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COHOMOTOPY OF LIE GROUPS

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1. Introduction

The purpose of this note is to study the set $\mathrm{Cdg}(X, n)=\mathrm{Cdg}([X, S^n])$ when $X=G$ is a compact simply connected simple Lie group, where

$$\mathrm{Cdg}: [X, S^n]\to \mathrm{Hom}(\pi_n(X), \pi_n(S^n))$$

assigns the induced homotopy homomorphism f_* to the homotopy class of a map $f: X\to S^n$. To estimate $\mathrm{Cdg}(X,n)$ we introduce an invariant $\mathrm{cdg}(X,n)$ and its stable version ${}^s\mathrm{cdg}(X,n)$, which are non-negative integers or infinity, such that ${}^s\mathrm{cdg}(G,3)$ was denoted by $\mathrm{cd}(G)$ in [9]. We denote by $\mathrm{cdg}_p(X,n)$ the exponent of a prime number p in the prime power decomposition of $\mathrm{cdg}(X,n)$ when $0<\mathrm{cdg}(X,n)<\infty$. For convenience' sake we set $\mathrm{cdg}_p(X,n)=0$ when $\mathrm{cdg}(X,n)=0$. We define ${}^s\mathrm{cdg}_p(X,n)$ similarly. We prove the following two theorems.

Theorem 1. *If G is a compact simply connected Lie group, that is, $G=G_1\times\cdots\times G_t$ with G_i a compact simply connected simple Lie group, then* $\mathrm{cdg}(G, n)$ *and* ${}^s\mathrm{cdg}(G, n)$ *are finite and the following seven statements are equivalent for any prime number p.*

(1) $\mathrm{cdg}_p(G, 3)=0$.
(2) ${}^s\mathrm{cdg}_p(G, 3)=0$.
(3) $\mathrm{cdg}_p(G_i, 3)=0$ *for all i.*
(4) ${}^s\mathrm{cdg}_p(G_i, 3)=0$ *for all i.*
(5) *G_i is p-regular for every i.*
(6) *G is p-regular.*
(7) $\mathrm{cdg}_p(G, n)=0$ *for all n.*

Theorem 2. *If G is a compact simply connected simple Lie group, then* $\mathrm{Cdg}(G, n)$ *is a subgroup of* $\mathrm{Hom}(\pi_n(G), \pi_n(S^n))$ *of maximal rank. Indeed* $\mathrm{Cdg}(G, n)$ *is* $\mathrm{cdg}(G,n)\boldsymbol{Z}\{s_1'\}\oplus c\boldsymbol{Z}\{s_2'\}$ *if $(G,n)=(Spin(4m), 4m-1)$ and* $\mathrm{cdg}(G,n)\cdot\mathrm{Hom}(\pi_n(G), \pi_n(S^n))$ *otherwise. Here $\pi_{4m-1}(Spin(4m))=\boldsymbol{Z}\{s_1\}\oplus\boldsymbol{Z}\{s_2\}$ and s_i' is the dual element to s_i; c is 1 if $m\leq 2$ and 2 if $m\geq 3$;* $\mathrm{cdg}(G, n)$ *is non-zero if and only if $n\in\{n_1, \cdots, n_r\}$, where $H^*(G; \boldsymbol{Q})\cong H^*(\prod_{i=1}^r S^{n_i}; \boldsymbol{Q})$.*

In this note all spaces are path-connected with base point and all maps preserve base point. Base point of any H-space is the unit of it. To simplify notation, we denote a map and its homotopy class by the same letter.

We define invariants $\mathrm{cdg}(X, n)$ and ${}^s\mathrm{cdg}(X, n)$ in §2, prove Theorems in §3, and give three results without proofs in §4.

2. Homotopy invariants

We will use the following notation and convention: We denote by $a|b$ that $b=ca$ for some integer c. For any subset A of $\boldsymbol{Z}$ which contains a non-zero, we denote by $\mathrm{GCD}(A)$ the greatest common divisor of the non-zero integers in A. For convenience' sake we set $\mathrm{GCD}(0)=0$, $k|\infty$ for any non-zero integer k, and $0\cdot\infty=0$, hence $\infty|0$. For any subset A of $\{k\in\boldsymbol{Z};\ k\geq 0\}\cup\{\infty\}$, we denote by $\mathrm{LCM}(A)$ the least common multiple of A (it may be ∞) if A is non-empty and contains neither 0 nor ∞, and 0 if A is empty or contains 0, and ∞ if A does not contain 0 but ∞. For any grop C, we denote by ${}^{ab}C$ the abelianization of C, that, is ${}^{ab}C$ is the quotient group of C by its commutator subgroup. Note that the canonical surjection $C\to{}^{ab}C/\mathrm{Tor}$ induces an isomorphism $\mathrm{Hom}(C, \boldsymbol{Z})\cong\mathrm{Hom}({}^{ab}C/\mathrm{Tor}, \boldsymbol{Z})$, where Tor denotes the torsion subgroup. The group C has the rank r, $\mathrm{rank}C=r$, if ${}^{ab}C/\mathrm{Tor}$ is a free abelian group of rank r. We denote by $\mathfrak{B}(C)$ the set of $x\in{}^{ab}C/\mathrm{Tor}$ which is not divisible by any integer ≥ 2.

Put $\{X, Y\}=\lim_{k\to\infty}[\Sigma^k X, \Sigma^k Y]$ and ${}^s\pi_n(X)=\{S^n, X\}$. Let ${}^s\mathrm{Cdg}$: $\{X, S^n\}\to\mathrm{Hom}({}^s\pi_n(X), {}^s\pi_n(S^n))$ be the stable version of Cdg. For any $\alpha\in\pi_n(X)$, we denote by $\mathrm{cdg}(X, n;\alpha)$ or $\mathrm{cdg}(\alpha)$ the non-negative generator of the subgroup of $\boldsymbol{Z}$ generated by the image of α^*: $[X, S^n]\to\pi_n(S^n)=\boldsymbol{Z}$. We define ${}^s\mathrm{cdg}(\alpha)$ similarly for any $\alpha\in{}^s\pi_n(X)$. If α, $\beta\in\pi_n(X)$ represent the same element in ${}^{ab}\pi_n(X)/\mathrm{Tor}$, then $\mathrm{cdg}(\alpha)=\mathrm{cdg}(\beta)$. Thus cdg can be defined on ${}^{ab}\pi_n(X)/\mathrm{Tor}$. Similarly ${}^s\mathrm{cdg}$ is defined on ${}^s\pi_n(X)/\mathrm{Tor}$.

DEFINITION 2.1.

$$\mathrm{cdg}(X, n) = \mathrm{LCM}\{\mathrm{cdg}(\alpha);\ \alpha\in\mathfrak{B}(\pi_n(X))\},$$
$${}^s\mathrm{cdg}(X, n) = \mathrm{LCM}\{{}^s\mathrm{cdg}(\alpha);\ \alpha\in\mathfrak{B}({}^s\pi_n(X))\}.$$

The invariant ${}^s\mathrm{cdg}(X, n)$ has been studied by several people when X is the Thom space of an n-dimensional vector bundle [8]. Note from [10] that $\mathrm{cdg}(Sp(n)/Sp(k), 4n-1)$ and $\mathrm{cdg}(U(n)/U(k), 2n-1)$ are James numbers [2] for $0\leq k<n$, though $\mathrm{cdg}(O(8)/O(1), 7)=6$ and the James number of $SO(8)=O(8)/O(1)$ is 1.

The invariant $\mathrm{cdg}(X, n)$ may be ∞, though it is finite if X is a finite CW-complex. Indeed we have

EXAMPLE 2.2. For each prime p, let $\alpha_1(3;p)$ and $\alpha_1(2;p)$ be generators of the p-components of $\pi_{2p}(S^3)$ and $\pi_{2p}(S^2)$, respectively [12]. Set $\alpha_1(n;p)=\Sigma^{n-3}\alpha_1(3;p)\in\pi_{n+2p-3}(S^n)$ and $X(n;p)=S^n\cup_{\alpha_1(n;p)}e^{n+2p-2}$ for $n\geq 3$, and set $X(2;p)=$

$S^2\cup_{\alpha_1(2;p)}e^{2p+1}$. Then $\mathrm{cdg}(X(n;p), n)=p$ and $\mathrm{cdg}(\prod_p X(n;p), n)\neq 0$, hence $\mathrm{cdg}(\prod_p X(n;p), n)=\infty$ by Proposition 2.6 below.

Proposition 2.3. *If $\pi_n(X)$ is of finte rank, then the following three assertions hold.*
(1) $\mathrm{cdg}(X, n)<\infty$, *and* $\mathrm{cdg}(X, n)=\mathrm{cdg}(\alpha)$ *for some* $\alpha\in\mathscr{B}(\pi_n(X))$ *if* $\mathscr{B}(\pi_n(X))$ *is non-empty.*
(2) $\mathrm{cdg}(X, n)\neq 0$ *if and only if* $\mathrm{rank}\,\pi_n(X)=\mathrm{rank}\,\langle\mathrm{Cdg}(X, n)\rangle\geq 1$, *where* $\langle\mathrm{Cdg}(X, n)\rangle$ *is the subgroup of* $\mathrm{Hom}(\pi_n(X), \pi_n(S^n))$ *generated by* $\mathrm{Cdg}(X, n)$.
(3) $\mathrm{cdg}(X, n)\neq 0$ *if and only if* $\mathrm{rank}\,\pi_n(X)\geq 1$ *and there exists an integer* $r\geq 1$ *such that* $r\cdot\mathrm{Hom}(\pi_n(X), \pi_n(S^n))\subset\langle\mathrm{Cdg}(X, n)\rangle$. *In the latter case* $\mathrm{cdg}(X, n)$ *is equal to the least of such* r.
Stable version also holds.

Proof. Put $t=\mathrm{rank}\,\pi_n(X)$ and $s=\mathrm{rank}\,\langle\mathrm{Cdg}(X, n)\rangle$. We denote by $\{a_1, \cdots, a_t\}$ and $\{\alpha_1, \cdots, \alpha_t\}$ a free basis of $\mathrm{Hom}(\pi_n(X), \pi_n(S^n))$ and its dual basis of ${}^{ab}\pi_n(X)/\mathrm{Tor}$, respectively.

First we prove (2). Suppose $\mathrm{cdg}(X, n)\neq 0$. Then trivially $t\geq 1$. To induce a contradiction, suppose $t>s$. Then we can take $\{a_1, \cdots, a_t\}$ satisfying $\langle\mathrm{Cdg}(X, n)\rangle\subset\langle a_1, \cdots, a_s\rangle$. It follows that $f_*(\alpha_t)=0$ for all $f\colon X\to S^n$, hence $\mathrm{cdg}(\alpha_t)=0$ and $\mathrm{cdg}(X, n)=0$. This is a contradiction. Hence $t=s$. Conversely suppose that $t=s\geq 1$ and $\mathrm{cdg}(X, n)=0$. Then we can take $\{\alpha_1, \cdots, \alpha_t\}$ satisfying $\mathrm{cdg}(\alpha_t)=0$. It follows that $\mathrm{Cdg}(X, n)\subset\langle a_1, \cdots, a_{t-1}\rangle$ so that $s\leq t-1$. This is a contradiction. Hence $\mathrm{cdg}(X, n)\neq 0$ if $t=s\geq 1$. This proves (2).

Next we prove (1). If $\mathrm{cdg}(X, n)=0$, then there is no problem. So suppose that $\mathrm{cdg}(X, n)\neq 0$. Then $t=s\geq 1$ as shown above. Choose $\{a_i; 1\leq i\leq t\}$ such that $\{k_i a_i; 1\leq i\leq t\}$ is a basis of $\langle\mathrm{Cdg}(X, n)\rangle$, where $k_i\geq 1$. Put $k=\mathrm{LCM}\{k_i\}$. Then

$$k=\mathrm{Min}\{r>0;\ r\cdot\mathrm{Hom}(\pi_n(X), \pi_n(S^n))\subset\langle\mathrm{Cdg}(X, n)\rangle\}$$

and hence $k\cdot\mathrm{Hom}(\pi_n(X), \pi_n(S^n))\subset\langle\mathrm{Cdg}(X, n)\rangle$, where Min denotes the minimum. Evaluating at any $\beta=\sum c_i\alpha_i\in{}^{ab}\pi_n(X)/\mathrm{Tor}$, we have $k\cdot\mathrm{GCD}\{c_i\}\boldsymbol{Z}\subset\mathrm{cdg}(\beta)\boldsymbol{Z}=\mathrm{GCD}\{k_i c_i\}\boldsymbol{Z}$ so that $\mathrm{cdg}(\beta)=\mathrm{GCD}\{k_i c_i\}\,|\,k\cdot\mathrm{GCD}\{c_i\}$. If $\beta\in\mathscr{B}(\pi_n(X))$, then $\mathrm{GCD}\{c_i\}=1$ and $\mathrm{cdg}(\beta)\,|\,k$, hence $\mathrm{cdg}(X, n)\,|\,k$. Set $d_i=k/k_i$ and $\alpha=\sum d_i\alpha_i$. Then $\mathrm{GCD}\{d_i\}=1$ and $\alpha\in\mathscr{B}(\pi_n(X))$. We then have $\mathrm{cdg}(\alpha)=k$, so $\mathrm{cdg}(X, n)=\mathrm{cdg}(\alpha)=k$. This proves (1) and a part of (3).

Other part of (3) follows immediately from (2). The same proof is valid for stable case. This completes the proof of Proposition 2.3.

Proposition 2.4. (1) *If all of the following five conditions are satisfied, then* $\mathrm{cdg}(X, n)$ *is non-zero.*
(i) *X is a finite CW-complex.*
(ii) $\mathrm{rank}\,\pi_n(X)\geq 1$.

(iii) *X is simply connected if $n \geq 2$.*

(iv) Image$\{\pi_n(X^{(n-1)}) \to \pi_n(X)\}$ *is a torsion, where $X^{(k)}$ is the k-skeleton of X.*

(v) *All attaching maps of $2n$-cells in $X/X^{(n-1)}$ are null homotopic if n is even.*

(2) *If X is a finite CW-complex with* $\operatorname{rank}^s \pi_n(X) \geq 1$, *then* ${}^s\mathrm{cdg}(X, n)$ *is non-zero.*

Proof. The assertions for $n=1$ can be proved by using the facts that the composite of $[X, S^1] \cong H^1(X) \cong \mathrm{Hom}(H_1(X), \boldsymbol{Z}) \cong \mathrm{Hom}(\pi_1(X), \boldsymbol{Z})$ is Cdg and that ${}^{ab}\pi_1(X) \cong {}^s\pi_1(X)$.

Suppose $n \geq 2$ and five conditions in (1).

First we shall show that Cdg is a surjection on $[X^{(n+1)}/X^{(n-1)}, S^n]$. This is trivial if X has no $(n+1)$-cell, so we assume that X has $(n+1)$-cells. We then have a cofibre sequence $\vee S^n \xrightarrow{p} \vee S^n \xrightarrow{i} X^{(n+1)}/X^{(n-1)}$ and the commutative diagram:

$$\begin{array}{ccccc} [\vee S^n, S^n] & \stackrel{p^*}{\leftarrow} & [\vee S^n, S^n] & \stackrel{i^*}{\leftarrow} & [X^{(n+1)}/X^{(n-1)}, S^n] \\ \mathrm{Cdg} \downarrow \cong & & \mathrm{Cdg} \downarrow \cong & & \downarrow \mathrm{Cdg} \\ \mathrm{Hom}(\pi_n(\vee S^n), \mathrm{Z}) & \underset{p_*^*}{\leftarrow} & \mathrm{Hom}(\pi_n(\vee S^n), \mathrm{Z}) & \underset{i_*^*}{\leftarrow} & \mathrm{Hom}(\pi_n(X^{(n+1)}/X^{(n-1)}), \mathrm{Z}) . \end{array}$$

In this diagram, the upper horizontal sequence is the same as the stable one and hence exact, i_*^* is a monomorphism, and $p_*^* \circ i_*^* = 0$. By chasing the diagram, it follows that the third Cdg is a surjection.

Given any $a \in \mathrm{Hom}(\pi_n(X^{(n+1)}/X^{(n-1)}), \boldsymbol{Z})$, choose $b\colon X^{(n+1)}/X^{(n-1)} \to S^n$ such that $\mathrm{Cdg}(b) = a$. By (v) and [1, 3.1], we can construct skeleton-wise a map f: $X/X^{(n-1)} \to S^n$ such that $f \circ i = k \circ b$ for some $k \neq 0$, where $i\colon X^{(n+1)}/X^{(n-1)} \subset X/X^{(n-1)}$. This implies that $\langle \mathrm{Cdg}(X/X^{(n-1)}, n) \rangle$ is of maximal rank, since

$$i_*^*\colon \mathrm{Hom}(\pi_n(X/X^{(n-1)}), \boldsymbol{Z}) \cong \mathrm{Hom}(\pi_n(X^{(n+1)}/X^{(n-1)}), \boldsymbol{Z}) .$$

By (iii) and a theorem of Blakers-Massey, $\pi_n(X, X^{(n-1)}) \cong \pi_n(X/X^{(n-1)})$. Then by (iv) the homomorphism

$$q_*^*\colon \mathrm{Hom}(\pi_n(X/X^{(n-1)}), \boldsymbol{Z}) \to \mathrm{Hom}(\pi_n(X), \boldsymbol{Z})$$

induced by the quotient map q has a finite cokernel. Therefore $\langle \mathrm{Cdg}(X, n) \rangle$ is of maximal rank, since $q_*^* \langle \mathrm{Cdg}(X/X^{(n-1)}, n) \rangle \subset \langle \mathrm{Cdg}(X, n) \rangle$. Hence $\mathrm{cdg}(X, n) \neq 0$ by Proposition 2.3. This proves (1). By almost the same proof as the above, we have (2).

The following two results can be proved easily. So we omit their proofs.

Proposition 2.5. (1) *If X is k-connected with $n \leq 2k+1$ and $\mathcal{B}({}^s\pi_n(X))$ is non-empty, then* ${}^s\mathrm{cdg}(X, n) \mid \mathrm{cdg}(X, n)$.

(2) *If* $\operatorname{rank} \pi_n(X) = \operatorname{rank} {}^s\pi_n(X) = 1$, *then* $m \cdot {}^s\mathrm{cdg}(X, n) \mid \mathrm{cdg}(X, n)$, *where*

the suspension $\sum^{\infty}$: ${}^{ab}\pi_n(X)/\mathrm{Tor}=\boldsymbol{Z}\to{}^{s}\pi_n(X)/\mathrm{Tor}=\boldsymbol{Z}$ *is multiplication by* m.

(3) *If* G *is a connected simple Lie group, then* $\mathrm{rank}\,\pi_3(G)=\mathrm{rank}\,{}^{s}\pi_3(G)=1$ *and*

$${}^{s}\mathrm{cdg}(\tilde{G}, 3)\,|\,m\cdot{}^{s}\mathrm{cdg}(G, 3)\,|\,\mathrm{cdg}(G, 3)$$

where $\tilde{G}$ *is a universal covering group of* G *and* m *is a non-zero integer defined as in* (2) *for* $X=G$.

We denoted $m\cdot{}^{s}\mathrm{cdg}(G, 3)$ in 2.5(3) by $\mathrm{cd}(G)$ in [9]. Hence ${}^{s}\mathrm{cdg}(G, 3)=\mathrm{cd}(G)$ if G is simple and simply connected.

Proposition 2.6. (1) *If* $\mathscr{B}(\pi_n(X_i))$ *is non-empty for* $i=1, 2$, *then* $\mathrm{LCM}\{\mathrm{cdg}(X_1, n), \mathrm{cdg}(X_2, n)\}\,|\,\mathrm{cdg}(X_1\times X_2, n)$. *Stable version also holds.*

(2) *If* ${}^{ab}\pi_n(X_1)$ *is a torsion, then* $\mathrm{cdg}(X_1\times X_2, n)=\mathrm{cdg}(X_2, n)$.

(3) *If* $\pi_n(X_i)$ *is of finite rank and* $\mathrm{cdg}(X_i, n)\neq 0$ *for* $i=1, 2$, *then* $\mathrm{cdg}(X_1\times X_2, n)=\mathrm{LCM}\{\mathrm{cdg}(X_1, n), \mathrm{cdg}(X_2, n)\}$.

(4) *If* X_i *is* $(n-1)$*-connected and* ${}^{s}\pi_n(X_i)$ *is of finite rank for* $i=1, 2$, *then* ${}^{s}\mathrm{cdg}(X_1\times X_2, n)\,|\,{}^{s}\mathrm{cdg}(X_1, n)\cdot{}^{s}\mathrm{cdg}(X_2, n)$.

3. Proof of Theorem

In this section G denotes a compact connected Lie group of type $\{n_1, \cdots, n_r\}$, that is, $H^*(G; \boldsymbol{Q})\cong H^*(\prod_{i=1}^{r} S^{n_i}; \boldsymbol{Q})$. As is well-known, n_i is odd and there are maps $f: \prod_i S^{n_i}\to G$ and $g: G\to\prod S^{n_i}$ which induce isomorphisms $\pi_*(\prod S^{n_i})\otimes\boldsymbol{Q}\cong\pi_*(G)\otimes\boldsymbol{Q}$ (see [7]). From this and Proposition 2.3 we have

Proposition 3.1. *The following five statements are equivalent.*

(1) $\mathrm{Cdg}(G, n)$ *is non-trivial.*

(2) $\mathrm{cdg}(G, n)$ *is non-zero.*

(3) $\mathrm{rank}\,\pi_n(G)=\mathrm{rank}\,\langle\mathrm{Cdg}(G, n)\rangle\geq 1$.

(4) $\mathrm{rank}\,\pi_n(G)\geq 1$.

(5) $n\in\{n_1, \cdots, n_r\}$.

Proof of Theorem 1. Numbers $\mathrm{cdg}(G, n)$ and ${}^{s}\mathrm{cdg}(G, n)$ are finite by Proposition 2.3. Put $A(n)=\{i;\ \mathrm{rank}\,\pi_n(G_i)\geq 1\}$ and define ${}^{s}A(n)$ similarly. Then $A(3)={}^{s}A(3)=\{i;\ 1\leq i\leq t\}$. We have ${}^{s}\mathrm{cdg}(G, 3)\,|\,\mathrm{cdg}(G, 3)$ by 2.5 (1). Thus (1) implies (2). We have $\mathrm{LCM}\{{}^{s}\mathrm{cdg}(G_i, 3)\}\,|\,{}^{s}\mathrm{cdg}(G, 3)$ and $\mathrm{cdg}(G, n)=\mathrm{LCM}\{\mathrm{cdg}(G_i, n);\ i\in A(n)\}$ by 2.6. Hence (2) implies (4), and (1) and (3) are equivalent. By Theorem 4.1 (1) of [9], (4) and (5) are equivalent. Trivially (5) implies (6), and (7) implies (1).

To prove that (6) implies (7), suppose (6). By Proposition 3.1, we may suppose that $n\in\{n_1, \cdots, n_r\}$. Then there is a p-equivalence $f: G\to S=\prod_{i=1}^{r} S^{n_i}$ so that $\mathrm{rank}\,\pi_n(G)=\mathrm{rank}\,\pi_n(S)=u$, say, and the image of $f_*: \pi_n(G)\to\pi_n(S)$ is of maximal rank. Let $\{\alpha_1, \cdots, \alpha_u\}$ be a free basis of $\pi_n(G)/\mathrm{Tor}$ and $\{a_1, \cdots, a_u\}$ its dual basis of $\mathrm{Hom}(\pi_n(G), \pi_n(S^n))$. Let $\{k_1, \cdots, k_u\}$ be positive integers and $\{\beta_1, \cdots, \beta_u\}$ a free basis of $\pi_n(S)$ such that $f_*(\alpha_i)=k_i\beta_i$. Then k_i is prime to p. Since $f^\sharp_* \circ \mathrm{Cdg}=\mathrm{Cdg}\circ f^*$: $[S, S^n]\to \mathrm{Hom}(\pi_n(G), \pi_n(S^n))$ and since Cdg is surjective on $[S, S^n]$, we have $\mathrm{Cdg}(G, n)\supset \mathrm{Image}(f^\sharp_*)=\oplus_{i=i}^u k_i\boldsymbol{Z}\{a_i\}$. Hence $\mathrm{Cdg}(G, n)$ contains $\mathrm{LCM}\{k_i\}\cdot \mathrm{Hom}(\pi_n(G), \pi_n(S^n))$ so that $\mathrm{cdg}(G, n)\,|\,\mathrm{LCM}\{k_i\}$ by Proposition 2.3 (3), therefore $\mathrm{cdg}_p(G, n)=0$. This implies (7) and completes the proof of Theorem 1.

EXAMPLE 3.2. For G non-simply connected, Theorem 1 does not hold in general: $\mathrm{cdg}(SO(3), 3)=2$ and ${}^s\mathrm{cdg}(SO(3), 3)=1$ (see [10]).

Recall that if G is simple then $n\in\{n_1, \cdots, n_r\}$ if and only if rank $\pi_n(G)$ is 1 or 2 and rank $\pi_n(G)=2$ if and only if $(\tilde{G}, n)=(Spin(4m), 4m-1)$ for $m\geq 2$. Then the following and Proposition 3.1 prove Theorem 2 except for the case $(G, n)=(Spin(4m), 4m-1)$.

Proposition 3.3 (James). *If n is odd, then the image of α^*: $[X, S^n]\to\pi_n(S^n)$ is a subgroup for every $\alpha\in\pi_n(X)$. In particular if n is odd and* rank $\pi_n(X)=1$, *then* $\mathrm{Cdg}(X, n)=\mathrm{cdg}(X, n)\cdot\mathrm{Hom}(\pi_n(X), \pi_n(S^n))$.

Proof. The first assertion can be proved by the method in [3, p.88]. The second assertion then follows, since $\alpha^*=ev_\alpha\circ\mathrm{Cdg}$ and ev_α is an isomorphism if rank $\pi_n(X)=1$ and α represents a generator of ${}^{ab}\pi_n(X)/\mathrm{Tor}=\boldsymbol{Z}$, where ev_α: Hom $(\pi_n(X), \pi_n(S^n))\to\pi_n(S^n)$ is the evaluation at α, that is, $ev_\alpha(\theta)=\theta(\alpha)$.

Let

$$Spin(4m-1)\xrightarrow{i} Spin(4m)\xrightarrow{p} S^{4m-1}$$

be the canonical bundle for $m\geq 1$. Then we have

$$\pi_{4m-1}(Spin(4m)) = \boldsymbol{Z}\{s_1\}\oplus\boldsymbol{Z}\{s_2\},$$
$$\mathrm{Hom}(\pi_{4m-1}(Spin(4m)), \pi_{4m-1}(S^{4m-1})) = \boldsymbol{Z}\{s_1'\}\oplus\boldsymbol{Z}\{s_2'\}$$

where s_1 is the image under i_* of a generator of $\pi_{4m-1}(Spin(4m-1))=\boldsymbol{Z}$ and s_2 is an element such that $p_*(s_2)$ is 2 if $m\geq 3$ and 1 if $m\leq 2$ (cf., [5]); s_j' is the dual element to s_j. Then the following completes the proof of Theorem 2.

Proposition 3.4. *The number* $\mathrm{cdg}(Spin(4m), 4m-1)$ *is non-zero and*

$$\mathrm{Cdg}\,(Spin(4m), 4m-1) = \mathrm{cdg}(Spin(4m), 4m-1)\boldsymbol{Z}\{s_1'\}\oplus c\boldsymbol{Z}\{s_2'\}$$

where c is 2 if $m\geq 3$ and 1 if $m\leq 2$.

Proof. If $m\leq 2$, then $Spin(4m)\approx Spin(4m-1)\times S^{4m-1}$ and the assertion can be obtained easily.

Suppose that $m\geq 3$. Then $s_2^*(p)=2$, hence $\mathrm{cdg}(s_2)=2$ by the following lemma.

Lemma 3.5. *If X is an H-space and n is odd with $n \neq 1$, 3, 7, then* $\mathrm{cdg}(\alpha)$ *is even for every* $\alpha \in \pi_n(X)$.

To simplify notations, we set $(G, n)=(Spin(4m), 4m-1)$. By definition, we have

$$\mathrm{Cdg}(G, n) \subset \mathrm{cdg}(s_1)\boldsymbol{Z}\{s_1'\} \oplus 2\boldsymbol{Z}\{s_2'\}.$$

Take any integers k_1 and k_2. Then there exists a map $f: G \to S^n$ such that Cdg $(f)=\mathrm{cdg}(s_1)k_1s_1'+2js_2'$ for some integer j. Let $I: G \to G$ be the inversion, that is, $I(A)=A^{-1}$. Then Cdg of the composition of

$$G \xrightarrow{d} G \times G \xrightarrow{1 \times f} G \times S^n \xrightarrow{g_{\pm}} S^n$$

is $\mathrm{cdg}(s_1)k_1s_1'+(2j \pm 2)s_2'$, where d is the diagonal map, g_+ the canonical action and $g_-=g_+ \circ (I \times 1)$. Inductively we then have $\mathrm{cdg}(s_1)k_1s_1'+2k_2s_2' \in \mathrm{Cdg}(G, n)$. Hence $\mathrm{Cdg}(G, n)=\mathrm{cdg}(s_1)\boldsymbol{Z}\{s_1'\} \oplus 2\boldsymbol{Z}\{s_2'\}$. Also $\mathrm{cdg}(s_1)$ is even from Lemma 3.5, hence $\mathrm{cdg}(G, n)=\mathrm{cdg}(s_1) \neq 0$ from Proposition 2.3(3) and the following lemma.

Lemma 3.6. $\mathrm{cdg}(s_1) \neq 0$.

Proof of 3.5. Let $g: X \to S^n$ be a map such that $g \circ \alpha=\mathrm{cdg}(\alpha) \in \pi_n(S^n)= \boldsymbol{Z}$. Then the degree of the composition of

$$S^n \xrightarrow{i_j} S^n \times S^n \xrightarrow{\alpha \times \alpha} X \times X \xrightarrow{\mu} X \xrightarrow{g} S^n$$

is $\mathrm{cdg}(\alpha)$ for $j=1$, 2, where i_j is the inclusion to the j-th factor and μ is the multiplication. Hence $\mathrm{cdg}(\alpha)^2[\iota_n, \iota_n]=[\mathrm{cdg}(\alpha)\iota_n, \mathrm{cdg}(\alpha)\iota_n]=0$, so $\mathrm{cdg}(\alpha)$ is even, because the Whitehead square $[\iota_n, \iota_n]$ of the identity map ι_n of S^n is of order 2.

Proof of 3.6. Set $n=4m-1$. Then the homomorphism $\pi_n(Spin(n))= \boldsymbol{Z} \to \pi_n(Spin(n+2))=\boldsymbol{Z}$ induced by the inclusion is multiplication by e, where e is 1 if $m \geq 3$ and 2 if $m \leq 2$. Thus we have $\mathrm{cdg}(Spin(n+1), n; s_1) \mid e \cdot \mathrm{cdg}(Spin(n+2), n)$. Since the latter number is non-zero by Proposition 3.1, so is the former.

This completes the proofs of Proposition 3.4 and Theorem 2.

REMARK 3.7 ([10]). By almost the same proof as the above, we can prove that $\mathrm{Cdg}(SO(m), n)$ is a subgroup of maximal rank. By using Proposition 4.1 below, we can prove that if G is simple but not necessarily simply connected, then $\mathrm{Cdg}(G, n)$ contains a subgroup of maximal rank.

4. Other results

We give three results. See [6] and [10] for their proofs. When we study Cdg(G, n) for non-simply connected G, the following is useful.

Proposition 4.1. *Let $q:H\to G$ be a finite covering homomorphism and m the least positive integer such that $x^m=1$ for all x in the kernel of q. Then we have*
(1) $m\cdot \mathrm{Cdg}(H, n)\subset q^*_* \,\mathrm{Cdg}(G, n)\subset \mathrm{Cdg}(H, n)$,
(2) $\mathrm{cdg}(\beta)\,|\,\mathrm{cdg}(q_*\beta)\,|\,m\cdot \mathrm{cdg}(\beta)$ *for every* $\beta\in\pi_n(H)$,
(3) $\mathrm{cdg}(H, n)\,|\,\mathrm{cdg}(G, n)\,|\,m\cdot\mathrm{cdg}(H, n)$ *for* $n\geq 2$,
(4) $\mathrm{cdg}(H, 1)\,|\,m$.

Let $\Xi: \pi_n(X)\to H_n(X)$ be the Hurewicz homomorphism. Put $PH_n(X)=\{x\in H_n(X);\ d_*(x)=x\otimes 1+1\otimes x\}$, where $d: X\to X\times X$ is the diagonal map. As is easily seen, $\Xi(\pi_n(X))\subset PH_n(X)$. It is known as a theorem of Cartan-Serre that $\Xi\otimes \boldsymbol{Q}: \pi_*(G)\otimes\boldsymbol{Q}\cong PH_*(G)\otimes\boldsymbol{Q}$. L. Simith[11] studied the problem: What is the smallest positive integer $N(G, n)$ such that $N(G, n)x$ is contained in the image of the modulo torsion Hurewicz homomorphism

$$\Xi: \pi_n(G)/\mathrm{Tor}\to PH_n(G)/\mathrm{Tor}$$

for every $x\in PH_n(G)/\mathrm{Tor}$?

Proposition 4.2. *If G is simple or simply connected, then* cdg(G, n) *is a multiple of* $N(G, n)$.

EXAMPLE 4.3. The number $N(G, n)$ has been determined for classical groups, G_2 and F_4 (see e.g., [4]). The first few values of the Smith's upper bound $N(n)$ of $N(G, n)$ are $N(3)=1$, $N(5)=2^2$, $N(7)=2^4\cdot 3$, $N(9)=2^6\cdot 3$, $N(11)=2^8\cdot 3^2\cdot 5$ (see[11]). If G is simple and simply connected, then $N(G, 3)=1$ and cdg (G, 3) is even except for $G=S^3$. We have $N(SU(3), 5)=\mathrm{cdg}(SU(3), 5)=2$; $N(Sp(2), 7)=\mathrm{cdg}(Sp(2), 7)=2^2\cdot 3=N(Sp(3), 7)$; $2^5\cdot 3\,|\,\mathrm{cdg}(Sp(3), 7)\,|\,2^8\cdot 3$; $N(SU(5), 9)=\mathrm{cdg}(SU(5), 9)=2^3\cdot 3$; $N(G_2, 11)=\mathrm{cdg}(G_2, 11)=2^3\cdot 3\cdot 5$.

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