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THE ARTINIAN Λ-MODULE AND THE PAIRING ON THE CYCLOTOMIC Z_l-EXTENSIONS

YASUMASA AKAGAWA

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Introduction

Let l be a prime number, $\boldsymbol{Z}_l$ the ring of the l-adic integers, and $\Lambda=\boldsymbol{Z}_l[[T]]$ the formal power series ring of indeterminate T over $\boldsymbol{Z}_l$. Let K be an algebraic number field containing ζ_1 (and $\sqrt{-1}$ if $l=2$) and $k_\omega=k(\zeta_\infty)=k(\zeta_n|n=1, 2, \cdots)$ the cyclotomic $\boldsymbol{Z}_l$-extension over k; $\zeta_n=\exp(2\pi i/l^n)$. Given an abelian extension M/k_ω which is Galois over k and restricted by some local conditions, we can regard the Galois group $\mathrm{Gal}(M/k_\omega)$ as a Noetherian Λ-module and develope the so-called Iwasawa theory. In this paper we shall treat such Noetherian Λ-modules comming from Galois groups and their (twisted) duals, which are regarded as Artinian Λ-modules naturally. The main instrument for the study is a pairing Ψ on some two Artinian Λ-modules X and Y. In [4] a pairing works effectively but our Ψ is different from this essentially, Ψ is actually defined on the whole $X\times Y$ and non-degenerate except Λ-divisible parts and a finite factor. So we shall know that X and Y have similar types of Artinian Λ-modules each other. Specially if we take the maximal unramified abelian l-extension over k_ω fully decomposed at every prime spot over (l) on the one hand and an l-ramified abelian l-extension which is maximal under a local condition such that any $\zeta_n\in k(\zeta_n)$ is written as a local norm from this field to $k(\zeta_n)$ at every spot on the other hand, the results will be most typical. Actually the arguments of this case will be used effectively in the study of Leopoldt's conjecture.

1. Noetherian Λ-modules

Throughout this paper we fix a prime number l. Let $\boldsymbol{Z}_l$ be the ring of the l-adic integers and $\Lambda=\boldsymbol{Z}_l[[T]]$ be the ring of formal power series of indeterminate T over $\boldsymbol{Z}_l$. It is well known that Λ is a local ring of Krull dimension 2, with the maximal ideal $\boldsymbol{m}=(l, T)$. A proper prime ideal $\boldsymbol{p}$ of Λ is always principal and written $\boldsymbol{p}=(l)$ or $\boldsymbol{p}=(P(T))$ by a distinguished polynominal $P(T)\in\boldsymbol{Z}_l[T]$, i.e. the one of the form $P(T)=T^n+a_{n-1}T^{n-1}+\cdots+a_0\equiv T^n \mod (l)$ in $\boldsymbol{Z}_l[T]$. The unit group $\Lambda^\times$ of Λ has a subgroup $(1+T)^{\boldsymbol{Z}_l}$ isomorphic to $\boldsymbol{Z}_l$ in the evident manner through multiplication-addition translation. Let Γ be a topological group isomorphic to $\boldsymbol{Z}_l$ with a generator γ: $\Gamma=\langle\gamma\rangle=\gamma^{\boldsymbol{Z}_l}$. A $\boldsymbol{Z}_l$-Γ-module is a Λ-module as it were, defining the action of γ on it to coincide with the multiplication map of $1+T$. Put $T_m=(1+T)^{l^m}-1\in\boldsymbol{Z}_l[T]$ a distinguished polynomial, and $\boldsymbol{Z}_l[[T_m]]=\Lambda_m\subset\Lambda$. Put $\gamma_m=\gamma^{l^m}$, $\Gamma_m=\langle\gamma_m\rangle\subset\Gamma$; $m=0, 1, \cdots$. A Λ-module or a $\boldsymbol{Z}_l$-Γ-module is a Λ_m-module or a $\boldsymbol{Z}_l$-Γ_m-module in the same time by the restrictions, making the correspondence $1+T_m\rightleftarrows\gamma_m$. A characterestic Λ_m-submodule of a Λ-module is a characteristic Λ-submosule as it were. From now on we treat only locally compact modules. For a Λ-module M, the torsion, the Λ-torsion, the divisibility, and the Λ-divisibility are denoted by

(1.1) $$\text{Tor } M=\{\sigma\in M\,|\,z\sigma=0 \text{ for some } z(\neq 0)\in\boldsymbol{Z}_l\}$$

(1.2) $$\Lambda\text{-tor } M=\{\sigma\in M\,|\,f(T)\sigma=0 \text{ for some } f(T)\,(\neq 0)\in\Lambda\}$$

(1.3) $$l^\infty M=\{\sigma\in M\,|\,\sigma=z\tau \text{ by a } \tau\in M \text{ for any } z(\neq 0)\in\boldsymbol{Z}_l\}$$

(1.4) $$\Lambda^\infty M=\{\sigma\in M\,|\,\sigma=f(T)\tau \text{ by a } \tau\in M \text{ for any } f(T)\,(\neq 0)\in\Lambda\}\,.$$

We shall denote the direct sum of two modules M and N by $M\dotplus N$ and that of r copies of M by $\dot{r}M$. A Λ-homomorphism φ: $M\rightarrow N$ with finite kernel and finite cokernel is called a pseudo-Λ-isomorphism, and denoted by φ: $M\succsim N$. Given M and N, when there is a φ: $M\succsim N$ we denote $M\succsim N$ and when $M\succsim N$ and $N\succsim M$, $M\overset{\sim}{\rightleftarrows}N$. When a non-negative integer r and a set of prime power ideals $\{\boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s}\}$ in Λ are given, we put

$$E(r;\boldsymbol{p}_1^{e_1},\cdots,\boldsymbol{p}_s^{e_s})=\dot{r}\Lambda\dotplus\Lambda/\boldsymbol{p}_1^{e_1}\dotplus\cdots\dotplus\Lambda/\boldsymbol{p}_s^{e_s}\,.$$

We shall call this typical Noetherian Λ-module an elementary Noetherian Λ-module and $\{r:\boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s}\}$ its invariant. Two elementary Noetherian Λ-modules are pseudo-Λ-isomorph (actually Λ-isomorph) only when their invariants coincide. Use an abbreviation $E(0;\,\boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s})=E(\boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s})$.

Theorem 1.1. (Iwasawa-Serre-Cohn and others [5]) *For a Noetherian Λ-module M there is an elementary Noetherian Λ-module*

$$E(M)=E(r;\,\boldsymbol{p}_1^{e_1},\cdots,\boldsymbol{p}_s^{e_s})$$

such that

$$M\succsim E(M)\,.$$

The invariant of $E(M)$ is uniquely determined depending only on M, not on φ: $M\succsim E(M)$. For any φ: $M\succsim E(M)$, Ker φ *coincides always with the characteristic Λ-module* Fin M *the maximal finite Λ-submodule of M.* □

The pseudo-Λ-isomorphism $M\succsim E(M)=E(r;\,\boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s})$ does not mean $E(M)\succsim M$. But, if $r=0$ we can compose $E(M)\succsim M$ easily. For example, if

$\varphi: M \simeq E(M)$ is injective with $r=0$ and l^cCoker $(\varphi: M \simeq E(M))=\{0\}$, $c\geq 0$, we can form a Λ-homomorphism $\varphi': E(M) \simeq M$ with trivial kernel and the cokernel such that l^cCoker $\varphi'=\{0\}$ also easily.

We call the invariant of $E(M)$ the invariant of M and denote it by inv M and define the characteristic polynomial of M by

$$f_M(T) = \prod P_i(T)^{e_i} \quad (\boldsymbol{p}_i^{e_i} = (P_i(T)^{e_i}) \in \text{inv } M,\ \boldsymbol{p}_i \neq (l))$$

and the essential exponent of M by

$$\boldsymbol{e}(M) = \max e_i \qquad (\boldsymbol{p}_i^{e_i} \in \text{inv } M,\ \boldsymbol{p}_i = (l))$$
$$(= 0 \text{ if there is no } \boldsymbol{p}_i = (l))\,.$$

When $\boldsymbol{e}(M)=0$ namely $|\text{Tor } M|<\infty$, M is said pseudo-torsion free. The minimal number $e(M)$ such that $l^{e(M)}\text{Tor } M=\{0\}$ is called exponent of M, e.g. $l^{\boldsymbol{e}(M)}M$ is pseudo-torsion free and $l^{e(M)}M$ is torsion free.

Theorem 1.2. (Iwasawa) *For a Noetherian Λ-module M, Λ-tor M is characterized as the maximal Λ-submodule (or Λ_m-submodule, $m\geq 0$) of M with finite $\boldsymbol{Z}_l$-rank therefore*

$$\Lambda_m\text{-tor } M = \Lambda\text{-tor } M \qquad \textit{for any} \quad m\geq 0\,.$$

Put $\deg f_M(T)=\lambda$. *Then*

(1.5) $$\Lambda\text{-tor } M/\text{Tor } M \cong \dot{\lambda}\boldsymbol{Z}_l \qquad (\textit{as } \boldsymbol{Z}_l\textit{-modules})\,.$$

Specially when M is pseudo-torsion free,

(1.6) $$T_{m'}\Lambda\text{-tor } M = l^{m'-m}T_m\Lambda\text{-tor } M$$

for every $m\gg 0$ (every sufficiently large $m\geq 0$) and $m'\geq m$ and (1.5) *can become precisely*

(1.7) $$\Lambda\text{-tor } M = (\Lambda\text{-tor } M)_{fr} \dotplus \text{Fin } M \qquad (\Lambda_m\textit{-direct})$$

for every $m\gg 0$ where $(\Lambda$-tor $M)_{fr}$ is a Λ_m-submodule of Λ-tor M (not unique) isomorphic to $\dot{\lambda}\boldsymbol{Z}_l$.

Proof. Only the last statement concerned to (1.7) will be required to prove. Since $|\text{Fin } M|<\infty$, there is an $m_0\geq 0$ such that $T_{m_0}(\Lambda\text{-tor } M) \subset l^{\boldsymbol{e}(M)}\Lambda\text{-tor } M$. When we take as $(\Lambda\text{-tor } M)_{fq}$ any $\boldsymbol{Z}_l$-direct complement of Fin M in Λ-tor M $(\simeq \dot{\lambda}\boldsymbol{Z}_l \dotplus \text{Fin } M)$ it is a Λ_m-submodule for $m\geq m_0$ therefore (1.7) will be obtained. □

For a Noetherian Λ-module M, $l^{\boldsymbol{e}(M)}M$ is pseudo-torsion free. In the remained part of this section we treat only pseudo-torsion free case.

Theorem 1.3. *For a pseudo-torsion free Noetherian* Λ*-module* M

$$M = M_{\Lambda tf} \dot{+} \Lambda\text{-tor } M \qquad (\Lambda_m\text{-direct}) \tag{1.8}$$

for every $m \gg 0$, *where* $M_{\Lambda tf}$ *is a* Λ_m*-torsion free* Λ_m*-submodule of* M (*not nesessarily unique*). *So, combining this with* (1.7),

$$M = M_{\Lambda tf} \dot{+} (\Lambda\text{-tor } M)_{fr} \dot{+} \text{Fin } M \qquad (\Lambda_m\text{-direct}) \tag{1.9}$$

for every $m \gg 0$.

Proof. Let $\varphi\colon M/\Lambda\text{-tor } M \cong \dot{r}_0\Lambda = \dot{r}\Lambda_m (m \geq 0, r = r_m = r_0 l^m)$. Since $|\text{Coker}\varphi| < \infty$, $T_m \text{Coker } \varphi = \{0\}$ for $m \gg 0$. Then by the elementary divisor theory we may put

$$\text{Im } \varphi = (l^{c_1}, T_m) \dot{+} \cdots \dot{+} (l^{c_r}, T_m) \subset \dot{r}\Lambda_m ; m \gg 0 . \tag{1.10}$$

Fix such an m and put $\max\{c_k\} = c$, $m + c = m'$. Take $\sigma_1, \cdots, \sigma_r$ and $\tau_1, \cdots, \tau_r \in M$ such that

$$\varphi(\sigma_k) = l^{c_k} \in (l^{c_k}, T_m) \quad \text{the } k\text{-th direct factor of (1.10)}$$
$$\varphi(\tau_k) = T_m \in \text{the same.}$$

Put $T_m\sigma_k - l^{c_k} \cdot \tau_k = \rho_k$ which is in Λ-tor M. From (1.6) we may assume, renewing m by a large one if necessary, $T_m(\Lambda\text{-tor } M) \subset 2l(\Lambda\text{-tor } M)$ accordingly

$$N_{m'm}(\Lambda\text{-tor } M) \subset l^c(\Lambda\text{-tor } M)$$

where

$$N_{m'm} = T_{m'} T_m^{-1} = 1 + (1 + T_m) + \cdots + (1 + T_m)^{l^c - 1} \in \boldsymbol{Z}_l[T_m] . \tag{1.11}$$

So, we can take $\rho'_k \in \Lambda$-tor M such that $N_{m'm}\rho_k = l^{c_k} \cdot \rho'_k$. Then

$$T_{m'}\sigma_k - l^{c_k}(N_{m'm}\tau_k + \rho'_k) = 0. \tag{1.12}$$

Put $r' = rl^c$ and determine $\sigma'_1, \cdots, \sigma'_{r'}, \tau'_1, \cdots, \tau'_{r'} \in M$ so that

$$\sigma'_{k+1} = \begin{cases} \sigma_{j+1} & \text{if } k = l^c j, \ 0 \leq j < r \\ T_m^i \tau_j & \text{if } k = i + l^c j, \ 1 \leq i < l^c, \ 0 \leq j < r \end{cases}$$

$$\tau'_{k+1} = \begin{cases} N_{m'm}\tau_{j+1} + \rho'_{j+1} & \text{if } k = l^c j, \ 0 \leq j < r \\ T_{m'}\sigma'_{k+1} & \text{if } k = i + l^c j, \ 1 \leq i < l^c, \ 0 \leq j < r \end{cases}$$

and then $c'_1, \cdots, c'_{j'} \geq 0$ by

$$c'_{k+1} = \begin{cases} c_{j+1} & \text{if } k = l^c j, \ 0 \leq j < r \\ 0 & \text{if } k = i + l^c j, \ 1 \leq i < l^c, \ 0 \leq j < r. \end{cases}$$

From (1.12)

$$T_{m'}\sigma'_k = l^{c'_k}\cdot\tau'_k;\ k=1, \cdots, r'$$

therefore

$$\langle\sigma'_1, \cdots, \sigma'_{r'}, \tau'_1, \cdots, \tau'_{r'}\rangle \cong (l^{c'_1}, T_{m'}) \dotplus \cdots \dotplus (l^{c'_{r'}}, T_{m'}) \subset \dot{r}'\Lambda_{m'}$$

namely this can be adopted as $M_{\Lambda tf}$, then (1.8) is $\Lambda_{m'}$-direct. □

We define $c=c(M)\geq 0$ by

$$l^c = \text{exponent of Coker } (\varphi: M/\Lambda\text{-tor } M \lesssim \dot{r}_m\Lambda_m);\ m \gg 0\,,$$

which is used already in the above proof. Every sufficiently large $m\geq 0$ will be said steadily large, when it admits the Λ_m-direct decomposition (1.7), T_m Fin $M=0$, $T_m\text{Coker}(\varphi: M/\Lambda\text{-tor } M \lesssim \dot{r}_m\Lambda_m)=0$, and $T_{m'}\Lambda\text{-tor } M = l^{m'-m}T_{m'}\Lambda\text{-tor } M \subset 2l\Lambda\text{-tor } M$ for any $m'\geq m$.

Proposition 1.4. *Let M be a torsion* free *Λ-torsion Λ-module. Then $M\cong\dot{\lambda}\boldsymbol{Z}_l$ as $\boldsymbol{Z}_l$-module. Let $E(M)=E(\boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s})$. Then there are Λ-submodules $M_1, \cdots, M_s\subset M$ such that $E(M_i)=E(\boldsymbol{p}_i^{e_i})$, $M_i\cap\Sigma_{j\neq i}M_j=\{0\}$ (so $\Sigma_i M_i=\dot{\Sigma}M_i$), and $|M\colon \Sigma_i M_i|<\infty$.*

Proof. The first assertion $M\cong\dot{\lambda}\boldsymbol{Z}_l$ is a direct consequence of Theorem 1.2. Fix a $\varphi\colon M\cong E(M)$ and decompose $E(M)=E(\boldsymbol{p}_1^{e_1})\dotplus\cdots\dotplus E(\boldsymbol{p}_s^{e_s})$. Put $M_i=\varphi^{-1}(\text{Im}\,\varphi\cap E(\boldsymbol{p}_i^{e_i}))$. The three properties for M_i will be easily checked. □

When $E(M)=E(\boldsymbol{p}^e)$ we say the Noetherian Λ-module M is pseudo-indecomposable. From the above arguments, pseudo-indecomposable torsionfree M is characterized as a Noetherian Λ-module such that $|\boldsymbol{p}^e M|<\infty$ but $|\boldsymbol{p}^{e-1}M|=\infty$ for some prime $\boldsymbol{p}=(P(T))$ $(\neq l\Lambda)$ in Λ and $e>0$. This e is determined by $\text{rank}_{\boldsymbol{Z}_l} M=e\cdot\deg P(T)$.

2. Artinian Λ-modules

Let $\boldsymbol{R}$ be the additive group of the real numbers, $\boldsymbol{Z}$ that of rational integers, and $\boldsymbol{T}=\boldsymbol{R}/\boldsymbol{Z}$ be the 1-torus. Let $\boldsymbol{T}_l=\boldsymbol{Q}_l/\boldsymbol{Z}_l$, $\boldsymbol{Q}_l$ being the l-adic rational numbers. From now on we fix a $\kappa\in 2l\boldsymbol{Z}_l$ and define an l-divisible group W by

$$W\cong\varinjlim_n \Lambda/(l^n, T\text{-}\kappa) \tag{2.1}$$

where the injective limit is given by the l-times map

$$\begin{aligned}&\Lambda/(l^n, T-\kappa)\to\Lambda/(l^{n+1}, T-\kappa)\\&(F(T) \bmod (l^n, T-\kappa)\mapsto lF(T) \bmod (l^{n+1}, T-\kappa))\end{aligned} \tag{2.2}$$

namely, $W\cong\boldsymbol{T}_l$ abstructly and $Tw=\kappa w$; $w\in W$. We denote for a Λ-module M

$$\hat{M} = \mathrm{Hom}\,(M, W)$$

which is a $\boldsymbol{Z}_l$-Γ-module, so a Λ-module by the usual right γ-action

$$x^\gamma(\sigma) = (x(\sigma^{\gamma^{-1}}))^\gamma = x((1+\bar{T})\sigma);\ x \in \hat{M},\ \sigma \in M \tag{2.3}$$
$$\text{where } \bar{T} = (1+\kappa)\,(1+T)^{-1} - 1 \in \Lambda\,.$$

For $F(T) \in \Lambda$ we denote $\bar{F}(T) = F(\bar{T})$. Then $F(T) \mapsto \bar{F}(T)$ defines an involutive automorphism (i.e. $\bar{\bar{F}}(T) = F(T)$) of Λ. Since Λ is a pro-l group, the Pontrijagin dual $M^* = \mathrm{Hom}\,(M, \boldsymbol{T})$ of a Λ-module M with left γ-action (i.e. $x^\gamma(\sigma) = (x(\sigma^\gamma))^{\gamma^{-1}} = x(\sigma^\gamma)$) can be identified to Hom $(M, \boldsymbol{T}_l)$ which is, regardless the Γ-action, equal to $\hat{M}$. When a $\boldsymbol{Z}_l$-Γ-module M is given, we made it a Λ-module identifying the action of γ to that of $(1+T)$-multiplication, conserving the same notation M. If we identify the action of γ to $(1+\bar{T})$-multiplication on the other hand, we obtain a new Λ-module which we shall denote by $\bar{M}$. From (2.3)

$$\hat{M} = \bar{M}^*\ (= (M^*)^- = (\bar{M})^* \text{ being the same})\,. \tag{2.4}$$

As we are treating always locally compact modules the following facts are held

i) $\hat{\hat{M}} = M$
ii) $\hat{M}$ is Artinian if and only if M is Noetherian
iii) $l^\infty \hat{M} = \hat{M}$ if and only if Tor $M = \{0\}$
iv) $\Lambda^\infty \hat{M} = \{0\}$ if and only if Λ-tor $M = M$.

When M is Noetherian Λ-module we denote

$$M(n) = M/l^n M;\ n \gg 0$$

and when X is Artinian

$$X(n) = \{x \in X \mid l^n x = 0\}\,;\ n \gg 0$$

(so $M(n) = (\hat{M}(n))^\wedge$). *E.g.* $\boldsymbol{Z}_l(n) \simeq \boldsymbol{T}_l(n) \simeq \boldsymbol{Z}/l^n\boldsymbol{Z}$. When F is Noetherian and Artinian in other words $|F| < \infty$, we use only $n \geq e(F)$, so there will come out no confusion. We call the typical Artinian Λ-module

$$\hat{E}(r;\ \boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s}) = (E(r;\ \boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s}))^\wedge$$
$$= \dot{r}\hat{\Lambda} \dotplus (\Lambda/\boldsymbol{p}_1^{e_1})^\wedge \dotplus \cdots \dotplus (\Lambda/\boldsymbol{p}^{e_s})^\wedge$$

an elementary Artinian Λ-module. We have streightfoward versions of Theorems 1.1$\sim$1.4 as follows.

Theorem 2.1. *For an Artinian Λ-module X there is an elementary Artinian Λ-module $E(X) = \hat{E}(r;\ \boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s})$ such that $E(X) \underset{\to}{\sim} X$. The invariant of $(E(X))^\wedge$ $\{r;\ \boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s}\}$ is uniquely determined dependig only on X but not on the choice of φ: $E(X) \underset{\to}{\sim} X$. For any φ: $E(X) \underset{\to}{\sim} X$, Im φ is always coincided with*

Cofin X *the minimal* Λ*-submodule of* X *with finite index.* □

We call the invariant of $(E(X))^\wedge$ the invariant of X and denote it by inv X namely under the notations of Theorem 2.1 inv $X=\{r; \boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s}\}$. The characteristic polynomial of X, the essential coexponent of X, and the coexponent of X are given by $f_X(T)=f_{\hat{X}}(T)=\Pi P_i(T)^{e_i}$ $(\boldsymbol{p}_i=(P_i(T)))$, $\boldsymbol{c}(X)=\max_{i\ \boldsymbol{p}_i=(l)} e_i$, $l^{c(X)}=$(the exponent of $X/l^\infty X$). When $\boldsymbol{c}(X)=0$, X is called pseudo-l-divisible.

Theorem 2.2. *For an Artinian* Λ*-module* X, $\Lambda^\infty X$ *is characterized as the minimal* Λ*-submodule* (*or* Λ_m*-submodule*, $m\geq 0$) *of* $l^\infty X$ *with the factor module of finite* $\boldsymbol{T}_l$*-rank so uniquely determined for any* $m\geq 0$ *by*

$$\Lambda_m^\infty X = \Lambda^\infty X;\ m\geq 0\,.$$

Put $\deg f_X(T)=\lambda$. *Then*

$$l^\infty X/\Lambda^\infty X \cong \dot{\lambda}\,\boldsymbol{T}_l \qquad (\textit{as } \boldsymbol{Z}_l\textit{-module})\,. \tag{2.6}$$

Specially if X *is pseudo-l-divisible,*

$$l^n \operatorname{Ker} T_{m'} = \operatorname{Ker} T_m;\ T_{m'},\ T_m \in \text{Endomorphlsm } (l^\infty X/\Lambda^\infty X)$$

for any $m\gg 0$ *and* $m'=m+n\geq m$, *and*

$$X/\Lambda^\infty X = (X/\Lambda^\infty X)_{fr} \dotplus \operatorname{Fin} X \qquad (\Lambda_m\textit{-direct}) \tag{2.7}$$

where $(X/\Lambda^\infty X)_{fr}$ *is the* Λ_m*-submodule of* $X/\Lambda^\infty X$ *isomorphic to* $\dot{\lambda}\boldsymbol{T}_l$ *and* Fin$X$ *is a maximal* $\boldsymbol{Z}_l$*-direct factor with finite order* (*not unique*), *so* $(X/\Lambda^\infty X)_{fr}=l^\infty(X/\Lambda^\infty X)$. □

Theorem 2.3. *For a pseudo-l-divisible Artinian* Λ*-module* X

$$X=\Lambda^\infty X \dotplus X_{\Lambda df} \qquad (\Lambda_m\textit{-direct}) \tag{2.8}$$

for every $m\gg 0$ *where* $X_{\Lambda df}$ *is a* Λ_m*-divisibility-free submodule of* X (*not unique*) *so, combining with* (2.7)

$$X = \Lambda^\infty X \dotplus l^\infty(X_{\Lambda df}) \dotplus \operatorname{Fin}X;\ m\gg 0\,. \tag{2.9}$$ □

Corollary 2.4. *When* X *is Artinian in general,*

$$X=(\Lambda^\infty X \dotplus l^\infty(X_{\Lambda df}))+(\text{bounded exponent}) \tag{2.10}$$ □

Theorem 2.5. *Let* X *be a* Λ*-divisibility-free and l-divisible Artinian* Λ*-module. Then* $X\cong\dot{\lambda}\boldsymbol{T}_l$; $\lambda=\deg f_X(T)$. *Fix a* φ: $E(X)\simeq X$ *and let* $E(X)=\hat{E}(\boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s})=\hat{E}(\boldsymbol{p}_1^{e_1})+\cdots, +\hat{E}(\boldsymbol{p}_s^{e_s})$. *When we put* $\varphi(\hat{E}(\boldsymbol{p}_i^{e_i})=X_i$, *we obtain three facts*: *i*) $E(X_i)=\hat{E}(\boldsymbol{p}_i^{e_i})$, ii) $X=X_1+\cdots+X_s$, *and* iii) $|X_i\cap\Sigma_{j\neq i}X_j|<\infty$; $i=1, \cdots, s$. □

As we have seen in Section 1, $E(X)\gtrsim X$ does not mean $X\gtrsim E(X)$. But if $\Lambda^{\infty}X=\{0\}$, after easy discussion we can form the inverse.

When $E(X)=\hat{E}(\boldsymbol{p}^e)$ we say the Artinian Λ-module X is pseudo-indecomposable, similarly as Noetherian case. The pseudo-indecomposable l-divisible Λ-module is characterized as an Artinean Λ-module such that $|\boldsymbol{p}^eX|<\infty$ but $|\boldsymbol{p}^{e-1}X|=\infty$ for some prime $\boldsymbol{p}=(P(T))\ (\neq(l))$ in Λ and $e>0$. Then $E(X)=\hat{E}(\boldsymbol{p}^e)$ and $X\gtrsim \boldsymbol{T}_l^{e\cdot\deg P(T)}$ abstractly.

3. Pairing

We denoted the l^n-torsion of an Artinian Λ-module X by

$$X(\mathrm{n}) = \{x\in X\,|\,l^n x = 0\}.$$

In this section X and Y are Artinian Λ-modules. Assume that there are pairing maps

$$\psi_n\colon X(n)\times Y(n)\to W(n)$$

at all $n\geq 1$ satisfying

$$\psi_n(x+x', y) = \psi_n(x, y)+\psi_n(x', y) \tag{3.1}$$
$$\psi_n(x, y+y') = \psi_n(x, y)+\psi_n(x, y')$$

$$\psi_n(lx'', y) = \psi_{n+1}(x'', y) \tag{3.2}$$
$$\psi_n(x. ly'') = \psi_{n+1}(x, y'')$$

for any $x, x'\in X(n), y, y'\in Y(n), x''\in X(n+1), y''\in Y(n+1)$. Then we call the set $\psi=\{\psi_n\}$ a pairing of $X\times Y$. When a topological group Δ acts on X, Y, and W and ψ satisfies further

$$\psi_n(x^\delta, y^\delta) = \psi_n(x, y)^\delta;\quad \delta\in\Delta \tag{3.3}$$

for $x\in X(n)$ and $y\in Y(n)$, we call ψ a Δ-pairing of $X\times Y$. A Γ-pairing is specially called Λ-pairing, for which (3.3) is equivalent to

$$\psi_n(F(T)x, y) = \psi_n(x, \bar{F}(T)y);\quad F(T)\in\Lambda \tag{3.4}$$

because, if (3.3), $\psi_n(Tx, y)=\psi_n((1+T)x, y)-\psi_n(x, y)=\psi_n(x, (1+T)^{-1}y)^\gamma-\psi_n(x, y)=\psi_n(x, \bar{T}y)$ and vise versa. Let $X'\subset X$ and $Y'\subset Y$ be Λ-submodules. We put

$$X'^{\perp}(\psi_n) = \{y\in Y(n)\,|\,\psi_n(x, y) = 0\ \text{ for any } x\in X'(n)\}$$
$$Y'^{\perp}(\psi_n) = \{x\in X(n)\,|\,\psi_n(x, y) = 0\ \text{ for any } y\in Y'(n)\}.$$

Since

$$X'^{\perp}(\psi_n)\subset X'^{\perp}(\psi_{n+1})\ \text{ and samely }\ Y'^{\perp}(\psi_n)\subset Y'^{\perp}(\psi_{n+1})$$

because of (3.2), we can define

$$X'^{\perp}(\psi) = \varinjlim_n X'^{\perp}(\psi_n) \subset Y \quad \text{and} \quad Y'^{\perp}(\psi) = \varinjlim_n Y'^{\perp}(\psi_n) \subset X$$

which are Λ-submodules respectively if ψ is Λ-pairing. In general

$$X'^{\perp}(\psi)(n) \supset X'^{\perp}(\psi_n)$$

and the equality is held if X' is divisible, because of (3.2). Similar facts will be held for Y'. When $l^d(Y^{\perp}(\psi)) = \{0\}$ for some $d \geq 0$, ψ is said left pseudo-nondegenerate and the minimal d_l of such d is called the left degeneracy of ψ. When $d_l = 0$, ψ is said left nondegenerate. The terminologies about right hand side will be used similarly. We put $\max\{d_1, d_r\} = d(\psi)$ and call it merely degeneracy of ψ.

Proposition 3.1. i) *Let X, X', Y, and Y' be Artinian Λ-modules. Assume there are Λ-homomorphisms*

$$\varphi_X\colon X \to X', \quad \varphi_Y\colon Y \to Y'.$$

If a Λ-pairing $\psi'\colon X' \times Y' \to W$ is given, we can define a Λ-pairing $\psi\colon X \times Y \to W$ by

$$\psi_n(x, y) = \psi'_n(\varphi_X(x), \varphi_Y(y)).$$

ii) *Assume both φ_X and φ_Y are surjective and there are $c \geq 0$ and $c' \geq 0$ such that*

$$l^c(\operatorname{Ker} \varphi_X) = \{0\} \text{ and } l^{c'}(\operatorname{Ker} \varphi_Y) = \{0\}.$$

If there exists a Λ-pairing $\psi\colon X \times Y \to W$, we define $\psi'_n\colon X'(n) \times Y'(n) \to W(n)$ by

$$\psi'_n(\varphi_X(x), \varphi_Y(y)) = \psi_n(l^c x, l^{c'} y).$$

Then ψ'_n is well-defined and $\psi' = \{\psi'_n\}$ is a Λ-pairing on $X' \times Y'$. The succession of this map $\psi \to \psi'$ after the one $\psi' \to \psi$ given in i) *coincides with $l^{c+c'}$-times map $\psi' \to l^{c+c'}\psi'$*

When specially X and Y are divisible (accordingly so are X' and Y'), $\psi' = 0$ will follow only if $\psi = 0$.

Proof. Only the last assetion will be required to prove. From the divisibilities of X and Y any $x \in X(n)$ and $y \in Y(n)$ have $l^{-c-c'}x \in X(n+c+c')$ and $l^{-c-c'}y \in Y(n+c+c')$. If $\psi' = 0$,

$$\begin{aligned} \psi_n(x, y) &= \psi_{n+c+c'}(l^{-c-c'}x, y) \\ &= \psi_{n+c+c'}(l^c(l^{-c-c'}x), l^{c'}(l^{-c-c'}y)) \\ &= \psi'_{n+c+c'}(\varphi_X(l^{-c-c'}x), \varphi_Y(l^{-c-c'}y)) = 0. \end{aligned}$$

□

Our interests are on the pseudo-nondegeneracy of ψ, so the discussion will be limitted in the case where X and Y are divisible.

Theorem 3.2. *Let X and Y be divisible Artinian Λ-modules and $f_X(T)$ and $\tilde{f}_Y(T)$ have no common prime factor. Then any Λ-pairing $\psi: X\times Y\to W$ is trivial.*

Proof. Case 1. One of X and Y is $\hat{\Lambda}$-free, say $X=\dot{r}\hat{\Lambda}$. Take $x\in X(n)$ and $y\in Y(n)$. Since both X and Y are injective limits of finite l-groups, there is $m\gg 0$ such that

$$T_m x=0, \quad T_m y=0, \quad \text{and} \quad T_m W(n)=0.$$

Since $\Lambda=\varprojlim_{m,n}(\Lambda/(l^n, T_m))$, we have $\hat{\Lambda}=\varinjlim_{m,n}(\Lambda/(l^n, T_m))^\wedge$ so

$$X(n)=\dot{r}(\varinjlim_m(\Lambda/(l^n, T_m))^\wedge).$$

Here $(\Lambda/(l^n, T_{m'}))^\wedge\cong(\Lambda/(l^n, T_{m'}))^*$ as Λ_m-modules if $m'>m$ because of $T_mW(n)=0$ and $\Lambda/(l^n, T_{m'})\cong \boldsymbol{Z}_l(n)[\Gamma(m')]$ a self-dual Λ_m-module. Put $\Gamma(m, m')=\Gamma^{l^m}/\Gamma^{l^{m'}}\subset\Gamma(m')=\Gamma/\Gamma^{l^{m'}}$. Since $\boldsymbol{Z}_l(n)[\Gamma(m')]^{\Gamma(m,m')}$ (the submodule of $\Gamma(m,m')$-invariant elements) coincides with the norm group $N_{\Gamma(m\ m')}\boldsymbol{Z}_l(n)[\Gamma(m')]$ we can write with $x'\in X(n)$ and $m'=m+n$,

$$x=N_{m'm}x'=[1+(1+T_m)+\cdots+(1+T_m)^{l^n-1}]x'.$$

So

$$\begin{aligned}\psi_n(x,y)&=\psi_n(N_{m'm}x', y)=\psi_n(x', \overline{N_{m'm}}y)\\&=\psi_n(x', l^n y)=0.\end{aligned}$$

Case 2. One of X and Y is Λ-divisible, say $\dot{r}\hat{\Lambda}\twoheadrightarrow X$ surjective. Think of this $\dot{r}\hat{\Lambda}\to X$ and $Y\xrightarrow{\text{id.}}Y$. From the results of Case 1 and Proposition 3.1, $l^c\psi=0$ if $l^c(\mathrm{Ker}\,(\dot{r}\hat{\Lambda}\to X))=0$. So, from (3.2) and the divisibilities of X and Y, $\psi=0$.
Case 3. $\Lambda^\infty X=\{0\}$ and $\Lambda^\infty Y=\{0\}$. Since $E(X)\twoheadrightarrow X$ and $E(Y)\twoheadrightarrow Y$ are both surjective from the divisivilities of X and Y, we have

$$\tilde{f}_X(T)X=\{0\}, \quad \tilde{f}_Y(T)Y=\{0\}.$$

From GCM $\{f_X(T), \tilde{f}_Y(T)\}=1$ we can find $A(T), B(T)\in\Lambda$ and $m\geq 0$ such that

$$A(T)\tilde{f}_X(T)+B(T)f_Y(T)=l^m.$$

Here, for any $x\in X(n)$ and $y\in Y(n)$ we take $l^{-m}x\in X(m+n)$ and $l^{-m}y\in Y(m+n)$ then

$$\begin{aligned}\psi_n(x,y)&=\psi_{m+n}(x, l^{-m}y)\\&=\psi_{m+n}(B(T)f_Y(T)l^{-m}x.\ l^{-m}y)\\&=\psi_{m+n}(B(T)l^{-m}x,\ l^{-m}\tilde{f}_Y(T)y)\\&=0.\end{aligned}$$

General case. Using Theorem 2.3 we decompose

$$X = X_{\Lambda df} \dot{+} \Lambda^{\infty} X, \quad Y = Y_{\Lambda df} \dot{+} \Lambda^{\infty} Y.$$

From the above results, the four restrictions $\psi|_{X_{\Lambda df} \times Y_{\Lambda df}}, \cdots$ etc. are all naught pairings. □

Corollary 3.3. *When X and Y are divisible and ψ: $X \times Y \to W$ is a Λ-pairing,*

$$Y^{\perp}(\psi) \supset \Lambda^{\infty} X, \quad X^{\perp}(\psi) \supset \Lambda^{\infty} Y. \qquad \square$$

By the similar calculations used in the above proof Case 3, the next theorem is easy therefore the proof is omitted.

Theorem 3.4. *Let X and Y be divisible Artinian pseudo-indecomposable Λ-modules such that $E(X)=\hat{E}(\boldsymbol{p}^e)$, $E(Y)=\hat{E}(\bar{\boldsymbol{p}}^f)$ with $e, f \geq 1$ where $\boldsymbol{p}$ is a prime in Λ. Then, for any Λ-pairing ψ: $X \times Y \to W$,*

$$Y^{\perp}(\psi) \supset \bar{\boldsymbol{p}}^f X \quad \text{and} \quad X^{\perp}(\psi) \supset \boldsymbol{p}^e Y.$$

Therefore if $e>f$ (or $e<f$) ψ is left (or right resp.) degenerate, accordingly if $e \neq f$, ψ is degenerate. □

Let

$$X = \Lambda^{\infty} X + (l^{\infty} X)_{\Lambda df} + (\text{bounded exponent})$$
$$Y = \Lambda^{\infty} Y + (l^{\infty} Y)_{\Lambda df} + (\text{bounded exponent})$$

as in Corollary 2.4. From Corollary 3.3

$$\psi|_{\Lambda^{\infty} X \times *} = 0 \text{ and } \psi|_{* \times \Lambda^{\infty} X} = 0.$$

Of course

$$\psi|_{(\text{bounded exp-}) \times *} \quad \text{and} \quad \psi|_{* \times (\text{bounded exp-})}$$

have both bounded exponents. So, about the pseudo-nondegeneracy of ψ only to investigate

$$\psi|_{(l^{\infty} X)_{\Lambda df} \times (l^{\infty} Y)_{\Lambda df}}$$

is interseting. When the last is pseudo-nondegenerate, we say ψ is essentially pseudo-nondegerate.

Theorem 3.5. *Let X and Y be divisible Λ-divisibility-free Artinian Λ-modules and ψ: $X \times Y \to W$ be a pseudo-nondegenerate Λ-pairing. When $E(X) = \hat{E}(\boldsymbol{p}_1^{e_1}, \cdots, \boldsymbol{p}_s^{e_s})$, $E(Y)$ is of the form*

$$E(Y) = \hat{E}(\bar{\boldsymbol{p}}_1^{e_1}, \cdots, \bar{\boldsymbol{p}}_s^{e_s}).$$

Put

$$X = X_1 + \cdots + X_s, \quad |X_i \cap \Sigma_{j \neq i} X_j| < \infty$$

where $E(X_i) = \hat{E}(p_i^{e_i})$ *the i-th direct factor of* $E(X)$ (*cf. Theoren* 2.5). *Then we can put*

$$Y = Y_1 + \cdots + Y_s, \quad |Y_i \cap \Sigma_{j \neq i} Y_j| < \infty$$

where $E(Y_i) = \hat{E}(\bar{p}_i^{e_i})$ *the i-th direct factor of* $E(Y)$ *and*

$$\psi|_{X_i \times Y_j} \text{ is } \begin{cases} \text{pseudo-nondegenerate} & \text{if } i = j \\ 0 & \text{if } i \neq j . \end{cases}$$

Proof. Let $E(X) = \hat{E}(p_1^{e_1}, \cdots, p^{e_s})$ and $E(Y) = \hat{E}(q_1^{f_1}, \cdots, q_t^{f_t})$. Put $X_2 + \cdots + X_s = X_1'(=0$ if $s=1)$. Then

$$X = X_1 + X_1' \quad \text{and} \quad l^e(X_1 \cap X_1') = 0 \qquad \text{for some} \quad e \geq 0 .$$

Put $Y_1 = l^\infty(X_1'^{\perp}(\psi))$ and $Y_1' = l^\infty(X_1^{\perp}(\psi))$. Since $l^e(X_1(n) \cap X_1'(n)) = 0$, it follows that

$$\begin{aligned} l^e Y(n) &\subset X(e)^{\perp}(\psi_n) \qquad (n \geq e) \\ &\subset X_1^{\perp}(\psi_n) + X_1'^{\perp}(\psi_n) \end{aligned}$$

and consequently

$$Y = l^e Y = Y_1 + Y_1' .$$

From this we know that $s \geq 2$ means $t \geq 2$. Interchanging X and Y, $s=1$ if and only if $t=1$. The proof will be done by the induction about s easily from here. □

4. Λ-modules comming from Galois theory of the cyclotomic Z_l-extension

We fix an algebraic number field k having a finite degree over the rational numer field Q and its algebraic closure k^{alg}/k. The algebraic closure of the local field $k_{\mathfrak{p}}$, the completion of k at a prime spot $\mathfrak{p}$, is obtained by the composite of $k_{\mathfrak{p}}$ and k^{alg}: $k_{\mathfrak{p}}^{alg} = k_{\mathfrak{p}} k^{alg}$. An algerbraic extension of k is always taken in k^{alg}/k and the local one in $k_{\mathfrak{p}}^{alg}/k_{\mathfrak{p}}$. We put

$$\zeta_n = \exp(2\pi i / l^n) \in k^{alg}; \quad n = 0, 1, \cdots .$$

For a local or global field F the rational integer $\nu \geq 0$ such that $\zeta_\nu \in F$ but $\zeta_{\nu+1} \notin F$ will be denoted by $\nu(F)$. When a Galois extension of a field has a pro-l group as its Galois group, we call this extension a Galois l-extension and a subfield of a Galois l-extension merely l-extension. Let $\infty > \nu(F) = \nu \geq 1$ (≥ 2 if $l=2$). We put $F_n = F(\zeta_{\nu+n})$; n$\geq$0, the cyclotomic cyclic extension of degree l^n and $F_\infty = F(\zeta_\infty)$

$=F(\zeta_n | n=1, 2, \ldots)$ the cyclotomic $\boldsymbol{Z}_l$-extension. Let $\mathrm{Gal}\,(F_\omega/F)=\Gamma=\langle\gamma\rangle$ and $\gamma: \zeta_n \mapsto \zeta_n^{1+\kappa}$, $\kappa\in 2l\boldsymbol{Z}_l$, $n=1, 2, \cdots$. We define an involutive automorphism $F(T)\to\bar{F}(T)$ in Λ as in Section 3. Assume we are given a Galois l-extension Ω/F containing F_ω. Put

$$M=\mathrm{Gal}\,(\Omega/F_\omega)/\mathrm{Gal}\,(\Omega/F_\omega)^c$$

where $\mathrm{Gal}\,(\Omega/F_\omega)^c$ denotes the commutator subgroup of $\mathrm{Gal}\,(\Omega/F_\omega)$. After any extending of γ in $\mathrm{Gal}\,(\Omega/F)$, via the inner automorphism $\sigma\mapsto\gamma^{-1}\sigma\gamma$, M becomes a $\boldsymbol{Z}_l$-Γ-module, accordingly a Λ-module. By Kummer theory we can identify

$$\hat{M}(n)=(\Omega^{l^n}\cap F_\omega^\times)/(F_\omega^\times)^{l^n}.$$

Therefore, noting that $((\Omega^{l^n}\cap F_\omega^\times)/(F_\omega^\times)^{l^n})^\Gamma=(\Omega^{l^n}\cap F^\times)/(F^\times)^{l^n}\langle\zeta_{\nu(F)}\rangle$ where $(*)^\Gamma$ means the subgroup of the Γ-invariant elements, we know

Lemma 4.1. (4.1) $(M/\bar{T}M)^\wedge(n)=(\Omega^{l^n}\cap F^\times)/(F^\times)^{l^n}\langle\zeta_{\nu(F)}\rangle$.
Therefore

$$(M/\bar{T}M)^\wedge=\varinjlim_n\ (\Omega^{l^n}\cap F^\times)/(F^\times)^{l^n}\langle\zeta_{\nu(F)}\rangle \tag{4.2}$$

being defined by the l-times map $(\Omega^{l^n}\cap F^\times)/(F^\times)^{l^n}\langle\zeta_{\nu(F)}\rangle\to(\Omega^{l^{n+1}}\cap F^\times)/(F^\times)^{l^{n+1}}\langle\zeta_{\nu(F)}\rangle$ *such that* $x \bmod (F^\times)^{l^n}\langle\zeta_{\nu(F)}\rangle\mapsto x^l \bmod (F^\times)^{l^{n+1}}\langle\zeta_{\nu(F)}\rangle$. □

When $\mathrm{Gal}\,(\Omega/F)$ is a free pro-l group with r free generators we call Ω/F a free pro-l extension of rank r.

Lemma 4.2. *Assume Ω/F is a free pro-l extension of rank r. Fix an $m\geq 0$ and put* $\mathrm{Gal}\,(F_m/F)=\Gamma(m)=\Gamma/\Gamma^{l^m}$. *Then*

$$M\cong(r-1)^{\cdot}\Lambda \tag{4.3}$$

$$\varprojlim_n\,((\Omega^{l^n}\cap F_m^\times)/(F_m^\times)^{l^n})\cong\langle\zeta_{\nu(F)+m}\rangle\times(r-1)^{\cdot}\boldsymbol{Z}_l[\Gamma(m)] \tag{4.4}$$

being defined by the canonical map $(\Omega^{l^{n+1}}\cap F_m^\times)/(F_m^\times)^{l^{n+1}}\to(\Omega^{l^n}\cap F_m^\times)/(F_m^\times)^{l^n}$ $(x \bmod (F_m^\times)^{l^{n+1}}\mapsto x \bmod (F_m^\times)^{l^n})$.

Proof. Take $\{\gamma, \sigma_1, \cdots, \sigma_{r-1}\}$ a free generator system of $\mathrm{Gal}\,(\Omega/F)$ so that γ is as above and $\sigma_i|_{F_\omega}=\mathrm{id.}$, $i=1, \cdots, r-1$. We know for the free pro-l group $\mathrm{Gal}\,(\Omega/F)$ and its normal subgroup $\mathrm{Gal}\,(\Omega/F_n)$ with finite cyclic factor group $\Gamma(n)=\Gamma/\Gamma^{l^n}$,

$$\mathrm{Gal}\,(\Omega/F_n)=\langle\gamma^{l^n}, \gamma^{-j}\sigma_i\gamma^j \,|\, 1\leq i\leq r-1, 0\leq j\leq l^n-1\rangle$$

a free pro-l group of rank $(r-1)l^n+1$. (Schreier's Theorem, regardless pro-l topology. To modify it in the case of pro-l group is an elememtary work.) Therefore

$$\mathrm{Gal}\,(\Omega/F_\omega)/\mathrm{Gal}\,(\Omega/F_n)^c\cong(r-1)^{\cdot}\boldsymbol{Z}_l[\Gamma(n)]\,.$$

Taking $\varprojlim_n$, we have

$$M \cong (r-1)^{\cdot}\Lambda\,.$$

The next (4.4) is a direct consequence of (4.1) and (4.3). □

Now, at each $\mathfrak{p}$ in k we shall fix a free pro-l extension $\Omega^{\mathfrak{p}}/k_{\mathfrak{p}}$ satisfying

$$\Omega^{\mathfrak{p}} \supset k_{\mathfrak{p}\omega}\,. \tag{4.5}$$

When $\mathfrak{p}$ is not on (l), $\Omega^{\mathfrak{p}}$ is necessarily the unramified $\boldsymbol{Z}_l$-extension. For any finite l-extension K/k and a prolongation $\mathfrak{P}\,|\,\mathfrak{p}$, we put

$$\Omega^{\mathfrak{P}} = \Omega^{\mathfrak{p}}K/K_{\mathfrak{P}}$$

which is also a free pro-l extension, because we can regard $\mathrm{Gal}\,(\Omega^{\mathfrak{P}}/K_{\mathfrak{P}}) \subset \mathrm{Gal}\,(\Omega^{\mathfrak{p}}/k_{\mathfrak{p}})$ with finite index. Let $\overline{K_{\mathfrak{P}}^{\times}} = \varprojlim_n K_{\mathfrak{P}}^{\times}/K_{\mathfrak{P}}^{\times l^n}$ the pro-l-closure of $K_{\mathfrak{P}}^{\times}$. Any element $\xi \in \overline{K_{\mathfrak{P}}^{\times}}$ is written as

$$\xi = \lim\,(\xi_n \bmod (K_{\mathfrak{P}}^{\times})^{l^n});\ \xi_n \in K_{\mathfrak{P}}^{\times}\,,\quad \xi_n \equiv \xi_{n+1} \bmod (K_{\mathfrak{P}}^{\times})^{l^n}\,.$$

We call ξ an $\Omega^{\mathfrak{P}}$-element if

$$K_{\mathfrak{P}\omega}(\sqrt[l^n]{\xi_n}) \subset \Omega^{\mathfrak{P}};\quad n = 1, 2, \dots\,.$$

The group of the $\Omega^{\mathfrak{P}}$-elements will be denoted by $E_{\mathfrak{P}}$, which is nothing but the left hand side of (4.4). Therefore

Proposition 4.3. *Let* rank $\mathrm{Gal}\,(\Omega^{\mathfrak{p}}/k_{\mathfrak{p}}) = r_{\mathfrak{p}}$. *Let* $k_{\mathfrak{p}m} = K_{\mathfrak{P}}$. *We have* $\overline{K_{\mathfrak{P}}^{\times}} \supset E_{\mathfrak{P}} \supset \langle \zeta_{\nu(\mathfrak{P})} \rangle$; $\nu(\mathfrak{P}) = \nu(K_{\mathfrak{P}})$, *and*

$$E_{\mathfrak{P}} \cong \langle \zeta_{\nu(\mathfrak{P})} \rangle \times (r_{\mathfrak{p}}-1)^{\cdot}\boldsymbol{Z}_l[\Gamma(m)] \qquad (\textit{direct})\,.$$

Regard $\overline{k_{\mathfrak{p}}^{\times}} \subset \overline{K_{\mathfrak{P}}^{\times}}$ *canonically, the former being composed of all the* $\mathrm{Gal}\,(K_{\mathfrak{P}}/k_{\mathfrak{p}})$-*invariant elements. Then* $E_{\mathfrak{p}} = E_{\mathfrak{P}} \cap \overline{k_{\mathfrak{p}}^{\times}} = N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}} E_{\mathfrak{P}}$. □

A local abelian l-extension $F/K_{\mathfrak{P}}$ will be called an $\Omega^{\mathfrak{P}}$-orthogonal extension if

$$E_{\mathfrak{P}} \subset \overline{N_{F/K_{\mathfrak{P}}}F^{\times}}\ (= \cap_{K_{\mathfrak{P}} \subset F' \subset F,\,[F' : K_{\mathfrak{P}}] < \infty} \overline{N_{F'/K_{\mathfrak{P}}} F'^{\times}} \subset \overline{K_{\mathfrak{P}}^{\times}}$$
$$\text{a compact subset)}$$

For example, if $\mathfrak{P}$ is not on (l), then $\Omega^{\mathfrak{P}} = K_{\mathfrak{P}\omega}$. When $\Omega^{\mathfrak{P}} = K_{\mathfrak{P}\omega}$, $E_{\mathfrak{P}} = \langle \zeta_{\nu(\mathfrak{P})} \rangle$ and an $\Omega^{\mathfrak{P}}$-orthogonal extension is the compound of all the $\boldsymbol{Z}_l$-extensions or one of its subextensions.

Proposition 4.4. *If* $\mathfrak{P}$ *is not on* (l), *an* $\Omega^{\mathfrak{P}}$-*orthogonal extension of* $K_{\mathfrak{P}}$ *is nothing but the cyclotomic* (*or samely, unramified*) $\boldsymbol{Z}_l$-*extension* $\Omega^{\mathfrak{P}}/K_{\mathfrak{P}}$ *or its subexten-*

sion. If $\mathfrak{P}$ is on (l), the maximal $\Omega^{\mathfrak{P}}$-orthogonal exensnion of $K_{\mathfrak{P}}$ is a $([K_{\mathfrak{P}}:\boldsymbol{Q}_l]+2-r_{\mathfrak{P}})$pl $\boldsymbol{Z}_l$-extension:

$$\mathrm{Gal}\,(\text{max. }\Omega^{\mathfrak{P}}\text{-orth.}/K_{\mathfrak{P}}) \cong ([K_{\mathfrak{P}}:\boldsymbol{Q}_l]+2-r_{\mathfrak{P}})^{\cdot}\boldsymbol{Z}_l$$

where $r_{\mathfrak{P}}=\mathrm{rank}\ \mathrm{Gal}\,(\Omega^{\mathfrak{P}}/K_{\mathfrak{P}})$. *In the case $k_{\mathfrak{p}}\subset K_{\mathfrak{P}}\subset k_{\mathfrak{p}\omega}=k_{\mathfrak{p}}(\zeta_\infty)$, an abelian extension $F/k_{\mathfrak{p}}$ is $\Omega^{\mathfrak{p}}$-orthogonal if and only if so is $K_{\mathfrak{P}}F/K_{\mathfrak{P}}$.*

Anyway, any abelian extension in $\Omega^{\mathfrak{P}}/K_{\mathfrak{P}}$ is $\Omega^{\mathfrak{P}}$-orthogonal.

Proof. We may treat only the case $\mathfrak{P}\,|\,(l)$. By Artin-Waples theorem

$$\overline{K_{\mathfrak{P}}^{\times}}/\langle\zeta_{\nu(\mathfrak{P})}\rangle \cong ([K_{\mathfrak{P}}:\boldsymbol{Q}_l]+1)^{\cdot}\boldsymbol{Z}_l\,.$$

Using the local class field theory and Lemma 4.2 we can determine the type of Gal (max. $\Omega^{\mathfrak{P}}$-orth.$/K_{\mathfrak{P}}$) as asserted. Since (after extension to $\overline{k_{\mathfrak{p}}^{\times}}$) norm residue symbol $(\xi, F/k_{\mathfrak{p}})=\mathrm{id.}$ for any $\xi\in E_{\mathfrak{p}}$ if and only if $F/k_{\mathfrak{p}}$ is $\Omega^{\mathfrak{p}}$-orthogonal, we can conclude our proof because $(\xi', K_{\mathfrak{P}}F/K_{\mathfrak{P}})=(N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}\xi', F/k_{\mathfrak{p}})$; $\xi'\in E_{\mathfrak{P}}$ and $N_{K_{\mathfrak{P}}/k_{\mathfrak{p}}}E_{\mathfrak{P}}=E_{\mathfrak{p}}$ by Proposition 4.3. □

Next we shall define global matters. From now on we fix k such that

$$\nu(K)\geq 1 \quad (\geq 2 \quad \text{if } l=2)\,.$$

Let K/k be a finite l-extension, again. If L/K is an l-extension and every $K_{\mathfrak{P}}L$ is in $\Omega^{\mathfrak{P}}$, then we say L/K is an Ω-extension. If M/K is an abelian l-extension and every $K_{\mathfrak{P}}M/K_{\mathfrak{P}}$ is an $\Omega^{\mathfrak{P}}$-orthogonal extension, we say M/K is an $\Omega^{\perp}$-extension. An abelian Ω-extension is always $\Omega^{\perp}$-extension by Proposition 4.3 and an $\Omega^{\perp}$-extension is always l-ramified, i.e. unramified at every $\mathfrak{P}$ not on (l). Noting that the compound of Ω-extensions is again an Ω-extension and samely for $\Omega^{\perp}$-extensions, we can define

$$\Omega^{ab}(K) = \text{the maximal abelian }\Omega\text{-extension of } K$$

$$\Omega^{\perp}(K) = \text{the maximal }\Omega^{\perp}\text{-extension of } K\,.$$

For infinite extension k_ω/k we put

$$\Omega^{ab}(k_\omega) = \cup_{n<\omega}\Omega^{ab}(k_n)$$

$$\Omega^{\perp}(k_\omega) = \cup_{n<\omega}\Omega^{\perp}(k_n)\,.$$

Since both $\Omega^{ab}(k_\omega)$ and $\Omega^{\perp}(k_\omega)$ are Galois over k and contained in the maximal abelian l-ramified l-extension $k^{(l)-ram}/k$,

$$M = \mathrm{Gal}\,(\Omega^{ab}(k_\omega)/k_\omega)$$

$$N = \mathrm{Gal}\,(\Omega^{\perp}(k_\omega)/k_\omega)$$

are Noetherian Λ-modules by Lemma 4.1. Further we put

$$X=\hat{M}$$

$$Y=\hat{N}$$

which are Artinian Λ-modules. We can set

$$X(n)=(\Omega^{ab}(k_\omega)^{l^n}\cap k_\omega^\times)/(k_\omega^\times)^{l^n}$$

$$Y(n)=(\Omega^\perp(k_\omega)^{l^n}\cap k_\omega^\times)/(k_\omega^\times)^{l^n}$$

by Kummer theory.

5. A pairing defined by the triple symbol

Here we shall define a pairing $\Psi: X\times Y\to W$ using the triple symbol ([1]). The symbol $(x, y, z\,|\,k)_{l^n}$ is defined when $\zeta_n\in k$, x and y are strictly orthogonal, and three elements x, y, and z are orthogonal in some conditions. Specially if $l=2$, the definitions are complicated, but if $\zeta_{n+2}\in k$ they are a little simpler (cf. Introduction of [1]). We shall recall them here. Take

$$\bar{x}=(x \bmod (k_\omega^\times)^{l^n})\in X(n),\quad x\in\Omega^{ab}(k_\omega)^{l^n}\cap k_\omega^\times$$

$$\bar{y}=(y \bmod (k_\omega^\times)^{l^n})\in Y(n),\quad y\in\Omega^\perp(k_\omega)^{l^n}\cap k_\omega^\times$$

and $m\gg 0$ so that $x, y, \zeta_n\in k_m$ (then $x\in\Omega^{ab}(k_m)^{l^n}\cap k_m^\times$ and $y\in\Omega^\perp(k_{m'})^{l^n}\cap k_m^\times$ for some $m'\geq m$. From Proposition 4.4 we have also $y\in\Omega^\perp(k_m)^{l^n}\cap k_m^\times$). Then three elements $\{x, y, \zeta_{\nu+m}\}\subset k_m^\times$ are orthogonal mod $(k_m^\times)^{l^n}$ i.e.

$$\left(\frac{x, y}{\mathfrak{p}}\right)_{l^n}=\left(\frac{y, \zeta_{\nu+m}}{\mathfrak{p}}\right)_{l^n}=\left(\frac{\zeta_{\nu+m}, x}{\mathfrak{p}}\right)_{l^n}=1$$

at any $\mathfrak{p}$ in k_m about Hilbert-Hasse symbol and specially $\{x, \zeta_{\nu+m}\}$ are strictly orthogonal mod $(k_m^\times)^{l^n}$, i.e. moreover

$$k_{m\mathfrak{l}}(\sqrt[l^n]{x}, \sqrt[l^n]{\zeta_{\nu+m}})\subset\Omega^{\mathfrak{l}}$$

at any $\mathfrak{l}\,|\,(l)$ in k_m. (Samely as the case $l\neq 2$, in case $l=2$ and $\zeta_{n+2}\in k_m$, we say x and $\zeta_{\nu+m}$ are strictly orthogonal mod $(k_m^\times)^{l^n}$ if some one in $x(k_m^\times)^{l^n}$ and the other in $\zeta_{\nu+m}(k_m^\times)^{l^n}$ are strictly orthogonal. When $l=2$, some more conditions than the above inclusion are required outside l for the strict orthogonality, but in the present case where $\zeta_{n+2}\in k_m$, we may check further only that x and $\zeta_{\nu+m}$ are orthogonal mod $(k_m^\times)^{2^{n+1}}$. These will be known easily if we compair the original definition of strict orthogonality and the present modified one. Of course x and $\zeta_{\nu+m}$ are orthogonal mod $(k_m^\times)^{2^{n+1}}$.) Since $y\in\Omega^\perp(k_m)^{l^n}\cap k_m^\times$ it follows that $(\xi, y\,|\,k_{m\mathfrak{q}})_{l^n}=1$ for $\xi\in(\Omega^{\mathfrak{q}})^{l^n}\cap k_{m\mathfrak{q}}^\times$. So, using the statements at p169 [1], (the l-independence of $\{x, \zeta_{\nu+m}\}$ is not essential as seen in ii) 3 [1]) the symbol in extended sense

$$(x,\ \zeta_{\nu+m},\ y;\ \zeta_n\,|\,k_m)_{l^n} \quad (=(x,\ \zeta_{\nu+m},\ y)_{l^n} \text{ by abbrev.})$$

can be defined. Fix an identification $W=\langle\zeta_\infty\rangle=\langle\zeta_n\,|\,n\geq 1\rangle$ correspomding $w_n=(1 \bmod (l^n,\ T-\kappa))\in W$ to ζ_n. We put

$$\Psi_n(\bar{x},\ \bar{y}) = (x,\ \zeta_{\nu+m},\ y)_{l^n}\,. \tag{5.1}$$

Denote the set of all the $\mathfrak{l}$ in k_m over (l) by $S(k_m)$ or simply by S.

Proposition 5.1. *By means of* (5.1) $\Psi_n(\bar{x},\ \bar{y})$ *is well-defined, namely the value* $(x, \zeta_{\nu+m}, y)_{l^n}$ *in* W *does not depend on the choice of* $m\geq 0$ *and* $x, y\in k_m$ *such that* ζ_n *(and* ζ_{n+2} *if* $l=2)\in k_m$, $\bar{x}=(x \bmod (k_\infty^\times)^{l^n})$, *and* $\bar{y}=(y \bmod (k_\infty^\times)^{l^n})$.

Proof. At first we fix an $m\geq 0$ as above and assume $\bar{x}$ is of order l^n, i.e.

$$x\notin(k_m^\times)^l\langle\zeta_{\nu+m}\rangle\,. \tag{5.2}$$

Put $k_{m+n}=K$. As it is shown in Proposition 1 [1] we can find $a\in K^\times$ satisfying

$$a^{1-\sigma}\equiv x \bmod (K^\times)^{k^n} \tag{5.3}$$

for $\sigma\in\mathrm{Gal}\,(K(\sqrt[l^n]{x})/k_m(\sqrt[l^n]{x}))$ such that $\zeta_{\nu+m+n}{}^\sigma=\zeta_n\zeta_{\nu+m+n}$

$$\begin{aligned}\mathrm{Gal}\,(K(\sqrt[l^n]{x},\ \sqrt[l^n]{a})/K)&\simeq\mathrm{Gal}\,(K(\sqrt[l^n]{x},\ \sqrt[l^n]{a})/k_m(\sqrt[l^n]{x}))\\&\simeq \boldsymbol{Z}_l(n)\times\boldsymbol{Z}_l(n)\end{aligned} \tag{5.4}$$

$$k_{m\mathfrak{l}}(\zeta_{\nu+m+n},\ \sqrt[l^n]{x},\ \sqrt[l^n]{a})\subset\Omega^{\mathfrak{l}} \text{ at any } \mathfrak{l}\in S\,. \tag{5.5}$$

Then the principal ideal (a) in K can be written as

$$(a)\equiv\mathfrak{a}\ (\text{mod } l^n\text{-power, mod } S) \qquad \text{in } K$$

where $\mathfrak{a}$ is an ideal in k_m, having no-S-factor, namely $(a)=\mathfrak{a}$ except l^n-th power ideal and S-factor in K. After these preliminary, the triple symbol is well-defined by

$$(x,\ \zeta_{\nu+m},\ y)_{l^n}=\left(\frac{y\,|\,k_m}{\mathfrak{a}}\right)_{l^n}$$

using the Hilbert symbol on the right hand side. Here we remark that the condition (5.4) is equivalent (under (5.3)) to the splitting of the canonical exact sequence

$$\begin{aligned}1\to\mathrm{Gal}\,(K(\sqrt[l^n]{x},\ \sqrt[l^n]{a})/K)&\to\mathrm{Gal}\,(K(\sqrt[l^n]{x},\ \sqrt[l^n]{a})/k_m)\\&\to\mathrm{Gal}\,(K/k_m)\to 1\end{aligned}$$

in other words

$$\sqrt[l^n]{a}^{\,\sigma^{l^n}-1}=1\,. \tag{5.6}$$

As far as we use (5.6) instead of (5.4), the first assumption (5.2) is of no use for the definition of triple symbol ([1], p175 ii) § 3) so (5.6) is more useful than (5.4). After m is fixed the choices of $x, y \in k_m$ are free by the multiplying of elements of $k_\infty^{l^n} \cap k_m^\times = (k_m^\times)^{l^n} \langle \zeta_{\nu+m} \rangle$ therefore x and y may be replaced by $x\zeta$ and $y\zeta'$; $\zeta, \zeta' \in (k_m^\times)^{l^n} \langle \zeta_{\nu+m} \rangle$. But, even this replacement we can use the same a because $x\zeta \equiv x \bmod (K^\times)^{l^n}$, therefore $\mathfrak{a}$ is reserved and

$$\left(\frac{\zeta'}{\mathfrak{a}}\right)_{l^n} = \left(\frac{\zeta'' \mid K}{\mathfrak{a}}\right)_{l^n}$$

using $\zeta'' \in (K^\times)^{l^n} \langle \zeta_{\nu+m+n} \rangle$ such that $N_{K/k_m} \zeta'' \equiv \zeta' \bmod (k_m^\times)^{l^n}$ and continuing the calculation

$$\begin{aligned} &= \Pi_{\mathfrak{P} \text{ in } \mathfrak{a}, \text{ in } K} \left(\frac{a, \zeta'' \mid K}{\mathfrak{P}}\right)_{l^n} \\ &= \Pi_{\mathfrak{P} \mid (l)} \left(\frac{\zeta'', a \mid K}{\mathfrak{P}}\right)_{l^n} \\ &= 1 \end{aligned}$$

by (5.5). Accordingly

$$\left(\frac{y\zeta'}{\mathfrak{a}}\right)_{l^n} = \left(\frac{y}{\mathfrak{a}}\right)_{l^n}.$$

Thus, we may show the independence of our symbol about the choice of m. Let $m' > m$. The remained task is to show

$$(5.7) \qquad (x, \zeta_{\nu+m'}, y; \zeta_n \mid k_{m'})_{l^n} = (x, \zeta_{\nu+m}, y; \zeta_n \mid k_m)_{l^n}.$$

Assume in a time being

$$(5.8) \qquad y \in \Omega^{ab}(k_m)^{l^n} \cap k_m^\times$$

samely as x. Since $\zeta_{\nu+m} = N_{k_{m'}/k_m} \zeta_{\nu+m'}$, from the transgression theorem of triple symbols ([1], Theorem 1 IV)) we have (5.7). When not necessarily (5.8) is held, let $a' \in K' = k_{m'+n}$ satisfy the equivalents of (5.3), (5.6), and (5.5), over $k_{m'}$. Put $L = k_m(\zeta_{\nu+m'+n}, \sqrt[l^n]{x}, \sqrt[l^n]{a}, \sqrt[l^n]{a'})$ (or $= k_m(\zeta_{\nu+m'+n+1}, \sqrt[l^{n+1}]{x}, \sqrt[l^n]{a}, \sqrt[l^n]{a'})$ if $l=2$). Since

$$k_{m\mathfrak{l}} L \subset \Omega^{\mathfrak{l}} \text{ at each } \mathfrak{l} \in S$$

we have

$$y \in N_{k_{m\mathfrak{l}} L / k_{m\mathfrak{l}}} (k_{m\mathfrak{l}} L)^\times \quad \text{at each} \quad \mathfrak{l} \in S$$

(*c.f.* Lemma 1 [1]) so, using the density theorem in the class field theory we can find $z \in L^\times$ such that

(5.9) $$N_{L/k_m} z \equiv y \bmod ((k_m{}_{\mathfrak{l}}L)^\times)^{l^n} \text{ at each } \mathfrak{l} \in S$$

(5.10) $$(z) = \mathfrak{Z} \pmod{S(L)},$$

$\mathfrak{Z}$ being a prime in L fully decomposed in L/k_m.

Put

$$N_{L/k_m} z = y' \in k_m .$$

Then from the definition we have easily

$$(x, \zeta_{\nu+m}, y'; \zeta_n | k_m)_{l^n} = \left(\frac{y'|k_m}{\mathfrak{a}}\right)_{l^n} = 1 ,$$

$$(x, \zeta_{\nu+m'}, y'; \zeta_n | k_{m'})_{l^n} = \left(\frac{y'|k_{m'}}{\mathfrak{a}'}\right)_{l^n} = 1 ,$$

of course after the checking of the posibility of definition. So, for (5.7) we may prove

(5.11) $$(x, \zeta_{\nu+m}, yy'^{-1}; \zeta_n | k_m)_{l^n} = (x, \zeta_{\nu+m'}, yy'^{-1}; \zeta_n | k_{m'})_{l^n} .$$

But in this time $\{x, \zeta_{\nu+m}, yy'^{-1}\}$ in k_m are strictly orthogonal mod$(k_m)^{l^n}$ by (5.9) and (5.10) accordingly so are $\{x, \zeta_{\nu+m'}, yy'^{-1}\}$ in $k_{m'}$. By the same reason as the case of (5.8) we can obtain (5.11). □

Now, our $\Psi_n: X(n) \times Y(n) \to W(n)$ satisfy (3.1) because of Theorem 1 [1]. When $\bar{x} = (x \bmod (k_\infty)^{l^{n+1}}) \in X(n+1)$ and $\bar{y} = (y \bmod (k_\infty)^{l^n}) \in Y(n)$, $l\bar{x} = (x \bmod (k_\infty)^{l^n}) \in X(n)$ and $\bar{y} = (y^l \bmod (k_\infty)^{l^{n+1}}) \in Y(n+1)$ therefore

$$\begin{aligned} \Psi_n(l\bar{x}, \bar{y}) &= (x, \zeta_{\nu+m}, y)_{l^n} \qquad (x, y \in k_m) \\ &= (x, \zeta_{\nu+m}, y^l)_{l^{n+1}} \\ &= \Psi_{n+1}(\bar{x}, \bar{y}) \end{aligned}$$

which means the former of (3.2). The latter will be obtained by the alternative arguments samely. As (3.3) follows from Theorem 1 III [1] we can conclude

Theorem 5.2. *Our $\Psi = \{\Psi_n\}$ is a Λ-pairing $X \times Y \to W$.* □

6. Quasi-nondegeneracy of Ψ

Lemma 6.1. *Let $\zeta_n \in k$ and an ideal $\mathfrak{a}$ in k have no S-factor. Assume*

(6.1) $$\left(\frac{y|k}{\mathfrak{a}}\right)_{l^n} = 1 \qquad \text{for any} \quad y \in \Omega^\perp(k)^{l^n} \cap k^\times .$$

Then there is an element $c \in k^\times$ such that

(6.2) $$(c) \equiv \mathfrak{a} \ (\text{mod } l^n\text{-th power, mod } S)$$

(6.3) $$k_{\mathfrak{l}}(\sqrt[l^n]{c})\subset\Omega^{\mathfrak{l}} \text{ at every } \mathfrak{l}\,|\,(l)\,.$$

Proof. Let the idele group of k be J_k, the principal idele group P_k, and the idele class group C_k. From the class field theory we can set

$$J_k^{l^n}\cap P_k=P_k^{l^n}$$

so the canonical sequence

$$1\to P_k/P_k^{l^n}\to J_k/J_k^{l^n}\to C_k/C_k^{l^n}\to 1$$

is exact. Any element $y\in\Omega^{\perp}(k)^{l^n}\cap k^{\times}$ defines an idele class character $\chi_y\in\hat{C}_k\subset\hat{J}_k$ by

$$\chi_y(\boldsymbol{x})=\Pi_{\text{all }\mathfrak{p}}\,(x_{\mathfrak{p}},y\,|\,k_{\mathfrak{p}})_{l^n};\ \boldsymbol{x}=(\cdots,x_{\mathfrak{p}},\cdots)\in J_k$$

using local Hilbert-Hasse symbol $(x_{\mathfrak{p}},y\,|\,k_{\mathfrak{p}})_{l^n}$. Define a character group $\bar{\mathfrak{X}}$ by

$$\bar{\mathfrak{X}}=\{\chi_y\in\hat{J}_k\,|\,y\in\Omega^{\perp}(k)^{l^n}\cap k^{\times}\}\subset\hat{C}_k\subset\hat{J}_k\,.$$

The class field theory again says the kernel of $\bar{\mathfrak{X}}$ in $C_k/C_k^{l^n}$ is $(\Pi_{\text{all }\mathfrak{p}}\,E_{\mathfrak{p}})C_k^{l^n}/C_k^{l^n}$. If $\boldsymbol{c}=(\cdots,c_{\mathfrak{p}},\cdots)\in J_k$ is such one that $(\boldsymbol{c})=\mathfrak{a}$ and $c_{\mathfrak{l}}=1$ at every $\mathfrak{l}\in S$, then (6.1) says $\boldsymbol{c}\in(\Pi\ E_{\mathfrak{p}})P_kJ_k^{l^n}$ so there is $c\in P_k\cap\boldsymbol{c}(\Pi\ E_{\mathfrak{p}})J_k^{l^n}$ which will satisfy (6.2) and (6.3) by itself. □

Proposition 6.2. *Take $\bar{x}=(x \bmod (k_{\omega}^{\times})^{l^n})\in X(n)$. Fix $m\geq 0$ such that $x\in k_m$ and an $e\geq 0$. If*

(6.4) $$l^e\psi_n(\bar{x},\bar{y})=0$$

far any $\bar{y}=(y \bmod (k_{\omega}^{\times})^{l^n})\in Y(n)$ defined in k_m (i.e. $y\in k_m$) then we can find $b\in K=k_{m+n}$ such that

(6.5) $$b^{1-\sigma}\equiv x^{l^e} \bmod (K^{\times})^{l^n}$$

for $\sigma\in\mathrm{Gal}(K/k_m)$, σ: $\zeta_{\nu+m+n}\mapsto\zeta_n\xi_{\nu+m+n}$, and

(6.6) $$K(\sqrt[l^n]{x},\sqrt[l^n]{b})\subset\Omega^{ab}(K)\,.$$

(Note that, in (6.4), m is fixed previousely and then $\bar{y}$ runs in $Y(n)$.)

Proof of Proposition 6.2. Take $a\in K$ and determine $\mathfrak{a}$ in k_m as in Proposition 5.1. From (6.4)

$$\left(\frac{y\,|\,k_m}{\mathfrak{a}}\right)_{l^n}^{l^e}=1 \qquad \text{for } \ y\in\Omega^{\perp}(k_m)^{l^n}\cap k_m^{\times}$$

namely

$$\left(\frac{y\,|\,k_m}{\mathfrak{a}^{l^e}}\right)_{l^n}=1\,.$$

From Lemma 6.1 there is $c \in k_m$ such that

$$(c) \equiv \mathfrak{a}^{l^e} \ (\text{mod } l^n\text{-th power})$$

$$k_{m_{\mathfrak{l}}} K(\sqrt[l^n]{c}) \subset \Omega^{\mathfrak{l}} \quad \text{at every} \quad \mathfrak{l} \in S(k_m)$$

So, we may put

$$\mathfrak{b} = a^{l^e} c^{-1}. \qquad \square$$

Proposition 6.3. *Assume* $\lambda(X) \neq 0$ *and fix two numbers* $n > e \geq \mathbf{e}(X)$. *Take an* $\bar{x} \in (l^\infty X)_{\Lambda df}(n)$ *such that* $l^e \bar{x} \neq 0$. *Then*

$$\Psi_n(\bar{x}, \bar{y}) \neq 0 \text{ for some } \bar{y} \in Y(n). \tag{6.7}$$

Proof. Let $m_0 \geq 0$ be the number such that any $m \geq m_0$ is steadily large. Since $(l^\infty X)_{\Lambda df} \cong \dot{\lambda} T_l$, we know for the given n and e, $|(l^\infty X)_{\Lambda df}(n-e)| < \infty$, so there is an $m \gg m_0$ such that

$$T_m (l^\infty X)_{\Lambda df}(n-e) = 0 \tag{6.8}$$

and $\bar{x}$ is defined in k_m *i.e.*

$$\bar{x} = (x \text{ mod } (k_\omega^\times)^{l^n});\ x \in k_m.$$

Assume on the contrary of (6.7)

$$\Psi_n(\bar{x}, \bar{y}) = 0 \quad \text{for every} \quad \bar{y} \in Y(n).$$

From Proposition 6.2 we can find a $b \in K = k_{m+n}$ s atisfying conditions (6.6) and (6.5) in other words, we can set $\bar{b} = (b \text{ mod } (k_\omega^\times)^{l^n}) \in X(n)$ such that

$$-T_m \bar{b} = \bar{x}.$$

These imply

$$l^e \bar{x} = -T_m l^e \bar{b} \in T_m(l^e \cdot X(n)). \tag{6.9}$$

On the other hand, from (6.8) and the Λ_m-direct decomposition

$$l^e X = (l^\infty X)_{\Lambda df} \dotplus \Lambda^\infty X \dotplus (\text{finite}) \qquad (\text{cf. Theorem 2.3})$$

we know

$$l^e (l^\infty X)_{\Lambda df}(n) \cap T_m(l^e \cdot X(n)) \subset (l^\infty X)_{\Lambda df}(n-e) \cap T_m((l^e X)(n-e)) = 0.$$

Since $l^e \bar{x} \neq 0$, this contradicts to (6.9). $\square$

With the alternative assertion to Proposition 6.3 interchanging X and Y, we obtain the next theorem.

Theorem 6.4. *Let $\Psi: X\times Y\rightarrow W$ be the Λ-pairing defined in Section* 5. *This Ψ has the left degeneracy $d_X\leq e(X)$ and the right $d_Y\leq e(Y)$, and consequently Ψ is essentially pseudo-nondegenerate.* □

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Department of Mathematics
Nara Women's University
Nara, 630 Japan