Suga, S.
Osaka J. Math.
28 (1991), 323-346

HIGHEST WEIGHT MODULES ASSOCIATED WITH CLASSICAL IRREDUCIBLE REGULAR PREHOMOGENEOUS VECTOR SPACES OF COMMUTATIVE PARABOLIC TYPE

SHUICHI SUGA

(Received April 20, 1990)

Introduction

Let $\mathfrak{g}$ be a classical complex simple Lie algebra. We assume $\mathfrak{g}$ has a $\boldsymbol{Z}$-gradation of the form:

$$\mathfrak{g} = \mathfrak{g}(-1)+\mathfrak{g}(0)+\mathfrak{g}(1).$$

Let G be a connected complex Lie group with Lie algebra $\mathfrak{g}$, and $G(0)$ the connected subgroup of G corresponding to the Lie subalgebra $\mathfrak{g}(0)$ of $\mathfrak{g}$. We further assume that the pairs $(G(0), \mathfrak{g}(\pm 1))$ are irreducible regular prehomogeneous vector spaces [6], [8]. Let $d\lambda$ be a 1-dimensional representation of the parabolic subalegebra $\mathfrak{p}=\mathfrak{g}(0)+\mathfrak{g}(1)$, and $\boldsymbol{C}_{d\lambda}$ its representation space. Let $U(\mathfrak{g})$ and $U(\mathfrak{p})$ be the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{p}$, respectively. We denote by $V(d\lambda)$ the generalized Verma module induced from $d\lambda$:

$$V(d\lambda) = U(\mathfrak{g})\otimes_{U}(\mathfrak{p})\, \boldsymbol{C}_{d\lambda},$$

and by $L(d\lambda)$ its irreducible quotient.

The purpose of this paper is to give a realization of the $U(\mathfrak{g})$-module $L(d\lambda)$ using the irreducible relative invariant polynomial f of the pair $(G(0), \mathfrak{g}(-1))$. As an application, we recover the reducibility criterion of $V(d\lambda)$ (due to Jantzen [2]) and show that it has a natural interpretation in terms of the zeros of the b-function [4], [8] of f.

Acknowledgement. The author would like to express his sincere thanks to Professor N. Kawanaka for his constant encouragement and valuable advice. He also would like to express his thanks to Professors T. Hirai and T. Nomura for informing him the result of Johnson [3].

1. Statement of main results

In this section we state our main results more precisely. Let $\mathfrak{g}$, $\mathfrak{g}(0)$, $\mathfrak{g}(\pm 1)$, $\mathfrak{p}$, $U(\mathfrak{g})$ and $U(\mathfrak{p})$ be as in the Introduction. In particular $\mathfrak{g}$ is a classical complex simple Lie algebra with $\boldsymbol{Z}$-gradation of the form:

$$\mathfrak{g} = \mathfrak{g}(-1)+\mathfrak{g}(0)+\mathfrak{g}(1)\,. \tag{1.1}$$

Let P be the normalizer of the parabolic subalgebra $\mathfrak{p}=\mathfrak{g}(0)+\mathfrak{g}(1)$ in G. Let $\tilde{P}$ be the universal covering group of P and π: $\tilde{P}\to P$ the projection homomorphism. We choose an open neighborhood $V\subset P$ of the identity element so that there exists a section σ: $V\to\tilde{P}$ of π.

We assume that the pairs $(G(0), \mathfrak{g}(\pm 1))$ are irreducible regular prehomogeneous vector spaces [6], [8]. Let f (resp. f^*) be the irreducible relative invariant polynomial on $\mathfrak{g}(-1)$ (resp. $\mathfrak{g}(1)$). By definition f is an irredicible polynomial on $\mathfrak{g}(-1)$ satisfying

$$f(Ad(k)\,x) = \chi(k)f(x) \quad (k\in G(0), x\in\mathfrak{g}(-1)) \tag{1.2}$$

for a 1-dimensional character χ of $G(0)$. We extend χ to P trivially. We also consider χ as a character of $\tilde{P}$ and denote it by the same letter.

Let N^- be the subgroup of G corresponding to $\mathfrak{g}(-1)$. We denote the inverse of the exponential map exp: $\mathfrak{g}(-1)\to N^-$ by log: $N^-\to\mathfrak{g}(-1)$. We set $O=N^-V$, which is an open subset of G. Let λ be an arbitrary 1-dimensional character of $\tilde{P}$. Let

$$H(\lambda) = \{h\colon O\to \boldsymbol{C}\colon h \text{ is holomorphic}, h(gq) = \lambda(\sigma(q))\,h(g), g\in O, q\in V\}\,. \tag{1.3}$$

We can identify $H(\lambda)$ with the space of holomorphic functions $H(N^-)$ on N^-. By differentiating the left G-translation on $H(\lambda)$, we get an algebra homomorphism φ: $U(\mathfrak{g})\to\mathcal{D}(N^-)$ from $U(\mathfrak{g})$ to the algebra $\mathcal{D}(N^-)$ of differential operators on N^- with holomorphic coefficients.

Let f_χ be the holomorphic function on O defined by

$$f_\chi(nq) = \chi^{-1/2}(\sigma(q))f(\log n) \quad n\in N^-, q\in P\,.$$

We denote the differentials of λ and χ by $d\lambda$ and $d\chi$, respectively. Let $\mu=\mu(\lambda)$ be the complex number defined by

$$d\lambda = \mu d\chi\,. \tag{1.4}$$

We consider the complex power $v^\lambda=f_\chi^{-\mu}$ of f_χ. Then $\varphi(U(\mathfrak{g}))$ acts on v^λ. Let $W(\lambda)=\varphi(U(\mathfrak{g})).\, v^\lambda$, a $U(\mathfrak{g})$-module generated by v^λ. Rigorously speaking, v^λ should be defined as follows. Let α be a variable. Let X be an open ball in $\{x\in N^-\colon f(\log(x))\neq 0\}$. Let $\mathcal{D}(N^-)\,[\alpha]$ be a polynomial ring with coefficients in $\mathcal{D}(N^-)$. Then $N_\alpha=\mathcal{D}(N^-)\,[\alpha]f_\chi^\alpha$ is a $\mathcal{D}(N^-)\,[\alpha]$-module on X. We define v^λ to be the image of f_χ^α in the quotient $\mathcal{D}(N^-)\,[\alpha]$-module $N_\alpha/(2\mu+\alpha)\,N_\alpha$.

Theorem 1.1. *$W(\lambda)$ is an irreducible highest weight $U(\mathfrak{g})$-module with highest weight λ. In other words, $W(\lambda)$ is isomorphic to $L(d\lambda)$.*

If λ is the highest weight of a finite dimensional $U(\mathfrak{g})$-module, then the above realization is a special case of the Borel-Weil Theorem. (See, for example, [5].)

As an application of Theorem 1.1, we give a reducibility criterion for the generalized Verma modules $V(d\lambda)$. To state this, let $f^*(D_x)$ be the linear differential operator with constant coefficients defined by

$$f^*(D_x)\exp\langle\xi, x\rangle = f^*(\xi)\exp\langle\xi, x\rangle\,, \quad \xi\in\mathfrak{g}(1),\ x\in\mathfrak{g}(-1)\,,$$

where $\langle\ ,\ \rangle$ is the Killing form on $\mathfrak{g}$. It is known [4], [6] that there exists a polynomial $b(s)$ such that

$$f^*(D_x)f(x)^s = b(s)f(x)^{s-1},\ s\in\boldsymbol{C}\,. \tag{1.5}$$

The polynomial $b(s)$ is called the b-ufnction of the relative invariant f.

Corollary 1.2. *If -2μ is a positive integer or a zero of $b(s)$, then $V(d\lambda)$ is reducible.*

This gives a new interpretation of a result of Jantzen [2] in this special case.

Theorem 1.1 is proved in Sections 4–7 by case-by-case consideration. Corollary 1.2 is proved in Section 8.

2. Irreducible Regular Prehomogeneous Vector Spaces of Commutative Parabolic Type

In this section we summarize a part of the results of [6] in a form convenient to our purpose (See also [8]). We also give explicit formulas for the irreducible relative invariant polynomials and the corresponding characters.

We retain the notations in Section 1. If $(G(0), \mathfrak{g}(\pm 1))$ are irreducible regular prehomogeneous vector spaces, then the pairs $(G(0), \mathfrak{g}(\pm 1))$ are called *irredicible regular prehomogeneous vector spaces of commutative parabolic type.* According to [6], if $\mathfrak{g}$ is classical, these are classified into the following four cases.

Case I.

$$\mathfrak{g} = sl(2n, \boldsymbol{C})\,.$$

We define the gradation $\mathfrak{g}=\mathfrak{g}(-1)+\mathfrak{g}(0)+\mathfrak{g}(1)$ by

$$\mathfrak{g}(-1) = \{\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}: C\in M_n(\boldsymbol{C})\}\,,$$

$$\mathfrak{g}(0) = \{\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}: A, D\in M_n(\boldsymbol{C}),\ \mathrm{tr}A+\mathrm{tr}D = 0\}\,,$$

$$\mathfrak{g}(1)=\{\begin{bmatrix}0 & B\\ 0 & 0\end{bmatrix}: B\in M_n(\boldsymbol{C})\}\ .$$

We set $G=SL(2n, \boldsymbol{C})$. Then

$$G(0)=\{\begin{bmatrix}A & 0\\ 0 & D\end{bmatrix}: A,\ D\in GL(n, \boldsymbol{C}),\ \det A\ \det D=1\}$$
$$\cong S(GL(n, \boldsymbol{C})\times GL(n, \boldsymbol{C}))\,.$$

The irreducible relative invariant f is given by

$$f(x)=\det C,\ x=\begin{bmatrix}0 & 0\\ C & 0\end{bmatrix}\in\mathfrak{g}(-1)\,.$$

The character χ defined by (1.2) is given by

$$\chi(g)=(\det A)^{-2}\,,\quad g=\begin{bmatrix}A & 0\\ 0 & D\end{bmatrix}\in G(0)\,.$$

Case II.

$$\mathfrak{g}=\mathrm{sp}(2n, \boldsymbol{C})=\{X\in M_{2n}(\boldsymbol{C}):\ {}^tXA_{2n}+A_{2n}X=0\}\ ,$$

where $A_{2n}=\begin{bmatrix}0 & 1_n\\ -1_n & 0\end{bmatrix}$ with the $n\times n$ identity matrix 1_n.

$$\mathfrak{g}(-1)=\{\begin{bmatrix}0 & 0\\ C & 0\end{bmatrix}: C\in M_n(\boldsymbol{C}),\ {}^tC=C\}\ ,$$

$$\mathfrak{g}(0)=\{\begin{bmatrix}A & 0\\ 0 & -{}^tA\end{bmatrix}: A\in M_n(\boldsymbol{C})\}\ ,$$

$$\mathfrak{g}(1)=\{\begin{bmatrix}0 & B\\ 0 & 0\end{bmatrix}: B\in M_n(\boldsymbol{C}),\ {}^tB=B\}\ .$$

We set $G=Sp(2n, \boldsymbol{C})=\{g\in GL(2n, \boldsymbol{C}):\ {}^tg\ A_{2n}\,g=A_{2n}\}$. Then

$$G(0)=\{\begin{bmatrix}A & 0\\ 0 & {}^tA^{-1}\end{bmatrix}: A\in GL(n, \boldsymbol{C})\}\cong GL(n, \boldsymbol{C})\,,$$

$$f(x)=\det C\quad\text{for}\quad x=\begin{bmatrix}0 & 0\\ C & 0\end{bmatrix}\in\mathfrak{g}(-1)\,,$$

$$\chi(g)=(\det A)^{-2}\quad\text{tor}\quad g=\begin{bmatrix}A & 0\\ 0 & {}^tA^{-1}\end{bmatrix}\in G(0)\,.$$

Case III.

$$\mathfrak{g}=so(4n, \boldsymbol{C})=\{X\in M_{4n}(\boldsymbol{C}):\ {}^tXS_{4n}+S_{4n}X=0,\ trX=0\}\ ,$$

where $S_{4n}=\begin{bmatrix} 0 & 1_{2n} \\ 1_{2n} & 0 \end{bmatrix}$.

$$\mathfrak{g}(-1)=\{\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}: C\in M_{2n}(\boldsymbol{C}),\ {}^tC=-C\}\ ,$$

$$\mathfrak{g}(0)=\{\begin{bmatrix} A & 0 \\ 0 & -{}^tA \end{bmatrix}: A\in M_{2n}(\boldsymbol{C})\}\ ,$$

$$\mathfrak{g}(1)=\{\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}: B\in M_{2n}(\boldsymbol{C}),\ {}^tB=-B\}\ .$$

We set $G=SO(4n,\boldsymbol{C})=\{g\in GL(4n,\boldsymbol{C}):{}^tg\,S_{4n}\,g=S_{4n},\ \det g=1\}$. Then

$$G(0)=\{\begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix}: A\in GL(2n,\boldsymbol{C})\}\simeq GL(2n,\boldsymbol{C})\,.$$

The irreducible relative invariant f is the Pfaffian defined by:

$$f^2(x)=\det C,\ x=\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\in\mathfrak{g}(-1)\,.$$

$$\chi(g)=(\det A)^{-1},\ g=\begin{bmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{bmatrix}\in G(0)\,.$$

Case IV.

$$\mathfrak{g}=so(n+2,\boldsymbol{C})=\{X\in M_{n+2}(\boldsymbol{C}):{}^tXs_{n+2}+s_{n+2}X=0,\ trX=0\}\ ,$$

where $s_{n+2}=\begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 1_n & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$.

$$\mathfrak{g}(-1)=\{\begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 2{}^tx & 0 \end{bmatrix}: {}^tx=(x_1,\cdots,x_n)\in\boldsymbol{C}^n\}\ ,$$

$$\mathfrak{g}(0)=\{\begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & -a \end{bmatrix}: a\in\boldsymbol{C},\ A\in M_n(\boldsymbol{C}),\ A+{}^tA=0,\ trA=0\}\ ,$$

$$\mathfrak{g}(1)=\{\begin{bmatrix} 0 & 2{}^tx & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}: {}^tx=(x_1,\cdots,x_n)\in\boldsymbol{C}^n\}\ .$$

We set

$$G = SO(n+2, \boldsymbol{C}) = \{g \in GL(n+2, \boldsymbol{C}) : {}^t g\, s_{n+2}\, g = s_{n+2},\ \det g = 1\}\ .$$

Then

$$G(0) = \{\begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{bmatrix} : a \in \boldsymbol{C}^\times, A \in SO(n, \boldsymbol{C})\} \cong SO(n, \boldsymbol{C}) \times GL(1, \boldsymbol{C})\ .$$

$$f(y) = {}^t xx = x_1^2 + \cdots + x_2^n \quad \text{for} \quad y = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 2{}^t x & 0 \end{bmatrix} \in \mathfrak{g}(-1)\ ,$$

$$\chi(g) = a^{-2} \quad \text{for} \quad g = \begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{bmatrix} \in G(0)\ .$$

Remark 2.1. Besides these four "classical" ones there exists an "exceptional" irrecucible regular prehomogeneous vector space of commutative parabolic type: $\mathfrak{g}$ is the simple Lie algebra of type E_7, $\mathfrak{g}(0)$ is of type E_6 and dim $\mathfrak{g}(\pm 1)$ $=27$.

See [6] for the details.

3. $W(\lambda)$ is a highest weight module

In this section we show that the $U(\mathfrak{g})$-module $W(\lambda)$ defined in Section 1 is a highest weight module with highest weight λ. We retain the notations in the previous sections.

Lemma 3.1. *For $a \in G(0) \cap V$ sufficiently near to the identity, we have*

$$a \cdot v^\lambda = \lambda(\sigma(a))\, v^\lambda\ .$$

Proof. For $n \in N^-$ and $q \in V$, we have

$$\begin{aligned}(a \cdot v^\lambda)(nq) &= v^\lambda(a^{-1} nq) \\ &= f_\chi^{-2\mu}(a^{-1}\, n\, a\, a^{-1}\, q) \\ &= \chi(\sigma(a^{-1}\, q))^\mu f^{-2\mu}(Ad(a^{-1})(\log(n))) \\ &= \chi(\sigma(q))^\mu\, \chi(\sigma(a))^{-\mu}\, \chi(\sigma(a))^{2\mu} f^{-2\mu}(\log(n)) \\ &= \lambda(\sigma(a))\, v^\lambda(nq)\ .\end{aligned}$$

Q.E.D.

Lemma 3.2. *The Lie subalgebra* $\mathfrak{g}(1)$ *annihilates* v^λ.

We first consider cases I—III simultaneously. In these cases the Lie subalgebras $\mathfrak{g}(-1)$, $\mathfrak{g}(0)$ and $\mathfrak{g}(1)$ are given in the following form:

$$\mathfrak{g}(-1)=\{\begin{bmatrix}0&0\\C&0\end{bmatrix}\in\mathfrak{g}\},\ \mathfrak{g}(0)=\{\begin{bmatrix}A&0\\0&D\end{bmatrix}\in\mathfrak{g}\}\quad\text{and}\quad\mathfrak{g}(1)=\{\begin{bmatrix}0&B\\0&0\end{bmatrix}\in\mathfrak{g}\}.$$

The subgroups $G(0)$ and N^- are

$$G(0)=\{\begin{bmatrix}A&0\\0&D\end{bmatrix}\in G\},\ N^-=\{\begin{bmatrix}1&0\\C&1\end{bmatrix}\in G\}.$$

Let ν be any complex number, we define a 1-dimensional character $\lambda=\lambda_\nu$ of $\tilde{P}$ by

$$\lambda(\tilde{g})=(\det A)^\nu\quad\text{for}\quad\pi(\tilde{g})=\begin{bmatrix}A&B\\0&D\end{bmatrix}\in P,\ \tilde{g}\in\pi^{-1}(V)\cap\tilde{P}.$$

Then v^λ is given by

$$v^\lambda(\begin{bmatrix}1&0\\C&1\end{bmatrix}q)=\lambda(\sigma(q))\,(\det(C))^\nu,\ q\in V.$$

(If we define the complex number μ by (1.4), then $\mu=-\nu/2$ (case I,II) or $\mu=-\nu$ (case III).)

Proof of Lemma 3.2 (cases I—III).

For $A=\begin{bmatrix}0&X\\0&0\end{bmatrix}\in\mathfrak{g}(1)$, $n=\begin{bmatrix}1&0\\C&1\end{bmatrix}\in N^-$ and $q\in V$, we have

$$\exp(-sA)\,nq=\begin{bmatrix}1&-sX\\0&1\end{bmatrix}\begin{bmatrix}1&0\\C&1\end{bmatrix}q$$
$$=\begin{bmatrix}1&0\\C(1-sXC)^{-1}&1\end{bmatrix}\begin{bmatrix}1-sXC&-sX\\0&1+sC(1-sXC)^{-1}X\end{bmatrix}q.$$

In the above calculations we assumed that s is small enough so that $(1-sXC)^{-1}$ exists. Now from the definition of v^λ, we have

$$\begin{aligned}v^\lambda(\exp(-sA)\,nq)&=\lambda(\sigma(q))\det\{C(1-sXC)^{-1}\}^\nu\det(1-sXC)^\nu\\&=\lambda(\sigma(q))\det(C)^\nu\\&=v^\lambda(nq)\end{aligned}$$

Differentiating this at $s=0$, we get the assertion of the lemma in cases I–III.

Q.E.D.

We consider the remaining case IV. In this case the subgroups $G(0)$ and N^- are given by

$$G(0)=\{\begin{bmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1}\end{bmatrix} : a\in \boldsymbol{C}^{\times},\ A\in SO(n,\boldsymbol{C})\}\cong SO(n,\boldsymbol{C})\times GL(1,\boldsymbol{C})$$

and

$$N^{-}=\{\begin{bmatrix} 1 & 0 & 0 \\ x & 1_n & 0 \\ {}^t xx & 2{}^t x & 1\end{bmatrix} : {}^t x=(x_1,\cdots,x_n)\in \boldsymbol{C}^n\}\ .$$

Let ν be any complex number, we define a 1-dimensional character $\lambda=\lambda_\nu$ of $\tilde{P}$ by

$$\lambda(\tilde{g})=a^{\nu}\quad \text{for}\quad \pi(\tilde{g})=g=\begin{bmatrix} a & * & * \\ 0 & A & * \\ 0 & 0 & a^{-1}\end{bmatrix}\in G(0),\ \tilde{g}\in\pi^{-1}(V)\cap\tilde{P}.$$

Then v^{λ} is given by

$$v^{\lambda}(\begin{bmatrix} 1 & 0 & 0 \\ x & 1_n & 0 \\ {}^t xx & 2{}^t x & 1\end{bmatrix} q)=\lambda(\sigma(q))\,(x_1^2+\cdots+x_n^2)^{\nu},\ q\in V\ .$$

(If we define the complex number μ by (1.4), then $\mu=-\nu/2$).

Proof of Lemma 3.2 (case IV).

$$\text{For } A=\begin{bmatrix} 0 & 2{}^t z & 0 \\ 0 & 0 & z \\ 0 & 0 & 0\end{bmatrix}\in \mathfrak{g}(1),\ n=\begin{bmatrix} 1 & 0 & 0 \\ x & 1_n & 0 \\ xx^t & 2{}^t x & 1\end{bmatrix}\in N^{-}\quad\text{and}\quad q\in V,$$

we have

$\exp(-sA)\,nq$

$$=\begin{bmatrix} 1 & 0 & 0 \\ \dfrac{(x-s({}^t xx)\,z)}{(1-2s{}^t zx+s^2({}^t zz)\,({}^t xx))} & 1_n & 0 \\ \dfrac{{}^t xx}{(1-2s{}^t zx+s^2({}^t zz)\,({}^t xx))} & * & 1\end{bmatrix}\begin{bmatrix} 1-2s{}^t zx+s^2({}^t zz)\,({}^t xx) & -2s{}^t z & s^2({}^t zz) \\ 0 & * & -sz \\ 0 & 0 & *\end{bmatrix} q\,.$$

In the above calculations we assumed that s is small enough so that $(1-2s{}^t zx+s^2({}^t zz)\,({}^t xx))^{-1}$ exists. From the definition of v^{λ}, we have

$$\begin{aligned} v^{\lambda}(\exp(-sA)nq) &= \lambda(\sigma(q))\left\{\frac{t(x-s({}^t xx)\,z)\,(x-s({}^t xx)\,z)}{(1-2s{}^t zx+s^2({}^t zz)\,({}^t xx))^2}\right\}^{\nu}(1-2s{}^t zx+s^2({}^t zz)\,({}^t xx)) \\ &= \lambda(\sigma(q))\,({}^t xx)^{\nu} \\ &= v^{\lambda}(nq)\,, \end{aligned}$$

from which the lemma follows. **Q.E.D.**

From Lemma 3.1 and Lemma 3.2 we have:

Proposition 3.4. *The $U(\mathfrak{g})$-module $W(\lambda)=\varphi(U(\mathfrak{g})).\, v^{\lambda}$ is a highest weight module with highest weight λ.*

4. The irreducibility of $W(\lambda)$ (Case I)

In this section we prove Theorem 1.1 in case I. We retain the notations in the previous sections. First we analyze the representation space $\varphi(U(\mathfrak{g})).\, v^{\lambda} = W(\lambda)$. From the definition (1.3), we can identify $H(\lambda)$ with the space $H(N^-)$ of holomorphic functions on N^-. The latter space can be idnetified with the space $H(\mathfrak{g}(-1))$ of holomorphic functions on $\mathfrak{g}(-1)$ via the exponential map. Let $(e_{jk})_{j,k=1,\cdots,n}$ be the $n\times n$ matrix units, and $(x_{jk})_{j,k=1,\cdots,n}$ the standard coordinate system on

$$\mathfrak{g}(-1)=\left\{\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}: C\in M_n(\boldsymbol{C})\right\}.$$

Then the set of matrices

$$E_{jk}=\begin{bmatrix} 0 & 0 \\ e_{jk} & 0 \end{bmatrix}, \quad j,k=1,2,\cdots,n \tag{4.1}$$

gives a basis of $\mathfrak{g}(-1)$. It is easy to see that E_{jk} acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as $-\dfrac{\partial}{\partial x_{jk}}$. By the Poincaré-Birkhoff-Witt theorem, we have

$$\begin{aligned}\varphi(U(\mathfrak{g}))\cdot v^{\lambda} &= \varphi(U(\mathfrak{g}(-1))\, U(\mathfrak{g}(0))\, U(\mathfrak{g}(1)))\cdot v^{\lambda} = \varphi(U(\mathfrak{g}(-1)))\cdot v^{\lambda} \\ &= \{Dv^{\lambda}\colon D\in\mathcal{D}_{\mathrm{const}}(\mathfrak{g}(-1))\}\,,\end{aligned}$$

where $\mathcal{D}_{\mathrm{const}}(\mathfrak{g}(-1))$ is the set of constant coefficient differential operators on $\mathfrak{g}(-1)$.

For the proof of Theorem 1.1, we introduce simple root vectors of $\mathfrak{g}$ explicitly. Let

$$E_{\alpha_j}=\begin{array}{r}\\ \\ j)\\ \\ \\ \\ \\ \end{array}\begin{bmatrix} 0 & 0 & & & & & & 0 \\ & \ddots & \ddots & & & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & & 0 & 0 & & \\ & & & & & 0 & \ddots & \\ & & & & & & \ddots & \ddots \\ 0 & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix},\quad 1\leq j\leq 2n-1\,,$$

and let $E_{\beta}=E_{\alpha_n}$.

Since $W(\lambda)$ is a highest weight module by Proposition 3.4, the following theorem yields its irreducibility.

Theorem 4.1. *Suppose $w \in W(\lambda)$ is annihilated by every $\varphi(E_{\alpha_j})$, $j=1, \cdots, 2n-1$. Then w is a scalar multiple of the highest weight vector v^λ.*

We need some lemmas for the proof of this Theorem. We can assume w is a weight vector. In particular, w may be considered as a weight vector with respect to the center of $\mathfrak{g}(0)$.

Let

$$z = \frac{1}{2}\begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix},$$

which is an element of the 1-dimensional center of $\mathfrak{g}(0)$. It is easy to check that $d\chi(z) = -n$.

Lemma 4.2. *Let μ be the complex number defined by* (1.4). *Then the element z acts on $H(\lambda) = H(\mathfrak{g}(-1))$*

$$\varphi(z) = \sum_{j,k=1}^{n} x_{jk} \frac{\partial}{\partial x_{jk}} + \mu n . \tag{4.2}$$

In other words, $\varphi(z)$ is essentially the Euler's differential operator.

Proof. For $q \in V$ and

$$m = \begin{bmatrix} 1_n & 0 \\ C & 1_n \end{bmatrix} \in N^-, \ C = (x_{jk}) ,$$

we have

$$\exp(-sz)\, mq = \begin{bmatrix} 1_n & 0 \\ e^s C & 1_n \end{bmatrix} \begin{bmatrix} e^{-s/2}\, 1_n & 0 \\ 0 & e^{s/2}\, 1_n \end{bmatrix} q .$$

Hence, for $h \in H(\lambda) = H(\mathfrak{g}(-1))$, we have

$$\begin{aligned} \{\exp(sz)\, h\}(mq) &= h(\exp(-sz)\, mq) \\ &= \lambda(\exp(-sz))\, h(e^s C) \\ &= \chi(\exp(-sz))^{\mu} h(e^s C) . \end{aligned}$$

Differentiating this at $s=0$, we get the lemma. Q.E.D.

As a function on $\mathfrak{g}(-1)$, v^λ has homogeneous degree $-2n\mu$. By Lemma 4.2 if the vector w in Theorem 4.1 is a weight vector with respect to $\varphi(z)$, then it must be a homogeneous function on $\mathfrak{g}(-1)$. Since w is a linear combination of various partial derivatives of v^λ, we can assume that it is a linear combination of the i-th derivatives of v^{λ} some fixed nonnegative integer i. Hence we can assume that

$$(4.3)\qquad w(x)=a(C)\,(\det C)^{-2\mu-i} \quad \text{for} \quad x=\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\in\mathfrak{g}(1) \quad (C=(x_{jk}))$$

where $a(C)$ is a homogeneous polynomial in x_{jk} whose degree is $(n-1)\,i$.

Let U be the maximal unipotent subgroup of $G(0)$ generated by $\{\exp(tE_{\alpha_j})\}_{j\neq n}$. Since w is annihilated by $\{\varphi(E_{\alpha_j})\}_{j\neq n}$, w is an $Ad(U)$-invariant function on $\mathfrak{g}(-1)$. Since $(\det C)^{-2\mu-i}$ is $Ad(U)$-invariant, $a(C)$ is also an $Ad(U)$-invariant polynomial. The following proposition on the ring of $Ad(U)$-invariant polynomials on $\mathfrak{g}(-1)$ is due to Johnson [3]. (See also Muller, Rubenthaler and Schiffmann [6].)

Proposition 4.3. *Let $\boldsymbol{C}[\mathfrak{g}(-1)]^U$ be the ring of $Ad(U)$-invariant polynomials on $\mathfrak{g}(-1)$. Then it is isomorphic to the polynomial ring with n indeterminates:*

$$\boldsymbol{C}[\mathfrak{g}(-1)]^U=\boldsymbol{C}[I_1,\cdots,I_n]\,.$$

Here I_j is given by the following formulas:

$$(4.4)\qquad I_j=\det\begin{bmatrix} x_{n-j+1\,1} & \cdots\cdots & x_{n-j+1\,j} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ x_{n\,1} & \cdots\cdots & x_{n\,j} \end{bmatrix},\quad j=1,2,\cdots,n\,.$$

We also need:

Lemma 4.4. *The simple root vector E_β acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as*

$$(4.5)\qquad \varphi(E_\beta)=\sum_{j,k=1}^{n} x_{jn}\,x_{1k}\frac{\partial}{\partial x_{jk}}+2\mu x_{1n}\,,$$

where $2\mu x_{1n}$ means a multiplication operator.

Proof. Using E_β instead of A in the calculations of the proof of Lemma 3.2, for $h\in H(\mathfrak{g}(-1))$, we have

$$\{\exp(sE_\beta)\,h\}(x_{jk})=(1\text{-}sx_{1n})^{-2\mu}\,h(x_{jk}+\frac{sx_{jn}\,x_{1k}}{1-tx_{1n}})\,.$$

Differentiating this at $s=0$, we get the lemma. Q.E.D.

Lemma 4.5. *If we write $\varphi(E_\beta)=D_\beta+2\mu x_{1n}$, then D_β acts on $det(x_{jk})$*

$$(4.6)\qquad D_\beta(\det(x_{jk}))=x_{1n}\det(x_{jk})\,.$$

Proof. By Lemma 3.2 we have

$$D_\beta(\det(x_{jk}))^{-2\mu} = -2\mu x_{1n} \det(x_{jk})^{-2\mu}.$$

Since D_β is a first-order differential operator, the left hand side of the above equation is

$$-2\mu \det(x_{jk})^{-2\mu-1} D_\beta(\det(x_{jk})),$$

from which the lemma follows. Q.E.D.

Proof of Theorem 4.1.

If $d\lambda=0$, then $v^\lambda=v^0$ corresponds to the function whose value is identically 1 on $\mathfrak{g}(-1)$. Hence $W(\lambda)=\varphi(U(\mathfrak{g}))\cdot v^\lambda$ is the 1-dimensional trivial $\mathfrak{g}$-module and Theorem is obvious in this case.

Suppose $d\lambda\neq 0$ and assume $w(x)$ is annihilated by all the $\varphi(E_{\alpha_j})$, $j=1, \cdots, 2n-1$. Recall (4.3) that we can assume

$$w = w(x) = a(C)(\det C)^{-2\mu-i}.$$

Here $a(C)$ is a homogeneous polynomial in $\{x_{jk}\}$ with homogeneous degree $(n-1)i$. If $i=0$, we have nothing to prove. Hence we assume $i>0$. By Proposition 4.3 we conclude $a(C)$ is an element of $\boldsymbol{C}[I_1, \cdots, I_n]$. By Lemma 4.5 we have

$$\begin{aligned}
\varphi(E_\beta)w(x) &= (D_\beta+2\mu x_{1n})(a(C)(\det C)^{-2\mu-i} \\
&= (D_\beta a)(C)(\det C)^{-2\mu-i}-(2\mu+i)a(C)(\det C)^{-2\mu-i-1}D_\beta(\det C) \\
&\quad +2\mu x_{1n}a(C)(\det C)^{-2\mu-i} \\
&= (D_\beta a)(C)(\det C)^{-2\mu-i}-(2\mu+i)x_{1n}a(C)(\det C)^{-2\mu-i}+2\mu x_{1n}a(C)(\det C)^{-2\mu-i} \\
&= \{(D_\beta a)(C)-ix_{1n}a(C)\}(\det C)^{-2\mu-i}.
\end{aligned}$$

Hence $\varphi(E_\beta)w(x)=0$ implies $(D_\beta a)(C)-ix_{1n}a(C)=0$. Since $a(C)$ has homogeneous degree $ni-i$, we write $a(C)$ in the following form:

$$a(C) = \sum_{0\leq m<i} a_m(I_1, \cdots, I_{n-1})I_n^m,$$

where $a_m(I_1, \cdots, I_{n-1})\in\boldsymbol{C}[I_1, \cdots, I_{n-1}]$. Then by Lemma 4.5

$$(D_\beta-ix_{1n})a(C) = \sum_{0\leq m<i}\{D_\beta a_m+(m-i)x_{1n}a_m\}I_n^m.$$

Hence we have

$$(D_\beta a)(I_1, \cdots, I_{n-1})+(m-i)x_{1n}a_m(I_1, \cdots, I_{n-1}) = 0 \quad (0\leq m<i).$$

We consider the coefficients of x_{1n} in the above equation. We can write D_β as follows:

$$D_\beta = x_{1n}^2 \frac{\partial}{\partial x_{1n}} + \sum_{k=1}^{n-1} x_{1n} x_{1k} \frac{\partial}{\partial x_{1k}} + \sum_{j=2}^{n} x_{jn} x_{1n} \frac{\partial}{\partial x_{jn}} + \sum_{\substack{j\neq 1 \\ k\neq 1}} x_{jn} x_{1k} \frac{\partial}{\partial x_{jk}} .$$

From the definition of $I_1, \cdots, I_{n-1}$ (4.4), $a_m(I_1, \cdots, I_{n-1})$ contains no $x_{11}, \cdots, x_{1n}, x_{2n}, \cdots, x_{nn}$. Hence the above description of D_β shows that the coefficient of x_{1n} in $(D_\beta a)(I_1, \cdots, I_{n-1})$ is equal to zero. This implies $a_m(I_1, \cdots, I_{n-1})$ is equal to zero for any m. Hence $w(x)$ must be zero. This completes the proof of Theorem 4.1. Q.E.D.

5. The irreducibility of $W(\lambda)$ (Case II)

In this section we set $\mathfrak{g}=sp(2n, \boldsymbol{C})$ and prove the irreducibility of $W(\lambda)$ in case II. We use the notations in Section 2. In particular the Lie subalgebra $\mathfrak{g}(-1)$ is given by

$$\mathfrak{g}(-1) = \left\{ \begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} : C \in M_n(\boldsymbol{C}), {}^tC = C \right\} .$$

As in the previous section we identify $H(\lambda)$ with $H(\mathfrak{g}(-1))$. Let $\{e_{jk}\}_{j\,k=1,\cdots,n}$ be the matrix units of $n\times n$ matrices and (x_{jk}) $1\leq j\leq k\leq n$ the standard coordinate system on $\mathfrak{g}(-1)$. Then the set of matrices

$$E_{jk} = \begin{bmatrix} 0 & 0 \\ e_{jk}+e_{kj} & 0 \end{bmatrix}, \quad 1\leq j\leq k\leq n$$

gives a basis of $\mathfrak{g}(-1)$. It is easy to see that E_{jk} acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as $-\frac{\partial}{\partial x_{jk}}$ for $j<k$, and as $-2\frac{\partial}{\partial x_{jj}}$ for $j=k$. Hence by the Poincaré-Birkhoff-Witt theorem, we have

$$\varphi(U(\mathfrak{g}))\, v^\lambda = \{Dv^\lambda : D \in \mathcal{D}_{\mathrm{const}}(g(-1))\} .$$

Let

$$E_{\alpha_j} = \begin{bmatrix} e_j & 0 \\ 0 & -{}^te_j \end{bmatrix}, 1\leq j\leq n-1, \quad \text{where} \quad e_j = e_{j\,j+1} .$$

Let

$$E_\beta = \begin{bmatrix} 0 & e_{nn} \\ 0 & 0 \end{bmatrix}.$$

We prove the following Theorem which yields the irreducibility of $W(\lambda)$.

Theorem 5.1. *Suppose $w\in W(\lambda)$ is annihilated by every $\varphi(E_{\alpha_j})$, $j=1, \cdots, n-1$ and $\varphi(E_\beta)$. Then w is a scalar multiple of the highest weight vector v^λ.*

We can assume that w is a weight vector and consider the weight of the central element

$$z = \frac{1}{2}\begin{bmatrix} 1_n & 0 \\ 0 & -1_n \end{bmatrix},$$

of $\mathfrak{g}(0)$. The following lemma can be proved by a calculation similar to that in the proof of Lemma 4.2.

Lemma 5.2. *Let μ be the complex number defined by* (1.4). *Then the element z acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as the following differential operator*:

$$\varphi(z) = \sum_{j\le k} x_{jk}\frac{\partial}{\partial x_{jk}} + \mu n\,. \tag{5.1}$$

As a function on $\mathfrak{g}(-1)$, v^λ has homogeneous degree $-2n\mu$. Hence if the vector w in Theorem 5.1 is a weight vector with respect to $\varphi(z)$, then it must be a homogeneous function on $\mathfrak{g}(-1)$. Hence w must be a linear combination of the i-th derivatives of v^λ for some fixed nonnegative integer i. Hence we can assume that

$$w(x) = a(C)(\det C)^{-2\mu-i} \quad \text{for} \quad x = \begin{bmatrix} 1_n & 0 \\ C & 1_n \end{bmatrix},\ {}^tC = C,\ C = (x_{jk}) \tag{5.2}$$

where $a(C)$ is a homogeneous polynomial in $\{x_{jk}\}_{j\le k}$ whose degree is $(n-1)\,i$.

Let U be the maximal unipotent subgroup of $G(0)$ determined by the simple root vectors $\{E_{\alpha_j}\}_{j=1,\cdots,n-1}$. Since w is annihilated by $\{\varphi(E_{\alpha_j})\}_{j=1,\cdots,n-1}$, then w is an $Ad(U)$-invariant function on $\mathfrak{g}(0)$. Since $(\det C)^{-2\mu-i}$ is $Ad(U)$-invariant, we conclude that $a(C)$ is also an $Ad(U)$-invariant polynomial. Here we need Johnson's result [3].

Proposition 5.3. *The ring of $Ad(U)$-invariant polynomials $\boldsymbol{C}[\mathfrak{g}(-1)]^U$ is isomorphic to the polynomial ring with n indeterminates*:

$$\boldsymbol{C}[g(-1)]^U = \boldsymbol{C}[I_1, \cdots, I_n]\,.$$

Here I_j are given by the following formulas:

$$I_j = \det\begin{bmatrix} x_{11}\cdots x_{1j} \\ \cdots\cdots\cdots \\ x_{1j}\cdots x_{jj} \end{bmatrix},\quad j = 1, 2, \cdots, n\,. \tag{5.3}$$

Lemma 5.4. *The simple root vector E_β acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as the differential operator*:

$$\varphi(E_\beta) = \sum_{j\le k} x_{jn}\,x_{kn}\frac{\partial}{\partial x_{jk}} + 2\mu x_{nn}\,. \tag{5.4}$$

Lemma 5.5. *If we write $E_\beta = D_\beta + 2\mu x_{nn}$, then*

$$D_\beta(\det C) = x_{nn}(\det C) \quad \text{for} \quad C = {}^tC,\ C = (x_{jk})\,. \tag{5.5}$$

We omit the proof.

Proof of the Theorem 5.1.

If $d\lambda=0$, then $W(\lambda)$ is the 1-dimensional trivial $\mathfrak{g}$-module and Theorem is obvious.

Suppose $d\lambda \neq 0$ and

$$w = w(x) = a(c)\,(\det C)^{-2\mu-i} \in W(\lambda),\ C = {}^tC = (x_{jk})$$

is annihilated by $\varphi(E_{\alpha_j})$ $j=1, \cdots, n-1$ and $\varphi(E_\beta)$. Here $a(C)$ is a homogeneous polynomial in $\{x_{jk}\}_{j\leq k}$ with homogeneous degree $(n-1)\,i$. We can assume $i>0$. By Proposition 5.3 we conclude $a(C)$ is an element of $\boldsymbol{C}[I_1, \cdots, I_n]$. We consider the action of the simple root vector E_β. By Lemma 5.6 we have

$$\begin{aligned}
\varphi(E_\beta)\,w(x) &= (D_\beta+2\mu x_{nn})\,a(C)\,(\det C)^{-2\mu-i} \\
&= (D_\beta\,a)\,(C)\,(\det C)^{-2\mu-i}-(2\mu+1)\,a(C)\,(\det C)^{-2\mu-i-1}\,D_\beta(\det C) \\
&\quad +2\mu x_{nn} a(C)\,(\det C)^{-2\mu-i} \\
&= (D_\beta\,a)(C)(\det C)^{-2\mu-i}-(2\mu+i)\,x_{nn}\,a(C)(\det C)^{-2\mu-i}+2\mu x_{nn}\,a(C)(\det C)^{-2\mu-i} \\
&= \{(D_\beta\,a)\,(C)-ix_{nn}\,a(C)\}\,(\det C)^{-2\mu-i}\,.
\end{aligned}$$

Hence $\varphi(E_\beta)\,w(x)=0$ imples $(D_\beta\,a)\,(C)-ix_{nn}\,a(C)=0$. Since $a(C)$ has homogeneous degree $ni-i$, we write $a(C)$ in the following form:

$$a(C) = \sum_{0\leq m<i} a_m(I_1, \cdots, I_{n-1})\,I_n^m\,,$$

where $a_m(I_1, \cdots, I_{n-1}) \in \boldsymbol{C}[I_1, \cdots, I_n]$. Then by Lemma 5.5

$$(D_\beta - ix_{nn})\,a(C) = \sum_{0\leq m<i} \{D_\beta\,a_m+(m-i)\,x_{nn}\,a_m\}\,I_n^m\,.$$

Hence we have

$$(D_\beta\,a_m)\,(I_1, \cdots, I_{n-1})+(m-i)\,x_{nn}\,a_m(I_1, \cdots, I_{n-1}) = 0 \quad 0\leq m<i\,.$$

We consider the coefficients of x_{nn} in the above equation. We can write D_β as follows:

$$D_\beta = x_{nn}^2\,\frac{\partial}{\partial x_{nn}} + \sum_{j=1}^{n-1} x_{jn}\,x_{nn}\,\frac{\partial}{\partial x_{jn}} + \sum_{1\leq j\leq k<n} x_{jn}\,x_{kn}\,\frac{\partial}{\partial x_{jk}}\,.$$

From the definition (5.3) of $I_1, \cdots, I_{n-1}$, $a_m(I_1, \cdots, I_{n-1})$ contains no $x_{1n}, x_{2n}, \cdots, x_{nn}$. Hence the above description of D_β shows that the coefficient of x_{nn} in $(D_\beta\,a)\,(I_1, \cdots, I_{n-1})$ is equal to zero. This implies $a_m(I_1, \cdots, I_{n-1})$ is equal to zero for any m. Hence $w(x)$ is zero. This completes the proof of Theorem 5.1.

Q.E.D.

6. The irredicibility of $W(\lambda)$ (Case III)

In this section we set $\mathfrak{g}=so(4n, \boldsymbol{C})$ and prove the irreducibility of $W(\lambda)$ in case III. We use the notations in Section 2. In particular the Lie subalgebra $\mathfrak{g}(-1)$ is given by

$$\mathfrak{g}(-1)=\{\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix} | C\in M_{2n}(\boldsymbol{C}),\ {}^tC=-C\}\ .$$

We identify $H(\lambda)$ with $H(\mathfrak{g}(-1))$ as in the previous sections. Let $\{e_{jk}\}_{j,k=1,\cdots,2n}$ be the matrix units of $2n\times 2n$ matrices, and (x_{jk}) $1\leq j<k\leq 2n$ the standard coordinate system on $\mathfrak{g}(-1)$. Then the set of matrices

$$E_{jk}=\begin{bmatrix} 0 & 0 \\ e_{jk}-e_{kj} & 0 \end{bmatrix},\ 1\leq j<k\leq 2n$$

gives a basis of $\mathfrak{g}(-1)$. It is easy to see that E_{jk} acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as $-\dfrac{\partial}{\partial x_{jk}}$. Hence by the Poincaré-Birkhoff-Witt theorem, we have

$$\varphi(U(\mathfrak{g}))\,v^{\lambda}=\{Dv^{\lambda}\colon D\in\mathcal{D}_{\mathrm{const}}(\mathfrak{g}(-1))\}\ .$$

Let

$$E_{\alpha_j}=\begin{bmatrix} e_j & 0 \\ 0 & -{}^te_j \end{bmatrix},\ 1\leq j\leq 2n-1 \quad \text{where} \quad e_i=e_{ii+1}\,.$$

Let

$$E_{\beta}=\begin{bmatrix} 0 & e_{2n\,2n-1}-e_{2n-1\,2n} \\ 0 & 0 \end{bmatrix}.$$

We prove the following Theorem which yields the irreducibility of $W(\lambda)$.

Theorem 6.1. *Suppose $w\in W(\lambda)$ is annihilated by every $\varphi(E_{\alpha_j})$, $j=1,\cdots, 2n-1$ and $\varphi(E_{\beta})$, then w is a scalar multiple of the highest weight vecotr v^{λ}.*

We can assume that w is a weight vector and consider the weight of the central element

$$z=\frac{1}{2}\begin{bmatrix} 1_{2n} & 0 \\ 0 & -1_{2n} \end{bmatrix},$$

of $\mathfrak{g}(0)$.

Lemma 6.2. *Let μ be the complex number defined by* (1.4). *Then the element z acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as the following differential operator*:

$$(6.1)\qquad \varphi(z)=\sum_{1\leq j<k\leq 2n} x_{jk}\frac{\partial}{\partial x_{jk}}+\mu n\,.$$

As a function on $\mathfrak{g}(-1)$, v^{λ} has homogeneous degree $-2n\mu$. Hence if the vector w in Theorem 6.1 is a weihgt vector with respect to the $\varphi(z)$, then it must be a homogeneous function on $\mathfrak{g}(-1)$. Hence w is a linear combination of the i-th derivatives of v^{λ} for some fixed nonnegative integer i. Hence we can assume that

$$(6.2)\qquad w(x)=a(C)\,(\det C)^{-\mu-i},\ x=\begin{bmatrix}1_{2n} & 0\\ C & 1_{2n}\end{bmatrix},\ {}^tC=-C,\ C=(x_{jk})$$

where $a(C)$ is a homogeneous polynomial in $\{x_{jk}\}_{j<k}$ whose degree is $(2n-1)\,i$.

Let U be the maximal unipotent subgroup of $G(0)$ determined by $\{E_{\alpha_j}\}_{j=1,\cdots,2n-1}$. Since w is annihilated by $\{\varphi(E_{\alpha_j})\}_{j=1,\cdots,2n-1}$, w is an $Ad(U)$-invariant function on $g(-1)$. Since $(\det C)^{-\mu-i}$ is $Ad(U)$-invariant, we conclude that $a(C)$ is an $Ad(U)$-invariant polynomial. Here we need Johnson's result [3].

Proposition 6.3. *The ring of $Ad(U)$-invariant polynomials $\boldsymbol{C}[g(-1)]^U$ is isomorphic to the polynomial ring with n indeterminates*:

$$\boldsymbol{C}\mathfrak{g}(-1)^U=\boldsymbol{C}[I_1,\cdots,I_n]\,.$$

Here I_j are given by the following formulas:

$$(6.3)\qquad I_j=\det\begin{bmatrix}0 & -x_{12} & \cdots & -x_{12j}\\ x_{12} & 0 & \cdots & -x_{22j}\\ \cdots & \cdots & \cdots & \cdots\\ x_{12j} & \cdots & \cdots & 0\end{bmatrix},\ j=1,2,\cdots,n\,.$$

Lemma 6.4. *The simple root vector E_β acts on $H(\lambda)=H\mathfrak{g}(-(-1))$ as*:

$$\varphi(E_\beta)=\sum_{1\le j<k\le 2n-2}(x_{j\,2n-1}\,x_{k\,2n}-x_{j\,2n}\,x_{k\,2n-1})\frac{\partial}{\partial x_{jk}}$$
$$+\sum_{j=1}^{2n-2}x_{j\,2n-1}\,x_{2n-1\,2n}\frac{\partial}{\partial x_{j\,2n-1}}+\sum_{j=1}^{2n-1}x_{j\,2n}\,x_{2n-1\,2n}\frac{\partial}{\partial x_{j\,2n}}+2\mu x_{1n-1\,2n}\,.$$

Lemma 6.5. *If we write $\varphi(E_\beta)=D_\beta+2\mu x_{2n-1\,2n}$, then $D_\beta(\det C)=2x_{2n-1\,2n}\,(\det C)$ for $C\in M_{2n}(C)$, ${}^tC=-C$.*

Proof of Theorem 6.1.

If $d\lambda=0$, then $W(\lambda)$ is the 1-dimensional trivial $\mathfrak{g}$-module and Theorem is obvious.

Suppose $d\lambda\neq 0$ and

$$w=w(x)=a(C)\,(\det C)^{-\mu-i}\in W(\lambda)\,,\quad C=-{}^tC=(x_{jk})$$

is annihilated by $\varphi(E_{\alpha_j})$ $j=1,\cdots,n-1$ and $\varphi(E_\beta)$. Here $a(C)$ is a homogeneous polynomial in $\{x_{jk}\}_{j<k}$ with homogeneous degree $(2n-1)\,i$. We can assume $i>0$. By Proposition 6.3, $a(C)$ is an element of $\boldsymbol{C}[I_1,\cdots,I_n]$. By Lemma 5.6, we have

$$\begin{aligned}\varphi(E_\beta)\,w(x) &= (D_\beta+2\mu x_{2n-1\,2n})\,a(C)\,(\det C)^{-\mu-i}\\ &= (D_\beta a)\,(C)\,(\det C)^{-\mu-i}-(\mu+i)\,a(C)\,(\det C)^{-\mu-i}\,D_\beta(\det C)\\ &\quad +2\mu x_{2n-1\,2n}\,a(C)\,(\det C)^{-\mu-i}\\ &= (D_\beta a)\,(C)\,(\det C)^{-\mu-i}-2(\mu+i)\,x_{2n-1\,2n}\,a(C)\,(\det C)^{-\mu-1}\\ &\quad +2\mu x_{2n-1\,2n}\,a(C)\,(\det C)^{-\mu-i}\\ &= \{(D_\beta a)\,(C)-2ix_{2n-1\,2n}\,a(C)\}\,(\det C)^{-\mu-i}\end{aligned}$$

Hence $\varphi(E_\beta)\,w(x)=0$ implies $(D_\beta\, a)\,(C)-2ix_{2n-1\,2n}\,a(C)=0$. Since $a(C)$ has homogeneous degree $2ni-i$, we write $a(C)$ in the following form:

$$a(C)=\sum_{0\le m<i} a_m(I_1, \cdots, I_{n-1})\,I_n^m\,,$$

where $a_m(I_1, \cdots, I_{n-1})\in \boldsymbol{C}[I_1, \cdots, I_n]$. Then by Lemma 6.5,

$$(D_\beta-2ix_{2n-1\,2n})\,a(C)=\sum_{0\le m<1}\{D_\beta\, a_m+2(m-i)\,x_{2n-1 2n}\,a_m\}\,I_n^m\,.$$

Hence we have

$$(D_\beta\, a_m)\,(I_1, \cdots, I_{n-1})+2(m-i)\,x_{2n-1\,2n}\,a_m(I_1, \cdots, I_{n-1})=0\,,\quad 0\le m<i\,.$$

We consider the coefficients of $x_{2n-1\,2n}$ in the above equation. Since

$$\begin{aligned}D_\beta &= \sum_{1\le j<k\le 2n-2}(x_{j\,2n-1}\,x_{k\,2n}-x_{j\,2n}\,x_{k\,2n-1})\frac{\partial}{\partial x_{jk}}\\ &\quad+\sum_{j=1}^{2n-2}x_{j\,2n-1}\,x_{2n-1\,2n}\frac{\partial}{\partial x_{j\,2n-1}}+\sum_{j=1}^{2n-1}x_{j\,2n}\,x_{2n-1\,2n}\frac{\partial}{\partial x_{j\,2n}}\end{aligned}$$

and from the definition of $I_1, \cdots, I_{n-1}$ (6.3), a_m contains no $x_{1\,2n-1}, \cdots, x_{2n-2\,2n-1}$, $x_{1\,2n}, \cdots, x_{2n-1\,2n}$. Hence the above description of D_β shows that the coefficient of $x_{2n-1\,2n}$ in $(D_\beta\, a)\,(I_1, \cdots, I_{n-1})$ is equal to zero. This implies $a_m(I_1, \cdots, I_{n-1})$ is equal to zero for any m. Hence $w(x)$ is in fact zero. This completes the proof of Theorem 6.1. Q.E.D.

7. Proof of the irredicibility of $W(\lambda)$ (Case IV)

In this section we set $\mathfrak{g}=so(n+2, \boldsymbol{C})$ and prove the irreducibility of $W(\lambda)$ in case IV. We use the notations in Section 2. In particular the Lie subalgebra $\mathfrak{g}(-1)$ is given by

$$\mathfrak{g}(-1)=\left\{\begin{bmatrix}0&0&0\\ x&0&0\\ 0&2^s x&0\end{bmatrix}\Big|\, x\in \boldsymbol{C}^n\right\}.$$

We identify $H(\lambda)$ with $H(\mathfrak{g}(-1))$. Let (x_j) $1\leq j\leqq n$ be the standard coordinate system on $\mathfrak{g}(-1)\simeq \boldsymbol{C}^n$ and $\{E_j\}_{j=1,\cdots,n}$ the standard basis of $\boldsymbol{C}^n$. It is easy to see that E_j acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as $-\dfrac{\partial}{\partial x_j}$. By the Poincaré-Birkhoff-Witt theorem, we have

$$\varphi(U(\mathfrak{g}))\, v^{\lambda} = \{Dv^{\lambda} \colon D \in \mathcal{D}_{\text{const}}(g(-1))\}\ .$$

We introduce the following notations of matrices:

$$\mathbf{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ E = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix},\ E' = \begin{bmatrix} 1 & -1 \\ i & i \end{bmatrix} \quad \text{and} \quad e = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

We set $l=[n/2]$ and, for $j=1, \cdots, l-1$, wet

$$E_{\alpha_j} = \begin{bmatrix} 0 & & & & & & 0 \\ & \mathbf{O} & & & & & \\ & & \ddots & & & & \\ & & & \mathbf{O} & E & & \\ & & & -{}^tE & \mathbf{O} & & \\ & & & & & \ddots & \\ & & & & & & \mathbf{O} \\ & 0 & & & & & & 0 \end{bmatrix}.$$

If n is odd, then we set

$$E_{\alpha_l} = \begin{bmatrix} 0 & & & & 0 \\ & \mathbf{O} & & & \\ & & \ddots & & \\ & & & \mathbf{O} & e \\ & & & -{}^te & 0 \\ & 0 & & & & 0 \end{bmatrix}.$$

If n is even, then we set

$$E_{\alpha_l} = \begin{bmatrix} 0 & & & & 0 \\ & \mathbf{O} & & & \\ & & \ddots & & \\ & & & \mathbf{O} & E' \\ & & & -{}^tE' & \mathbf{O} \\ & 0 & & & & 0 \end{bmatrix}.$$

Let

$$E_{\beta} = \begin{bmatrix} 0 & {}^te & 0 \\ & \mathbf{O} & \\ 0 & & \ddots & 0 \end{bmatrix}.$$

We prove the following Theorem.

Theorem 7.1. *Suppose $w \in W(\lambda)$ is annihilated by every $\varphi(E_{\alpha_j}), j=1, \cdots, l$, and $\varphi(E_\beta)$, then w is a scalar miltiple of the highest weight vector v^λ.*

We can assume that w is a weight vector and consider the weight of the central element

$$z = \begin{bmatrix} 1 & & \\ & 0 & \\ & & -1 \end{bmatrix},$$

of $\mathfrak{g}(0)$.

Lemma 7.2. *Let μ be the complex number defined by* (1.4). *Then the element z acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as the following differential operator*:

$$\varphi(z) = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + 2\mu \,. \tag{7.1}$$

As a function on $\mathfrak{g}(-1)$, v^λ has homogeneous degree -4μ. Hence if the veator w in Theorem 7.1 is a weight vector with respect to the $\varphi(z)$, then it is a homogeneous function on $\mathfrak{g}(-1)$. Hence w is a linear combination of the i-th derivatives of v^λ for some nonnegative integer i. Hence we assume that

$$w(y) = a(x)\,({}^t xx)^{-2\mu-i} \quad \text{for} \quad y = \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 2{}^t x & 0 \end{bmatrix}, \; x = (x_j) \tag{7.2}$$

where $a(x)$ is a homogeneous polynomial in $\{x_j\}$ whose homogeneous degree is i.

Let U be the maximal unipotent subgroup of $G(0)$ determined by the simple root veators $\{E_{\alpha_j}\}_{j=1,\cdots,l}$. Since w is annihilated by $\{\varphi(E_{\alpha_j})\}_{j=1,\cdots,l-1}$, w is an $Ad(U)$-invariant function on $\mathfrak{g}(-1)$. Since $({}^t xx)^{-2\mu-i}$ is $Ad(U)$-invariant, we conclude that $a(x)$ is also an $Ad(U)$-invariant polynomial. Here we need Johnson's result [3].

Proposition 7.3. *The ring of $Ad(U)$-invariant polynomials $\boldsymbol{C}[\mathfrak{g}(-1)]^U$ is isomorphic to the polynomial ring with two indeterminates*;

$$\boldsymbol{C}[\mathfrak{g}(-1)]^U = \boldsymbol{C}[I_1, I_2] \,.$$

Here I_1 and I_2 are given by the following formulas:

$$I_1 = x_1 + \sqrt{-1}\, x_1,\; I_2 = {}^t xx \,. \tag{7.3}$$

Lemma 7.4. *The simple root vector E_β acts on $H(\lambda)=H(\mathfrak{g}(-1))$ as the following differential operator*:

$$\varphi(E_\beta) = 2(x_1 - \sqrt{-1}\, x_2)(\varphi(z)+2\mu) - I_2 \left(\frac{\partial}{\partial x_1} - \sqrt{-1}\frac{\partial}{\partial x_2}\right) \tag{7.4}$$

where $\varphi(z)$ *is given by Lemma* 7.2 *and* $I_2={}^t xx$.

If we set $J=x_1-\sqrt{-1}\,x_2$ and $D=\frac{\partial}{\partial x_1}-\sqrt{-1}\,\frac{\partial}{\partial x_2}$, then $DI_2=2J$.

Proof of Theorem 7.1.

If $d\lambda=0$, then $W(\lambda)$ is the 1-dimensional trivial $\mathfrak{g}$-module and Theorem is obvoius.

Suppose $d\lambda\neq 0$ and $w=w(y)=a(x)\,({}^t xx)^{-2\mu-i}$ is annihilated by all $\varphi(E_{\alpha_j})$ $j=1, \cdots, l$ and $\varphi(E_\beta)$. Here $a(x)$ is a homogeneous polynomial in $\{x_j\}$ with homogeneous degree i. We can assume $i>0$. By Proposition 7.3 we conclude $a(x)$ is an element of $\boldsymbol{C}[I_1, I_2]$. By Lemma 7.4

$$\begin{aligned}\varphi(E_\beta)\,w(y) &= \{2J(\varphi(z)+2\mu)-I_2\,D\}\;a(x)\,I_2^{-2\mu-i}\\ &= 2J(-4\mu-i+2\mu)\,a(x)\,I_2^{-2\mu-i}-(Da)\,(x)\,I_2^{-2\mu-i+1}+2(2\mu+i)\,JI_2^{-2\mu-1}\,a(x)\\ &= (Da)\,(x)\,I_2^{\lambda-i+1}\end{aligned}$$

Hence $\varphi(E_\beta)\,w(x)=0$ implies $(Da)\,(x)=0$. Since $a(x)$ has homogeueous degree i, we can write it as the following form:

$$a(x)=\sum_{1\leq k\leq[i/2]} a_k\,I_2^k\,I_1^{i-2k}\,.$$

Then

$$\begin{aligned}(Da)\,(x) &= \sum_{1\leq k\leq[i/2]} a_k\{D(I_2^k)\,I_1^{i-2k}+I_2^k\,D(I_1^{i-2k})\\ &= \sum_{1\leq k\leq[i/2]} 2a_k\,I_2^{k-1}\,I_1^{i-2k-1}(kJI_1+iI_2-2kI_2)\\ &= \sum_{1\leq k\leq[i/2]} 2a_k\,I_2^{k-1}\,I_1^{i-2k-1}\{(i-k)\,(x_1^2+x_2^2)+(i-2k)\,(x_3^2+\cdots+x_n^2)\}\,.\end{aligned}$$

Since $i>k$ in the summation of the above equation, we conclude that $(Da)\,(x)=0$ implies $a(x)=0$. This completes the proof of Theorem 7.1. Q.E.D.

8. Reducibilities of generalized Verma modules

In this section we discuss the reducibilites of Verma Modules induced from the maximal parabolic subalgebra $\mathfrak{p}$ (Corollary 1.2). This gives a representation theoretic interpretation of the zeros of the b-function.

Let $d\lambda$ be a one dimensional representation of $\mathfrak{p}$. Let $\boldsymbol{C}_{d\lambda}$ be the representation space of $d\lambda$. We define generalized Verma module $V(d\lambda)$ by

$$V(d\lambda)=U(\mathfrak{g})\otimes_{U(\mathfrak{p})}\boldsymbol{C}_{d\lambda}\,.$$

Jantzen [2] gave a reducibility criterion for $V(d\lambda)$ using his formulas for determinants of contravariant forms. (Jantzen's result is, in fact, much more general.)

Let μ be the complex number defined by (1.4). By our construction of the irreducible highest weight module $W(\lambda)$, we can recover Jantzen's result in the following form:

Coroltary 8.1. *If -2μ is a positive integer or a zero of the b-function $b(s)$ of f, then $V(d\lambda)$ is reducible.*

Proof. If -2μ is a positive integer, then the highest weight vector $v^{\lambda}=f^{-2\mu}$ of $W(\lambda)$ is a polynomial function on $g(1)$. Hence $W(\lambda)$ becomes finite dimensional and $V(d\lambda)$ is reducible.

Now we consider the case when $b(-2\mu)=0$. In case I the equation (1.5) is given explicitly by the following Capelli's identity (See Weyl [10]):

$$(\det(\partial/\partial x_{jk}))\det(x_{jk})^s = s(s+1)\cdots(s+n-1)\det(x_{jk})^{s-1} \qquad (8.1)$$

$$(\text{i.e.}\quad b(s) = s(s+1)\cdots(s+n-1)).$$

Suppose $V(d\lambda)$ is irreducible. Then $W(\lambda)$ and $V(d\lambda)$ are isomorphic. But then by the Poincaré-Birkhoff-Witt theorem, $W(\lambda)$ is isomorphic to $U(\mathfrak{g}(-1))$ as vector spaces. Then

$$(-1)^n \varphi(E_{1\sigma(1)})\,\varphi(E_{2\sigma(2)})\cdots\varphi(E_{n\sigma(n)})\,v^{\lambda} = \frac{\partial^n}{\partial x_{1\,\sigma(1)}\,\partial x_{2\,\sigma(2)}\cdots\partial x_{n\,\sigma(n)}}\det(x_{jk})^{-2\mu}$$

must be linearly independent, where σ runs over the set S_n of all permutations of $\{1, 2, \cdots, n\}$. (Recall (4.1) that E_{ij}'s are basis elements of $\mathfrak{g}(-1)$.) Since $b(-2\mu)=0$, this contradicts to the Capelli's identity. Hence $V(d\lambda)$ is also reducible in this case.

Case II–IV can be treated completely analogously, the role of (8.1) being played by the following formulas:

Case II

$$\det\begin{bmatrix} \frac{\partial}{\partial x_{11}} & \frac{1}{2}\frac{\partial}{\partial x_{12}} & \cdots & \frac{1}{2}\frac{\partial}{\partial x_{1n}} \\ \frac{1}{2}\frac{\partial}{\partial x_{12}} & \frac{\partial}{\partial x_{22}} & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{1}{2}\frac{\partial}{\partial x_{1n}} & \cdots & \cdots & \frac{\partial}{\partial x_{nn}} \end{bmatrix} f(x_{ij})^s = s(s+1/2)\cdots(s+(n-1)/2)f(x_{ij})^{s-1}$$

where $f(x_{ij})$ is the determinant of the $n\times n$ symmetric matrix (x_{ij}).

Case III

$$f(\partial/\partial x_{ij})f(x_{ij})^s = s(s+2)\cdots(s+2n-2)f(x_{ij})^{s-1},$$

where $f(x_{ij})$ is the Pfaffian of the $2n\times 2n$ antisymmetric matrix (x_{ij}) and given explicitly by the following formula:

$$f(x_{ij}) = \sum_{\substack{\sigma\in S_{2n} \\ \sigma(2i-1)<\sigma(2i) \\ \sigma(2i-1)<\sigma(2i+1)}} \operatorname{sgn}(\sigma)\, x_{\sigma(1)\sigma(2)}\cdots x_{\sigma(2n-1)\sigma(2n)}\,.$$

Case IV

$$\Big(\sum_{i=1}^{n}\Big(\frac{\partial}{\partial x_i}\Big)^2\Big)\Big(\sum_{i=1}^{n} x_i^2\Big)^s = 4s(s+(n-2)/2)\Big(\sum_{i=1}^{n} x_i^2\Big)^{s-1}. \qquad \text{Q.E.D.}$$

9. Final remarks

1. Using the result of Jantzen [2] (see also Enright, Howe and Wallach [1]), one can verify that the statement of Corollary 8.1 is true in the exceptional case (mentioned in Remark 2.1) also.

2. Let (G_0, K_0) be an irreducible Hermitian symmetric pair of tube type. Let $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$) be the Lie algebra of G_0 (resp. K_0), and $\mathfrak{g}_0=\mathfrak{k}_0\oplus\mathfrak{p}_0$ the Cartan decomposition of $\mathfrak{g}_0$. By convention we delete the subscript o to denote complexified Lie algebras. So we have the decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ of the compexified Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{k}_0$ has the 1-dimensional center $Z=\boldsymbol{R}z$, where the eigenvalues of z under the adjoint action on $\mathfrak{p}$ are $\pm i$. Let

$$\mathfrak{p}^+ = \{x\in\mathfrak{p}\,|\,[z, x] = ix\} \quad \text{and} \quad \mathfrak{p}^- = \{x\in\mathfrak{p}\,|\,[z, x] = -ix\}\,.$$

If we set $\mathfrak{g}(-1)=\mathfrak{p}^-$, $\mathfrak{g}(0)=\mathfrak{k}$ and $\mathfrak{g}(1)=\mathfrak{p}^+$, then we have a $\boldsymbol{Z}$-gradation

$$\mathfrak{g} = \mathfrak{g}(-1)+\mathfrak{g}(0)+\mathfrak{g}(-1)\,.$$

Then the pairs $(G(0), \mathfrak{g}(\pm 1))$ are irreducible regular prehomogeneous vector spaces of commutative parabolic type. If λ, a 1-dimensional character of $\mathfrak{g}(0)$, corresponds to a zero of the b-function, then it is known that the $\mathfrak{g}_0$-module $W(\lambda)$ is unitarizable. See [1], [7] and [9].

References

[1] T.J. Enright, R. Howe and N.R. Wallach: *A classification of unitary highest weight modules,* P.C. Trombi ed., Representation Theory of Reductive Groups, 97–143 Progress in Math. 40 Birkhauser Boston 1983.

[2] J.C. Jantzen: *Kontravatiante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren,* Math. Ann. **226** (1977), 53–65.

[3] K.D. Johnson: *On a ring of invariant polynomials on a Hermitian symmetric space,* J. of Algebra **67** (1981), 72–81.

[4] T. Kimura: *The b-functions and holonomy diagrams of irreducible regular preho-*

mogeneous vector spaces, Nagoya Math. J. **85** (1982), 1–80.

[5] A. Knapp: Representation Theory of Semisimple Groups. An Overview Based on Examples, Princeton Univ. Press, Princeton New Jwesey, 1986.

[6] I. Muller, H. Rubenthaler and G. Schiffmann: *Structure des espaces préhomogènes associés a certaines algèbres de Lie graduées,* Math. Ann. **274** (1986), 95–123.

[7] H. Rossi and M. Vergne: *Analytic continuation of the holomorphic discrete series of a semisimple Lie groups,* Acta Math. **136** (1976), 1–59.

[8] M. Sato and T. Kimura: *A classification of irreducible prehomogeneous vector spaces and their relative invariants*. Nagoya Math. J. **65** (1977), 1–155.

[9] N.R. Wallach: *Analytic continuation of the discrete series I, II,* Trans. Amer. Math. Soc. **251** (1979), 1–37.

[10] H. Weyl: The Classical Groups, Princeton Univ. Press, Princeton New Jersey, 1946.

Department of Information and Computer Sciences
Kumamoto National College of Technology
Nishigoshi-machi Kumamoto 861–11 Japan

Current Address
Division of Junior College
Ryukyu University
Nishihara-cho Okinawa 903–01 JAPAN