Cordero, M.
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# SEMIFIELD PLANES OF ORDER $p^4$ THAT ADMIT A $p$-PRIMITIVE BAER COLLINEATION

MINERVA CORDERO

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## 1. Introduction

Let $\pi$ denote a semifield plane of order $q^2$ and kernel $\mathcal{K}\cong GF(q)$ where $q$ is a prime power $p^r$. A $p$-primitive Baer collineation of $\pi$ is a collineation $\sigma$ which fixes a Baer subplane of $\pi$ pointwise and whose order is a $p$-primitive divisor of $q-1$ (i.e. $|\sigma|\,|q-1$ but $|\sigma|\not| p^i-1$ for $1\leq i<r$). A semifield plane of order $p^4$ and kernel $GF(p^2)$, where $p$ is an odd prime, is called a *$p$-primitive semifield plane* if it admits a $p$-primitive Baer collineation.

In [7], Hiramine, Matsumoto and Oyama presented a construction method (which was extended by Johnson in [9]) by which translation planes of order $q^4$ and kernel $\supseteq GF(q^2)$, $q=p^r$, are obtained from arbitrary translation planes of order $q^2$ and kernel $GF(q)$. The class of $p$-primitive semifield planes is precisely the class of planes obtained when this method is applied to the Desarguesian plane of order $p^2$ (see Johnson [10], Theorem 2.1).

In this article we study some properties of $p$-primitive semifield planes and determine necessary and sufficient conditions for isomorphism within this class. The main result is on the number of nonisomorphic $p$-primitive semifield planes

**Theorem 4.2.** *For any odd prime $p$, there are $\left(\frac{p+1}{2}\right)^2$ nonisomorphic $p$-primitive semifield planes of order $p^4$.*

We show that of these, $\frac{p+1}{2}$ are Hughes-Kleinfeld semifield planes and one is a Dickson semifield plane. Also, the Boerner-Lantz semifield planes of order $p^4$ are shown to be $p$-primitive semifield planes. Each of the remaining planes is either a Generalized twisted field plane or is a new plane.

Further properties of $p$-primitive semifield planes, including an explicit representation of the autotopism group will be reported elsewhere. This work is part of the author's Ph.D. dissertation at the University of Iowa which was written under the supervision of Professor Norman L. Johnson and the author wishes to thank Prof. Johnson for his encouragement and many discussions on the subjeat.

## 2. Properties of $p$-primitive semifield planes

In [7] Hiramine, Matsumoto and Oyama introduced the following construction method:

Let $\pi$ denote a translation plane of order $q^2$ and kernel $GF(q)$, where $q$ is a prime power $p^r (p>2)$, with matrix spread set

$$\left\{\begin{bmatrix} x & y \\ g(x,y) & h(x,y)\end{bmatrix} : \ x, y \in \mathcal{K} \cong GF(q)\right\}$$

where $g$ and $h$ are mappings from $\mathcal{K}\times\mathcal{K}$ into $\mathcal{K}$. Let $\mathcal{F}=GF(q^2)\supset\mathcal{K}$. Take an element $t\in\mathcal{F}-\mathcal{K}$ with $t^2\in\mathcal{K}$ and define a mapping $f$: $\mathcal{F}\to\mathcal{F}$ by

$$f(x+yt) = g(x,y)-h(x,y)t$$

for $x, y\in\mathcal{K}$. Then

$$\left\{\begin{bmatrix} u & v \\ f(v) & u^q\end{bmatrix} : \ u, v\in GF(q^2)\right\}$$

represents a matrix spread set of a translation plane, $\pi(f)$, or order $q^4$ and kernel $GF(q^2)$. In [10] Johnson showed that if $\pi$ is a semifield plane then $\pi(f)$ is a semifield plane which admits a $p$-primitive Baer collineation; and conversely, if a semifield plane of order $q^2$ and kernel $\supseteq\mathcal{K}\cong GF(q)$, $q=p^r$, admits a $p$-primitive Bear collineation, then $q$ is a square and coordinates may be chosen so the matrix spread set for $\pi$ may be represented in the form

$$\left\{\begin{bmatrix} u & v \\ f(v) & u^{\sqrt{q}}\end{bmatrix} : \ u, v\in \mathcal{K}\cong GF(q)\right\}.$$

Now if $\pi(f)$ is a $p$-primitive plane (so order $(\pi(f))=p^4$) then $\pi$ is a semifield plane of order $p^2$, hence $\pi$ is Desarguesian. We conclude that if $\pi$ is a semifield plane of order $p^4$ and kernel $GF(p^2)$ then $\pi$ is a $p$-primitive semifield plane if and only if $\pi$ is obtained from the construction method of Hiramine, Matsumoto and Oyama applied to the Desarguesian plane of order $p^2$; and this occurs if and only if $\pi$ admits a matrix spread set of the form

$$\left\{\begin{bmatrix} u & v \\ f(v) & u^p\end{bmatrix} : \ u, v\in GF(p^2)\right\}$$

where $f$ is an additive function on $GF(p^2)$. Therefore, $f(v)=f_0v+f_1v^p$ for some $f_0, f_1\in GF(p^2)$. (See e.g. [14]). We shall denote this plane by $\pi(f)$ or $\pi(f_0, f_1)$. In the following proposition we give conditions on the function $f$ that give a matrix spread of a $p$-primitive semifield plane.

**Proposition 2.1.** *Let $f$: $GF(p^2)\to GF(p^2)$ be given by $f(u)=f_0u+f_1u^p$ where*

*$f_0=a_0+a_1t$, $f_1=b_0+b_1t$, $a_0, a_1, b_0, b_1 \in GF(p)$ and let $\theta$ be a nonsquare in $GF(p)$ such that $t^2=\theta$. Then $\pi(f)$ is a p-primitive semifield plane if and only if*

$$a_0^2-(a_1^2-b_1^2)\theta$$

*is a nonsquare in $GF(p)$.*

Proof. First, since we must have the determinant of the difference of any two distinct matrices in the spread must be $\neq 0$, i.e.

$$\det\left[\begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix} - \begin{bmatrix} w & z \\ f(z) & w^p \end{bmatrix}\right] \neq 0$$

for every $u, v, w, z \in GF(p^2)$ such that $(u,v)\neq(w,z)$, we need $(u-w)(u-w)^p-(v-z)(f(v)-f(z))\neq 0$. Since $f$ is additive this is equivalent to

(1) $u^{p+1}-vf(v)\neq 0$ for every $u, v \in GF(p^2)$, $(u,v)\neq(0,0)$.

Let $t\in GF(p^2)-GF(p)$ such that $t^2=\theta\in GF(p)$. Let $GF(p^2)=GF(p)[t]$. Then if $v=x+yt$ for $x, y \in GF(p)$ we have

$$v^2 = x^2+2xyt+\theta y^2$$

and

$$v^{p+1} = x^2-y^2\theta \, .$$

Since $u^{p+1}\in GF(p)$ for every $u\in GF(p^2)$, (1) becomes

(2) $z-f_0v^2-f_1v^{p+1}\neq 0$ for every $(z,v)\in GF(p)\times GF(p^2)-\{(0,0)\}$.

Let $f_0=a_0+a_1t$ and $f_1=b_0+b_1t$ for $a_0, a_1, b_0, b_1 \in GF(p)$. Then (2) becomes

$$z-(a_0+a_1t)(x^2+z^2\theta+2xzt)-(b_0+b_1t)(x^2-z^2\theta)\neq 0$$

So

$$z-(a_0x^2+a_0z^2\theta+2a_1xz\theta+b_0x^2-b_0z^2\theta)$$
$$-(2a_0xz+a_1x^2+a_1z^2\theta+b_1x^2-b_1z^2\theta)t\neq 0 \, .$$

Hence the $t$-component above must be $\neq 0$. When $z=0$ and $x\neq 0$ we have

$$(a_1+b_1)x^2\neq 0 \text{ for every } x\in GF(p)-\{0\} \text{; so}$$

$a_1+b_1\neq 0$.

When $z\neq 0$, dividing by $z^2$ we get, letting $w=\dfrac{x}{z}$,

$$(a_1+b_1)w^2+(a_1-b_1)\theta+2a_0w\neq 0 \text{ for every } w\in GF(p) \, .$$

Therefore, the discriminant

$$4a_0^2-4(a_1+b_1)(a_1-b_1)\theta \text{ is a nonsquare in } GF(p) \, .$$

Hence, we must have $a_0^2-(a_1^2-b_1^2)\theta$ is a nonsquare in $GF(p)$. Conversely, if

$a_0, a_1, b_1$ satisfy this condition and if we let

$$f_0 = a_0 + a_1 t \text{ and } f_1 = b_0 + b_1 t \text{ for some } b_0 \in GF(p),$$

then the function $f: GF(p^2) \to GF(p^2)$ given by

$$f(v) = f_0 v + f_1 v^p$$

gives a matrix spread set for a $p$-primitive semifield plane $\pi(f)$.

In the next proposition, further properties of the function $f$ are studied.

**Proposition 2.2.** *Let $\pi(f_0, f_1)$ be a p-primitive semifield plane. Then,*

(*i*) *$f_0$ and $f_1$ cannot belong both to $GF(p)$. In particular, if $f_0 = 0$ then $f_1 \notin GF(p)$.*

(*ii*) *If $f_1 = 0$ then $f_0$ is a nonsquare in $GF(p^2)$.*

(*iii*) *If $f_0 \neq 0$ and $f_1 \neq 0$ then $f_0^{p+1} \neq f_1^{p+1}$.*

Proof. (i) follows directly from (2.1) for if $f_0 = a_0 + a_1 t$ and $f_1 = b_0 + b_1 t$ both belong to $GF(p)$ then $a_1 = 0 = b_1$ and in (2.1), we will have $a_0^2$ is a nonsquare in $GF(p)$. If $f_1 = 0$ then $\Delta = \det \begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix} = u^{p+1} - f_0 v^2$. So if $f_0$ is a square in $GF(p^2) - \{0\}$, say $f_0 = b^2$ then for $u = 1$ and $v = \frac{1}{b}$ we will have $\Delta = 0$. Thus $f_0$ cannot be a square in $GF(p^2)$; this proves (ii). Suppose now that $f_0 \neq 0$, $f_1 \neq 0$ and $f_0^{p+1} - f_1^{p+1} = 0$. If $f_0 = a_0 + a_1 t$ and $f_1 = b_0 + b_1 t$ we have

$$f_0^{p+1} - f_1^{p+1} = a_0^2 - (a_1^2 - b_1^2)\theta - b_0^2 = 0\,.$$

So $a_0^2 - (a_1^2 - b_1^2)\theta = b_0^2$ and this contradicts (2.1). Thus, $f_1^{p+1} \neq f_0^{p+1}$.

## 3. The isomorphism theorem

Let $\pi(f_0, f_1)$ and $\pi(F_0, F_1)$ be $p$-primitive semifield planes. The following theorem determines necessary and sufficient conditions on the functions $f = (f_0, f_1)$ and $F = (F_0, F_1)$ for the planes $\pi(f)$ and $\pi(F)$ to be isomorphic.

**Theorem 3.1.** *Two p-primitive semifield planes $\pi(f_0, f_1)$ and $\pi(F_0, F_1)$ are isomorphic if and only if one of the following is satisfied:*

(*i*) $F_0 = ac^{p-1} f_0$ *and* $F_1 = af_1$

*or*

(*ii*) $F_0 = ac^{p-1} f_0^p$ *and* $F_1 = af_1^p$

*for some $a \in GF(p) - \{0\}$ and $c \in GF(p^2) - \{0\}$.*

*In particular, $\pi(0, f_1) \cong \pi(F_0, F_1)$ if and only if $F_0 = 0$ and $F_1 = af_1$ or $F_1 = af_1^p$ for some $a \in GF(p)$.*

Proof. Any isomorphism of translation planes is a bijective semilinear map of one plane into the other (as vector spaces over their kernels). Thus, $\pi(f)$ and $\pi(F)$ are isomorphic if and only if there exists a semilinear transformation.

$$\sigma\begin{bmatrix} A & D \\ C & B \end{bmatrix}$$

which sends a spread set of $\pi(f)$ onto a spread set of $\pi(F)$, where $\sigma$ is an automorphism of $GF(p^2)$ and $A,B,C,D$ are $2\times 2$ nonsingular matrices over $GF(p^2)$.

The elation axis $(O,X)$ is sent to $(O,X)$ and we may assume that $(X,O)$ is sent into $(X,O)$ because, since $\pi$ is a semifield plane, the elation group is transitive on the components not equal to $(O,X)$. Thus $D=O$ and $C=O$.

Let $A=\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ and let $d=a_1a_4-a_2a_3$. Suppose first that $\sigma=1$. If $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ sends the component $\left(X, X\begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix}\right)$ of $\pi(f)$ into the component $\left(X, X\begin{bmatrix} x & y \\ F(y) & x^p \end{bmatrix}\right)$ of $\pi(F)$ then

$$\frac{1}{d}\begin{bmatrix} a_4 & -a_2 \\ -a_3 & a_1 \end{bmatrix}\cdot\begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix}\cdot\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}=\begin{bmatrix} x & y \\ F(y) & x^p \end{bmatrix}$$

i.e.

$$\frac{1}{d}\begin{bmatrix} b_1a_4u-b_1a_2f(v)+b_3a_4v-b_3a_2u^p & b_2a_4u-b_2a_2f(v)+b_4a_4v-b_4a_2u^p \\ -b_1a_3u+b_1a_1f(v)-b_3a_3v+b_3a_1u^p & -b_2a_3u+b_2a_1f(v)-b_4a_3v+b_4a_1u^p \end{bmatrix}$$
$$=\begin{bmatrix} x & y \\ F(y) & x^p \end{bmatrix}.$$

From here, we get that

$$\left[\frac{b_1a_4u-b_1a_2f(v)+b_3a_4v-b_3a_2u^p}{d}\right]^p=\frac{-b_2a_3u+b_2a_1f(v)-b_4a_3v+b_4a_1u^p}{d}$$

and

$$F\left(\frac{b_2a_4u-b_2a_2f(v)+b_4a_4v-b_4a_2u^p}{d}\right)=\frac{-b_1a_3u+b_1a_1f(v)-b_3a_3v+b_3a_1u^p}{d}.$$

Therefore, the following conditions must be satisfied in order for $\begin{bmatrix} A & O \\ O & B \end{bmatrix}$ to be an isomorphism of $\pi(f_0,f_1)$ into $\pi(F_0,F_1)$:

(1) $\dfrac{b_2a_3}{d}=\left[\dfrac{b_3a_2}{d}\right]^p,$

(2) $\frac{b_4a_1}{d}=\left[\frac{b_1a_4}{d}\right]^p$,

(3) $\frac{b_2a_1f_0-b_4a_3}{d}=-\left[\frac{b_1a_2}{d}\right]^p f_1^p$,

(4) $\frac{b_2a_1f_1}{d}=\left[\frac{b_3a_4}{b}\right]^p-\left[\frac{b_1a_2}{d}\right]^p f_0^p$,

(5) $F_0\left[\frac{b_2a_4}{d}\right]-F_1\left[\frac{b_4a_2}{d}\right]^p=\frac{-b_1a_3}{d}$,

(6) $-F_0f_0\left[\frac{b_2a_2}{d}\right]+F_0\left[\frac{b_4a_4}{d}\right]-F_1f_1^p\left[\frac{b_2a_2}{d}\right]^p=\left[\frac{b_1a_1}{d}\right]f_0-\frac{b_3a_3}{d}$,

(7) $-F_0f_1\left[\frac{b_2a_2}{d}\right]-F_1f_0^p\left[\frac{b_2a_2}{d}\right]^p+F_1\left[\frac{b_4a_4}{d}\right]^p=\left[\frac{b_1a_1}{d}\right]f_1$,

(8) $-F_0\left[\frac{b_4a_2}{d}\right]+F_1\left[\frac{b_2a_4}{d}\right]^p=\left[\frac{b_3a_1}{d}\right]$.

Now we consider the following cases:

Case I: $a_2=0$
Case II: $a_1=0$
Case III: $a_1\neq 0$ and $a_2\neq 0$

**Case I:** $a_2=0$

Since $d=a_1a_4\neq 0$, we must have $a_1\neq 0$ and $a_4\neq 0$. From (1) and (3), we have that $a_3=0$ and $b_2f_0=0$. Thus $b_2=0$ or $b_2\neq 0$ and $f_0=0$.

Suppose $b_2=0$; hence $b_1\neq 0$ and $b_4\neq 0$. Conditions (1), (3) and (5) are trivially satisfied and (2) becomes

$$(2)' \quad \left[\frac{b_1}{a_1}\right]^p=\left[\frac{b_4}{a_4}\right].$$

From (4), we get $b_3=0$; thus (8) is satisfied trivially. Substituting (2)′ into (7) we get

$$F_1=\left[\frac{a_1}{a_4}\right]^{p+1}f_1.$$

Substituting $a=\left[\frac{a_1}{a_4}\right]^{p+1}$ in (6), we get

$$F_0=a\left[\frac{a_4}{b_1}\right]^{p-1}f_0.$$

If we let $c = \dfrac{a_4}{b_1}$, we have

$$F_0 = ac^{p-1}f_0 \text{ and } F_1 = af_1 .$$

Suppose now that $b_2 \neq 0$. Then $f_1 = 0$ and $f_0 \neq 0$. (1) and (3) are trivially satisfied and (2) and (4) become, respectively

$$(2)' \quad \left[\frac{b_4}{a_4}\right] = \left[\frac{b_1}{a_1}\right]^p, \text{ and}$$

$$(4)' \quad f_1 = \frac{a_4}{b_2}\left[\frac{b_3}{a_1}\right]^p.$$

From (5), we get $F_0 = 0$ and then (6) is trivially satisfied. Hence, (7) becomes

$$(7)' \quad F_1\left[\frac{b_4}{a_1}\right]^p = \frac{b_1}{a_4}f_1$$

and (8) gives $F_1 = \left[\dfrac{a_1}{b_2}\right]^p\left[\dfrac{b_3}{a_4}\right]$. From (4)′ and (8), we get $F_1 = \left[\dfrac{a_1}{a_4}\right]^{p+1} f_1^p$. If $b_1 \neq 0$ then $b_4 \neq 0$ by (2)′ and from (7)′ we have $F_1 = \left[\dfrac{a_1}{b_4}\right]^p\left[\dfrac{b_1}{a_4}\right]f_1$. Solving in (2)′ for $b_4$ and replacing it in this last equation, we get $F_1 = \left[\dfrac{a_1}{a_4}\right]^{p+1} f_1$. From (4)′ and (8)′ we get $\dfrac{F_1}{f_1^p} = \left[\dfrac{a_1}{a_4}\right]^{p+1}$, i.e. $F_1 = \left[\dfrac{a_1}{a_4}\right]^{p+1} f_1^p$. From this last two expressions for $F_1$, we get $f_1^p = f_1$ and this implies $f_1 \in GF(p)$. But this contradicts (2.2) (i). Therefore, we must have $b_1 = 0$. It follows from (2)′ that $b_4 = 0$ and now (7)′ is trivially satisfied. Let $a = \left[\dfrac{a_1}{a_4}\right]^{p+1}$; then $a \in GF(p)$ and $F_1 = af_1^p$.

Thus we have proved that if

$$\Gamma = \begin{pmatrix} a_1 & 0 & & \\ a_3 & a_4 & & \\ & & b_1 & b_2 \\ & & b_3 & b_4 \end{pmatrix}$$

is an isomorphism of $\pi(f_0, f_1)$ into $\pi(F_0, F_1)$ then $a_3 = 0$, and if $b_2 = 0$ then $b_3 = 0$ and

$$F_0 = ac^{p-1}f_0 \text{ and } F_1 = af_1$$

where $a = \left[\dfrac{a_1}{a_4}\right]^{p+1} \in GF(p)$, $c = \dfrac{a_4}{b_1}$ and $\left[\dfrac{b_1}{a_1}\right]^p = \dfrac{b_4}{a_4}$. If $b_2 \neq 0$, then $b_1 = 0 = b_4$,

$f_0=0=F_0$, $F_1=af_1^p$ where $a=\left[\frac{a_1}{a_4}\right]^{p+1}\in GF(p)$ and $a_1,a_4,b_2,b_3$ satisfy the condition $\frac{a_4}{b_2}=\left[\frac{a_1}{b_3}\right]^p f_1$. Also, if $F=(F_0,F_1)$ and $f=(f_0,f_1)$ are related as above then $\Gamma$ is an isomorphism of $\pi(f)$ into $\pi(F)$ for the corresponding choices of $a_i$'s and $b_i$'s.

**Case II:** $a_1=0$

In this case, $d=-a_2\,a_3\neq 0$ and so we must have $a_2\neq 0$ and $a_3\neq 0$. From (1) and (2), we have (1)$'$ $\frac{b_2}{a_2}=\left[\frac{a_3}{b_3}\right]^p$ and $b_1\,a_4=0$. Thus $b_1=0$ or $b_1\neq 0$ and $a_4=0$.

Suppose $b_1=0$. Thus $b_3\neq 0$ and $b_2\neq 0$. From (3) and (4) we get $b_4=0$ and $a_4=0$ then (7) gives $F_0=0$ and substituting (1)$'$ in (6), we get $f_1^p\,F_1=\left[\frac{a_3}{a_2}\right]^{p+1}$, so $F_1=\left[\frac{a_3}{a_2}\right]^{p+1}\frac{1}{f_1^{p+1}}\cdot f_1$; letting $a=\left[\frac{a_3}{a_2 f_1}\right]^{p+1}$ we have $F_1=a\,f_1$ and $a\in GF(p)$.

If $f_0\neq 0$, in (7) we have (7)$'$ $F_1=-\left[\frac{a_3}{b_2}\right]^{p-1}\frac{f_1}{f_0^p}F_0$ and substituting this into (6) we get

$$F_0\left[\frac{f_0^{p+1}-f_1^{p+1}}{f_0^p}\right]=\frac{a_3^{p+1}}{a_2^2\,b_3^{p-1}}\,.$$

Since $f_0^{p+1}-f_1^{p+1}\neq 0$ by (2 2) (iii), we can solve for $F_0$ and then replacing this in (7)$'$ we get

$$F_1=-\left[\frac{a_3}{a_2}\right]^{p+1}\cdot\frac{f_1}{f_0^{p+1}-f_1^{p+1}}\,.$$

Let $a=-\left[\frac{a_3}{a_2}\right]^{p+1}\frac{1}{f_0^{p+1}-f_1^{p+1}}$ and $c=\frac{a_2 t}{b_3}f_0$. Then $a\in GF(p)$, $F_0=ac^{p-1}f_0$ and $F_1=af_1$. Suppose now that $b_1\neq 0$ and $a_4=0$. Then (3) becomes

$$(3)'\ f_1^p=\frac{b_4}{a_2}\left[\frac{a_3}{b_1}\right]^p$$

and from (4) we have $f_0=0$. From (5) we get $b_4\neq 0$ and

$$(5)'\ F_1=\frac{b_1}{a_2}\left[\frac{a_3}{b_4}\right]^p.$$

Now (6) becomes

$$(6)'\ F_1\left[\frac{b_2}{a_3}\right]^p f_1^p=\left[\frac{b_3}{a_2}\right]$$

and from (8) we have $F_0=0$. Thus (7) is satisfied. Now $b_2=0$ for if $b_2\neq 0$ then substituting (3)$'$ and (1)$'$ in (5)$'$ and (6)$'$ respectively, we get $F_1 f_1=\left[\frac{a_3}{a_2}\right]^{p+1}=$

$F_1 f_1^p$ which implies $f_1 \in GF(p)$; but $f_0=0$ implies $f_1 \notin GF(p)$ (2 2) (i). From this contradiction, we get that $b_2=0$ and from (1) we have $b_3=0$. Solving (3)' for $a_3$ and replacing it into (5)' we get

$$F_1 = \left[\frac{b_1}{b_4}\right]^{p+1} f_1^p .$$

Let $a=\left[\frac{b_1}{b_4}\right]^{p+1}$, so $a \in GF(p)$ and $F_1=af_1^p$. Therefore, in this case, we proved that if $\Gamma=\begin{bmatrix} 0 & a_2 & & \\ a_3 & a_4 & & \\ & & b_1 & b_2 \\ & & b_3 & b_4 \end{bmatrix}$ is an isomorphism of $\pi(f_0, f_1)$ into $\pi(F_0, F_1)$ then $a_4=0$. If $b_1=0$ then $b_4=0$; in this case if $f_0 \neq 0$, then $F_0=ac^{p-1}f_0$ and $F_1=af_1$ where $a=-\left[\frac{a_3}{a_2}\right]^{p+1} \frac{1}{f_0^{p+1}-f_1^{p+1}} \in GF(p)$, and $c=\frac{a_2 t f_0}{b_3}$; if $f_0=0$, then $F_0=0$, and $F_1=af_1$ where $a=\left[\frac{a_3}{a_2 f_1}\right]^{p+1} \in GF(p)$. In either case, $a_2$, $a_3$, $b_2$, $b_3$ satisfy the condition $(b_2/a_2)=(b_3/a_3)^p$. If $b_1 \neq 0$ then $b_2=b_3=0, f_0=0=F_0$ and $F_1=af_1^p$ where $a=\left[\frac{b_1}{b_4}\right]^{p+1}$. Also $a_2$, $a_3$, $b_1$, $b_4$ satisfy the condition $\left[\frac{b_4}{a_2}\right]^p=\left[\frac{b_1}{a_3}\right]f_1$. Again if $F=(F_0, F_1)$ and $f=(f_0, f_1)$ are related as above then $\Gamma$ provides an isomorphism between $\pi(F)$ and $\pi(f)$.

**Case III:** $a_1 \neq 0$ and $a_2 \neq 0$

Let $A_i=\frac{a_i}{d}$. From (1) and (2), we get

$$(1)' \quad b_3 = \frac{b_2^p A_3^p}{A_2}$$

$$(2)' \quad b_4 = \frac{b_1^p A_4^p}{A_1}$$

In (3), we have $b_2 A_1 f_0 + (b_1^p)\left[A_2^p f_1^p - \frac{A_4^p A_3}{A_1}\right]=0$. If $f_0 \neq 0$ then we get $b_2=\frac{b_1^p}{A_1 f_0}\left[\frac{A_4^p A_3}{A_1} - A_2^p f_1^p\right]$.

Let $C=\frac{A_4^p A_3 - A_1 A_2^p f_1^p}{A_1^2 f_0}$. Then

$$b_2 = C b_1^p$$

$$b_3 = \frac{C^p b_1 A_3^p}{A_2}$$

$$b_4 = \frac{b_1^p A_4^p}{A_1}$$

Thus $b_1 \neq 0$. Substituting these in (4) and dividing by $A_2^2 A_1^{2p}$, we get

$$f_0^{p+1} - f_0^{p+1} + \frac{A_3 A_4^p}{A_2^p A_1}(f_1^p + f_1) - \frac{A_3^2 A_4^{2p}}{A_1^2 A_2^{2p}} = 0\,.$$

Letting $u = \dfrac{A_3 A_4^p}{A_1 A_2^p}$, we have

$$u^2 - (f_1^p + f_1)\,u - (f_0^{p+1} - f_1^{p+1}) = 0\,.$$

Therefore, the discriminant, $D$, of this equation has to be a square in $GF(p^2)$. If $f_0 = u_0 + u_1 t$ and $f_1 = v_0 + v_1 t$ for some $u_0$, $u_1$, $v_0$, $v_1 \in GF(p)$, $t \in GF(p^2) - GF(p)$, $t^2 = \theta \in GF(p)$, then $D = u_0^2 - (u_1^2 - v_1^2)\,\theta$ and by (2.1) $D$ is a nonsquare in $GF(p)$. Therefore, if $a_1 \neq 0$ and $a_2 \neq 0$ we must have $f_0 = 0$.

Now substituting (1)′ and (2)′ in (3) and (4) respectively we get

$$b_1^p \left[\frac{A_2^p A_1 f_1^p - A_4^p A_3}{A_1}\right] = 0$$

and

$$b_2 \left[\frac{A_1 A_2^p f_1 - A_3 A_4^p}{A_2^p}\right] = 0\,.$$

If $b_1 \neq 0$ and $b_2 \neq 0$ we get $f_1^p = \dfrac{A_4^p A_3}{A_2^p A_1} = f_1$ which implies $f_1 \in GF(p)$. Since this contradicts (2.2) (i), we must have that $b_1 = 0$ or $b_2 = 0$.

Suppose $b_1 = 0$. Then $b_2 \neq 0$ and from (2)′ we get $b_4 = 0$. From (4) we get (4)′ $f_1 = \dfrac{A_3 A_4^p}{A_1 A_2^p}$ and this implies that $A_3 \neq 0$ and $A_4 \neq 0$, so from (5) we have $F_0 = 0$, so (7) is satisfied. Now in (6) we have (6)′ $F_1 f_1^p = \left[\dfrac{A_3}{A_2}\right]^{p+1}$. Solving in (1)′ for $b_3$ and replacing it in (8), we get (8)′ $F_1 = \left[\dfrac{A_3}{A_4}\right]^p \dfrac{A_1}{A_2}$. Combining (4)′ and (8)′, we get $F_1 f_1 = \left[\dfrac{A_3}{A_2}\right]^{p+1}$ and this equals $F_1 f_1^p$ by (6)′. Therefore, $f_1^p = f_1$; this implies $f_1 \in GF(p)$ which contradicts (2.2) (i). Therefore, $b_1 = 0$ is not possible and we must consider the case $b_2 = 0$.

Suppose $b_2 = 0$. Then from (1)′ we get $b_3 = 0$. Now, replacing (2) into (3) and (5) we get

$$f_1^p = \frac{A_4^p A_3}{A_1 A_2^p} \quad \text{and} \quad F_1 = \frac{A_3 A_1^p}{A_2^p A_4}\,.$$

From these, we get $F_1 = \left[\dfrac{A_1}{A_4}\right]^{p+1} f_1^p(*)$. From (6), we get that $F_0 = 0$; now (7) becomes $F_1 = \left[\dfrac{A_1}{A_4}\right]^{p+1} f_1$; combining this with $(*)$ we get $f_1^p = f_1$ and this implies that $f_1 \in GF(p)$; but this contradices (2.2) (i). Therefore, the case $b_2 = 0$ is not possible either and we conclude that there is no isomorphism with $a_1 \neq 0$ and $a_2 \neq 0$; this completes the case $\sigma=1$.

Suppose now that $\sigma \neq 1$. The semilinear transformation $\sigma \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$ with $\sigma$: $x \mapsto x^p$, the Frobenius automorphism, induces an isomorphism of $\pi(f_0, f_1)$ into $\pi(f_0^p, f_1^p)$:

$$\begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix}^\sigma = \begin{bmatrix} u^\sigma & v^\sigma \\ f(v)^\sigma & u^{p\sigma} \end{bmatrix} = \begin{bmatrix} x & y \\ F(y) & x^p \end{bmatrix}$$

with $F(y)=F(v^p)=F_0 v^p+F_1 v=f(v)^p=f_0^p v^p+f_1^p v$; so $F_0=f_0^p$ and $F_1=f_1^p$. Therefore, two $p$-primitive planes $\pi(f_0, f_1)$and $\pi(F_0, F_1)$ are isomorphic if and only if there exists linear isomorphism of $\pi(F_0, F_1)$ into $\pi(f_0, f_1)$ or $\pi(f_0^p, f_1^p)$. This completes the proof of the theroem.

**Corollary 3.2.** *All the planes $\pi(f_0, f_1)$ with $f_1=0$ constitute an isomorphism class with $\frac{p^2-1}{2}$ elements. The planes in this class are Dickson semifield planes.*

Proof. Let $\pi(f_0, f_1)$ and $\pi(F_0, F_1)$ be $p$-primitive semifield planes with $f_1=0$ and $F_1=0$. Then by (2.2) (ii), $f_0$ and $F_0$ are nonsquares in $GF(p^2)$; thus there exists $d \in GF(p^2)$ such that $F_0=d^2 f_0$.

By (3.1), $\pi(f_0, 0) \cong \pi(F_0, 0)$ if and only if there exists $a \in GF(p)$, $c \in GF(p^2)$ such that $F_0=ac^{p-1} f_0$ or $F_0=ac^{p-1} f_0^p$. Let $a \in GF(p)$ with $|a|=p-1$ and $c \in GF(p^2)$ with $|c|=p^2-1$; then $|c^{p-1}|=p+1$ and $|ac^{p-1}|=(p^2-1)/2$. It follows that $ac^{p-1}$ is a generator of the subgroup of squares in $GF(p^2)$; hence $d^2$ is a power of $ac^{p-1}$ and therefore $\pi(f_0, 0) \cong \pi(F_0, 0)$.

Conversely, it follows directly from the theorem that if $\pi(F_0, F_1) \cong \pi(f_0, 0)$ then $F_1=0$.

If $f_1=0$ then $\pi=\pi(f_0, f_1)$ has matrix spread set

$$\left\{ \begin{bmatrix} u & v \\ f_0 v & u^p \end{bmatrix} : u, v \in GF(p^2) \right\}$$

and the product is given by

$$(x, y) \cdot (u, v) = (xy+y f_0 v,\ xv+yu^p)$$

This is the product in [5, p. 241] with $\alpha=\beta=1$ and $\sigma$: $x \mapsto x^p$ and therefore $\pi$ is a Dickson semifield plane.

**Corollary 3.3.** *There are $\frac{p+1}{2}$ nonisomorphic $p$-primitive semifield planes $\pi(f_0, f_1)$ with $f_0=0$. The number of planes isomorphic to $\pi(f_0, f_1)$ is $p-1$ if $f_1^{p-1}=-1$ and is $2(p-1)$ if $f_1^{p-1} \neq -1$. All the planes $\pi(0, f_1)$ are Hughes-Kleinfeld semi-*

*field planes.*

Proof. $\pi(F_0, F_1)\cong\pi(0, f_1)$ if and only if $F_0=0$ and $F_1=af_1$ or $F_1=af_1^p$ for some $a\in GF(p)-\{0\}$. By (2.2) (i), $f_1\notin GF(p)$; hence if $f_1=b_0+b_1t$, then $b_1\neq 0$. Taking $a=\frac{1}{b_1}$, we have $\pi(0, b_0+b_1t)\cong\pi\left(0, \frac{b_0}{b_1}+t\right)$.

Now $\pi(0, b+t)\cong\pi(0, c+t)$ if and only if $b+t=a(c+t)$ or $b+t=a(c+t)^p=a(c-t)$ for some $a\in GF(p)-\{0\}$. In the first case $a=1$ and $b=c$ and in the second $a=-1$ and $b=-c$. Thus the number of nonisomorphic planes with $f_0=0$ is $\frac{p-1}{2}+1=\frac{p+1}{2}$.

Suppose $\pi(0, F_1)\cong\pi(0, f_1)$. Then $F_1=af_1$ or $F_1=af_1^p$ for some $a\in GF(p)-\{0\}$. Now, $af_1=bf_1^p$ for some $a, b\in GF(p)-\{0\}$ if and only if $f_1^{2(p-1)}=1$. Since $f_1\notin GF(p)$, we have $af_1=bf_1^p$ for some $a, b\in GF(p)-\{0\}$ if and only if $f_1^{p-1}\neq -1$. Hence the number of planes isomorphic to $\pi(0, f_1)$ is $2(p-1)$ if $f_1^{p-1}\neq -1$ and is $p-1$ if $f_1^{p-1}=-1$.

If $\pi=\pi(f_0, f_1)$ is a $p$-primitive semifield plane with $f_0=0$ then the matrix spread set of $\pi$ is of the form

$$\left\{\begin{bmatrix} u & v \\ f_1v^p & u^p\end{bmatrix} : u, v\in GF(p^2)\right\}$$

and

$$\begin{aligned}(x, y)\cdot(u, v) &= (x, y)\begin{bmatrix} u & v \\ f_1v^p & u^p\end{bmatrix}\\ &= (xu+yf_1v^p, xv+yu^p)\end{aligned}$$

for $x, y, u, v\in GF(p^2)$. This is the product in a semifield of all Knuth types (i)–(iv). In [8] Hughes and Kleinfeld showed that a semifield of order $q^2$ and kernel $GF(q)$ is of all four types if and only if $\mathcal{N}_m=\mathcal{N}_r=\mathcal{N}_l\cong GF(q)$; a semifield plane corresponding to a semifield with this property is called a Hughes-Kleinfeld semifield plane.

**Corollary 3.4.** *If* $\pi(F_0, F_1)\cong\pi(f_0, f_1)$ *and* $F_0=f_0\neq 0$, *then* $F_1=\pm f_1$ *or* $F_1=\pm f_1^p$. *Conversely, if* $F_0=f_0\neq 0$ *and* $F_1=\pm f_1$ *or* $F_1=\pm f_1^p$ *then* $\pi(F_0, F_1)\cong\pi(f_0, f_1)$.

Proof. Suppose $\pi(F_0, F_1)\cong\pi(f_0, f_1)$ and $F_0=f_0\neq 0$. Then, from (3.1) there exist $a\in GF(p)-\{0\}$ and $c\in GF(p^2)-\{0\}$ such that $F_0=ac^{p-1}f_0$ and $F_1=af_1$ or $F_0=ac^{p-1}f_0^p$ and $F_1=af_1^p$. Let $F_0=ac^{p-1}f_0$. Then $ac^{p-1}=1$ and this implies that $a^{p+1}=1$. Therefore, $|a|$ divides $p+1$. But $|a|$ divides $p-1$; hence $|a|$ divides 2 and consequently $a=\pm 1$. Thus, $F_1=\pm f_1$ or $F_1=\pm f_1^p$. If $F_0=ac^{p-1}f_0^p$ then, since $F_0=f_0$, we have $a(cf_0)^{p-1}=1$ and again, we obtain $a^{p+1}=1$; by the same argument as above, we obtain the result. The converse follows directly.

In 1984 Boerner-Lantz defined a class of semifields of order $q^4$ as follows:
Let $S=\{\alpha+\beta x \mid \alpha, \beta \in GF(9)\}$ and $x \notin GF(9)$. Define addition on $S$ to be the usual vector addition. If multiplication $\cdot$ is defined on $S$ by

$$(\alpha+\beta x)\cdot(\gamma+\delta x) = \alpha\gamma+\beta(\delta^3 a-\delta_1)+(\alpha\delta+\beta\gamma^3)\, x$$

where $\delta=\delta_1+\delta_2\, a$, $a \notin GF(3)$, $a^2=2a+1$ and $\delta_1, \delta_2 \in GF(3)$, then $(S, +, \cdot)$ is a semifield. Now this is generalized for $p>3$ as follows: Let $q=p^r$ with $p>3$. Choose $\sigma \in GF(q)$ such that $x^2-\sigma$ is irreducible over $GF(q)$ and $1+4\sigma$ is a nonsquare. Let $a \notin GF(q)$ be a root of $x^2=\sigma$ and $S=\{\alpha+\beta s \mid \alpha, \beta \in GF(q^2)\}$ where $s \notin GF(q^2)$. Define addition on $S$ to be the usual vector addition. If multiplication is defined on $S$ by

$$(\alpha+\beta s)\cdot(\gamma+\delta s) = \alpha\gamma+\beta(\delta^q a-\delta_1)+(\delta+\beta x^q)\, s$$

where $\delta=\delta_1+\delta_2 a$, $\delta_1, \delta_2 \in GF(q)$, then $(S, +, \cdot)$ is a semifield of dimension 2 over $GF(q^2)$. Boerner-Lantz [3]. In the next corollary we show that the semifield planes of order $p^4$ associated to the Boerner-Lantz semifields are $p$-primitive semifield planes.

**Corollary 3.5.** *Let $\pi=\pi(f_0, f_1)$ be a $p$-primitive semifield plane with $f_0 \neq 0$ and $f_1 \neq 0$. Then the number of planes isomorphic to $\pi$ is $p^2-1$ if $f_1^{2(p-1)}=1$ and is $2(p^2-1)$ if $f_1^{2(p-1)} \neq 1$. The semifield planes of Boerner-Lantz of order $p^4$ are $p$-primitive with $f_0 \neq 0$ and $f_1 \neq 0$, and for $p>3$, $f_1^{2(p-1)} \neq 1$.*

Proof. By (3.1), $\pi(F_0, F_1) \cong \pi(f_0, f_1)$ if and only if there exist $a \in GF(p)-\{0\}$ and $c \in GF(p^2)-\{0\}$ such that

$$F_0 = ac^{p-1} f_0 \quad \text{and} \quad F_1 = a f_1$$

or

$$F_0 = ac^{p-1} f_0^p \quad \text{and} \quad F_1 = a f_1^p\,.$$

Thus, there are $p^2-1$ or $2(p^2-1)$ planes isomorphic to $\pi(f_0, f_1)$. Now, if $ac^{p-1} f_0 = bd^{p-1} f_0^p$ and $a f_1 = b f_1^p$ for some $a, b \in GF(p)-\{0\}$ and $c, d \in GF(p^2)-\{0\}$, then $f_1^{p-1}=\dfrac{a}{b}$ and this implies $f_1^2 \in GF(p)$ and hence $f_1^p=1$.

Conversely, if $f_1^{2(p-1)}=1$, then $f_1^{p-1}=\pm 1$. If $f_1^{p-1}=1$, then $ac^{p-1} f_0^p=a(cf_0)^{p-1} f_0$ and $a f_1^p=a f_1$. If $f_1^{p-1}=-1$ then $ac^{p-1} f_0^p=a(f_0\, cw)^{p-1} f_0$ and $a f_1^p=-a f_1$ for some $w \in GF(p^2)$ such that $w^{p-1}=-1$. Therefore, there are $p^2-1$ planes isomorphic to $\pi(f_0, f_1)$ if and only if $f_1^2 \in GF(p)$.

For $p=3$, the product for the Boerner-Lantz semifield is given by

$$(x, y)\,(u, v) = (x, y)\begin{bmatrix} u & v \\ v^3 a - v_1 & u^p \end{bmatrix} \text{ where } a \notin GF(3)\,,$$

$a^2=2a+1$ and $v=v_1+v_2a$ for $v_1, v_2 \in GF(3)$.

Letting $t=a-1$ we have $v^3a-v_1=(1+t)v+2v^3$ and the semifield plane of Boerner-Lantz is $p$-primitive with matrix spread set

$$\left\{\begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix} : u, v \in GF(3^2)\right\}$$

where $f(v)=(1+t)v+2v^3$. For $p>3$,

$$(x, y)(u, v) = (x, y)\begin{bmatrix} u & v \\ v^p t-v_1 & u^p \end{bmatrix}$$

where $v=v_1+v_2 t$, $v_1, v_2 \in GF(p)$. Now $t^p=-t$ so that $v^p t-v_1=v^p t-\frac{1}{2}v-\frac{1}{2}v^p=\left(-\frac{1}{2}\right)v+\left(t-\frac{1}{2}\right)v^p$. Letting $f(v)=\left(-\frac{1}{2}\right)v+\left(-\frac{1}{2}+t\right)v^p$ we have

$$(x, y)(u, v) = (x, y)\begin{bmatrix} u & v \\ f(v) & u^p \end{bmatrix}.$$

Therefore, the semifield planes of Boerner-Lantz of order $p^4$ are $p$-primitive semifield planes. Moreover $f_0=-\frac{1}{2}\neq 0$, $f_1=-\frac{1}{2}+t\neq 0$ and $f_1^2=\frac{1}{4}-t+t^2$; so $f_1^2 \notin GF(p)$.

In [9], Johnson showed that in general the semifield planes of Boerner-Lantz of order $q^4$ may be obtained from the construction method of Hiramine, Matsumoto and Oyama from the Desarguesian planes. Then he obtains another class of planes from the Desarguesian ones and he conjectures that these two classes are not isomorphic. In fact this is the case because any plane in this second class has $f_0=0$ and by (3.3) it is a Hughes-Kleinfeld semifield plane. Therefore, it is not isomorphic to the planes of Boerner-Lantz (since for those planes $f_0\neq 0$).

## 4. On the number of non-isomorphic $p$-primitive semifield planes

Let $\pi(f_0, f_1)$ be a $p$-primitive semifield plane with $f_0=a_0+a_1 t$, $f_1=b_0+b_1 t$, $a_0, a_1, b_0, b_1 \in GF(p)$. By (2.1), $a_0^2-(a_1^2-b_1^2)\theta$ is a nonsquare in $GF(p)$ where $\theta$ is a nonsquare in $GF(p)$ such that $t^2=\theta$. The proof of the following proposition depends on this fact.

**Proposition 4.1.** *For any prime $p$, $p>2$, there are $\frac{(p^2-p)^2}{2}$ functions $f$ such that $\pi(f)$ is a $p$-primitive semifield plane.*

Proof. Let $f(u)=f_0 u+f_1 u^p$, $f_0=a_0+a_1 t$, $f_1=b_0+b_1 t$, $a_0, a_1, b_0, b_1 \in GF(p)$. Let $\theta$ be an arbitrary but fixed nonsquare in $GF(p)$ and let $t^2=\theta$. By (2.1), $\pi(f_0, f_1)$ is a $p$-primitive semifield plane if and only if $a_0^2-(a_1^2-b_1^2)\theta$ is a nonsquare in $GF(p)$.

Let $W$ be a nonsquare in $GF(p)$. Then the number of solutions in $GF(p)$ of the equation

$$a_0^2-(a_1^2-b_1^2)\,\theta = W$$

is $p^2-p$. (See Dickson [6, p. 48]). Since $b_0$ is arbitrary in $GF(p)$ and there are $\frac{p-1}{2}$ nonsquares in $GF(p)$, we have that there are $(p^2-p)\left(\frac{p-1}{2}\right)p=\frac{(p^2-p)^2}{2}$ $p$-primitive semifield planes.

It follows from the proof that there are $(p^2-p)\left(\frac{p-1}{2}\right)$ $p$-primitive semifield planes $\pi(f_0, f_1)$ with $f_1=b_0+b_1\,t$ and $b_0=0$ and $p-1$ with $f_0=0$ and $b_0=0$. If $f_1=b_0+b_1\,t\neq 0$ and $b_1=0$ the condition is now $a_0^2-a_1^2\,\theta$ nonsquare in $GF(p)$ and using Dickson [6, p. 46], we conclude that there are $(p^2-1)\left(\frac{p-1}{2}\right)$ $p$-primitive semifield planes with $f_1=b_0\in GF(p)$, and consequently $f_0\neq 0$. These remarks and 4.1 will be used in the proof of the following result.

**Theorem 4.2.** *For any odd prime $p$, there are $\left(\frac{p+1}{2}\right)^2$ nonisomorphic $p$-primitive semifield planes of order $p^4$.*

Proof. First, by (3.3) there are $\frac{p+1}{2}$ nonisomorphic $p$-primitive semifield planes $\pi(f_0, f_1)$ with $f_0=0$ and the number of isomorphic planes to $\pi(0, f_1)$ for fixed $f_1$ is $p-1$ if $f_1^2\notin GF(p)$ and is $2(p-1)$ if $f_1^2\in GF(p)$.

Second, by (3.2) there are $\frac{p^2-1}{2}$ $p$-primitive semifield planes with $f_1=0$ and they are all isomorphic.

It remains to determine the number of nonisomorphic $p$-primitive planes with $f_0\neq 0$ and $f_1\neq 0$. Let $f_0=a_0+a_1\,t\neq 0$, $f_1=b_0+b_1\,t\neq 0$ and suppose that $f_1^2\in GF(p)$; hence $b_0=0$ or $b_1=0$.

By (3.2), and the remarks after (4.1) we get that the number of $p$-primitive semifield planes with $f_0\neq 0$, $f_1=b_0+b_1\,t\neq 0$ and $b_0=0$ is

$$(p^2-p)\left(\frac{p-1}{2}\right)-(p-1)-(p+1)\left(\frac{p-1}{2}\right)=(p-3)\left(\frac{p^2-1}{2}\right)$$

(the left hand side is (# of planes with $b_0=0$)$-$(# of planes with $f_0=0$ and $b_0=0$) $-$(# of planes with $f_1=0$)).

By (3.5), there are $p^2-1$ isomorphic planes to a fixed plane with $f_0\neq 0$ and $f_1^2\in GF(p)$. Thus, there are $\frac{p-3}{2}$ nonisomorphic planes with $f_1^2\in GF(p)$ and $b_0=0$.

By the remarks after (4.1), there are $(p^2-1)\left(\frac{p-1}{2}\right)$ $p$-primitive semifield planes with $b_1=0$ and $f_0\neq 0$ and by (3.5) there are $p^2-1$ isomorphic planes to each one; hence, the number of nonisomorphic planes with $b_1=0$ is $\frac{p-1}{2}$ and therefore there are $\frac{p-3}{2}+\frac{p-1}{2}=p-2$ nonisomorphic $p$-primitive semifield planes with $f_0\neq 0$ and $f_1^2\in GF(p)$. Suppose that $f_0\neq 0$ and $f_1^2\notin GF(p)$. Then, by (3.5), there are $2(p^2-1)$ $p$-primitive semifield planes isomorphic to a plane $\pi(f_0, f_1)$ with $f_0\neq 0$ and $f_1^2\notin GF(p)$ and by (4.1) there are $\frac{(p^2-p)^2}{2}$ $p$-primitive semifield planes; hence there are $\frac{(p-1)(p-3)(p^2-1)}{2}$ $p$-primitive semifield planes with $f_0\neq 0$ and $f_1^2\notin GF(p)$ and these divide into $\frac{(p-1)(p-3)}{4}$ isomorphism classes. Having considered all the possibilities we conclude that there are $\left(\frac{p+1}{2}\right)^2$ nonisomorphic $p$-primitive semifield planes of order $p^4$.

## 5. Classification of $p$-primitive semifield planes

Presently there are nine classes of proper (non-Desarguesian) semifield planes, namely: the semifield planes of Dickson [5, p. 241], Knuth four types [12] (these include the Hughes-Kleinfeld planes [8]), Knuth of characteristic 2 [12], Kantor [11], Sandler [13], Boerner-Lantz [3] and the two classes discovered by Albert called twisted field planes [1] and generalized twisted field planes [2] and the commutative semifields of Cohen and Ganley.

Now we answer the following question: of the known classes of semifield planes, which one contains $p$-primitive semifield planes?

By (3.2), if $\pi=\pi(f_0, f_1)$ is a $p$-primitive semifiled plane with $f_1=0$ then $\pi$ is a Dickson semifield plane and if $f_0=0$, $\pi$ is a Hughes-Kleinfeld semifield plane by (3.3). By (3.5), the Boerner-Lantz semifield planes of order $p^4$, $p$-primitive with $f_0\neq 0$, $f_1\neq 0$ and for $p>3$, $f_1^{2(p-1)}\neq 1$. Of the other known classes, the only one which could contain $p$-primitive semifield planes is the class of Generalized twisted field planes: the twisted field planes and Sandler semifield planes are of dimension 4 over the left nucleus and the Knuth and Kantor semifields planes are of characteristic 2. If a Knuth type (i), (ii), (iii) or (iv) semifield plane $\pi$ is $p$-primitive then $\mathcal{N}_m=\mathcal{N}_l=\mathcal{N}_r\cong GF(p^2)$ and thus it is a Hughes–Kleinfeld semifield plane. The $p$-primitive semifields are not commutative. So they do not belong to the class constructed by Cohen and Ganley.

For $p=3$ there are four nonisomorphic $p$-primitive semifield planes; two of these are Hughes-Kleinfeld semifield planes, one is a Dickson semifield plane and the other is the plane of Boerner-Lantz of order 81. For $p\geq 5$ we say that a $p$-primitive semifield plane is of *type IV* if $f_0\neq 0$ and $f_1^{2(p-1)}\neq 0, 1$, and of *type V* if $f_0\neq 0$ and $f_1^{2(p-1)}=1$. A $p$-primitive semifield plane of type IV which is not a Boerner-Lantz semifield plane and any plane of type V is either a Generalized twisted field plane or is a new plane. The distinction of these two cases is currently under investigation.

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Department of Mathematics
Texas Tech University
Lubbock, Texas 79409
U. S. A.