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Osaka J. Math.
28 (1991), 285–294

ON FINITELY PSEUDO-FROBENIUS RINGS

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(Received April 19, 1990)

In this paper we are concerned with FPF rings and GFC rings. In section 2 we provide some results about these rings; we show that every right GFC ring is essentially bounded (Proposition 4) and give a characterization of right FPF rings (Theorem 11). Finally, we present examples to illustrate Theorem 11.

1. Preliminaries

Throughout this paper R will always denote an associative ring with identity and all R-modules will be unital.

If every finitely generated faithful right R-module is a generator of the category mod-R of right R-modules then R is said to be *right finitely pseudo-Frobenius* (*right FPF*). Following [2], R is said to be *generated by faithful cyclic* (*right GFC*) if every faithful cyclic right R-module is a generator of mod-R. Right FPF rings are obviously right GFC and the class of right FPF rings includes right PF rings and Dedekind domains.

Let M be a right R-module, X (resp. S) a subset of M (resp. R), A a right ideal of R and n a positive integer. Then we denote by $r_R(X)$ (resp. $l_R(S)$) the right (resp. left) annihilator of M (resp. S) in R, by $Tr_R(M)$ the trace ideal of M, i.e., $Tr_R(M)=\Sigma\{\mathrm{Im}(f)\mid f\in\mathrm{Hom}_R(M, R)\}$ and by $Z_r(M)$ the singular submodule of M, i.e., $Z_r(M)=\{x\in M\mid r_R(x)$ is essential in $R_R\}$. Further we denote by $M^{(n)}$ the direct sum of n copies of M. By ideals we will mean two-sided ideals of R.

Let τ be a hereditary torsion theory for mod-R. Then we denote by $L(\tau)$ the Gabriel topology corresponding to τ and by $\tau(M)$ the τ-torsion submodule of M. Set $B/A=\tau(R/A)$. If A is an ideal of R then we see that B becomes an ideal; hence in particular, $\tau(R)$ is an ideal of R. A submodule N of M is *τ-closed* in M if M/N is τ-torsionfree. We let G denote the Goldie torsion theory for mod-R. We then note that M is G-torsionfree if and only if $Z_r(M)=0$, i.e., M is right non-singular.

We refer to [8] for all the torsion-theoretic notions used in this paper.

The following easy result will be used repeatedly without reference throughout the sequel.

Lemma. *For a right ideal A of R, $Tr_R(R/A)=l_R(A)R$.*

2. FPF (GFC) rings

A submodule N of a right R-module M is *essentially closed* in M if it has no proper essential extensions inside M, or equivalently there exists a submodule L of M such that N is maximal with respect to $N \cap L=0$. We note that every G-closed submodule of M is essentially closed in it. Further, it is easy to show that if $L \leqq N \leqq M$ are right R-modules such that L is essentially closed in M and N is essential in M then N/L is essential in M/L.

Now, the following result is easy.

Lemma 1. *An ideal I of R is G-closed in R_R if and only if it is essentially closed in R_R and R/I is right non-singular over R/I.*

Lemma 2. *Let I be an ideal of R and A a right ideal of R such that $I+A$ is essential in R. If R/A is a generator of mod-R then I is essential in R_R.*

Proof. Assume that R/A is a generator of mod-R, that is, $l_R(A)R=R$. Then there exists a finite number of elements $a_i \in l_R(A)$ and $b_i \in R$ $(i=1, \cdots, n)$ such that $1=\sum_{i=1}^{n} a_i b_i$. Setting $B=\{x \in R \mid b_i x \in I+A \text{ for all } i=1, \cdots, n\}$, we see from the essentiality of $I+A$ that B is an essential right ideal of R. It then follows that I is essential in R_R, because $B \leqq I$.

The following result shows that if R is right GFC then $Z_r(R)$ contains all nilpotent one-sided ideals of R.

Proposition 3. *Assume that R is right GFC, and let A be a nilpotent right ideal of R. Then $r_R(A)$ is essential in R_R.*

Proof. Let n be the nilpotent index of A. The assertion is clear for $n=1$.

Now let $n>1$ and assume that the assertion is true for every nilpotent right ideal of R with nilpotent index $n'<n$. Choose a right ideal B of R maximal with respect to $B \leqq r_R(A^2)$ and $B \cap r_R(A)=0$. Then $B \oplus r_R(A)$ is essential in $r_R(A^2)$. Since A^2 has nilpotent index $\leqq n-1$, the induction hypothesis assures that $r_R(A^2)$ is essential in R_R. Thus $B \oplus r_R(A)$ is essential in R. On the other hand, we have $Ar_R(R/B) \leqq B \cap r_R(A)=0$; hence $r_R(R/B) \leqq B \cap r_R(A)=0$. Since R is right GFC, R/B is a generator of mod-R. It now follows from Lemma 2 that $r_R(A)$ is essential in R_R.

If every essential right ideal of R contains an ideal essential in R as a right ideal then R is said to be *right essentially bounded*. By [3, Proposition 1.3B], every essential right ideal of a right FPF ring contains a non-zero ideal. On the other hand, by [4, Corollary 2.2.], a left Noetherian, right FPF and right order in a QF ring is right essentially bounded. However, we see that every right GFC ring is right essentially bounded. To show this, let A be an essential right ideal of a right GFC ring R, and choose a right ideal B of R maximal with respect to $B \leqq A$ and $r_R(R/A) \cap B = 0$. We then see that $r_R(R/A) \oplus B$ is essential in R, and further that R/B is faithful; hence it is a generator of mod-R. Now Lemma 2 shows that $r_R(R/A)$ is essential in R_R, as desired. Thus we have the following result.

Proposition 4. *Every right GFC ring is right essentially bounded.*

From the above two Propositions, we obtain the following result.

Corollary 5. *Assume that R is right GFC. Then an ideal I of R is G-closed in R_R if and only if it is a semiprime ideal which is essentially closed in R_R.*

Proof. Assume that I is G-closed in R_R. To show that I is a semiprime ideal of R, let J be an ideal of R such that $I \leqq J$ and $J^2 \leqq I$. Choose a right ideal A of R such that $A \leqq J$ and $A \cap I = 0$. Since R/I is a non-singular right R-module, so is A. On the other hand, $A^2 \leqq A \cap J^2 \leqq A \cap I = 0$; hence Proposition 3 implies $A \leqq Z_r(R)$. Thus we have $A = 0$, which shows that I is essential in J_R. Since I is essentially closed in R_R by Lemma 1, we must have $I = J$. Therefore, I is indeed a semiprime ideal of R.

Conversely, assume that I is a semiprime ideal which is essentially closed in R_R, and set $\bar{R} = R/I$. According to Lemma 1, it suffices to show that $\bar{R}_{\bar{R}}$ is non-singular. Let $x + I \in Z_r(\bar{R})$, and set $A = \{a \in R \mid xa \in I\}$. Then A is an essential right ideal of R, and $r_{\bar{R}}(x + I) = A/I$. By Proposition 4, A contains an ideal H essential in R_R. Set $\bar{H} = (H + I)/I$. Since I is essentially closed in R_R, the essentiality of H implies that $\bar{H}$ is essential in $\bar{R}_{\bar{R}}$. Now, $(l_{\bar{R}}(\bar{H}) \cap \bar{H})^2 \leqq l_{\bar{R}}(\bar{H})\, \bar{H} = 0$; hence we see that $l_{\bar{R}}(\bar{H}) = 0$, because $\bar{R}$ is a semiprime ring. Thus we have $x + I \in l_{\bar{R}}(\bar{H}) = 0$, from which we conclude that $\bar{R}_{\bar{R}}$ is non-singular.

Immediately, Corollary 5 implies the following result which is a generalization of [2, Proposition 2.5] and [3, Theorem 3.3].

Corollary 6. *A right GFC ring is right non-singular if and only if it is a semiprime ring.*

By [8, Proposition VI, 6.2], we have $G(R) = \{x \in R \mid x + Z_r(R) \in Z_r(R/Z_r(R))\}$. Thus [3, Theorem 5.1] shows that if R is right FPF then $G(R)$ is a direct summand of R as a right ideal and $R/G(R)$ is a non-singular right FPF ring. More generally we have the following result.

Proposition 7. *Assume that R is right FPF, and let I be an ideal which is G-closed in R_R. Then*

(1) *I is a direct summand of R_R.*

(2) *R/I is a right and left non-singular right FPF ring.*

Proof. (1) Choose a right ideal A of R maximal with respect to $A\cap I=0$. Then $R/A\oplus R/I$ is finitely generated faithful; hence by assumption, $R=Tr_R(R/A\oplus R/I)=Tr_R(R/A)+Tr_R(R/I)=l_R(A)R+l_R(I)$. Set $\bar{R}=R/I$ and $\bar{A}=(A\oplus I)/I$. Then, observing that I is essentially closed in R_R by Lemma 1 and that $A\oplus I$ is essential in R, we see that $\bar{A}$ is an essential right ideal of $\bar{R}$. Since $\bar{A}\leqq r_{\bar{R}}(x+I)$ for every $x\in l_R(A)$, it follows from the essentiality of $\bar{A}$ and Lemma 1 that $l_R(A)\leqq I$. Thus we obtain $R=I+l_R(I)$. Writing $1=a+b$ where $a\in I$ and $b\in l_R(I)$, we see that a is an idempotent of R and $I=aR$. Consequently, I is a direct summand of R_R.

(2) Let M be a finitely generated faithful right $\bar{R}$-module and set $X=I\oplus M$. Since $r_R(X)=r_R(I)\cap r_R(M)=r_R(I)\cap I$, we see from (1) that $r_R(X)=0$; hence X is a finitely generated faithful right R-module. Thus by assumption, in particular, X generates R/I, while (1) says $\mathrm{Hom}_R(I_R, (R/I)_R)=0$. It then follows that M generates R/I as a right R-module and so does as a right (R/I)-module. Therefore we conclude that R/I is a right FPF ring. Moreover, Lemma 1 and [3, Theorem 3.6] imply that R/I is a right and left non-singular ring.

As consequences of Proposition 7, we obtain the following results.

Corollary 8. *If R is right FPF then every G-closed right ideal of R is a right annihilator ideal of R.*

Proof. Given any G-closed right ideal A of R, choose a right ideal C of R maximal with respect to $C\leqq r_R\, l_R(A)$ and $A\cap C=0$. If $C=0$ then we see from the G-closedness of A that $A=r_R\, l_R(A)$, which completes the proof. Thus it is enough to show that $C=0$.

Choose a right ideal B of R maximal with respect to $A\leqq B$ and $B\cap C=0$. Since C is non-singular and R/B is an essential extension of C, we see that B is G-closed in R; hence $G(R)\leqq B$. On the other hand, observing that $B\oplus C$ is essential in R and that it is contained in $r_R\, l_R(B)$, we see that $r_R\, l_R(B)$ is essential in R; hence $l_R(B)\leqq G(R)$. Thus we have $l_R(B)\leqq r_R(R/B)$, which implies $Tr_R(R/B)\leqq r_R(R/B)$. Since B is G-closed in R_R and hence so is $r_R(R/B)$, Proposition 7 shows that R is a direct sum of $r_R(R/B)$ and a right ideal of R generated by R/B; hence in particular, we have $R=r_R(R/B)+Tr_R(R/B)$. It then follows $R=r_R(R/B)$, that is, $B=R$, from which C must be zero, as desired.

Corollary 9. *Assume that R is right FPF. If M is a finitely generated non-singular right R-module with finite Goldie dimension then $End_R(M)$ is a two-sided order in a semisimple ring.*

Proof. Since $r_R(M)$ is G-closed in R_R and M is non-singular as a right

$R/r_R(M)$-module, without loss of generality we may assume by Proposition 7 that M is faithful and R is non-singular. It then follows that R is isomorphic to a direct summand of a finite direct sum of copies of M; hence R_R has finite Goldie dimension, because M has finite Goldie dimension. Now, we see from Corollary 6 and [3, Corollary 3.16C] that R is a semiprime right and left Goldie ring. Therefore, [6, Theorems 2.2.15 and 2.2.17] show that $End_R(M)$ is a two-sided order in a semisimple ring.

Let τ be a hereditary torsion theory for mod-R. Then τ is *stable* if the τ-torsion class is closed under injective envelopes, and $L(\tau)$ is *bounded* if it contains a cofinal subset consisting of ideals of R. We note from [8, Proposition VI, 7.3] that G is stable, and from [8, Chapter VI, Section 6.3] that if R is right non-singular then $L(G)$ consists of all the essential right ideals of R; hence R is right essentially bounded if and only if $L(G)$ is bounded.

To provide a characterization of right FPF rings, we need the following result.

Lemma 10. *Let τ be a stable hereditary torsion theory for mod-R such that $L(\tau)$ is bounded. For a finitely generated right R-module M, the following conditions are equivalent:*

(1) $r_R(M)\leqq\tau(R)$.

(2) $r_R(M/\tau(M)) = \tau(R)$.

Proof. First we shall show $r_R(\tau(M))\in L(\tau)$. To this end, choose a submodule N of M maximal with respect to $\tau(M)\cap N=0$. Observing that τ is stable and that M/N is an essential extension of $\tau(M)$, we see that M/N is τ-torsion. Since M is finitely generated, $M/N=x_1R+\cdots+x_nR$ for a finite number of elements $x_1,\cdots,x_n\in M/N$. Further, since M/N is τ-torsion and $L(\tau)$ is bounded, there exist ideals $I_i\in L(\tau)$ $(i=1,\cdots,n)$ such that $I_i\leqq r_R(x_i)$ for each i. We then see that $\bigcap_{i=1}^{n} I_i\in L(\tau)$ and $\bigcap_{i=1}^{n} I_i\leqq r_R(M/N)\leqq r_R(\tau(M))$, from which we conclude $r_R(\tau(M))\in L(\tau)$.

(1) $\Rightarrow$ (2). Since $L\tau(R)=0$ for every τ-torsionfree right R-module L, we always have $\tau(R)\leqq r_R(M/\tau(M))$. Conversely, according to (1), we have $r_R(M/\tau(M))\,r_R(\tau(M))\leqq r_R(M)\leqq\tau(R)$. Now, noting that $R/\tau(R)$ is τ-torsionfree and that $r_R(\tau(M))\in L(\tau)$ as is seen above, we see $r_R(M/\tau(M))\leqq\tau(R)$. Thus we obtain $r_R(M/\tau(M))=\tau(R)$.

(2) $\Rightarrow$ (1) is clear.

In [7] Kobayashi has provided a characterization of non-singular right FPF rings. Now we state a characterization of right FPF rings, a part of which is an extension of [7, Theorem 1].

Theorem 11. *The following conditions on R are equivalent:*

(1) *R is right FPF.*

(2) (i) *For every finitely generated non-singular right R-module M, R is a direct sum of $r_R(M)$ and a right ideal generated by M.*

(ii) *$L(G)$ is bounded.*

(iii) *Every finitely generated faithful right R-module generates $G(R)$.*

(3) (i) *For every finitely generated right ideal A of R such that $r_R(A)$ is G-closed in R_R, R is a direct sum of $r_R(A)$ and a right ideal generated by A.*

(ii) *$L(G)$ is bounded.*

(iii) *For every finitely generated faithful right R-module M such that $G(M)$ is a direct summand of M, $G(M)$ generates $G(R)$.*

(iv) *Every finitely generated non-singular right R-module can be embedded into a free right R-module.*

Proof. (1) $\Rightarrow$ (2). (2) (i) follows from Proposition 7, and (2) (iii) is clear. To show (2) (ii), let $A \in L(G)$ and set $I=r_R(R/A)$, $G(R/I)=J/I$ and $M=(R/A)\oplus J$. Then J is an ideal which is G-closed in R_R. It follows from Proposition 7 that $J=eR$ for some idempotent e of R and $r_R(M)=I \cap r_R(J) \leqq eR \cap (1-e)R=0$; hence M is finitely generated faithful. According to (1), M is a generator of mod-R, in particular, M generates $(1-e)R$. However, $\mathrm{Hom}_R(M, (1-e)R)=\mathrm{Hom}_R(R/A, (1-e)R)\oplus \mathrm{Hom}_R(J, (1-e)R)=0\oplus 0=0$, from which we see $e=1$. Thus $G(R/I)=R/I$, that is, $r_R(R/A)=I \in L(G)$. Therefore, $L(G)$ is bounded.

(2) $\Rightarrow$ (3). First we shall assume (2) and show the following

Claim 1. *For every ideal I which is G-closed in R_R, R/I is a right FPF ring.*

Set $\bar{R}=R/I$ and let $\bar{G}$ denote the Goldie torsion theory for mod-$\bar{R}$. Since $\bar{R}$ is a right non-singular ring by Lemma 1, $L(\bar{G})$ consists of all the essential right ideals of $\bar{R}$. First we show that $L(\bar{G})$ is bounded. Let $\bar{A}=A/I \in L(\bar{G})$. Then A is essential in R; hence $A \in L(G)$. According to (2) (ii), there exists an ideal J of R such that $J \leqq A$ and $J \in L(G)$. If $\bar{B}=B/I$ is a right ideal of $\bar{R}$ such that $\bar{J} \cap \bar{B}=0$ where $\bar{J}=(J+I)/I$, then $\bar{B}\cdot\bar{J}=\bar{B}\cap\bar{J}=0$, that is, $B\cdot J \leqq I$, from which we have $B \leqq I$, because $(R/I)_R$ is non-singular. Thus $\bar{J}$ is essential in $\bar{R}_{\bar{R}}$, which shows that $L(\bar{G})$ is bounded. Now, we turn to the proof of Claim 1. Let M be a finitely generated faithful right $\bar{R}$-module. We must show that M is a generator of mod-$\bar{R}$. Since $M/\bar{G}(M)$ is a faithful right $\bar{R}$-module by Lemma 10 and M obviously generates $M/\bar{G}(M)$, we may assume that $M_{\bar{R}}$ is non-singular; hence it is non-singular as an R-module, also. According to (2) (i), M generates $R/r_R(M)=\bar{R}_{\bar{R}}$, from which we conclude that $\bar{R}$ is right FPF. This completes the proof of Claim 1.

(3) (i) is immediate from (2) (i) and Claim 1.

To show (3) (iii), let M be a finitely generated faithful right R-module. By (2) (iii), we obtain an exact sequence $M^{(n)} \to G(R) \to 0$, and further it splits, because $G(R)_R$ is projective by (2) (i). Thus we may assume that $M^{(n)}=G(R)\oplus N$ for some integer n and some submodule N of $M^{(n)}$. It now follows $G(M)^{(n)}=G(M^{(n)})=G(R)\oplus G(N)$, from which we see that $G(M)$ generates $G(R)$.

Finally, to show (3) (iv), let M be a finitely generated non-singular right R-module. Then M is finitely generated non-singular as a right $R/r_R(M)$-module, while (2) (i) implies $R=r_R(M)\oplus A$ for some right ideal A of R. It then follows from Claim 1, [3, Theorem 3.12] and [5, Theorem 5.17] that M is embedded into $(R/r_R(M))^{(n)}\cong A^{(n)}\leqq R_R^{(n)}$ for some integer n.

(3) $\Rightarrow$ (1). First we shall assume (3) and show the following

Claim 2. (1) *$G(R)$ is a direct summand of R as a right ideal.*

(2) *For every finitely generated non-singular right R-module M such that $r_R(M)=G(R)$, M generates $R/G(R)$.*

Let M be a finitely generated non-singular right R-module such that $r_R(M)=G(R)$. By (3) (iv), we obtain an exact sequence $0\rightarrow M\xrightarrow{f}R^{(n)}$ for some integer n. Let p_i: $R^{(n)}\rightarrow R$ be the i-th projection ($i=1, \cdots, n$) and set $A=\sum_{i=1}^{n}p_i f(M)$. Then A is a finitely generated right ideal of R and $r_R(A)=G(R)$; hence (3) (i) says that R is a direct sum of $G(R)$ and a right ideal B generated by A. Since M obviously generates A, it also generates $B\cong R/G(R)$, which completes the proof of Claim 2.

To show that R is right FPF, let M be a finitely generated faithful right R-module, and choose a submodule N of M maximal with respect to $N\cap G(M)=0$. Since M/N is an essential extension of $G(M)$, it is G-torsion; hence setting $X=M/G(M)\oplus M/N$, we see that $G(X)=M/N$ and that X is finitely generated faithful. It now follows from (3) (iii) that $G(X)=M/N$ generates $G(R)$. On the other hand, by (3) (ii) and Lemma 10, we have $r_R(X/G(X))=G(R)$; hence Claim 2(2) shows that $X/G(X)\cong M/G(M)$ generates $R/G(R)$. Since M obviously generates both M/N and $M/G(M)$ and since $R\cong G(R)\oplus(R/G(R))$ by Claim 2(1), M generates R. This completes the proof of the theorem.

Assume that R is non-singular right FPF and let M be a finitely generated non-singular right R-module. It then follows from Theorem 11 that $R=r_R(M)\oplus A$ where A is a right ideal of R generated by M. Since R is a semiprime ring by Corollary 6, we see $\mathrm{Hom}_R(M, r_R(M))=0$, which implies $A=Tr_R(M)$. Thus $R=r_R(M)\oplus Tr_R(M)$ as ideals. Therefore, as a consequence of Theorem 11, we obtain the following result, in which (1)$\Leftrightarrow$(3) is due to [7, Theorem 1] (c.f. [5, Theorem 5.17]).

Corollary 12. *For a right non-singular ring R, the following conditions are equivalent:*

(1) *R is right EPF.*

(2) (i) *For every finitely generated non-singular right R-module M, $R=r_R(M)\oplus Tr_R(M)$ as ideals.*

(ii) *R is right essentially bounded.*

(3) (i) *For every finitely generated right ideal A of R, $R=r_R(A)\oplus Tr_R(A)$ as ideals.*

(ii) *R is right essentially bounded.*

(iii) *Every finitely generated non-singular right R-module can be embedded into a free right R-module.*

We call a ring homomorphism $\psi\colon R\to S$ a *flat epimorphism* if it is an epimorphism in the category of rings (or equivalently, the natural homomorphism $S\otimes_R S\to S$ is an isomorphism by [8, Chapter XI, Section 1]) and S is flat as a right R-module. We note that if both $\psi\colon R\to S$ and $\zeta\colon S\to T$ are flat epimorphisms then so is $\zeta\psi\colon R\to T$. For the Goldie torsion theory G for mod-R, we denote by Q_G the ring of quotients of R with respect to G and by $\varphi\colon R\to Q_G$ the canonical ring homomorphism.

Now assume that R is right FPF, and set $Q=Q_G$. Since $\varphi(R)\cong R/G(R)$ is projective as a right R-module by Proposition 7, we see that $\varphi\colon R\to\varphi(R)$ is a flat epimorphism. We also note from [8, Chapter IX, Sections 1 and 2] that $\mathrm{Hom}_R((Q/\varphi(R))_R, Q_R)=0$ and Q_R is injective and non-singular, and from Theorem 11 that if $x\in Q$ then $\varphi(R)+x\varphi(R)$ can be embedded into $R^{(n)}$ (in fact, into $\varphi(R)^{(n)}$) for some integer n. Now, following the same argument as in the proof of $(a)\Rightarrow(b)$ of [5, Theorem 5.17], we see that if $x\in Q$ and $J=\{\varphi(r)\in\varphi(R)\mid\varphi(r)x\in\varphi(R)\}$ then $QJ=Q$. It then follows from [5, Theorem 3.9] that the inclusion map $\varphi(R)\to Q$ is a flat epimorphism. Thus we have the following result.

Corollary 13. *If R is right EPF then $\varphi\colon R\to Q_G$ is a flat epimorphism.*

Finally, we present examples to illustrate Theorem 11.

Example 1. *There exists a ring satisfying the conditions* (2) (*ii*) *and* (*iii*) ((3) (*ii*), (*iii*) *and* (*iv*)) *of Theorem* 11, *but not FPF.*

Proof. Set $R=\{(x,y)\in Z\times Z\mid x\equiv y \pmod 2\}$ where Z is the ring of integers. Then R is a commutative semiprime Noetherian ring; hence it satisfies (2) (ii) and (iii) ((3) (ii), (iii) and (iv)) of Theorem 11.

Now, set $A=(2,0)R\oplus(0,2)R$. Then A is finitely generated faithful, but $Tr_R(A)=A\neq R$; hence R is not FPF.

Example 2. *There exists a ring satisfying the conditions* (2) (*i*) *and* (*iii*) ((3) (*i*), (*iii*) *and* (*iv*)) *of Theorem* 11, *but not FPF.*

Proof. Let R be a simple principal ideal domain but not a skew field (c.f.

[6, Proposition 1.3.8]). Then R satisfies the conditions (2) (i) and (iii) ((3)(i), (iii) and (iv)) of Theorem 11, while $L(G)$ is not bounded; hence R is not FPF by Theorem 11.

EXAMPLE 3. *There exists a ring satisfying the conditions* (2) (*i*) *and* (*ii*) ((3) (*i*), (*ii*) *and* (*iv*)) *of Theorem* 11, *but not FPF.*

Proof. Let F be a field and set $R=\begin{bmatrix} F & F[x]/(x^2) \\ 0 & F[x]/(x^2) \end{bmatrix}$. Then $Z_r(R)=\begin{bmatrix} 0 & (x)/(x^2) \\ 0 & (x)/(x^2) \end{bmatrix}$ and it is essential in R_R; hence R_R is G-torsion, from which it trivially satisfies the conditions (2) (i) and (ii) ((3) (i), (ii) and (iv)) of Theorem 11.

Now, set $A=\begin{bmatrix} F & F[x]/(x^2) \\ 0 & 0 \end{bmatrix}$. Then A is a faithful right ideal generated by $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, but $Tr_R(A)=A\neq R$; hence R is not right FPF.

EXAMPLE 4. *There exists a ring satisfying the conditions* (3) (*i*), (*ii*) *and* (*iii*) *of Theorem* 11, *but not FPF.*

Proof. Let F be a field, and set $F=F_i$ for $i=1, 2, \cdots$, and $R=\{x=(x_i)\in \prod_{i=1}^{\infty} F_i \mid$ there exists an integer n such that $x_n=x_i$ for all but finitely many $i\}$. Then R is a commutative von Neumann regular ring which is not self-injective, and it then satisfies the conditions (3) (i), (ii) and (iii) of Theorem 11. But, [5, Theorem 3.12] and Theorem 11 imply that R is not FPF.

Acknowledgement. The author would like to express his thanks to Professor Y. Kurata for his valuable advice in preparing this paper,

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