h-\tau)-T(t'-h)||\leq\frac{K}{\mu}\Big(\frac{\tau}{t'-h-\tau}\Big)^{\mu}\leq\frac{K}{\mu}\Big(\frac{\tau}{t-h-\tau}\Big)^{\mu},\\
&||T(t-h-\tau)-T(t-h)||\leq\frac{K}{\mu}\Big(\frac{\tau}{t-h-\tau}\Big)^{\mu},
\end{aligned}
$$

which imply

$$(4.13)\quad ||T(t'-h-\tau)-T(t-h-\tau)-T(t'-h)+T(t-h)||\leq\frac{2K}{\mu}\Big(\frac{\tau}{t-h-\tau}\Big)^{\mu}.$$

Combining (4.12) and (4.13) we get

$$(4.14)\quad ||T(t'-h-\tau)-T(t-h-\tau)-T(t'-h)+T(t-s)||\leq\frac{2K}{\mu}\frac{(t'-t)^{\kappa}\,\tau^{\mu-\kappa}}{(t-h-\tau)^{\mu}}\,.$$

In view of (4.14), Proposition 4.2, and Lemma 4.1

$$||I_3||\leq\int_0^{t-h}\frac{2K}{\mu}\frac{(t'-t)^{\kappa}\,\tau^{\mu-\kappa}}{(t-h-\tau)^{\mu}}\,||B(h)||\,\frac{C_n}{\tau}\,d\tau$$

$$= \frac{2K}{\mu} C_n \|B(h)\| (t'-t)^{\kappa} \int_0^{t-h} (t-h-\tau)^{-\mu} \tau^{\mu-\kappa-1} d\tau$$
$$= 2K C_n \|B(h)\| B(1-\mu, \mu-\kappa) \mu^{-1} (t'-t)^{\kappa} (t-h)^{-\kappa},$$

where $B(\cdot, \cdot)$ is the Beta function.

From (4.6), Proposition 4.2, and Lemma 4.1, we have

$$\|I_4\| \leq \frac{KC_n}{\kappa} \|B(h)\| \left(\frac{t'-t}{t-h}\right)^{\kappa}.$$

We estimate I_5 in case of $n=1$ and $n>1$ separately. First, we consider the case $n=1$.

$$\begin{aligned} I_5 &= \int_{t'-h}^{h} (B(t'-\tau)-B(t-\tau)-B(h)+B(t-t'+h)) A_1 W(\tau) d\tau \\ &\quad + (B(h)-B(t-t'+h)) \int_{t'-h}^{h} A_1 W(\tau) d\tau \\ &= I_5^1 + I_5^2. \end{aligned}$$

Noting $t'-h<\tau<h<t$, $0<t-\tau<t'-\tau<h$, $0<t-t'+h<h$, and using Lemma 4.1 we get analogously to (4.14)

$$\|B(t'-\tau)-B(t-\tau)-B(h)+B(t-t'+h)\| \leq 2C_\mu (t'-t)^{\kappa} (t-\tau)^{-\mu} (\tau-t'+h)^{\mu-\kappa}.$$

Hence

$$\begin{aligned} \|I_5^1\| &\leq \int_{t'-h}^{h} 2C_\mu (t'-t)^{\kappa} (t-\tau)^{-\mu} (\tau-t'+h)^{\mu-\kappa} C_1 \tau^{-1} d\tau \\ &\leq 2C_\mu C_1 (t'-t)^{\kappa} \int_{t'-h}^{t} (t-\tau)^{-\mu} (\tau-t'+h)^{\mu-\kappa-1} d\tau \\ &= 2C_\mu C_1 B(1-\mu, \mu-\kappa) (t'-t)^{\kappa} (t-t'+h)^{-\kappa} \\ &\leq 2C_\mu C_1 B(1-\mu, \mu-\kappa) (t'-t)^{\kappa} (t-h)^{-\kappa}. \end{aligned}$$

As is easily seen

$$\|I_5^2\| \leq C_\kappa C_1 (t'-t)^{\kappa} (t-t'+h)^{-\kappa}.$$

Therefore

$$\|I_5\| \leq \{2C_\mu C_1 B(1-\mu, \mu-\kappa) + C_\kappa C_1\} (t'-t)^{\kappa} (t-h)^{-\kappa}.$$

In case $n>1$ we have

$$\begin{aligned} \|I_5\| \leq \int_{t'-h}^{nh} C_\kappa \left(\frac{t'-t}{t-\tau}\right)^{\kappa} C_n \frac{d\tau}{\tau} &\leq \frac{C_\kappa C_n (t'-t)^{\kappa}}{(n-1)h} \int_{t-h}^{t} (t-\tau)^{-\kappa} d\tau \\ &\leq \frac{C_\kappa C_n (t'-t)^{\kappa}}{(n-1)(1-\kappa)h^{\kappa}} \\ &\leq \frac{C_\kappa C_n}{(n-1)(1-\kappa)} \left(\frac{t'-t}{t-h}\right)^{\kappa}. \end{aligned}$$

Therefore

$$||I_5|| \leq \text{const. } (t'-t)^{\kappa} (t-h)^{-\kappa} .$$

From Lemma 4.1 and Proposition 4.2

$$||I_6|| \leq \int_{t-h}^{t'-h} ||B||_{\infty} \frac{C_n}{\tau} d\tau = C_n ||B||_{\infty} \log \frac{t'-h}{t-h} \leq \frac{C_n ||B||_{\infty}}{\kappa} \left(\frac{t'-t}{t-h}\right)^{\kappa} .$$

Therefore for $nh \leq t < t' \leq (n+1) h$, we have

$$||V_0(t') - V_0(t)|| \leq \text{const. } (t'-t)^{\kappa} (t-h)^{-\kappa} ,$$

from which it follows that

$$||V(t') - V(t)|| \leq \text{const. } (t'-t)^{\kappa} (t-h)^{-\kappa} . \tag{4.15}$$

It is not difficult to show that for $n>1$

$$\begin{aligned} V(nh+0) &= A_0\, T(nh) + \int_0^{(n-1)h} T(nh-h-\tau)\, B(h)\, A_1\, W(\tau)\, d\tau \\ &\quad + \int_{(n-1)h}^{nh} B(nh-\tau)\, A_1\, W(\tau)\, d\tau \\ &= V(nh-0) , \end{aligned}$$

where we used $A_1\, W(\tau) = V(\tau)$ in the second integral. Hence $V(t)$ is continuous at $t=nh$, $n=1, 2, \cdots$. Using this and (4.15) we have for $(n-1)h < t \leq nh < t' \leq (n+1)h$, $n>1$,

$$\begin{aligned} ||V(t') - V(nh)|| &\leq \text{const.} \left(\frac{t'-nh}{nh-h}\right)^{\kappa} \leq \text{const. } \left(\frac{t'-t}{t-h}\right)^{\kappa} \\ ||V(nh) - V(t)|| &\leq \text{const. } \left(\frac{nh-t}{t-h}\right)^{\kappa} \leq \text{const. } \left(\frac{t'-t}{t-h}\right)^{\kappa} . \end{aligned}$$

Thus (4.15) holds for $h < t < t' < nh$, $t'-t < h$, $n>1$ with const. independent on t and t'. From the formular (4.11)

$$\begin{aligned} V(t') - V(t) &= V_0(t') - V_0(t) \\ &\quad + \int_{nh}^{t'} B(t'-\tau)\, A_1\, A_0^{-1}\, V(\tau)\, d\tau - \int_{nh}^{t} B(t-\tau)\, A_1\, A_0^{-1}\, V(\tau)\, d\tau \\ &= V_0(t') - V_0(t) \\ &\quad + \int_t^{t'} B(t'-\tau)\, A_1\, A_0^{-1}\, V(\tau)\, d\tau \\ &\quad + \int_{nh}^{t} (B(t'-\tau) - B(t-\tau))\, A_1\, A_0^{-1}\, V(\tau)\, d\tau . \end{aligned}$$

Noting $V(t) = A_0\, W(t)$, for $t>0$ we have established that there exists a constant $C_{n,\kappa}>0$ such that

$$\|A_0 W(t')-A_0 W(t)\| \leq C_{n,\kappa}(t'-t)^{\kappa}(t-h)^{-\kappa}.$$

for $h<t<t'\leq nh$, $t'-t<h$ for $n<1$. By the same way, we get

$$\|A_0(W(t')-W(t))A_0^{-1}\| \leq C_{n,\kappa}(t'-t)^{\kappa}.$$

It follows from (4.5), Proposition 4.2, and

$$\frac{d}{dt}W(t)=A_0 W(t)+\int_{-h}^{0}(a(s)-a(-h))A_1 W(t+s)\,ds+a(-h)\int_{-h}^{0}A_1 W(t+s)\,ds$$

that $\|dW(t)/dt\| \leq C_n/t$, $h<t\leq nh$. Hence the proof of Proposition 4.1 is complete.

5. Hölder continuity of the solution semigroup

Since in our case the solution of (2.3), (2.4) is represented by

$$u(t)=W(t)g^0+\int_{-h}^{0}W(t+s)[F_1 g^1](s)\,ds, \tag{5.1}$$

the solution semigroup $S(t)$ can be expressed in terms of $W(t)$.

Applying Proposition 4.1 to the equation (2.3) in the space V^* and noting that A_0+c_1 is an isomorphism from V to V^* we get

$$\|W(t')-W(t)\|_{B(V^*)} \leq C_n(t'-t), \tag{5.2}$$

$$\|W(t')-W(t)\|_{B(V^*,V)} \leq C_{n,\kappa}(t'-t)^{\kappa}(t-h)^{-\kappa}, \tag{5.3}$$

$$\|W(t')-W(t)\|_{B(V)} \leq C_{n,\kappa}(t'-t)^{\kappa}, \tag{5.4}$$

for $h<t<t'\leq nh$, $n=1,\cdots$, and $0<\kappa<\rho$.

Similarly applying the same Proposition to (2.3) in H

$$\|W(t')-W(t)\|_{B(H)} \leq C_n(t'-t) \tag{5.5}$$

for $h<t<t'\leq nh$. Using $(V, V^*)_{1/2,2}=H$ and the well known interpolation inequality we get from (5.2) and (5.3)

$$\|W(t')-W(t)\|_{B(V^*,H)} \leq C_{n,\kappa}(t'-t)^{(1+\kappa)/2}(t-h)^{-\kappa/2}, \tag{5.6}$$

and from (5.3) and (5.4)

$$\|W(t')-W(t)\|_{B(H,V)} \leq C_{n,\kappa}(t'-t)^{\kappa}(t-h)^{-\kappa/2}. \tag{5.7}$$

Proposition 5.1. *The solution semigroup* $S(t)$ *for* (2.3), (2.4) *is Hölder continuous in* $(3h, \infty)$ *in the operator norm.*

Proof. Let $u(t)$ be the solution of (2.3), (2.4), and $3h<t<t'$. Using (5.1), (5.5), and (5.6)

$$
\begin{aligned}
|u(t')-u(t)| &\leq |(W(t')-W(t))g^0| \\
&\quad +\int_{-h}^{0}\int_{-h}^{\sigma} |(W(t'-\sigma+\xi)-W(t-\sigma+\xi))\,a(\xi)\,A_1 g^1(\sigma)|\,d\xi\,d\sigma \\
&\leq C_n(t'-t)|g^0| \\
&\quad +C_{n,\kappa}\int_{-h}^{0}\int_{-h}^{\sigma} (t'-t)^{(\kappa+1)/2}(t-\sigma+\xi-h)^{-\kappa/2} H_0\|A\|_{B(V,V^*)}\|g^1(\sigma)\|\,d\xi\,d\sigma \\
&\leq C_n(t'-t)|g^0| \\
&\quad +C_{n,\kappa} H_0\|A_1\|_{B(V,V^*)}(t'-t)^{(\kappa+1)/2}\frac{(t-h)^{1-\kappa/2}}{1-\dfrac{\kappa}{2}}\int_{-h}^{0}\|g^1(\sigma)\|\,d\sigma \\
&\leq \text{const. } \{(t'-t)|g^0|+(t'-t)^{(\kappa+1)/2}(\int_{-h}^{0}\|g^1(\sigma)\|^2\,d\sigma)^{1/2}\}\ .
\end{aligned}
$$

With the aid of (5.3), (5.7) for $s\in[-h, 0)$

$$
\begin{aligned}
\|u(t'+s)-u(t+s)\| &= \|(W(t'+s)-W(t+s))g^0\| \\
&\quad +\|\int_{-h}^{0}\int_{-h}^{\sigma} (W(t'+s-\sigma+\xi)-W(t+s-\sigma+\xi))\,a(\xi)\,A_1 g^1(\sigma)\,d\xi\,d\sigma\| \\
&\leq C_{n,\kappa}(t'-t)^{\kappa}(t+s-h)^{-\kappa/2}|g^0| \\
&\quad +C_{n,\kappa}\int_{-h}^{0}\int_{-h}^{\sigma} (t'-t)^{\kappa}(t+s-\sigma+\xi-h)^{-\kappa} H_0\|A\|_{B(V,V^*)}\|g^1(\sigma)\|\,d\xi\,d\sigma \\
&\leq C_{n,\kappa}(t'-t)^{\kappa}(t+s-h)^{-\kappa/2}|g^0| \\
&\quad +C_{n,\kappa} H_0\|A_1\|_{B(V,V^*)}(t'-t)^{\kappa}\frac{(t+s-h)^{1-\kappa}}{1-\kappa}\int_{-h}^{0}\|g^1(\sigma)\|\,d\sigma \\
&\leq D(t'-t)^{\kappa}\{(t'+s-h)^{-\kappa/2}|g^0|+(t+s-h)^{1-\kappa}(\int_{-h}^{0}\|g^1(\sigma)\|^2\,d\sigma)^{1/2}\}
\end{aligned}
$$

where $D=\max\{C_{n,\kappa}, h^{1/2}(1-\kappa)^{-1}C_{n,\kappa}H_0\|A_0\|_{B(V,V^*)}\}$, hence

$$
\begin{aligned}
\int_{-h}^{0}\|u(t'+s)-u(t+s)\|^2\,ds &\leq 2D^2(t'-t)^{2\kappa}\{\int_{-h}^{0}(t+s-h)^{-\kappa}\,ds\,|g^0|^2 \\
&\quad +\int_{-h}^{0}(t+s-h)^{2(1-\kappa)}\,ds\int_{-h}^{0}\|g^1(\sigma)\|^2\,d\sigma\} \\
&\leq \text{const. }(t'-t)^{2\kappa}\|(g^0, g^1)\|^2\ .
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\|S(t')g-S(t)g\|^2 &= |u(t')-u(t)|^2+\int_{-h}^{0}\|u(t'+s)-u(t+s)\|^2\,ds \\
&\leq \text{const. }(t'-t)^{2\kappa}\|g\|^2\ .
\end{aligned}
$$

Note that $t+s-\sigma+\xi>h$ since $t>3h$. Therefore for any $T>3h$ there exists a constant C_T such that

$$\|S(t')-S(t)\|\leq C_T(t'-t)^{\kappa}\ .$$

Hence $S(t)$ is Hölder continuous on $(3h, \infty)$ in the operator norm.

6. Proof of the main theorem

In view of Proposition 5.1 the assumptions of Proposition 3.1 are satisfied for the problem (2.12), (2.13). Just as Theorems 4.2 and 8.1 of [7] it can be shown that F^* maps $D(A_T)$ to $D(A^*)$ and $A^* F^* = F^* A_T$ on $D(A_T)$, and F^* is an isomorphism from $Z^T_{\bar{\lambda}_j}$ to $Z^*_{\bar{\lambda}_j}$. Hence, the second statement of Proposition 3.1 is equivalent to

(ii)$'$ For each $j=1, \cdots, N$

$$\{\phi \in Z^T_{\bar{\lambda}_j} : \Phi_0^* [(A_T - \bar{\lambda}_j)^n \phi]^0 = 0, n = 0, \cdots, m_j - 1\} = \{0\} .$$

By the same manner as the proof of proposition 7.2 of [7] we have

$$Z^T_{\bar{\lambda}_j} = \ker (\bar{\lambda}_j - A_T)^{m_j} = \{(\phi_0^0, \exp (\bar{\lambda}_j \cdot) \sum_{i=0}^{m_j-1} (-\cdot)^i \phi_i^0/i!) : \\ \sum_{i=n}^{m_j-1} (-1)^{i-n} \Delta_T^{(i-n)}(\bar{\lambda}_j)\, \phi_i^0/(i-n)! = 0, n = 0, \cdots, m_j - 1\} . \tag{6.1}$$

Using $A_T(\phi^0, \phi^1) = (A_0^* \phi^0 + \int_{-h}^0 a(s) A_1^* \phi^1(s)\, ds, \dot{\phi}^1)$ we see that for the elements in the bracket of the right side of (6.1)

$$(\bar{\lambda}_j - A_T)^n (\phi_0^0, \exp (\bar{\lambda}_j \cdot) \sum_{i=0}^{m_j-1} (-\cdot)^i \phi_i^0/i!) \\ = (\phi_n^0, \exp (\bar{\lambda}_j \cdot) \sum_{i=n}^{m_j-1} (-\cdot)^{i-n} \phi_i^0/(i-n)!) \tag{6.2}$$

for $n=0, \cdots, m_j-1$. Combining (6.1) and (6.2) we see that (ii)$'$ is equivalent to the second statement of the theorem.

The author wishes to express his deep gratitude to the referee for his careful reading the manuscript and valuable comments.

References

[1] G. Da Prato and A. Lunardi: *Stabilizability of integrodifferential parabolic equations*, J. Integral Eq. and Appl. **2**(2) (1990), 281–304.

[2] G. Di Blasio, K. Kunisch, and E. Sinestrari: *L^2-regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. appl. **102** (1984), 38–57.

[3] G. Di Blasio, K. Kunisch, and E. Sinestrari: *Stability for abstract linear functional differential equations*, Israel J. Math. **50** (1985), 231–263.

[4] J. M. Jeong: *Spectral properties of the operator associated with a retarded functional differential equation in Hilbert space*, Proc. Japan Acad. **65A** (1989), 98–101.

[5] T. Kato: Pertubation Theory, 2nd ed., Springer, Berlin-Heidelberg-New York, 1976.

[6] R. Nagel: One-parameter Semigroups of Positive Operators, Lecture Notes in Mathematics 1184, Springer-Verlag Berlin-Heidelberg New York Tokyo, 1986.

[7] S. Nakagiri: *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math. **25** (1988), 353–398.

[8] S. Nakagiri, M. Yamamoto, and G.D. Lee: *Spectral mode controllability, observability and identifiability for isolated spectrum points*, preprint.

[9] H. Tanabe: Equations of Evolution, Pitman-London, 1979.

[10] H. Tanabe: *On fundamental solution for differential equation with time delay in Banach space*, Proc. Japan Acad. **64A** (1988), 131–134.

[11] H. Tanabe: *Structural operators for linear delay-differential equations in Hilbert space*, Proc. Japan Acad. **64A** (1988), 265–266.

[12] K. Yosida: Functional Analysis, 3rd ed., Springer Berlin-Göttingen-Heidelberg, 1980.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560
Japan