Fujio, M.
Osaka J. Math.
28 (1991), 163–211

SUPER G-STRUCTURES OF FINITE TYPE

MITSUHIKO FUJIO

(Received February 23, 1990)

0. Introduction

Many important differential geometric structures on manifolds such as Riemannian metrics, complex structures etc, can be grasped as G-structures. A G-structure is defined as a reduction of the structure group of the linear frame bundle of a manifold to a linear Lie group G. In other words, a G-structure on a manifold can be regarded as a system of partial differential equations of first order for local coordinates of the manifold. This formulation of G-structures can be generalized to supermanifolds.

Some of the concrete differential geometric structures which can be regarded as G-structures defined on supermanifolds, have been studied extensively. For example, many authors studied Reimannian supermanifolds (cf. [11]), since metric is one of the most fundamental objects in geometry and physics. Super Hamiltonian structure was formulated by Kostant ([5]) in order to study the geometric quantization on supermanifolds. Super CR-structure (a structure on a real subsupermanifold induced from an ambient complex superspace) was studied by Schwarz ([10]) and by Rosly and Schwarz ([6] and [7]) in relevance to supergravity.

The notion of G-structure on supermanifolds was first introduced in [10] and used in [6] and [7] although general theory of G-structures on supermanifolds is not developed there.

In this paper, we investigate general G-structures of finite type on supermaniolds and show the following theorem:

Main Theorem. *Let G be a linear Lie supergroup of finite type with the connected body. Then the equivalence problem of G-structures can be reduced to the equivalence problem of complete parallelisms, that is, $\{e\}$-structures.*

Our main theorem is a generalization of a well-known theorem (Theorem 0 in § 1) concerning manifolds and the outline of the proof goes similarly as that of usual manifolds. Many facts in super-geometry are formally the same as in the usual geometry, although their verifications are mostly non-trivial.

Differential geometric structures defined on supermanifolds are richer than those defined on usual manifolds as can be seen by the following facts:

There exists a close relationship between G-structures on supermanifolds and those on their bodies, which are usual manifolds). In [1], it was shown that a G-structure on a supermanifold M naturally induces, on its body M_b, a $\tilde{G}$-structure, where $\tilde{G}$ is a Lie group which is canonically associated to G and that the equivalences of G-structures induce those of $\tilde{G}$-strucutres.

The converse, however, is not true. More precisely, the association of a G-structures on a supermanifold to the $\tilde{G}$-structure on its body induced correspondece between differential invariants of the G-structure and that of the $\tilde{G}$-structure, but this correspondence is not one-to-one generally. An example of such degeneration of correspondence of differential invariants is given in [2], where a super analogue of the classical geometry of webs is studied.

We remark that in [1], supergeometry was formulated using categorical terminologies, which was first introduced by Schwarz [9] in order to make the theory of supermanifolds independent of the choice of the ground supernumber algebra. A systematic development of this formulation was given by the author in [3].

In this paper we follow de Witt [11] as to fundamental notions of supermanifolds for the sake of brevity, but we take, as the supernumber algebra, a Grassmann algebra Λ which is algebraically generated by countablly many elements over the real number field $\boldsymbol{R}$.

We give a brief review of the theory of G-structures on manifolds in § 1. In § 2, we generalize the notion of G-structures to supermanifolds and formulate the equivalence problems. Sections 3, 4 and 5 are devoted to introducing basic notions and to investigating them. In § 6, we introduce the notion of being of finite type for Lie superalgebras and G-structures and, in § 7, give a proof of our main theorem. As a typical example of G-structures of finite type, we consider OSp-structure (Riemannian supermetrics) in § 8 and we classify them in § 9 for the transitive structures. Basic results for bilinear forms on Λ-modules is given in § 10.

The author wishes to thank Professor T. Tsujishita for valualbe suggestions and constant encouragement. He would also like to express his gratitude to Professors H. Ozeki and S. Murakami for their help and encouragement for publishing this paper.

1. Review of the theory of G-structures

To illustrate our main theorem, we give a brief overview of the theory of G-structures on manifolds following Singer and Sternberg ([8]).

One of the most important problems in the theory of G-structures is to determine whether two G-structures are isomorphic or not. This is called *the* *equivalence problem* of G-structures. To solve the local equivalence problem of G-structures, we must find a complete system of differential invariants for each G-structure.

In their approach, Singer and Sternberg gave a series of differential invariants, which are systematically associated to each G-structure, and called each invariant *a structure function* of the G-structure. More precisely, they difined *the first order structure function* of a G-structure as the torsion of a suitably chosen G-connection, which is independent of the choices.

The first order structure function contains only the data of first order of the G-structure. To obtain the higher order data, they also gave geometric construction of prolongations. Namely, they defined *the first prolongation of a G-structure $P \to M$* as a $G^{(1)}$-structure $P^{(1)}$ on P, where $G^{(1)}$, called *the first prolongation of G*, is a Lie group determined both by the Lie algebra of G and by its inclusion into the general linear group. By continuing this procedure, *the k-th prolongations of P and G* are also defined for each positive integer k.

After a prolongation, the first order structure function of the $G^{(1)}$-structure gives us finer data of a given G-structure, which is called *the second order structure function of the G-structure.* By repeating prolongations, one obtains *the higher order structure functions* which generally contain new data of the G-structure. Moreover, the equivalence problem of G-structures is reduced to that of the prolongated ones.

Suppose now G is *of finite type*, that is, the k-th prolongation of the linear Lie group G is the unit group $\{e\}$ for some positive integer k. Then we obtain an $\{e\}$-structure, that is, a complete parallelism, by prolongating finitely many times.

Now we can summarize the above arguments as the following theorem:

Theorem 0. *Let G be a linear Lie group of finite type. Then the equivalence problem of G-structures can be reduced to the equivalence problem of complete parallelisms, that is, of $\{e\}$-structures.*

A typical example of linear Lie groups of finite type is $O(n)$ in $GL(n; \boldsymbol{R})$. In fact, it is known that $O(n)^{(1)} = \{e\}$. On the other hand, an $O(n)$-structure on an n-dimensional manifolds M can be identified with a Riemannian metric of M and an equivalence between $O(n)$-structures is nothing but an isometry. In this case, the first prolongation of a given $O(n)$-structure can be naturally regarded as its Levi-Civita's connection and hence Theorem 0 corresponds to the classical theorem for the relationship between Riemannian metrics and their Levi-Civita's connections (cf. Kobayashi [4]).

2. G-structures on supermanifolds

In this section, we generalize the notion of G-structure to supermaniolfds.

In what follows Λ denotes the algebra of supernumbers. Moreover, we fix non-negative integers m and n, and denote by M an $m|n$-dimensional supermanifold and, by V, the $m|n$-dimensional standard free Λ-module $\Lambda^{m|n}$.

2.1. Linear frame bundles. For a point x of M, we call a pure basis $u=(X_1, X_2, \cdots, X_{m+n})$ of the tangnet space $T_x(M)$ *a linear frame of* M *at* x. Since $T_x(M)$ is a free Λ-module of dimension $m|n$, a linear frame u can be identified with the Λ-linear isomorphism defined by

$$u\colon V\ni e_\mu \to X_\mu \in T_x(M) \qquad (\mu = 1, \cdots, m+n), \tag{2.1}$$

where e_μ's denote the standard basis of V. We denote by $L_x(M)$ the set of all linear frames of M at x.

Since the general linear group $GL(V)=GL(m|n;\Lambda)$ is the group of automorphisms of the Λ-module V, each element $a\in GL(V)$ acts on $L_x(M)$ from the right by

$$R_a\colon L_x(M)\ni u \to u\circ a\in L_x(M). \tag{2.2}$$

It is easily verified that the right action of $GL(V)$ on $L_x(M)$ is transitive and free.

By collecting $L_x(M)$:

$$L(M) = \bigcup_{x\in M} L_x(M),$$

we obtain a principal $GL(V)$-bundle $\pi\colon L(M)\to M$, where the projection π is defined by

$$\pi\colon L_x(M)\ni u \to x\in M, \tag{2.3}$$

The space $L(M)$ has a natural supermanifold structure. We call $L(M)$ *the linear frame bundle of* M.

On $L(M)$, we define a V-valued 1-form θ as follows. Let u be a point in $L(M)$ and put $x=\pi(u)$. The differential π_* of the projection is an even Λ-linear surjection of $T_u(L(M))$ onto $T_x(M)$ and the linear frame u is an isomorphism of V onto $T_x(M)$. Hence we can define θ at u by

$$\theta_u = u^{-1}\circ\pi_*\colon T_u(L(M)) \to V. \tag{2.4}$$

It can be verified that the V-valued 1-form θ thus defined has even parity and is smooth. We call θ *the canonical* 1*-form of* $L(M)$. As in the non-super case, the canonical 1-form θ has the following property.

Proposition 1. *Let* θ *be the canonical* 1*-form of* $L(M)$. *Then*

$$R_a{}^*\theta = a^{-1}\circ\theta$$

for $a\in GL(V)$.

Corollary 2. *Let θ be the canonical* 1*-form of $L(M)$. Then the V-valued* 2*-form $d\theta$ satisfies*

$$R_a{}^*d\theta = a^{-1}\circ d\theta$$

for $a \in GL(V)$.

2.2. Definition of *G*-structures. Let G be a Lie subsupergroup in $GL(V)$. We call a G-reduction $P \to M$ of the linear frame bundle $L(M) \to M$ *a G-structure on M*. Namely, a G-structure $P \to M$ is a reduced subbundle of $L(M)$ with the structure group G. For a G-structure $P \to M$, we will use the same notations for the projection and the canonical 1-form of $L(M)$ restricted to P, if there is no danger of confusions.

EXAMPLES.

a) $G = GL(V)$. In this case, $L(M)$ itself is the unique $GL(V)$-structure on M.

b) $G = \{e\}$. Let $P \to M$ be an $\{e\}$-structure. Then to each $x \in M$, a pure basis $X_1(x), \cdots, X_{m+n}(x)$ of the tangnet space $T_x(M)$ is uniquely assigned and hence P corresponds to *a complete parallelism on M.*

c) Let g be a non-degenerate supersymmetric bilinear form on V and $OSp(V)$ be the orthosymplectic group of V:

$$OSp(V) = \{a \in GL(V) \colon g(au, av) = g(u, v) \text{ for } u, v \in V\} .$$

Then an $OSp(V)$-structure $P \to M$ on M corresponds to a pseudo-Riemannian supermetric on M. In fact, for a gvien $OSp(V)$-structure $P \to M$, there exists a unique non-degenerate quadratic form $g(x)$ of $T_x(M)$ for each point x of M, which makes all the linear frames in the fibre P_x orthosymplectic bases of $T_x(M)$. In this case, the odd dimension n of M is neccesarily even. We will discuss $OSp(V)$-structures in more detail in section 8.

d) In contrast to c), let g be now a non-degenerate *anti*-supersymmetric bilinear form on V and

$$SpO(V) = \{a \in GL(V) \colon g(au, av) = g(u, v) \text{ for } u, v \in V\} .$$

In this case, an $SpO(V)$-structure on M corresponds to a non-degenerate 2-form ω on M. This is a generalization of almost symplectic structures to supermanifolds. In tihs case, the even dimension m of M is even. We call such a structure *an almost supersymplectic structure* or *an almost super Hamiltonian structure.* When the 2-form ω is closed, we call the structure *a supersymplectic structure* or *a super Hamiltonian structure.*

e) Let Λc be the complexification of the algebra of supernumbers Λ. If V is given a Λc-module structure, we denote by $GL(V; \Lambda c)$ the group of Λc-linear automorphisms of V. Then both the even and odd dimensions of

V are even. A $GL(V; \Lambda c)$-structure on M corresponds to a tensor field J on M of type (1,1) and of even parity such that $J^2=-1$. Such a tensor filed J is called *an almost complex structure of the supermanifold M*. Moreover, if M is a complex supermanifold and J is the natural almost complex structure of M then J is called *a complex structure*.

2.3. Equivalence problems. To state the equivalence problem of G-structures on supermanifolds, we need the notion of *isomorphisms* of G-structures.

Let M and N be supermanifolds of the same dimension $m|n$. For G-structures $P \to M$ and $Q \to N$, we say that P is *isomorphic* as a G-structure to Q if there exists a diffeomorphism f: $M \to N$ such that its differential f_*: $L(M) \to L(N)$ induces a diffeomophism of P onto Q. Such a diffeomorphism f is called *an isomorphism of G-structure from P to Q* and we say that the G-structures P and Q are *equivalent*. An isomorphism of a G-structure $P \to M$ to itself is called an *automorphism* of $P \to M$.

The equivalence problem of G-structures is to determine whether two supermanifolds with G-structures are equivalent or not.

Since for a given G-structure $P \to M$ and for an arbitrary superdomain U of M, the restriction of P to U is a G-structure on U, the equivalence problem can be localized. That is, for given G-structures $P \to M$ and $Q \to N$ and for a given pair (x, y) of points $x \in M$ and $y \in N$, we say that P and Q are *locally equivalent at* (x, y) if there exist open neighbourhoods U of x and W of y and an isomorphism f between $P|_V$ and $Q|_W$ such that $f(x)=y$. We will mainly consider the local equivalence problems.

2.4. Transitivity and flatness. A G-structure $P \to M$ is called *transitive* if it admits for an arbitrary pair (u, v) of u, $v \in P$ an automorphism f: $M \to M$ satisfying $f_*(u)=v$. The notion of *local transitivity* is similarly defined.

Let $\boldsymbol{E}^{m|n}$ be the $m|n$-dimensional Euclidean superspace. In other words, $\boldsymbol{E}^{m|n}$ is the even part of V. Since the linear frame bundle $L(\boldsymbol{E}^{m|n})$ is trivial:

$$L(\boldsymbol{E}^{m|n}) = \boldsymbol{E}^{m|n} \times GL(m|n; \Lambda),$$

we can define a natural G-structure on $\boldsymbol{E}^{m|n}$ by

$$\boldsymbol{E}^{m|n} \times G \to \boldsymbol{E}^{m|n} \tag{2.5}$$

for every Lie subsupergroup $G \subset GL(m|n; \Lambda)$. We call this *the standard flat G-structure*. A G-structure is called *flat* if it is isomorphic to the flat G-structure restricted to a superdomain of $\boldsymbol{E}^{m|n}$. Also, the notion of *local flatness* is defined.

We note that the standard flat G-structure is transitive, and that a locally flat G-structure is localy transitive.

3. Structure functions

We generalize the notion of the first order structure function of Singer and Sternberg ([8]) to G-structures on supermanifolds. As in section 2, M and V denote, respectively, an $m|n$-dimensional supermanifold and the $m|n$-dimensional standard free Λ-module. Moreover, we fix a Lie subsupergroup G in $GL(V)$ and conisder a G-structure $P \to M$. We denote the Lie superalgebra of G by $\mathfrak{g}$.

3.1. Infinitesimal actions. Since the Lie supergroup G acts on P from the right, it induces infinitesimal action of $\mathfrak{g}$ on P, that is, a homomorphism of Lie superalgebras:

$$\mathfrak{g} \in A \to A^* \in \mathfrak{X}(P),$$

where $\mathfrak{X}(P)$ denotes the Lie superalgebra of smooth vector fields on P. Moreover, since the action of G on P is free, the induced even Λ-linear map

$$\sigma_u \colon \mathfrak{g} \ni A \to A^*(u) \in T_u(P) \tag{3.1}$$

is injective for each point u of P. On the other hand, since $\pi \circ R_a = \pi$ for a in G, the image of σ_u is contained in the kernel of $\pi_* \colon T_u(P) \to T_x(M)$, where $x = \pi(u)$. By counting the dimensions, we have the following proposition.

Proposition 3. *Let $\pi \colon P \to M$ be a G-structure. Then for a point u of P, the following sequence of Λ-modules is exact:*

$$(0) \to \mathfrak{g} \xrightarrow{\sigma_u} T_u(P) \xrightarrow{\pi_*} T_x(M) \to (0),$$

where $x = \pi(u)$.

We call the kernel of the projection π_* *the vertical subspace* at u and denote it by $\mathfrak{g}_u$. By the above proposition, the vertical subspace $\mathfrak{g}_u$ can be canonically identified with the Lie superalgebra $\mathfrak{g}$ through σ_u.

The following proposition is the infinitesimal version of Proposition 1 and Corollary 2:

Proposition 4. *Let θ be the canonical 1-form of the G-structure $\pi \colon P \to M$. Then*

$$L_{A^*}\theta = -A \circ \theta, \tag{3.2}$$

and

$$i_{A^*}d\theta = -A \circ \theta, \tag{3.3}$$

for $A \in \mathfrak{g} \subset \mathfrak{gl}(V)$, where L_{A^*} and i_{A^*} denote the Lie derivative and inner product with respect to the vector field $A^* \in \mathfrak{X}(P)$, respectively.

3.2. Horizontal subspaces. Let u be a point of P. We call a complementary submodule to $\mathfrak{g}_u$ in $T_u(P)$ a *horizontal subspace at u*. That is, a horizontal subspace H_u is a Λ-submodule of $T_u(P)$ such that

$$T_u(P) = \mathfrak{g}_u \oplus H_u . \tag{3.4}$$

By Proposition 3, the projection, restricted to H_u,

$$\pi_* |_{H_u} : H_u \xrightarrow{\sim} T_x(M) \qquad (x = \pi(u)) . \tag{3.5}$$

is an isomorphism. Thus a horizontal subspace H_u is an $m|n$-dimensional free Λ-module. Moreover, by the definition of the canonical 1-form,

$$\theta_u |_{H_u} : H_u \xrightarrow{\sim} V \tag{3.6}$$

is also an isomorphism.

Hence for a given horizontal subspace H_u at u, we can define a Λ-linear map by

$$B_{H_u} = (\theta_u |_{H_u})^{-1} : V \to T_u(P) . \tag{3.7}$$

It is immediate from the definition that the Λ-linear map B_{H_u} is injective and has even parity. We call this map *the horizontal lift for the horizontal subspace* H_u.

Now we describe the subset Hor_u of $\mathrm{Hom}_\Lambda(V, T_u(P))$ consisting of all the horizontal lifts at u. We note that Hor_u is contained in the even subspace $\mathrm{Hom}_\Lambda(V, T_u(P))_0$.

Let H_u and H'_u be horizontal subspaces at u. Then by the definition of the canonical 1-form, we have

$$\pi_*(B_{H_u}(v)) = u(v) = \pi_*(B_{H'_u}(v)) ,$$

for v in V. This implies that the image of the difference $B_{H'_u} - B_{H_u}$ lies in the kernel of the projection π_*, that is, in the vertical subspace $\mathfrak{g}_u$ at u. Since $\mathfrak{g}_u$ is canonically isomorphic to the Lie superalgebra $\mathfrak{g}$ by σ_u, there exists a map $S_{H'_u H_u} : V \to \mathfrak{g}$ satisfying

$$\sigma_u(S_{H'_u H_u}(v)) = B_{H'_u}(v) - B_{H_u}(v) \qquad (v \in V) . \tag{3.8}$$

It is easily verified that $S_{H'_u H_u}$ is an even Λ-linear map of V into $\mathfrak{g}$.

Conversely, if we fix a horizontal subspace H_u then, for an arbitrary $S \in \mathrm{Hom}_\Lambda(V, \mathfrak{g})_0$, a horisontal subspace H'_u satisfying

$$\sigma_u(S(v)) = B_{H'_u}(v) - B_{H_u}(v) \qquad (v \in V) .$$

is uniquely determined and is given by

$$H'_u = \mathrm{Im}(B_{H_u} + \sigma_u \circ S) . \tag{3.9}$$

Thus the subset Hor_u of $\mathrm{Hom}_\Lambda(V, T_u(P))_0$ is parametrized by the Λ_0-module $\mathrm{Hom}_\Lambda(V, \mathfrak{g})_0$.

Moreover, the space Hor_u carries the structure of affine space with the fundamental vector space $\mathrm{Hom}_\Lambda(V, \mathfrak{g})_0$ by the action defined by

$$(3.10)\qquad \mathrm{Hor}_u \times \mathrm{Hom}_\Lambda(V, \mathfrak{g})_0 \ni (B_{H'_u} S) \\ \rightarrow B_{H_u} + \sigma_u \circ S \in \mathrm{Hor}_u \subset \mathrm{Hom}_\Lambda(V, T_u(P))_0 .$$

Thus we obtain the following proposition:

Proposition 5. *Let u be a point of P. Then the subset* Hor_u *of* Hom_Λ *$(V, T_u(P))$ consisting of all the horizontal lifts at u is an affine subspace with the fundamental vector space*

$$\mathrm{Hom}_\Lambda(V, \mathfrak{g})_0 \simeq \mathrm{Hom}_\Lambda(V, \mathfrak{g}_u)_0 .$$

3.3. Structure functions. In order to define the (first order) structure function of the G-structure P, we first consider the exterior differential $d\theta$ of the canonical 1-form θ.

Let u be a point of P. Since θ_u is an even Λ-linear map of $T_u(P)$ into V, $d\theta_u$ is an even anti-supersymmetric bilinear form on $T_u(P)$ with values in V, that is,

$$d\theta_u \in (V \otimes_\Lambda \wedge^2(T_u{}^*(P)))_0 .$$

Let H_u be a horizontal subspace at u. We denote by c_{H_u} the pullback of $d\theta_u$ by the horizontal lift B_{H_u}: $V \rightarrow T_u(P)$:

$$(3.11)\qquad c_{H_u} = B_{H_u}{}^*(d\theta_u) \in (V \otimes_\Lambda \wedge^2(V^*))_0 ,$$

that is,

$$c_{H_u}(v, w) = d\theta_u(B_{H_u}(v), B_{H_u}(w)) \qquad (v, w \in V) .$$

Now we determine how c_{H_u} depends on the choice of the horizontal subspace H_u. Let H_u and H'_u be horizontal subspaces at u. Then

$$\begin{aligned} c_{H'_u}(v, w) - c_{H_u}(v, w) &= d\theta_u(B_{H'_u}(v), B_{H'_u}(w)) - d\theta_u(B_{H_u}(v), B_{H_u}(w)) \\ &= d\theta_u(B_{H'_u}(v) - B_{H_u}(v), B_{H'_u}(w)) + d\theta_u(B_{H_u}(v), B_{H'_u}(w) - B_{H_u}(w)) \\ &= d\theta_u(S_{H'_u H_u}{}^*(v), B_{H'_u}(w)) + d\theta_u(B_{H_u}(v), S_{H'_u H_u}{}^*(w)) \end{aligned}$$

(by using (3.3) and the definition of horizontal lift)

$$\begin{aligned} &= -S_{H'_u H_u}(v) \circ \theta_u(B_{H'_u}(w)) + (-1)^{|v|\cdot|w|} S_{H'_u H_u}(w) \circ \theta_u(B_{H_u}(v)) \\ &= -S_{H'_u H_u}(v) w + (-1)^{|v|\cdot|w|} S_{H'_u H_u}(w) v . \end{aligned}$$

Thus we have

$$(3.12)\qquad c_{H'_u}(v, w)-c_{H_u}(v, w) = -S_{H'_uH_u}(v)w+(-1)^{|v|\cdot|w|}S_{H'_uH_u}(w)v\,.$$

We rewrite this equation compactly by introducing a new operator. Let

$$\mathcal{A}\colon V^*\otimes_\Lambda V^*\ni\phi\otimes\psi\to \tfrac{1}{2}(\phi\otimes\psi-(-1)^{|\phi|\cdot|\psi|}\psi\otimes\phi)\in\wedge^2V^*$$

be the super-alternating operator and put

$$\partial = \mathrm{Id}_v\otimes\mathcal{A}\colon V\otimes_\Lambda V^*\otimes_\Lambda V^* = \mathfrak{gl}(V)\otimes_\Lambda V^*\to V\otimes_\Lambda(\wedge^2V^*)\,.$$

It is clear that both operators are even. Then for S in $\mathfrak{gl}(V)\otimes_\Lambda V^*$, we have

$$\partial S(v, w) = -\tfrac{1}{2}(S(v)w-(-1)^{|v|\cdot|w|}S(w)v)\qquad (v, w\in V)\,.$$

If we denote the restrictions of ∂ to the submodule $\mathfrak{g}\otimes_\Lambda V^*$ of $\mathfrak{gl}(V)\otimes_\Lambda V^*$ and to the even subspaces of these Λ-modules by the same letter ∂, then the equation (3.12) can be rewritten as

$$(3.13)\qquad c_{H'_u}(v, w)-c_{H_u}(v, w) = 2\partial S_{H'_uH_u}(v, w)\qquad (v, w\in V)\,,$$

or equivalently, as bilinear forms,

$$(3.14)\qquad c_{H'_u}-c_{H_u} = 2\partial S_{H'_uH_u}\,.$$

Now we consider the following exact sequence of Λ-modules:

$$(3.15)\qquad (0)\to\partial(\mathfrak{g}\otimes_\Lambda V^*)\xrightarrow{\iota} V\otimes_\Lambda(\wedge^2V^*)$$
$$\xrightarrow{\rho}(V\otimes_\Lambda(\wedge^2V^*))_0/\partial(\mathfrak{g}\otimes_\Lambda V^*)_0\to(0)\,,$$

where ι and ρ respectively denote the inclusion and the quotient maps. Since both ι and ρ are even, by taking even subspaces of this sequence, we obtain the following sequence of Λ_0-modules which is also exact:

$$(3.16)\qquad (0)\to\partial(\mathfrak{g}\otimes_\Lambda V^*)_0\xrightarrow{\iota}(V\otimes_\Lambda(\wedge^2V^*))_0$$
$$\xrightarrow{\rho}(V\otimes_\Lambda(\wedge^2V^*))_0/\partial(\mathfrak{g}\otimes_\Lambda V^*)_0\to(0)\,,$$

where we used the canonical isomorphism

$$[V\otimes_\Lambda(\wedge^2V^*)/\partial(\mathfrak{g}\otimes_\Lambda V^*)]_0\xrightarrow{\sim}(V\otimes_\Lambda(\wedge^2V^*))_0/\partial(\mathfrak{g}\otimes_\Lambda V^*)_0\,.$$

The equation (3.14) tells us that the difference $c_{H'_u}-c_{H_u}$ lies in $\partial(\mathfrak{g}\otimes_\Lambda V^*)_0$. Hence c_{H_u} and $c_{H'_u}$ coincide modulo $\partial(\mathfrak{g}\otimes_\Lambda V^*)_0$. Namely, $\rho(c_{H_u})=\rho(c_{H'_u})$. Thus $\rho(c_{H_u})$ is independent of the choice of the horizontal space H_u and we obtain a function

(3.17) $c\colon P \ni u \to$

$$c(u) \in (V \otimes_\Lambda (\wedge^2 V^*))_0 / \partial(\mathfrak{g} \otimes_\Lambda V^*)_0 \subset V \otimes_\Lambda (\wedge^2 V^*) / \partial(\mathfrak{g} \otimes_\Lambda V^*)$$

defined by

$$c(u) = \rho(c_{H_u}) \qquad (u \in P), \tag{3.18}$$

where H_u is an arbitrary horizontal subspace at u. We call this function c *the (first order) structure function of the G-structure P.*

The following proposition is immediate from the definition of the structure function.

Proposition 6. *Let $P \to M$ and $Q \to N$ be G-structures. If $f\colon M \to N$ is an isomorphism of P onto Q then*

$$c_Q \circ f_* = c_P$$

where c_P and c_Q are respectively the structure functions of P and Q.

Corollary 7. *The structure function of a locally transitive G-structure is constant.*

3.4. G-connections. In order to interpret the structure function of the G-structure P, we use the concept of a connection in P. Here we mean, by a *connection in P*, a G-invariant horizontal distribution $\Gamma = \{H_u\}$ on P. More precisely, a connection Γ smoothly assignes a horizontal subspace $H_u \subset T_u(P)$ for each $u \in P$, so that

$$R_{a*} H_u = H_{u \cdot a} \qquad (u \in P,\ a \in G). \tag{3.19}$$

If it is clear which G-structure we are considering, say P, we call a connection in P simply by *a G-connection.*

As in the usual theory of connection, we can define, to each connection Γ, a $\mathfrak{g}$-valued 1-form ω satisfying

$$\omega(A^*) = A \qquad (A \in \mathfrak{g}) \tag{3.20}$$

and

$$R_a{}^* \omega = Ad(a)^{-1} \circ \omega \qquad (a \in G). \tag{3.21}$$

The relation between Γ and ω is given by

$$H_u = \{X \in T_u(P)\colon \omega_u(X) = 0\} \qquad (u \in P).$$

We call ω *the connection form of the conuection* Γ.

A connection Γ induces an operator D acting on differential forms on P: For a p-form φ, *the exterior covariant differential $D\varphi$ of φ* is defined by

$$(D\phi)_u(X_1, \cdots, X_{p+1}) = (d\phi)_u(hX_1, \cdots, hX_{p+1}) \qquad (X \in T_u(P))$$

for $u \in P$, where h is the projection

$$h: T_u(P) \to H_u . \tag{3.22}$$

As usual, the exterior differentials $D\theta$ of the canonical 1-form θ and $D\omega$ of the connection form ω are called *the torsion form* and *the cuvature form of the connection* Γ, respectively. We will denote by Θ the torsion form $D\theta$ and by Ω the curvature form $D\omega$. By the definition and Corollary 2, the torsion form Θ is a V-valued 2-form satisfying

$$R_a{}^*\Theta = a^{-1}\circ\Theta \qquad (a \in G) . \tag{3.32}$$

Similarly, by the definition and (3.21), the curvature form Ω is a $\mathfrak{g}$-valued 2-form and satisfies

$$R_a{}^*\Omega = Ad(a)^{-1}\circ\Omega \qquad (a \in G) . \tag{3.24}$$

Now we return to the study of the structure function of the G-structure P. Suppose that a connection $\Gamma = \{H_u\}$ in P is given. Let X be a tangent vector at u and $v = \theta_u(X) \in V$. Then we have

$$hX = B_{H_u}(v) \in H_u .$$

Hence we immediately obtain

$$\Theta_u(X, Y) = c_{H_u}(\theta_u(X), \theta_u(Y)) \qquad (X, Y \in T_u(P)) . \tag{3.25}$$

If we define a $V \otimes_\Lambda (\wedge^2 V^*)$-valued function $\tilde{c}$ on P by

$$\tilde{c}(u) = c_{H_u} ,$$

then (3.25) can be rewritten as

$$\Theta(X, Y) = \tilde{c}(\theta(X), \theta(Y)) \qquad (X, Y \in \mathfrak{X}(P)) . \tag{3.26}$$

Moreover, if we denote by $B(v)$ $(v \in V)$ the horizontal vector field on P defined by

$$(B(v))_u = B_{H_u}(v) \in H_u \qquad (u \in P) ,$$

then (3.26) implies that

$$\rho(\Theta(B(v), B(w))) = c(v, w) \qquad (v, w \in V) . \tag{3.27}$$

Thus we can regard the structure function of the G-structure P as the torsion form of the G-connection Γ modulo $\mathrm{Im}\,\partial$. We note that this interpretation does not depend on the choice of the connection. As a direct consequence of this interpretation we obtain the following proposition.

Proposition 8. *If a G-structure P admits a G-connection with zero torsion then its structure function is identically zero.*

Corollary 9. *The structure function of a locally flat G-structure is zero.*

Proof. By Proposition 6, it suffices to consider the standard flat G-structure $P=\boldsymbol{E}^{m|n}\times G\to\boldsymbol{E}^{m|n}$. It is easily verified that the torsion form of the natural connection in P is zero. Hence, by Proposition 8, the structure function of P is zero. q.e.d.

As we will seen in section 5, the curvature form is essentially the second order structure function.

3.5. Behavior of the structure function under the G-action. Finally, we describe the behavior of the structure function of the G-structure P under the right G-action on P. Let H_u be a horizontal subspace at $u\in P$. Then $R_{a*}H_u$ $(a\in G)$ is a horizontal subspace at $u\cdot a$. The horizontal lifts for these horizontal subspaces are related by

$$R_{a*}\circ B_{H_u}\circ a=B_{R_{a*}H_u}\,. \tag{3.28}$$

In fact, for an arbitrary $v\in V$, by the definition of horizontal lift and Proposition 1, it follows that

$$\begin{aligned}\theta_{u\cdot a}(R_{a*}(B_{H_u}((av)))&=(R_a{}^*\theta)_u(B_{H_u}(av))\\&=a^{-1}(\theta_u(B_{H_u}(av)))\\&=a^{-1}(av)=v\,.\end{aligned}$$

Moreover, since $R_{a*}\circ B_{H_u}\circ a$ is $R_{a*}H_u$-valued, we have

$$R_{a*}\circ B_{H_u}\circ a=(\theta_{u\cdot a}|_{R_{a*}H_u})^{-1}=B_{R_{a*}H_u}\,.$$

It follows from (3.28) and Corollary 2 that for $v,\ w\in V$

$$\begin{aligned}c_{R_{a*}H_u}(v,\ w)&=d\theta_{u\cdot a}(B_{R_{a*}H_u}(v),\ B_{R_{a*}H_u}(w))\\&=d\theta_{u\cdot a}(R_{a*}(B_{H_u}(av)),\ R_{a*}(B_{H_u}(aw)))\\&=(R_a{}^*d\theta)_u(B_{H_u}(av),\ B_{H_u}(aw))\\&=a^{-1}(d\theta_u(B_{H_u}(av),\ B_{H_u}(aw)))\\&=a^{-1}(c_{H_u}(av,\ aw))\,.\end{aligned}$$

Hence we have

$$c_{R_{a*}H_u}=\eta(a^{-1})c_{H_u}\,, \tag{3.29}$$

where η denotes the G-action on $V\otimes_\Lambda V^*\otimes_\Lambda V^*$ naturally induced from the one on V, that is,

$$(3.30)\qquad \eta(a)\colon V\otimes_\Lambda V^*\otimes_\Lambda V^*\ni v\otimes w^*\otimes z^* \to (av)\otimes(w^*\circ a^{-1})\otimes(z^*\circ a^{-1})\in V\otimes_\Lambda V^*\otimes_\Lambda V^*$$

for $a\ni G$.

We note that, since the super-alternating operator commutes with the G-action on $V^*\otimes_\Lambda V^*$, the operator ∂ also commutes with $\eta(a)$. Hence the submodule $V\otimes_\Lambda(\wedge^2 V^*)$ of $V\otimes_\Lambda V^*\otimes_\Lambda V^*$ is G-invariant. Moreover, the action of $\eta(a)$ on the submodule $\mathfrak{g}\otimes_\Lambda V^*$ is given by

$$\eta(a)(A\otimes v^*) = (Ad(a^{-1})A)\otimes(v^*\circ a^{-1}) \qquad (A\in\mathfrak{g},\ v^*\in V^*)\,.$$

Thus $\mathfrak{g}\otimes_\Lambda V^*$ is also a G-invariant submodule. Consequently, the operator

$$\partial\colon \mathfrak{g}\otimes_\Lambda V^* \to V\otimes_\Lambda(\wedge^2 V^*)$$

is a G-equivariant map and hence a G-action

$$\underline{\eta}(a)\colon V\otimes_\Lambda(\wedge^2 V^*)/\partial(\mathfrak{g}\otimes_\Lambda V^*) \to V\otimes_\Lambda(\wedge^2 V^*)/\partial(\mathfrak{g}\otimes_\Lambda V^*)$$

satisfying

$$(3.31)\qquad \underline{\eta}(a)\circ\rho = \rho\circ\eta(a)$$

is induced.

By combining (3.29) and (3.31), we conclude that

$$(3.32)\qquad c(u\cdot a) = \underline{\eta}(a^{-1})c(u) \qquad (u\in P,\ a\in G)\,.$$

This equatoin describes the behavior of the structure function under the G-action as we desired.

4. Prolongations

In this section, we introduce the notion of prolongation of G-structure. It will be defined as a reduced subbundle of the linear frame bundle of the G-structure. We will use the same notations as in the previous sections. Moreover, in what follows, we assume that the linear Lie supergroup G is of dimension $r|s$.

4.1. Distinguished linear frames. Let u be a point of the G-structure P. Since the tangent space $T_u(P)$ at u is a free Λ-module of dimension $(m+r)|(n+s)$, a linear frame w at u is a Λ-linear isomorphism

$$w\colon \Lambda^{(m+r)|(n+r)} \to T_u(P)\,,$$

where $\Lambda^{(m+r)|(n+s)}$ is the $(m+r)|(n+s)$-dimensional standard free Λ-module. By fixing a pure basis of $\mathfrak{g}$, we regard the linear frame w as an isomorphism

$$w\colon V\oplus\mathfrak{g}\to T_u(P)\,. \tag{4.1}$$

Then the structure group of the linear frame bundle $L(P)$ of P is identified with $GL(V\oplus\mathfrak{g})$.

Now we select a distinguished class of linear frames at a point $u\in P$ as follows. A linear frame at u is determined by its values on the subspaces V and $\mathfrak{g}$. On the other hand, there are two canonical Λ-linear maps associated to u defined, respectively, on V and $\mathfrak{g}$: One of them is u itself:

$$u\colon V\to T_x(M)\,,$$

and the other is the one induced from the infinitesimal action:

$$\sigma_u\colon \mathfrak{g}\ni A\to A^*(u)\in\mathfrak{g}_u\subset T_u(P)\,.$$

The linear frames at u that we choose are those compatible with these two maps: We call a linear frame w at u *distinguished* if it satisfies

$$\pi_*\circ(w|_V)=u \tag{4.2}$$

and

$$w|_g=\sigma_u\,. \tag{4.3}$$

We denote by $D_u(P)$ the set of all distinguished linear frames at $u\in P$. Then there is a natural correspondence between $D_u(P)$ and the set Hor_u of all horizontal lifts at u. In fact, it follows immediately from the definition that for a distinguished linear frame w at u, the image $w(V)$ is a horizontal subspace at u and hence $w|_V$ is the horizontal lift for $w(V)$. On the other hand by means of (4.3), a distinguished linear frame w at u is determined by its values on V. Thus we have a one-to-one correspondence between $D_u(P)$ and Hor_u defined by

$$D_u(P)\ni w\leftrightarrow w|_V\in\mathrm{Hor}_u\,. \tag{4.4}$$

Moreover, under the correspondence (4.4), a transitive and free action of $\mathrm{Hom}_\Lambda(V,\mathfrak{g})_0$ on $D_u(P)$ is induced from the action on Hor_u (cf. Proposition 5). It is easily verified that this action of $\mathrm{Hom}_\Lambda(V,\mathfrak{g}_0)$ on $D_u(P)$ is realized by the Abelian Lie subsupergroup

$$\left\{\tilde{S}=\begin{pmatrix}\mathrm{Id}_v & 0\\ S & \mathrm{Id}_{\mathfrak{g}}\end{pmatrix}\in GL(V\oplus\mathfrak{g})\colon S\in\mathrm{Hom}_\Lambda(V,\mathfrak{g})_0\right\}, \tag{4.5}$$

of $GL(V\oplus\mathfrak{g})$. We will denote this Lie supergroup by $\widetilde{(\mathfrak{g}\otimes_\Lambda V^*)_0}$ and identify it with $(\mathfrak{g}\otimes_\Lambda V^*)_0=\mathrm{Hom}_\Lambda(V,\mathfrak{g})_0$ by the isomorphism

$$\sim\colon \mathrm{Hom}_\Lambda(V,\mathfrak{g})_0\ni S\to\tilde{S}=\begin{pmatrix}\mathrm{Id}_v & 0\\ S & \mathrm{Id}_{\mathfrak{g}}\end{pmatrix}\in\widetilde{(\mathfrak{g}\otimes_\Lambda V^*)_0}\subset GL(V\oplus\mathfrak{g})\,. \tag{4.6}$$

Finally, we define *the distinguished linear frame bundle of P* by

$$D(P) = \bigcup_{u\in P} D_u(P).$$

Then $D(P)$ is a subbundle of the linear frame bundle $L(P)$ with the structure group $(\mathfrak{g}\otimes_\Lambda V^*)_0$. Thus we have the following proposition:

Proposition 10. *Let $P\to M$ be a G-structure. Then the distinguished linear frame bundle*

$$D(P)\to P$$

is a $\widetilde{(\mathfrak{g}\otimes_\Lambda V^*)}_0$-structure on P, where $\widetilde{(\mathfrak{g}\otimes_\Lambda V^*)}_0$ is the Abelian Lie subsupergroup of $GL(V\oplus\mathfrak{g})$ identified with $(\mathfrak{g}\otimes_\Lambda V^*)_0=\mathrm{Hom}_\Lambda(V,\mathfrak{g})_0$ by (4.6).

The following proposition asserts that the construction of the distinguished frame bundles of G-structures is canonical:

Proposition 11. *Let $P\to M$ and $Q\to N$ be G-structures and $f\colon M\to N$ an isomorphism of G-structure from P onto Q. Then the diffeomorphism*

$$f_*\colon P\to Q$$

is an isomorphism of $\widetilde{(\mathfrak{g}\otimes_\Lambda V^*)}_0$-structure of $D(P)$ onto $D(Q)$.

4.2. Lifting the values of the structure function. To define the notion of prolongation, we consider, as in the non-super theory of G-structures, the lifting of the values of the structure function

$$c\colon P\in u\to c(u)\in(V\otimes_\Lambda(\wedge^2V^*))_0/\partial(\mathfrak{g}\otimes_\Lambda V^*)_0$$

to $V\otimes_\Lambda(\wedge^2V^*)$. For this, it suffices to give a splitting of the exact sequence (3.15). A splitting of a short exact sequence of Λ-modules exists whenever all the modules are free:

Lemma 12. *Let*

$$(0)\to F_1\xrightarrow{\ \phi\ }F_2\xrightarrow{\ \psi\ }F_2\to(0)$$

be an exact sequence of free Λ-modules of finite rank. Then there exists a free submodule C of F_2 satisfying.

$$F_2=\phi(F_1)\oplus C.$$

Namely, an exact sequence of free Λ-modules of finite dimension splits.

In order to apply this lemma to (3.15), since $V\otimes_\Lambda(\wedge^2V^*)$ is already free, we must only show that both of $\partial(\mathfrak{g}\otimes_\Lambda V^*)$ and $V\otimes_\Lambda(\wedge^2V^*)/\partial(\mathfrak{g}\otimes_\Lambda V^*)$ are free.

However, by virtue of the following lemma, it suffices that one of them is free:

Lemma 13. *Let F be a free Λ-module of finite dimension. For an arbitrary Λ-module E and an injective Λ-linear map $\phi: E \to F$, the following three conditions are equivalent:*

i) *E is a free Λ-module.*

ii) *The image $\phi(E)$ is a free submodule of F.*

iii) *The quotient $F/\phi(E)$ is a free Λ-module.*

Thus our problem is reduced to the question whether $\partial(\mathfrak{g}\otimes_\Lambda V^*)$ is free or not. We call a Lie subsupergroup G of $GL(V)$ *admissible* if its Lie superalgebra $\mathfrak{g}$ satisfies the condition that $\partial(\mathfrak{g}\otimes_\Lambda V^*)$ is free.

We note that a usual Lie supergroup $G\subset GL(V)$ is admissible: A Lie subsupergroup G of $GL(V)$ is called *real* if its Lie superalgebra admits a pure basis consisting of real linear combinations of the standard basis of $\mathfrak{gl}(V)=\mathfrak{gl}(m|n;\Lambda)$. Here the standard basis of $\mathfrak{gl}(m|n;\Lambda)$ means the one consisting of elements of the form:

$$E^\mu{}_\nu = \begin{pmatrix} & \vdots & \\ \cdots & 1 & \cdots \\ & \vdots & \end{pmatrix} \begin{matrix} \\ <\mu \\ \\ \end{matrix} \qquad (\mu, \nu = 1, \cdots, m+n)\,.$$
$$\underset{\nu}{\wedge}$$

The following proposition gives a sufficient condition for a linear Lie supergroup G to be admissible.

Proposition 14. *A real Lie subsupergroup of $GL(V)$ is admissible.*

It is rather easy to check whether G is real or not:

Proposition 15. *If a Lie subsupergroup G of $GL(V)$ admits a system of defining equations consisting of smooth functions defined over $\boldsymbol{R}$ then G is real.*

We note that each smooth function on the $m|n$-dimensional Euclidean superspace $\boldsymbol{E}^{m|n}$ can be regarded as an element of the $\boldsymbol{Z}_2$-graded algebra $\Lambda\otimes C^\infty(\boldsymbol{R}^m)\otimes\wedge\boldsymbol{R}^n$ (cf. [11]). A smooth function on $\boldsymbol{E}^{m|n}$ is called defined over $\boldsymbol{R}$ if it is contained in the subalgebra $C^\infty(\boldsymbol{R}^m)\otimes\wedge\boldsymbol{R}^n$. This notion can be obviously extended to the functions on $GL(V)$.

By Proposition 15, the Lie subsupergroups in the examples of section 2 are all real and hence, by Proposition 14, admissible.

In what follows, we assume that the Lie supergroup G is admissible.

We can then lift the values of the structure function c as follows:

By the assumption, the exact sequence (3.15) is of free Λ-modules. Hence, by Lemma 12, there exists a free submodule C of $V\otimes_\Lambda(\wedge^2 V^*)$ such that

$$V \otimes_\Lambda (\wedge^2 V^*) = \partial(\mathfrak{g} \otimes_\Lambda V^*) \oplus C . \tag{4.7}$$

We call C *a complement to* $\partial(\mathfrak{g} \otimes_\Lambda V^*)$ *in* $V \otimes_\Lambda (\wedge^2 V^*)$.

If we take such a complement C then we can regard the structure function c as C-valued. Thus, through C, the values of c can be lifted to the Λ-module $V \otimes_\Lambda (\wedge^2 V^*)$. Of course, this lifting depends on the choice of the complement C. Finally, we note that since the Λ-submodule C is isomorphic to the quotient module $V \otimes_\Lambda (\wedge^2 V^*)/\partial(\mathfrak{g} \otimes_\Lambda V^*)$, its even subspace C_0 is isomorphic to $[V \otimes_\Lambda (\wedge^2 V^*)/\partial(\mathfrak{g} \otimes_\Lambda V^*)]_0$ as a Λ_0-module. Thus the lifted values of c lie in $C_0 \subset (V \otimes_\Lambda (\wedge^2 V^*))_0$.

4.3. Prolongations. Now, we are in the position to define the prolongation of a G-structure P. To do this, by taking a complement C to $\partial(\mathfrak{g} \otimes_\Lambda V^*)$ in $V \otimes_\Lambda (\wedge^2 V^*)$, we lift the range of the structure function c to $(V \otimes_\Lambda (\wedge^2 V^*))_0$, which we denote by $\tilde{c}$. For each horizontal subspace H_u ($u \in P$), we can select a class of horizontal subspaces H_u satisfying

$$c_{H_u} = \tilde{c}(u) , \tag{4.8}$$

or equivalently,

$$c_{H_u} \in C_0 . \tag{4.9}$$

Through the one-to-one correspondence (4.4), the horizontal subspaces of this class defines a subset of the space $D_u(P)$ of all the distinguished linear frames at u. We denote this subset by $P^{(1)}{}_u$. Finally, we define

$$P^{(1)} = \bigcup_{u \in P} P^{(1)}{}_u ,$$

and call it *the first prolongation of the G-structure P*.

It is clear that the first prolongation $P^{(1)}$ is a subbundle of the distinguished linear frame bundle $D(P)$. Moreover, its structure group is equal to the even subspace of the kernel of

$$\partial \colon \mathfrak{g} \otimes_\Lambda V^* \to V \otimes_\Lambda (\wedge^2 V^*) . \tag{4.10}$$

In fact, suppose that two horizontal subspaces H_u and H'_u satisfy the equation (4.8). Then

$$\partial S_{H'_u H_u} = \tfrac{1}{2}(c_{H'_u} - c_{H_u}) = 0 ,$$

where $S_{H'_u H_u}$ is the unique element of $\mathrm{Hom}_\Lambda(V, \mathfrak{g})_0$, satisfying (3.8). Hence we have

$$S_{H'_u H_u} \in (\ker \partial)_0 .$$

If we denote by w_{H_u} and $w_{H'_u}$ the distinguished linear frames corresponding to H_u and H'_u, respectively, then $w_{H'_u} = w_{H_u} \cdot \tilde{S}_{H'_u H_u}$, where

$$\tilde{S}_{H'_u H_u} = \begin{pmatrix} \mathrm{Id}_V & 0 \\ S_{H'_n H_u} & \mathrm{Id}_{\mathfrak{g}} \end{pmatrix} \in \widetilde{(\mathfrak{g} \otimes_\Lambda V^*)_0} \subset GL(V \oplus \mathfrak{g}) .$$

Here $\widetilde{(\mathfrak{g}\otimes_\Lambda V^*)_0}$ is the Lie subsupergroup (4.5) of $GL(V\oplus\mathfrak{g})$. Thus the structure group of $P^{(1)}$ is

$$\widetilde{(\ker \partial)_0} = \left\{ \tilde{S} = \begin{pmatrix} \mathrm{Id}_V & 0 \\ S & \mathrm{Id}_{\mathfrak{g}} \end{pmatrix} \in \mathrm{Hom}_\Lambda(V, \mathfrak{g})_0 \colon S \in (\ker \partial)_0 \right\} .$$

By our assumption that G is admissible, this is a Lie subsupergroup of $GL(V\oplus\mathfrak{g})$. We denote this Lie subsupergroup by $G^{(1)}$ and call it *the first prolongation of the linear Lie subsupergroup* $G\subset GL(V)$. Moreover, we denote its Lie superalgebra by $\mathfrak{g}^{(1)}$ and call it *the first prolongation of the linear Lie superalgebra* $\mathfrak{g}\subset\mathfrak{gl}(V)$. We note that $\mathfrak{g}^{(1)}$ and $G^{(1)}$ can be naturally identified with the kernel of Λ-linear map (4.10) and its even subspace, respectively.

As a summary, the first prolongation

$$P^{(1)} \to P$$

of the G-structure P is a $G^{(1)}$-structure on P.

Our construction of prolongation depends on the choice of the complement C to $\partial(\mathfrak{g}\otimes_\Lambda V^*)$ in $V\otimes_\Lambda(\wedge^2 V^*)$. To investigate the equivalence problem of G-structures for a concrete linear Lie supergroup G, we must choose a suitable complement C, which is G-invariant if possible. Thus, in what follows when we refer to prolongations of G-structures, we assume that a complement C to $\partial(\mathfrak{g}\otimes_\Lambda V^*)$ is fixed unless otherwise is stated.

REMARK. As we have seen in the construction of the prolongation of a G-structure, the assumption that the structure group G is admissible is necessary for the first prolongation $G^{(1)}$ of G to be a 'regular' Lie supergroup.

4.4. Naturality of prolongations. At first, we will see that the equivalence problem of G-structures is reduced to that of the prolongated ones. Nemely, by combining Propositions 6 and 7, we immediately obtain the following proposition.

Proposition 16. *Suppose that a complement C to $\partial(\mathfrak{g}\otimes_\Lambda V^*)$ in $V\otimes_\Lambda(\wedge^2 V^*)$ is given and fixed. Let $P\to M$ and $Q\to N$ be G-structures and $f\colon M\to N$ an isomorphism of G-structure from P onto Q. Then the diffeomcrphism*

$$f_*\colon P \to Q$$

is an isomorphism of $G^{(1)}$-structure of $P^{(1)}$ onto $Q^{(1)}$, that is, an isomorphism of the prolongations of the G-structures P and Q.

From this proposition, we can conclude that by fixing a complement C to $\partial(\mathfrak{g}\otimes_{\Lambda}V^{*})$ in $V\otimes_{\Lambda}(\wedge^{2}V^{*})$, the procedure of prolongation preserves the condition that G-structures are isomorphic.

The converse is true whenever the body of G is connected. *We will hereafter assume the body of G is connected.*

Let $\pi_{P}\colon P\to M$ and $\pi_{Q}\colon Q\to N$ be G-structures and $f^{(1)}\colon P\to Q$ be an isomorphism of $G^{(1)}$-structure from $P^{(1)}$ onto $Q^{(1)}$. Then it follows that

$$(f^{(1)})_{*}A^{*P}=A^{*Q}\qquad(A\in\mathfrak{g}) \tag{4.11}$$

and

$$(f^{(1)})^{*}\theta_{P}=\theta_{Q}\,, \tag{4.12}$$

where A^{*P} and A^{*Q} denote the infinitesimal actions of $A\in\mathfrak{g}$ on P and Q, respectively, and the canonical 1-forms on P and Q are denoted by θ_{P} and θ_{Q} respectively.

By (4.11) and connectivity of the body of G, $f^{(1)}$ commutes with the action of G, whence $\pi_{Q}\circ f^{(1)}$ is constant along each fibre of P. There exist thus a diffeomorphism $f\colon M\to N$ satisfying

$$f\circ\pi_{P}=\pi_{Q}\circ f^{(1)}\,.$$

On the other hand, by (4.12), $f^{(1)}$ coincides with the differential of f:

$$f_{*}=f^{(1)}\,. \tag{4.13}$$

This implies that f is an isomorphism of G-structure from P to Q. Thus we have the converse of Proposition 16:

Proposition 17. *Let $P\to M$ and $Q\to N$ be G-structures and $f^{(1)}\colon P\to Q$ an isomorphsim of $G^{(1)}$-structure from $P^{(1)}$ onto $Q^{(1)}$. If the body of G is connected, then there exists an isomorphism of G-structure*

$$f\colon M\to N$$

from P onto Q such that

$$f_{*}=f^{(1)}$$

Propositions 16 and 17 can be summarized as follows:

Theorem 18. *If G-structures are isomorphic then their prolongations are also isomorphic. Conversely, if the body of G is connected, the equivalence of prolongated structures implies the equivalence of the underlying G-structures.*

This theorem is the first step to our Main Theorem stated in the introduction.

5. Higher order prolongations

In this section, we introduce the notion of higher order prolongations and structure functions. By applying Theorem 18 successively, we can reduce the equivalence problem of G-structures to those of the prolongated structures. The merit of this procedure is that the higher we prolongate the structures the finer data we can get for each original G-structure as the higher order structure functions.

5.1. Higher order prolongations. The first prolongation of a G-structure is a $G^{(1)}$-structure. This procedure can be continued and we obtain higher order prolongations as follows.

We first define the notion of the second prolongations of linear Lie supergroups and their Lie superalgebras. Let G be a Lie subsupergroup of $GL(V)$ and $\mathfrak{g}\subset\mathfrak{gl}(V)$ be its Lie superalgebra. Recall that the first prolongations $g^{(1)}$ of g and $G^{(1)}$ of G can be, respectively, defined as the kernel of the operator

$$\partial=\mathrm{Id}_V\otimes_\Lambda \mathcal{A}\colon \mathfrak{g}\otimes_\Lambda V^*\to V\otimes_\Lambda(\wedge^2 V^*)\,,$$

and as its even subspace, where

$$\mathcal{A}\colon V^*\otimes_\Lambda V^*\to \wedge^2 V^*$$

is the super-alternating operater.

Generalizing this, we consider, for Λ-modules W and W', the operator

$$\partial=\mathrm{Id}_W\otimes_\Lambda \mathcal{A}\colon W\otimes_\Lambda W'\otimes_\Lambda W'\to W\otimes_\Lambda(\wedge^2 W')$$

and for an arbitrary submodule U of $W\otimes_\Lambda W'$, we define *the first prolongation* $U^{(1)}$ of U as the kernel of the restriction

$$\partial\colon U\otimes_\Lambda W\to W\otimes_\Lambda(\wedge^2 W')\,.$$

Thus the first prolongation $(\mathfrak{g}^{(1)})^{(1)}$ of $\mathfrak{g}^{(1)}\subset\mathfrak{g}\otimes_\Lambda V^*$ is defined as the kernel of

$$\partial\colon \mathfrak{g}^{(1)}\otimes_\Lambda V^*\to \mathfrak{g}\otimes_\Lambda(\wedge^2 V^*)\,,$$

which is called *the second prolongation of* $\mathfrak{g}$ and is denoted by $\mathfrak{g}^{(2)}$. Furthermore, we define *the second prolongation of* G by

$$G^{(2)}=(\mathfrak{g}^{(2)})_0\,.$$

If $G^{(1)}$ is admissible, then the second prolongation $G^{(2)}$ is an Abelian Lie supergroup identified with the Lie subsupergroup

$$\left\{\begin{pmatrix}\mathrm{Id}_V & 0 & 0\\ 0 & \mathrm{Id}_\mathfrak{g} & 0\\ T & 0 & \mathrm{Id}_{\mathfrak{g}^{(1)}}\end{pmatrix}\colon T\in(\mathfrak{g}^{(2)})_0\right\} \tag{5.1}$$

of $GL(V\oplus\mathfrak{g}\oplus\mathfrak{g}^{(1)})$ and its Lie superalgebra can be identified with $\mathfrak{g}^{(2)}$. In this case, the first prolongation

$$(P^{(1)})^{(1)} \to P^{(1)}$$

of $P^{(1)}$ can be defined, by taking a complement of $\partial(\mathfrak{g}^{(1)}\otimes_\Lambda V^*)$ in $\mathfrak{g}\otimes_\Lambda(\wedge^2 V^*)$, as a $G^{(2)}$-structure on $P^{(1)}$. We call this *the second prolongation of the G-structure* P and denote it by $\boldsymbol{P}^{(2)}$.

Now we generally define higher order prolongations, by induction. For a non-negative integer k, *the* $(k+1)$*-th prolongations* $\mathfrak{g}^{(k+1)}$ *and* $G^{(k+1)}$ are respectively defined by

$$\mathfrak{g}^{(k+1)} = (\mathfrak{g}^{(k)})^{(1)} \tag{5.2}$$

and by

$$G^{(k+1)} = (\mathfrak{g}^{(k+1)})_0\,. \tag{5.3}$$

For $k=0$, we define by

$$\mathfrak{g}^{(-1)} = V$$

and

$$\mathfrak{g}^{(0)} = \mathfrak{g}\,.$$

We will often denote G also by $G^{(0)}$.

We can again identify $G^{(k+1)}$ with the Abelain Lie subsupergroup

$$\left\{\begin{pmatrix} \mathrm{Id}_V & 0 \cdots\cdots\cdots & 0 \\ 0 & \mathrm{Id}_{\mathfrak{g}} \quad 0 \cdots & 0 \\ \vdots\ \ddots & \ddots \qquad \ddots & \\ 0 & \ddots \qquad \ddots & 0 \\ T & 0 \cdots 0 & \mathrm{Id}_{\mathfrak{g}^{(k)}} \end{pmatrix} : T\in(\mathfrak{g}^{(k+1)})_0\right\} \subset GL(V\oplus\mathfrak{g}\oplus\cdots\oplus\mathfrak{g}^{(k)}) \tag{5.4}$$

and its Lie superalgebra with $\mathfrak{g}^{(k+1)}$ whenever G, $G^{(1)}$, $\cdots$, $G^{(k)}$ are all admissble.

In what follows, we assume that not only G but also all the prolongations $G^{(k)}$ $(k=1, 2, \cdots)$ *are admissible.*

It is clear that real linear Lie supergroups always satisfy this assumption.

Under the above assumption and by taking complements C^{h+1} to $\partial(\mathfrak{g}^{(h)}\otimes_\Lambda V^*)$ in $\mathfrak{g}^{(h-1)}\otimes_\Lambda(\wedge^2 V^*)$ $(h=0, 1, \cdots, k)$, the $(k+1)$-th prolongation $P^{(k+1)}$ of a G-structure P can be defined for a non-negative integer k by

$$P^{(k+1)} = (P^{(k)})^{(1)} \to P^{(k)} \tag{5.5}$$

as a $G^{(k+1)}$-structure on $P^{(k)}$.

We summarize the main ingredients of the $(k+1)$-th prolongations of the G-structure $P \to M$ for a non-negative integer k in the following table. For the case $k=0$, $P^{(0)}$ stands for P.

$(k+1)$-th prolongation	$P^{(k+1)} \to P^{(k)}$
Operator ∂	$\partial\colon \mathfrak{g}^{(k)} \otimes_{\Lambda} V^* \to \mathfrak{g}^{(k-1)} \otimes_{\Lambda} (\wedge^2 V^*)$
Lie superalgebra $\mathfrak{g}^{(k+1)}$	$\mathfrak{g}^{(k+1)} = \ker \partial \subset \mathfrak{g}^{(k)} \otimes_{\Lambda} V^*$
Lie supergroup $G^{(k+1)}$	$G^{(k+1)} = (\mathfrak{g}^{(k+1)})_0 \subset GL(V \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(k)})$
Complement C^{k+1}	$C^{k+1} \oplus \partial(\mathfrak{g}^{(k)} \otimes_{\Lambda} V^*) = \mathfrak{g}^{(k-1)} \otimes_{\Lambda} (\wedge^2 V^*)$

5.2. Higher order structure functions. Since the k-th prolongation $P^{(k)} \to P^{(k-1)}$ of the G-structure $\pi\colon P \to M$ is a $G^{(k)}$-structure on $P^{(k-1)}$, we can consider its (first order) structure function. We call it *the $(k+1)$-th order structure function of the G-structure P* and denote it by $c^{(k+1)}$:

$$c^{(k+1)}\colon P^{(k)} \to (V \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(k-1)}) \otimes_{\Lambda} (\wedge^2 (V \oplus \mathfrak{g} \cdots \oplus \mathfrak{g}^{(k-1)})^*) / \partial(\mathfrak{g}^{(k)} \otimes_{\Lambda} (V \oplus \mathfrak{g} \cdots \oplus \mathfrak{g}^{(k-1)})^*) .$$

The $(k+1)$-th order structure function $c^{(k+1)}$ contains the data of k-th one $c^{(k)}$. To see this, we consider, for the sake of simplicity, the first prolongation $\pi^{(1)}\colon P^{(1)} \to P$ and the second order structure function

$$c^{(2)}\colon P^{(1)} \to (V \oplus \mathfrak{g}) \otimes_{\Lambda} (\wedge^2 (V \oplus \mathfrak{g}^*) / \partial(\mathfrak{g}^{(1)} \otimes_{\Lambda} (V \oplus \mathfrak{g})^*) .$$

and then we give a precise description for the second order structure function in the rest of this section.

5.3. Relation between the canonical 1-forms. To begin with, we describe the relation between the canonical 1-forms $\theta^{(1)}$ on $P^{(1)}$ and θ on P. Fix $w \in P^{(1)}$ and put $u = \pi^{(1)}(w)$. Since w is a distinguished linear frame at u, as an isomorphism of $V \oplus \mathfrak{g}$ onto $T_u(P)$, it can be expressed as

$$w = B_{H_u} \circ p_V + \sigma_u \circ p_{\mathfrak{g}} , \tag{5.6}$$

where H_u is the w-image of V and p_V and $p_{\mathfrak{g}}$ respectively denote the projections

$$p_V\colon V \oplus \mathfrak{g} \to V , \quad p_{\mathfrak{g}}\colon V \oplus \mathfrak{g} \to \mathfrak{g} ,$$

and w is regarded as an isomorphism of $V \oplus \mathfrak{g}$ onto $T_u(P)$. Moreover, since $\theta^{(1)}$ is a $(V \oplus \mathfrak{g})$-valued 1-form, it can be decomposed into V-component $\theta^{(1)}{}_V$ and $\mathfrak{g}$-component $\theta^{(1)}{}_{\mathfrak{g}}$. By definition of $\theta^{(1)}$, we have

$$(\theta^{(1)}{}_V)_w = p_V \circ w^{-1} \circ \pi^{(1)}{}_*, \quad (\theta^{(1)}{}_{\mathfrak{g}})_w = p_{\mathfrak{g}} \circ w^{-1} \circ \pi^{(1)}{}_* .$$

Let $X \in T_w(P^{(1)})$. Then its $\pi^{(1)}{}_*$-image can be uniquely expressed as

$$(5.7) \qquad \pi^{(1)}{}_*(X) = B_{H_u}(v) + \sigma_u(A) \qquad (v \in V,\ A \in \mathfrak{g}) .$$

By applying w^{-1} to this, we have

$$(5.8) \qquad w^{-1}(\pi^{(1)}{}_*(X)) = v + A \in V \oplus \mathfrak{g} .$$

This implies that

$$(5.9) \qquad (\theta^{(1)}{}_V)_w(X) = v , \quad (\theta^{(1)}{}_{\mathfrak{g}})_w(X) = A .$$

On the other hand, by applying θ_u to (5.7), we have

$$(5.10) \qquad \theta_u(\pi^{(1)}{}_*(X)) = v .$$

Thus we have the following relation between $\theta^{(1)}$ and θ:

$$(5.11) \qquad \theta^{(1)}{}_V = \pi^{(1)*}\theta .$$

In contrast to the V-component of $\theta^{(1)}$, there is no intrinsic expression for the $\mathfrak{g}$-component $\theta^{(1)}{}_{\mathfrak{g}}$. We can however locally express it as follows (cf. 5.21).

Let

$$\phi\colon P^{(1)}|_U \ni w \to (\pi^{(1)}(w), \tilde{\psi}(w)) \in U \times G^{(1)}$$

be a local triviality of $P^{(1)}$ over a superdomain U of P and

$$\tau\colon U \ni u \to \phi^{-1}(u, \tilde{0}) \in P^{(1)}|_U$$

be the canonical local cross section of P over U, where $\tilde{0} = e$ denotes the unit element of $G^{(1)}$. Write the element $\tilde{\psi}(w) \in G^{(1)}$ in the form:

$$(5.12) \qquad \tilde{\psi}(w) = \tilde{S} = \begin{pmatrix} \mathrm{Id}_V & 0 \\ S & \mathrm{Id}_{\mathfrak{g}} \end{pmatrix} \in GL(V \oplus \mathfrak{g}) \qquad (S \in (\mathfrak{g}^{(1)})_0) .$$

We will denote the element $S \in (\mathfrak{g}^{(1)})_0$ in (5.12) by $\psi(w)$. Then ψ is a $(\mathfrak{g}^{(1)})_0$-valued function on $P^{(1)}|_U$ satisfying

$$\widetilde{(\psi(w))} = \tilde{\psi}(w) .$$

Moreover, we will often use the notation ψ_w for $\psi(w)$.

By using the cross section τ, we define a horizontal distribution $\{H_u\}$ on $U \subset P$ as follows. Let $u \in U$. Then

$$\tau(u)\colon V \oplus \mathfrak{g} \to T_u(P)$$

is a distinguished linear frame at u. Hence the horizontal subspace H_u at u

is defined by

$$H_u = (\tau(u))(V) \subset T_u(P) . \tag{5.13}$$

Since $\tau(u)$ is in $P^{(1)}$, the horizontal subspace H_u satisfies

$$c_{H_u} = \widetilde{c^{(1)}}(u) \in C^1 , \tag{5.14}$$

where $\widetilde{c^{(1)}}$ is the lift of the first order structure function $c^{(1)}$ with respect to the complement C^1 to $\partial(\mathfrak{g}\otimes_\Lambda V^*)$ in $V\otimes_\Lambda(\wedge^2 V^*)$.

We note that the horizontal distribution $\{H_u\}$ is not necessarily a connection in $U\subset P$. In fact, $R_{a*}H_u$ is different from $H_{u\cdot a}$ for $u\in U$ and $a\in G$ in general. We will later consider the case when $\{H_u\}$ is a connection.

By using (5.6) and (5.13), we can express $\tau(u)$ as a distinguished linear frame by:

$$\tau(u) = B_{H_u}\circ p_V + \sigma_u \circ p_{\mathfrak{g}} . \tag{5.15}$$

Furthermore, for a general $w\in P^{(1)}$ satisfying $\pi^{(1)}(w)=u\in U$, it can be expressed as

$$w = B_{H_u}\circ p_V + \sigma_u\circ(p_{\mathfrak{g}} + \psi_w\circ p_V) . \tag{5.16}$$

since $w=\tau(u)\cdot\tilde{\psi}(w)$.

Now we introduce a $\mathfrak{g}$-valued 1-form ω on U as follows. For $u\in U$, we define ω_u by

$$\omega_u = p_{\mathfrak{g}}\circ(\tau(u)^{-1}) \colon T_u(P) \to \mathfrak{g} . \tag{5.17}$$

or equivalently,

$$\omega_u\circ\tau(u) = p_{\mathfrak{g}} \colon V\oplus\mathfrak{g} \to \mathfrak{g} . \tag{5.18}$$

Then we have

$$\omega_u(A^*{}_u) = \omega_u(\sigma_u(A)) = A \qquad (A\in\mathfrak{g})$$

and

$$\omega_u(B_{H_u}(v)) = 0 \qquad (v\in V) .$$

By applying $\pi^{(1)}{}_*$ from the right to the both sides of (5.17), we obtain

$$(\theta^{(1)}{}_{\mathfrak{g}})_{\tau(u)} = \pi^{(1)*}\omega_u . \tag{5.19}$$

Hence, by combining (5.11) and (5.39), we have

$$\theta^{(1)}{}_{\tau(u)} = \underbrace{\pi^{(1)*}\theta_u}_{V\text{-component}} + \underbrace{\pi^{(1)*}\omega_u}_{\mathfrak{g}\text{-component}} = \pi^{(1)*}(\theta_u + \omega_u) . \tag{5.20}$$

For a general $w\in P^{(1)}$ satisfying $\pi^{(1)}(w)=u\in U$, it follows from this and Proposition 1 that,

$$\begin{aligned}\theta^{(1)}{}_w &= \theta^{(1)}{}_{\tau(u)\cdot\tilde\psi(w)}\\ &= \tilde\psi(w)^{-1}\circ\theta^{(1)}{}_{\tau(u)}\circ(R_{\tilde\psi(w)*})^{-1}\\ &= \tilde\psi(w)^{-1}\circ((\theta_u+\omega_u)\circ\pi^{(1)}{}_*)\circ(R_{\tilde\psi(w)*})^{-1}\\ &= (\tilde\psi(w)^{-1}\circ(\theta_u+\omega_u))\circ(\pi^{(1)}{}_*\circ(R_{\tilde\psi(w)*})^{-1})\\ &= (\theta_u+\omega_u-\psi_w\circ\theta_u)\circ\pi^{(1)}{}_* .\end{aligned}$$

Thus we have

$$\theta^{(1)} = \underbrace{\pi^{(1)*}\theta}_{V\text{-component}}+\underbrace{\pi^{(1)*}\omega-\psi\circ\pi^{(1)*}\theta}_{\mathfrak{g}\text{-component}} . \tag{5.21}$$

5.4. Description of the second order structure function. Now we can describe the second order structure function.

First, we note that, by means of the local triviality, we can introduce a natural horizontal subspace K_w at $w\in P^{(1)}|_U$ by

$$K_w = R_{\tilde\psi(w)*}(\tau_*(T_u(P))) \qquad (u = \pi^{(1)}(w)) .$$

Then the horizontal lift B_{K_w} for K_w satisfies

$$\pi^{(1)}{}_*\circ B_{K_m} = w = B_{H_u}\circ p_V+\sigma_u\circ(p_{\mathfrak{g}}+\psi_w\circ p_V)$$

and hence for $Z=v+A\in V\oplus\mathfrak{g}$,

$$\pi^{(1)}{}_*\circ B_{H_u}(Z) = B_{H_u}(v)+(A+\psi_w(v))^*{}_u . \tag{5.22}$$

For each $Z\in V\oplus\mathfrak{g}$, we define a horizontal vector field $B(Z)$ by

$$(B(Z))_w = B_{K_w}(Z)\in K_w\subset T_w(P^{(1)}) .$$

Now we are ready to calculate $d\theta^{(1)}$ using (5.21). Since ψ is constant along the cross section τ, we have

$$L_{(B(Z))}\psi = 0 , \qquad Z\in V\oplus\mathfrak{g} . \tag{5.23}$$

Thus for $Z=v+A$, $Z'=v'+A'\in V\oplus\mathfrak{g}$, we have

$$\begin{aligned}(5.24)\quad & 2d(\psi_\cdot\circ\pi^{(1)*}\theta)(B(Z), B(Z'))\\ &= L_{B(Z)}(\psi_\cdot(v'))-(-1)^{|Z|\cdot|Z'|}L_{B(Z')}(\psi_\cdot(v))-\psi_\cdot\circ\pi^{(1)*}\theta([B(Z), B(Z')])\\ &= -\psi_\cdot\circ\pi^{(1)*}\theta([B(Z), B(Z')])\\ &= \psi_\cdot\circ(2d(\pi^{(1)*}\theta)(B(Z), B(Z'))\\ &\qquad -L_{B(Z)}(\pi^{(1)*}\theta(B(Z'))+(-1)^{|Z|\cdot|Z'|}L_{B(Z')}(\pi^{(1)*}\theta(B(Z)))\\ &=2\psi_\cdot\circ\tilde c(v, v')\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
(5.25)\quad & d(\pi^{(1)*}\theta)_w(B_{K_w}(Z), B_{K_w}(Z')) \\
&= d\theta_u(B_{H_u}(v)+(A+\psi_w(v))^*{}_u, B_{H_u}(v')+(A'+\psi_w(v'))^*{}_u) \\
&= d\theta_u(B_{H_u}(v), B_{H_u}(v')) \\
&\quad +(-1)^{|Z|\cdot|Z'|}\tfrac{1}{2}(A'+\psi_w(v'))\circ\theta_u(B_{H_u}(v))-\tfrac{1}{2}(A+\psi_w(v))\circ\theta_u(B_{H_u}(v')) \\
&\quad -\tfrac{1}{2}(A+\psi_w(v))\circ\theta_u((A'+\psi_w(v'))^*{}_u) \\
&= c_{H_u}(v, v')+(-1)^{|Z|\cdot|Z'|}\tfrac{1}{2}(A'+\psi_w(v'))(v)-\tfrac{1}{2}(A+\psi_w(v))(v') \\
&= c_{H_u}(v, v')-\tfrac{1}{2}(Av'-(-1)^{|Z|\cdot|Z'|}A'v) \\
&\quad -\tfrac{1}{2}(\psi_w(v)(v')-(-1)^{|Z|\cdot|Z'|}\psi_w(v')(v)) \\
&= c_{H_u}(v, v')-\tfrac{1}{2}(Av'-(-1)^{|Z|\cdot|Z'|}A'v)-\partial\psi_w(v, v') \\
&= c_{H_u}(v, v')-\tfrac{1}{2}(Av'-(-1)^{|Z|\cdot|Z'|}A'v)
\end{aligned}
$$

and

$$
\begin{aligned}
(5.26)\quad & d(\pi^{(1)*}\omega_w)(B_{K_w}(Z), B_{K_w}(Z')) \\
&= d\omega_u(B_{H_u}(v)+(A+\psi_w(v))^*{}_u, B_{H_u}(v')+(A'+\psi_w(v'))^*{}_u) \\
&= d\omega_u(B_{H_u}((v), B_{H_u}(v')) \\
&\quad +(-1)^{|Z|\cdot|Z'|}\tfrac{1}{2}(A'+\psi_w(v'))\circ\omega_u(B_{H_u}(v))-\tfrac{1}{2}(A+\psi_w(v))\circ\omega_u(B_{H_u}(v')) \\
&\quad -\tfrac{1}{2}\,ad(A+\psi_w(v))\circ\omega_u((A'+\psi_w(v'))^*{}_u) \\
&= d\omega_u(B_{H_u}(v), B_{H_u}(v'))-\tfrac{1}{2}[A+\psi_w(v), A'+\psi_w(v')]\,.
\end{aligned}
$$

By combining (5.24), (5.25) and (5.26), we have

$$
\begin{aligned}
(5.27)\quad c_{K_w}(Z, Z') &= d\theta^{(1)}{}_w(B_{K_w}(Z), B_{K_w}(Z')) \\
&= c_{H_u}(v, v')-\tfrac{1}{2}(Av'-(-1)^{|Z|\cdot|Z'}A'v)-\psi_w\circ c_{H_u}(v, v') \\
&\quad +d\omega_u(B_{H_u}(v), B_{H_u}(v'))-\tfrac{1}{2}[A+\psi_w(v), A'+\psi_w(v')]
\end{aligned}
$$

for $Z=v+A$, $Z'=v'+A'\in V\oplus\mathfrak{g}$.

If we define $\widetilde{c^{(2)}}(w)$ by

$$
\widetilde{c^{(2)}}(w) = c_{K_w}\in(V\oplus\mathfrak{g})\otimes_\Lambda(\wedge^2(V\oplus\mathfrak{g})^*)\,,
$$

then the equation (5.27) can be rewritten as

$$
\begin{aligned}
(5.28)\quad \widetilde{c^{(2)}}(w)(Z, Z') &= \widetilde{c^{(1)}}(u)(v, v')-\tfrac{1}{2}(Av'-(-1)^{|Z|\cdot|Z'|}A'v)-\psi_w\circ c_{H_u}(v, v') \\
&\quad +d\omega_u(B_{H_u}(v), B_u(v'))-\tfrac{1}{2}[A+\psi_w(v), A'+\psi_w(v')]\,.
\end{aligned}
$$

We note that this may not lie in the complement C^2 to $\partial(\mathfrak{g}^{(1)}\otimes_\Lambda V^*)$, as in section 3.

Thus we have shown that the second structure function $c^{(2)}$ contains the data of the first structure function $c^{(1)}$, since $\widetilde{c^{(2)}}(w)$ represents the class $c^{(2)}(w)$.

By induction, we obtain the following proposition:

Proposition 19. *The* $(k+1)$*-th order structure function* $c^{(k+1)}$ *contains the structure functions* $c^{(h)}$ $(h=1, \cdots, k)$.

Corollary 20. *If the* $(k+1)$*-th order structure function* $c^{(k+1)}$ *is constat for some integer* k*, then the lower structure functions* $c^{(h)}$ $(h=1, \cdots, k)$ *are all constant.*

In (5.28), by taking $\tau(u)$ as w, we obtain:

$$(5.29) \qquad \widetilde{c^{(2)}}(\tau(u))(Z, Z') = \widetilde{c^{(1)}}(u)(v, v') - \tfrac{1}{2}(Av' - (-1)^{|Z|\cdot|Z'|}A'v) + d\omega_u(B_{H_u}(v), B_{H_u}(v')) - \tfrac{1}{2}[A, A'] .$$

The equation (5.28) is obtained from (5.29) by the $G^{(1)}$-action on the second order structure functions. Moreover, (5.29) gives information corresponding to the natural decomposition

$$\begin{aligned}(V \oplus \mathfrak{g}) \otimes_{\Lambda} (\wedge^2 (V \oplus \mathfrak{g})^*) = {} & V \otimes_{\Lambda} (\wedge^2 V^*) \oplus V \otimes_{\Lambda} (V^* \otimes \mathfrak{g}^* + \mathfrak{g}^* \otimes_{\Lambda} V^*) \\ & \oplus V \otimes_{\Lambda} (\wedge^2 \mathfrak{g}^*) \oplus \mathfrak{g} \otimes_{\Lambda} (\wedge^2 V^*) \\ & \oplus \mathfrak{g} \otimes_{\Lambda} (V^* \otimes_{\Lambda} \mathfrak{g}^* + \mathfrak{g}^* \otimes_{\Lambda} V^*) \oplus \mathfrak{g} \otimes_{\Lambda} (\wedge^2 \mathfrak{g}^*)\end{aligned}$$

as in the following table.

$V \otimes_{\Lambda} (\wedge^2 V^*)$	$c^{(1)}(u)(v, v')$
$V \otimes_{\Lambda} (V^* \otimes_{\Lambda} \mathfrak{g}^* + \mathfrak{g}^* \otimes_{\Lambda} V^*)$	$-\tfrac{1}{2}(A'v - (-1)^{\lvert Z\rvert \cdot \lvert Z'\rvert} A'v)$
$V \otimes_{\Lambda} (\wedge^2 \mathfrak{g}^*)$	0
$\mathfrak{g} \otimes_{\Lambda} (\wedge^2 V^*)$	$d\omega_u(B_{H_u}(v), B_{H_u}(v'))$
$\mathfrak{g} \otimes_{\Lambda} (V^* \otimes_{\Lambda} \mathfrak{g}^* + \mathfrak{g}^* \otimes_{\Lambda} V^*)$	0
$\mathfrak{g} \otimes_{\Lambda} (\wedge^2 \mathfrak{g}^*)$	$-\tfrac{1}{2}[A, A']$

From this table, it can be seen that the really new data, if exist, appear in the $\mathfrak{g} \otimes_{\Lambda} (\wedge^2 V^*)$-component of the second order structure function. In fact, the $V \otimes_{\Lambda} (\wedge^2 V^*)$-component is the first order structure function itself and the other two non-trivial components encode, respectively, the $\mathfrak{g}$-action on V and Lie superbracket of $\mathfrak{g}$, which do not contain specific information of each G-structure. We will give an interpretation of the $\mathfrak{g} \otimes_{\Lambda} (\wedge^2 V^*)$-component for a special case in the next subsection.

5.5. Structure functions and *G*-connections. Finally, we consider the case when *the horizontal distribution* $\{H_u\}$ *defined by* (5.13) *is a connection* in $U \subset P$. We may assume that U is the inverse image by π of a superdomain in M, and that the $\mathfrak{g}$-valued 1-form ω is its connection form. Then we can rewrite (5.29) as

(5.30) $\widetilde{c^{(2)}}(\tau(u))(Z, Z')$

$$= \widetilde{c^{(1)}}(u)(v, v') - \tfrac{1}{2}(Av' - (-1)^{|Z|\cdot|Z'|}A'v) + \Omega_u(B_{H_u}(v), B_{H_u}(v')) - \tfrac{1}{2}[A, A'],$$

where Ω is the curvature form of the connection. The new data in the second order structure function is the curvature form of the connection.

We note that a necessary and sufficient condition for the existence of a (local) conneation in P such that each horizontal subspace of the connection satisfies (5.14) is that the image of the lifted first order structure function $\widetilde{c^{(1)}}$ is contained in a G-invariant submodule of $V \otimes_\Lambda (\wedge^2 V^*)$. For this, it suffises that the complement C' to $\partial(\mathfrak{g} \otimes_\Lambda V^*)$ in $V \otimes_\Lambda (\wedge^2 V^*)$ is G-invariant. Hence we obtain the following sufficient condition for the first prolongation to be a G-conneation:

Proposition 21. *Let $P \to M$ be a G-structure and suppose that the following two conditions are satisfied:*

i) $\mathfrak{g}^{(1)} = 0$

ii) *There exists a G-invariant complement C to $\partial(\mathfrak{g} \otimes_\Lambda V^*)$ in $V \otimes_\Lambda (\wedge^2 V^*)$. Then the first prolongation $P^{(1)} \to P$ with respect to C is a connection in P. Moreover, the $V \otimes_\Lambda (\wedge^2 V^*)$-component and the $\mathfrak{g} \otimes_\Lambda (\wedge^2 V^*)$-component of the second order structure function $c^{(k)}$ of P respectively correspond to the torsion form and the curvature form of the connection.*

6. G-structures of finite type

In this section, we introduce the condition of finiteness, under which our method is effective and the local equivalence problem can be soleved.

6.1. Notion of finite type. We define the notion of finiteness. A Lie subsuperalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is called *of finite type* if $\mathfrak{g}^{(k)} = (0)$ for some non-negative integer k. It is called *of infinite type* if it is not of finite type. A linear Lie supergroup G is called *of finite type* or *of infinite type* according to the type of its Lie superalgebra. Finally, a G-structure is called *of finite type* if its structure group G is of finite type.

It is easily verified that

$$\mathfrak{g}^{(k)} = \mathfrak{g} \otimes_\Lambda \underbrace{V^* \otimes_\Lambda \cdots \otimes_\Lambda V^*}_{k\text{-times}} \cap V \otimes_\Lambda S^{(k+1)}(V^*), \tag{6.1}$$

where $S^{(k+1)}(V^*)$ denotes the super-symmetric tensor product of V^* of degree $k+1$. Then it is clear that if $\mathfrak{g}^{(k)} = 0$ for some k then $\mathfrak{g}^{(h)} = 0$ for any integer h greater than k. From this fact, *the rank of a linear Lie superalgebra* $\mathfrak{g}$ of finite type can be defined as the minimal non-negative integer k satisfying

$$\mathfrak{g}^{(k)} = (0).$$

The ranks of linear Lie supergroups and G-structures of finite type are similarly defined.

REMARK. For historical reasons, we do not call a G-structure of infinite type even if its structure group G is of infinite type. This is due to the following facts: For some G-structures, the equivalence problem can be reduced to the one for a subgroup H of G by fixing a value of the structure functions. The subgroup H may be of finite type, though the original structure group G is of infinite type. We will not however take up the method of reduction in this paper. For details of the method of reduction of usual G-structures, see [9].

EXAMPLES. We examine finiteness for the Lie supergroup G appeared in the examples of section 2.

a) Since $\mathfrak{g}=\mathfrak{gl}(V)=V\otimes_{\Lambda}V^*$, it is clear that $GL(V)$ is of infinite type.

b) In contrast to a), we have $\mathfrak{g}=(0)$ and thus the unit group $\{e\}$ is of finite type and its rank is 0. As we will see later, the equivalence problem of finite type G-structures can be reduced to that of this structure.

c) In this case, the Lie superalgebra of $OSp(V)$ is given by

$$\mathfrak{osp}(V) = \{A\in\mathfrak{gl}(V)\colon g(Au, v)+(-1)^{|A|\cdot|u|}g(u, Av) = 0 \text{ for } u, v\in V\}\,.$$

Then it can be shown that $\mathfrak{osp}(V)^{(1)}=(0)$. In fact, if we define

$$T_v\colon V\ni u\to T_v(u) = T(v, u)\in V$$

for $T\in\mathfrak{osp}(V)^{(1)}$ and $v\in V$, then

$$T_v\in\mathfrak{osp}(V)$$

and

$$T_v(u) = (-1)^{|v|\cdot|u|}T_u(v) \qquad (v, u\in V)\,.$$

By using these facts and supersymmetricity of g, we have for u, v, $w\in V$,

$$\begin{aligned}
g(T_v(u), w) &= (-1)^{|v|\cdot|u|}g(T_u(v), w)\\
&= -(-1)^{|v|\cdot|u|+(|T|+|u|)\cdot|v|}g(v, T_u(w))\\
&= -(-1)^{|T|\cdot|v|+|v|\cdot(|T|+|u|+|w|)}g(T_u(w), v)\\
&= -(-1)^{|v|(|u|+|w|)+|u||w|}g(T_w(u), v)\\
&= (-1)^{|v|\cdot(|u|+|w|)+|u|\cdot|w|+(|T|+|w|)\cdot|u|}g(u, T_w(v))\\
&= (-1)^{|v|\cdot(|u|+|w|)+|T|\cdot|u|+|u|\cdot(|T|+|w|+|v|)}g(T_w(v), u)\\
&= (-1)^{|v|\cdot|w|+|u|\cdot|w|+|w|\cdot|v|}g(T_v(w), u)\\
&= -(-1)^{|u|\cdot|w|+(|T|+|v|)\cdot|w|}g(w, T_v(u))\\
&= -(-1)^{(|u|+|T|+|v|)\cdot|w|+|w|\cdot(|T|+|v|+|u|)}g(T_v(u), w)\\
&= -g(T_v(u), w)\,.
\end{aligned}$$

But since w is arbitrary and g is non-degenerate, we have

$$T_v(u) = 0 \qquad (u, v \in V).$$

Thus we have $T=0$ and $\mathfrak{osp}(V)^{(1)}=(0)$. Hence $OSp(V)$ is of finite type and its rank is 1.

Both of d) and e) are of infinite type, which will be shown in the next subsection.

6.2. A criterion of infiniteness. In general, for a Lie subsupergroup G of $GL(m|n; \Lambda)$, its body G_b is a Lie subgroup of $GL(m; \boldsymbol{R})\times GL(n; \boldsymbol{R})$ considered as a subgroup of $GL(m+n; \boldsymbol{R})$ by

$$GL(m; \boldsymbol{R})\times GL(n; \boldsymbol{R})\ni(a, b) \to \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL(m+n; \boldsymbol{R}).$$

Furthermore, suppose that the body G_b is the product Lie group

$$G_b = H\times K \qquad (H\subset GL(m; \boldsymbol{R}), K\subset GL(n; \boldsymbol{R})). \tag{6.2}$$

Then we can naturally regard the Lie algebra of H, which we deote by $\mathfrak{h}$, is contained in the even subspace of $\mathfrak{g}$. More precisely, suppose that the body V_b of V is a $\boldsymbol{Z}_2$-graded real vector space such that the even subspace $(V_b)_0$ on which $\mathfrak{h}$ acts is m-dimensional. Then

$$\mathfrak{h}\subset(V_b)_0\otimes_{\boldsymbol{R}}(V_b)_0^*\cap\mathfrak{g}\subset\mathfrak{g}_0. \tag{6.3}$$

Now we consider the operator

$$\begin{array}{cccc} \delta_H\colon & \mathfrak{h}\otimes_{\boldsymbol{R}}(V_b)_0^* & \to & (V_b)_0\otimes_{\boldsymbol{R}}(\wedge^2(V_b)_0^*) \\ & \cap & & \cap \\ & (\mathfrak{g}\otimes_\Lambda V^*)_0 & & (V\otimes_\Lambda(\wedge^2 V^*))_0 \end{array} \tag{6.4}$$

corresponding to ∂ in non-super case. It is clear that δ_H is the restriction of ∂. Thus, if the first prolongation of $\mathfrak{h}$ in non-super sence is non-trivial then that of $\mathfrak{g}$ in super sence is also non-trivial. Consequently, we have:

Proposition 22. *Let G be a Lie subsupergroup of $GL(m|n; \Lambda)$ such that its body G_b is the product Lie group:*

$$G_b = H\times K \qquad (H\subset GL(m; \boldsymbol{R}), K\subset GL(n; \boldsymbol{R})).$$

If the first factor $H\subset GL(m; \boldsymbol{R})$ of G_b is of infinite type in non-super sence, then G is infinite type in super sence.

REMARK. We note that even if K is of infinite type in the usual sense, G is not necessarily of infinite type. This is due to the fact that the restriction of ∂ to $\mathfrak{k}\otimes_{\boldsymbol{R}}(V_b)_1^*$ is the symmetrization map with values in $(V_b)_1\otimes_{\boldsymbol{R}}S^2(V_b)_1^*$

and nothing to do with the ∂ operator in the non-super case.

Now we return to the examples d) and e) in section 2. For d), if V is of dimension $2m|n$ then

$$(SpO(V))_b = Sp(m; \boldsymbol{R}) \times O(p, q), \qquad p+q = n,$$

where (p, q) is the signature of the restriction of the anti-supersymmetric bilinear form ω to the odd subspace V_1. We note that ω restricted to V_1, is a *symmetric* (*in usual sence*) bilinear form. For e), if V is of $2m|2n$-dimension then

$$(GL(V, \Lambda_c))_b = GL(m; \boldsymbol{C}) \times GL(n; \boldsymbol{C}).$$

For both of d) and e), it is known (see [8], for example) that the first factors (contained in $GL(2m; \boldsymbol{R})$) are of infinite type in non-super sence. Thus, by Proposition 21, we conclude that these Lie supergroups are of infinite type.

Finally, we remark that if the dimension of V is $m|2n$ then the body of $OSp(V)$ is

$$(OSp(V))_b = O(p, q) \times Sp(n; \boldsymbol{R}), \qquad p+q = m,$$

and hence the second factor is of infinite type. However, we already know that $OSp(V)$ is of finite type. This example shows that G can be of finite even if the second factor of its body is of infinite type in the usual sense. This is due to the fact explained in the above remark.

7. Main Theorem

7.1. Main Theorem. Now we are in the position to prove the main theorem:

Main Theorem. *Let G be a linear Lie supergroup of finite type with the connected body. Then the equivalence problem of G-structures can be reduced to the equivalence problem of complete parallelisms, that is, $\{e\}$-structures.*

Proof. By Theorem 18, the equivalence problem of the G-structures can be reduced to that of prolongated ones. But by definition, the k-th prolongations of the G-structures are complete parallelisms, where k is the rank of the Lie superalgebra $\mathfrak{g}$ of G. q.e.d.

Remark. In the main theorem, we assumed that the prolongations of the structure Lie supergroup G up to the rank of G are admissible.

7.2. Equivalence problem of complete parallelisms. The equivalence problem of complete parallelisms on supermanifolds can be solved similarly as that on usual manifolds. We will explain it only for the special case where the structure functions are constant. For a solution of this problem in general case, on usual manifolds, see [9].

Let $\pi: P \to M$ be an $\{e\}$-structure on M and $X_1, \cdots, X_{m+n}$ be the global frame field, that is, the complete parallelism over M corresponding to P. Then the (first order) structure function of P is essentially the Lie superbracket among X_μ's. In fact, it can be easily verified that for $x \in M$,

$$c^{(1)}(u)(e_\mu, e_\nu) = -\tfrac{1}{2}u^{-1}([X_\mu, X_\nu](x)) \qquad (\mu, \nu = 1, \cdots, m+n),$$

where e_μ's denote the standard basis of V and u is the unique linear frame at x:

$$u = (X_1(x), \cdots, X_{m+n}(x)) \qquad (x \in M).$$

Now suppose that the (first order) structure function of $P \to M$ is constant. Then, the set of vector fields $X_1, \cdots, X_{m+n}$ generates an $m|n$-dimensional Lie superalgebra.

Then the local equivalence problem of complete parallelisms with constant structure functions can be solved:

Theorem 23. *A necessary and sufficient condition for two complete parallelisms with constant structure functions to be locally equivalent is that their structure functions coincide.*

Proof. Let $P \to M$ and $Q \to N$ be $\{e\}$-structures with constant structure functions corresponding to complete parallelisms $X_1, \cdots, X_{m+n}$ over M and $Y_1, \cdots, Y_{m+n}$ over N, respectively.

It is clear that if P and Q are locally equivalent then their structure functions coincide.

Conversely, suppose that their structure functions coincide. We consider the product supermanifold $M \times N$. Then we can naturally regard the vector fileds X's and Y's as defined on $M \times N$. It is clear that $[X_\mu, Y_\nu]=0$ $(\mu, \nu = 1, \cdots, m+n)$. By combining this with our assumption, it follows that the distribution on $M \times N$ generated by the vector fields $X_\mu - Y_\mu$ $(\mu=1, \cdots, m+n)$ are in involution, that is, they are closed under the Lie superbracket. Then, by virtue of the super version of the Fröbenius Theorem, we can take an integral supermanifold through $(x, y) \in M \times N$ for arbitrary $x \in M$ and $y \in N$. It is easy to see that this integral supermanifold is the graph of a local isomorphism of $\{e\}$-structures at (x, y). q.e.d.

For G-structures of finite type, it follows from this theorem and Corollary 20, we obtain the following.

Proposition 24. *Let G be a linear Lie supergroup of finite type with the rank k, Then a necessary and sufficient condition for two G-structures with the constant $(k+1)$-th structure functions to be locally equivalent is that their $(k+1)$-th structure functions coincide.*

8. OSp-structures

In this section, we consider the OSp-structure as an example of G-structures of finite type. For this, we assume that the odd dimension n of the supermaniforld M (and of the standard Λ-module V) is an even number $2r$.

8.1. Riemannian supermanifolds and OSp-structures. Let p, q be non-negative integers satisfying $p+q=m$. A supersymmetric covariant tensor g of degree 2 on a supermanifold M is called *a pseudo-Riemannian supermetric of signature* (p, q) *on* M, if for each point $x \in M$, g_x is a non-degenerate qudratic form of signature (p, q) on the tangent space $T_x(M)$. When g is positive definite, that is, $q=0$, we call g *a Riemannian supermetic on* M. A supermanifold with a pseudo-Riemannian supermetric is called *a pseudo-Riemannian supermanifold*, and *Riemannian* if the supermetric is Riemannian.

Let g be a pseudo-Riemannian supermetric of singature (p, q) on M. Then, by taking orthosymplectic bases for g_x at each $x \in M$, we obtain an $OSp(p, q|2r)$-structure $OSp(M)$ on M. The $OSp(p, q|2r)$-strucutre $OSp(M) \to M$ is also called *the orthosymplectic linear frame bundle of the pseudo-Riemannian supermanifold* M.

8.2. The first prolongations of OSp-structures. To begin with, recall that the orthosymplectic group $OSp(p, q|2r)$ is of finite type of rank 1, since the operator

$$\partial \colon \mathfrak{osp}(p, q|2r) \otimes_{\Lambda} V^* \to V \otimes_{\Lambda} (\wedge^2 V^*) \tag{8.1}$$

is injective. On the other hand, both of $\mathfrak{osp}(p, q|2r)$ and $\wedge^2 V^*$ have the same dimension $m'|n'$, where

$$m' = \tfrac{1}{2}(m(m-1)+n(n+1)), \qquad n' = mn. \tag{8.2}$$

Hence the domain and the target of the operator ∂ have the same dimension. Hence the operator ∂ is an isomorphism.

The surjectivity of ∂ trivially guarantees the unique existence of an $OSp(p, q|2r)$-invariant complement to $\partial(\mathfrak{osp}(p, q|2r) \otimes_{\Lambda} V^*)$ in $V \otimes_{\Lambda} (\wedge^2 V^*)$. The prolongation with respect to this complement gives, by the injectivity of ∂, a complete parallelism on $OSp(M)$, which is a connection in $OSp(M)$ by Proposition 21.

Moreover, by our construction, the prolongation of $OSp(M)$ consists of distinguished linear frame w such that

$$c_{H_u} = 0, \tag{8.3}$$

where H_u is the horizontal subspace at $u = \pi^{(1)}(w)$ corresponding to w. From (3.27), it follows then that the torsion of the $OSp(p, q|2r)$-connection is zero.

We note that this is the unique $OSp(p, q|2r)$-connection such that the torsion form is zero. In fact, if an $OSp(p, q|2r)$-connection in $OSp(M)$ has zero torsion, the equation (8.3) must be satisfied by its horizontal subspaces at any point $u \in OSp(M)$. But the equation (8.3) determines a unique horizontal subspace at u, since ∂ is injective. Hence it coincides with the prolongation.

Thus we have the following theorem.

Theorem 25. *A first prolongation of $OSp(p, q|2r)$-structures is uniquely determined . This is the unique connection in $OSp(M)$ with zero torsion.*

From this theorem, we obtain the super-version of the classical theorem of Levi-Civita:

Corollary 26. *Let M be a pseudo-Riemannian supermanifold with the supermetric g. Then there is a unique connection in $OSp(M)$ with zero torsion*

In the terminology of covariant differentiation, we have:

Corollary 27. *Let M be a psuedo-Riemannian supermanifold with the supermetric g. Then there is a unique operator ∇: $\mathfrak{X}(M) \to \mathfrak{X}(M)$ such that, for arbitrary vector fields $X, Y, Z \in \mathfrak{X}(M)$,*

$$L_X(g(Y, Z)) = g(\nabla_X Y, Z) + (-1)^{|X|\cdot|Y|} g(Y, \nabla_X Z), \tag{8.4}$$

$$\nabla_X Y - (-1)^{|X|\cdot|Y|} \nabla_Y X - [X, Y] = 0. \tag{8.5}$$

We note that (8.4) means that the connection is metrical, that is, it preserves the supermetic. Moreover, the left hand side of (8.5) is the torsion tensor.

We call the unique connection in $OSp(M)$ obtained by prolongating $OSp(M)$ *the Levi-Civita's connection of the pseudo-Riemannian supermanifold M.* We will often identify Levi-Civita's connections with their covairant differentiation ∇ as above.

8.3. Curvature tensors. Since the first prolongation of an $OSp(p, q|2r)$-structure is an $OSp(p, q|2r)$-connection with zero torsion, its first order structure function is identically zero. Hence we cannot distinguish individual $OSp(p, q|2r)$-structures by their first order structure functions. However, since $OSp(p, q|2r)$-structures are of finite type of rank 1, all of their differential data are contained in the curvature form of their Levi-Civita's connections.

This curvature form defines a tensor, which is called *the curvature tensor on* M. Explicitly, the curvature tensor R on M is given by

$$R(X, Y)Z = \nabla_Y(\nabla_Y Z) - (-1)^{|X|\cdot|Y|} \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z \tag{8.6}$$

for $X, Y, Z \in \mathfrak{X}(M)$. By (8.4) and (8.5), the curvature tensor R is skew supersymmetric:

$$(8.7)\qquad g(R(X, Y)Z, W)+(-1)^{|Z|\cdot(|X|+|Y|)}g(Z, R(X, Y)W)=0 \quad (X, Y, Z, W\in\mathfrak{X}(M)).$$

Moreover, in a similar way as in the non-super case, we obtain the Bianchi's identity:

$$(8.8)\qquad (-1)^{|X|\cdot|Z|}R(X, Y)Z+(-1)^{|Y|\cdot|X|}R(Y, Z)X+(-1)^{|Z|\cdot|Y|}R(Z, X)Y=0$$

9. Classification of transitive *OSp*-structures

In this section, we classify transitive *OSp*-structures for the positive definite (i.e. $q=0$) case, by determining their complete systems of differential invariants. As in the previous section, the odd dimension n is assumed to be an even number $2r$.

9.1. Transitive *OSp*-structures. Since all the differential invarints for $OSp(p, q|2r)$-structures are contained in the second order structure function, the equivalence problem of $OSp(p, q|2r)$-structures can be solved, in principle, by comparing their second order structure functions, that is, their curvatures. Although it is difficult to judge whether a system of function on one space can be induced from that on another space or not, we can solve it for the transitive $OSp(p, q|2r)$-structures. This is owing to the following:

Since the first prolongation of each $OSp(p, q|2r)$-structure is an $OSp(p, q|2r)$-connection, it is diffeomorphic, by $\pi^{(1)}$, to the $OSp(p, q|2r)$-structure itself. Therefore, for transitive $OSp(p, q|2r)$-structures not only their first order structure functions but also the second order ones are constant by Corollary 7.

In the following, for the sake of simplicity, we consider the transitive $OSp(m|n)$-structures. Generalizations to $OSp(p, q|2r)$-structures are strainghtforward.

Examples. We give typical examples of transitive $OSp(m|n)$-structures.

a) Let $M=\boldsymbol{E}^{m|n}$ be the $m|n$-dimensional Euclidean superspace and $V=\Lambda^{m|n}$ be the $m|n$-dimensional standard free Λ-module with the nondegenerate quadratic form:

$$(9.1)\qquad \langle v, w\rangle = v^{st}Qw=\sum_{\mu,\nu=1}^{m+n}(-1)^{|v|\cdot|\mu|}v^{\mu}\cdot Q^{\mu}{}_{\nu}\cdot w^{\nu},$$

where

$$(9.2)\qquad Q=(Q^{\mu}{}_{\nu})=\begin{bmatrix} I_m & O & O \\ O & O & -I_r \\ O & I_r & O \end{bmatrix}.$$

Since M is the even part of the strandard free Λ-module $V=\Lambda^{m|n}$, the tangent space $T_z(M)$ at each point $z\in M$ can be naturally identified with V. By this identification, a Riemannian supermetric g on M is induced from $\langle\ ,\ \rangle$. With respect to the standard coodinate system $z^1, \cdots, z^{m+n}$ of M, the supermetric g can be expressed as

$$g = \sum_{\mu,\nu=1}^{m+n} (-1)^{|\mu|+|\nu|} Q^{\mu}{}_{\nu} dz^{\mu} \otimes dz^{\nu} . \tag{9.3}$$

It can be verified that the vector fields

$$\partial_{\mu} = \partial/\partial z^{\mu} \qquad (1 \leqq \mu \leqq m+n)$$

form an orthosympletic basis at each point of M, because of our convention of contractions:

$$(\partial_{\mu}, dz^{\nu}) = (-1)^{|\mu|}(dz^{\nu}, \partial_{\mu}) = \delta^{\mu}{}_{\nu} .$$

Therefore, the supermetric g corresponds to the standard flat $OSp(m|n)$-structure:

$$P = \boldsymbol{E}^{m|n} \times OSp(m|n) \rightarrow \boldsymbol{E}^{m|n} .$$

There are two actions on M: the one is the translations of M and the other is the $OSp(m|n)$-action on V restricted to M. It is clear that the both actions preserve g. Thus $P \rightarrow M$ is a transitive $OSp(m|n)$-structure.

We note that on M, a transitive $OSp(p, q|2r)$-structure is constructed similarly. We call the supermetric corresponding to this structure *the standard supermetric of signature* (p, q) *of the Euclidean superspace.*

b) Let k be an even element of Λ. Moreover, we assume that k is invertible. Hence $\mathcal{E}(k)$ is a non-zero real number (cf. Appendix). We denote the sign of $\mathcal{E}(k)$ by s.

Let h be the standard supermetric of the $m+1|n$-dimensional Euclidean superspace $\boldsymbol{E}^{m+1|n}$. We assume that the signature of h is $(m+1, 0)$ if $s=1$, and is $(m, 1)$ if $s=-1$. Similarly, on the $m+1|n$-dimensional free Λ-module $V = \Lambda^{m+1|n}$, we consider the non-degenerate quadratic form $\langle\ ,\ \rangle$ of the same signatute as h.

We define an $m|n$-dimensional subsupermanifold M of $\boldsymbol{E}^{m+1|n}$ by

$$M = \{z \in \boldsymbol{E}^{m+1|n} : \langle z, z \rangle = k^{-1}\} . \tag{9.4}$$

It can be verified that the tangent space $T_z M$ at each $z \in M$ is an $m|n$-dimensional regular submodule of $T_z \boldsymbol{E}^{m+1|n} = V$. Moreover, the quadratic form $\langle\ ,\ \rangle$ restricted to $T_z M$ is non-degenerate and of signature $(m, 0)$. Hence a Remannian supermetric g on M is defined.

We note that since $\mathcal{E}(s \cdot k)$ is a positive real number, there is an even element k' such that $\mathcal{E}(k')$ is positive and $(k')^2 = s \cdot k$. Then M contains the element

$$O = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (k')^{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} < m+1 . \tag{9.5}$$

We call O the origin of M. Since M is defined as the level hypersurface, it is invariant under the action of $OSp(V)$.

We note that

$$OSp(V) = \begin{cases} OSp(m+1|n) & (s=1) \\ OSp(m, 1|n) & (s=-1) . \end{cases}$$

The action of $OSp(V)$ obviously preserves the supermetric g of M. Moreover, it can be shown that $OSp(V)$ acts on M transitively. On the other hand, the isotropic subgroup at O is canonically isomorphic to $OSp(m|n)$. This group acts on the set of all orthosymplectic linear frames at O transitively. Hence we obtain a transitive $OSp(m|n)$-structure on M.

Finally, we give a local expression of g at the origin O. Let $z^1, \cdots, z^m, t, z^{m+1}, \cdots, z^{m+n}$ be the standard coordinate system of $\boldsymbol{E}^{m+1|n}$. We can take $z^1, \cdots, z^{m+n}$ as a local coordinate system of M around the origin O. Moreover, around O, M can be expressed as the graph of the function

$$t = \rho(z) = [s\cdot(k^{-1} - \sum_{\mu,\nu=1}^{m+n} z^\mu \cdot Q^\mu{}_\nu \cdot z^\nu)]^{1/2} . \tag{9.6}$$

Hence, around O, by using $Q^\nu{}_\mu = (-1)^{|\nu|\cdot|\mu|} Q^\mu{}_\nu$, we obtain

$$dt = d\rho = -(s/\rho)\cdot(\sum_{\mu,\nu=1}^{m+n} z^\mu \cdot Q^\mu{}_\nu dz^\nu) . \tag{9.7}$$

To give simpler description, we introduce the notation $z^\dagger$ which is defined by

$$z^{\mu\dagger} = \sum_{\nu=1}^{m+n} Q^\mu{}_\nu \cdot z^\nu \qquad (1 \leqq \mu \leqq m+n) . \tag{9.8}$$

Then the above equality can be rewritten as

$$dt = d\rho = -(s/\rho)\cdot(\sum_{\mu=1}^{m+n} (-1)^{|\mu|} z^{\mu\dagger} dz^\mu) \tag{9.9}$$

Since the supermetric h of $\boldsymbol{E}^{m+1|n}$ is

$$h = \sum_{\mu,\nu=1}^{m+n} (-1)^{|\mu|+|\nu|+1} Q^\mu{}_\nu dz^\mu \otimes dz^\nu + s\cdot dt \otimes dt .$$

the induced supermetric g can be expressed as

$$(9.10)\qquad g=\sum_{\mu,\nu=1}^{m+n}(-1)^{|\mu|+|\nu|}Q^{\mu}{}_{\nu}dz^{\mu}\otimes dz^{\nu}+s\cdot d\rho\otimes d\rho$$
$$=\sum_{\mu,\nu=1}^{m+n}(-1)^{|\mu|+|\nu|}(Q^{\mu}{}_{\nu}+s\cdot\rho^{-2}\cdot z^{\nu\dagger}\cdot z^{\mu\dagger})dz^{\mu}\otimes dz^{\nu}\,.$$

9.2. Sectional curvatures. In order to interpret the constancy of the second order structure functions in terms of curvature, we generalize the notion of sectional curvature to supermanifolds.

Let W be a free Λ-module. We define *the total dimension of a free $s|t$-dimensional submodule $U\subset W$* by $s+t$. A free submodule of W is called *a superplane in W* if its total dimension is 2.

Let M be a Riemannian supermanifold and g be its supermetric. We will define sectional curvature $\kappa_x(\Pi)$ at $x\in M$ for each regular superplane Π in T_xM. We note that every regular superplane in T_xM is of pure dimensional, i.e. $2|0$ or $0|2$ because its odd dimension must be even. Hence we may call the regular superplane Π even or odd according to its dimension. We denote by $|\Pi|$ the parity of Π. The regular superplane Π adimits a basis u, v both of which have the parity $|\Pi|$. Then the sectional curvature for Π is defined by

$$(9.11)\qquad \kappa_x(\Pi)=g(R(u,v)v,u)/\{g(u,u)\cdot g(v,v)-g(u,v)^2\}\,.$$

This is well defined by the following lemma:

Lemma 28. *Let M be a Riemannian supermanifold and g be its supermetric. Let u, v be homogeneous elements in $T_x(M)$ ($x\in M$). For even supernumbers a, b, c and $d\in\Lambda_0$, define*

$$u'=u\cdot a+v\cdot b\,,\quad v'=u\cdot c+v\cdot d\,.$$

Then we have

$$g(R(u',v')v',u')=\Delta^2\cdot g(R(u,v)v,u)$$

and

$$g(u',u')\cdot g(v',v')-g(u',v')^2=\Delta^2\cdot\{g(u,u)\cdot g(v,v)-g(u,v)^2\}\,,$$

where

$$\Delta=a\cdot d-b\cdot c\,.$$

The sectional curvature $\kappa_x(\Pi)$ is always an even element of Λ.

9.3. Superspaces of constant curvature. Let M be a Riemannian supermanifold and k be an even element of Λ. We call M *a superspace of constant curvature k* if the sectional curvatures of all the regular superplanes at an arbitrary point of M are equal to k.

Then we obtain the following theorem:

Proposition 29. *A Riemannian supermanifold M is a superspace of constant curvature if and only if its second order structure function is constant.*

Sketch of proof. To begin with, we note that it sufficies to show that for every $x \in M$, the sectional curvatures $\kappa_x(\Pi)$ does not depend on Π if and only if the second order structure function is constant along the fibre over x.

For this reason, we may consider the curvature tensor R_x at a fixed point $x \in M$. By taking an orthosymplectic basis $Z_1, \cdots, Z_{m+n}$ of $T_x M$, we put

$$(9.12) \qquad R_{ABDC} = g(R_x(Z_A, Z_B)Z_D, Z_C) \qquad (1 \leqq A, B, B, D \leqq m+n).$$

Although the R_{ABDC}'s contain the full data of the curvatre R_x, they are not independent because of the supersymmetricity of g, the skew supersymmetricity of R_x (8.7) and the Bianchi's identity (8.8). We can show that

$$(9.13) \qquad R_{ABDC} \qquad (A<B,\ C<D,\ A \leqq C,\ B \leqq D).$$

constitute a set of generators of R_{ABDC}'s.

Suppose now that the second order structure function is constant along the fibre over x. This is equivalent to the condition that R_{ABDC}'s are independent of the choice of orthosymplectic basis Z_A's. In particular, for each regular superplane Π at x, one can choose an orthosymplectic basis Z_A's such that

$$(9.14) \qquad \kappa_x(\Pi) = \begin{cases} R_{1221} & (\text{if } |\Pi| = 0) \\ -R_{m+1\ m+1+r\ m+1+r\ m+1} & (\text{if } |\Pi| = 1). \end{cases}$$

Hence, we see that the regular superplanes of the same parity have the same sectional curvature. To show that the constant sectional curvatures for the even and odd superplanes do agree, we use the following lemma:

Lemma 30. *Suppose that the sectional curvatures of even and odd regular superplanes at x are respectively equal to constants κ and λ. Then for an arbitrary orthosymplectic basis $Z_1, \cdots, Z_{m+n}$ at x, the generators of R_{ABDC}'s satisfy that*

$$R_{ijji} = R_{i\alpha\alpha' i} = R_{\alpha\alpha'\alpha'\alpha} = R_{\alpha\beta\beta'\alpha'} \qquad (1 \leqq i, j \leqq m,\ m+1 \leqq \alpha, \beta \leqq m+r),$$
$$R_{ABCD} = 0 \qquad (\textit{otherwise}),$$

where α' denotes $\alpha+r$. In particular, the constants κ and λ are equal.

This lemma is verified by applying the generators of $OSp(m|n)$ to the equalities $R_{ijji} = \kappa$, $R_{\alpha\alpha'\alpha'\alpha} = \lambda$.

Thus all the sectional curvatures are equal to a constnat, whence M is of constant curvature.

Conversely, if all the sectional curvatures are equal to a constant, then by the above lemma, the R_{ABDC}'s are independent of the choice of orthosymplectic basis. This implies that the second order structure function is constant along the fibre. q.e.d.

Theorem 31. *An $OSp(m|n)$-structure $P \to M$ is locally transitive if and only if M is a superspace of constant curvature. Moreover, two locally transitive $OSp(m|n)$-structures are locally equivalent if and only if their constant curvatures coincide.*

9.4. Superspace forms. Now we determine the values of hte constant curvatures of the transitive $OSp(m|n)$-structures given in the examples. Because of the homogenuity, it suffices to compute it only at the origin O. Here for example a), we mean by the origin O, the point of $\boldsymbol{E}^{m|n}$ such that all of its coordinates are zero.

At first, we note that for both a) and b), the supermetric g is expressed in the following form around O:

$$g = \sum_{\mu,\nu=1}^{m+n} (-1)^{|\mu|+|\nu|}(Q^{\mu}{}_{\nu} + \phi(z)\cdot{}^{\nu\dagger}\cdot z^{\mu\dagger})dz^{\mu}\otimes dz^{\nu}\,, \tag{9.15}$$

where

$$\phi(z) = \begin{cases} 0 & (\text{for } a) \\ s\cdot\rho^{-2} & (\text{for } b)\,. \end{cases} \tag{9.16}$$

In general, for the supermetric g expressed in local coordinate system as

$$g = \sum_{\mu,\nu=1}^{m+n} g_{\mu\nu}(z)dz^{\mu}\otimes dz^{\nu}\,, \tag{9.17}$$

the functions $\Gamma^{\lambda}{}_{\mu\nu}$ defined by

$$\nabla_{\mu}(\partial_{\nu}) = \sum_{\lambda=1}^{m+n} \Gamma^{\lambda}{}_{\mu\nu}\,\partial_{\lambda} \qquad (1\leqq \mu,\ \nu\leqq m+n)\,, \tag{9.18}$$

can be explicitly written as follows:

$$\begin{aligned} \Gamma^{\lambda}{}_{\mu\nu} = {} & (-1)^{|\lambda|\cdot(|\lambda|+|\mu|+|\nu|)}1/2 \sum_{\chi=1}^{m+n} \partial_{\nu} g^{\mu y} \\ & \times\{(-1)^{|\mu|\cdot(|\mu|+|\nu|)}\partial_{\mu}g_{\nu\chi} + (-1)^{|\nu|\cdot(|\nu|+|\chi|)}\partial_{\nu}g_{\chi\mu} - (-1)g^{|\chi|\cdot(|\chi|+|\mu|)}\partial_{\chi}g_{\mu\nu}\}\,, \end{aligned} \tag{9.19}$$

where $(g^{\mu\nu})$ denotes the inverse matrix of $(g_{\mu\nu})$.

When g is of the form (9.35), it can be shown that

$$\begin{aligned} \partial_{\chi}g_{\mu\nu} = {} & (-1)^{|\mu|+|\nu|}\{[\partial_{\chi}\phi - ((-1)^{|\nu|+|\chi|} + (-1)^{|\chi|+|\mu|})\phi^2\cdot z^{\chi\dagger}]\cdot z^{\nu\dagger}\cdot z^{\mu\dagger} \\ & + \phi\cdot[g_{\nu\chi}\cdot z^{\mu\dagger} + (-1)^{|\mu|\cdot(|\nu|+|\chi|)}g_{\chi\mu}\cdot z^{\nu\dagger}]\} \end{aligned}$$

and

$$\Gamma^{\lambda}{}_{\mu\nu} = (-1)^{|\lambda|\cdot(|\lambda|+|\mu|+|\nu|)+|\chi|}\tfrac{1}{2}\sum_{\chi=1}^{m+n} g^{\lambda\chi}$$
$$\times[z^{\chi\dagger}\cdot(z^{\nu\dagger}\cdot\partial_\mu\phi+\partial_\nu\phi\cdot z^{\mu\dagger}+2\phi\cdot Q^{\mu}{}_{\nu})-\partial_\chi\phi\cdot z^{\nu\dagger}\cdot z^{\mu\dagger}]\,.$$

We obtain immediately from this expression that if $m\geqq 2$

$$(R(\partial_1, \partial_2)\partial_2, \partial_1)(O) = \phi(O)$$

and if $n\geqq 2$

$$(R(\partial_{m+1}, \partial_{m+1+r})\partial_{m+1+r}, \partial_{m+1})(O) = -\phi(O)\,.$$

Consequently, for the both of a) and b), the value of the constant cuvature is equal to

$$\phi(O) = \begin{cases} 0 & (\text{for } a) \\ k & (\text{for } b)\,. \end{cases} \tag{9.20}$$

We call the transitive $OSp(m|n)$-structure with the constant curvature k constructed in the examples *the superspace form with the constant curvature k.* We note that $k=0$ for the example a).

As a corollary to Theorem 31, we obtain the following theorem:

Theorem 32. *Let M be a Riemannian supermanifold with the constant curvature k. If k is invertible or zero, then M is locally isometric to the superspace form of the same dimension with the constant curvature k.*

Remark. Our construction of superspace forms is a natural generalization of that for usual space forms. For the cases of non-zero constant curvatures, the fact that they are invertible is essensially used for the construction. Hence this construction can not be applied to superspace forms with non-zero but not invertible constant curvatures. Finally, we note that if they can be constructed, the induced structures on their bodies (cf. Introduction) would be flat Riemannian structures i.e. Euclidean spaces.

10. Appendix

We give a brief review of quadratic forms on a free Λ-module V.

10.1. Definitions. Let g be a bilinear form on V with the homogeneous parity $|g|$. A bilinear form g is called supersymmetric if it satisfies

$$g(v, w) = (-1)^{|v||w|}g(w, v) \qquad (v, w\in V)\,. \tag{10.1}$$

Notion of anti-supersymmetric bilinear form is similarly defined. Let g be a bilinear form. Then for each $v\in V$ a Λ-linear form $v^*\in V^*$ is defined by

$$v^*\colon V\ni w\to g(v, w)\in\Lambda\,. \tag{10.2}$$

It is clear that the parity of v^* is equal to $|v|+|g|$. Hence the duality map

$$\delta_g \colon V \ni v \to v^* \in V^* \tag{10.3}$$

is a right Λ-linear map with the parity $|g|$. If the duality map δ_g is an isomorphism, the bilinear form g is called *non-degenerate*. More generally, when the image of the duality map δ_g of every free submodule of V is a free submodule of V^*, we call g *regular*.

In what follows, we will consider only supersymmetric bilinear forms of even parity, which will be called *quadratic forms*. Let g be a quadratic form on V. For a submodule W of V, *the orthogonal complement $W^\perp$ to W* is defined by

$$W^\perp = \{v \in V \colon g(w, v) = 0 \quad \text{for } w \in W\} . \tag{10.4}$$

It is clear that the orthogonal complement $W^\perp$ is a submodule of V. For its freeness, we obtain, immediately from the definition, the following lemma.

Lemma 33. *Let g be a regular quadratic form on V. Then for an arbitrary free submodule $W \subset V$, the orthogonal complement $W^\perp$ is also a free submodule of V.*

We call the orthogonal complement $V^\perp$ of the total space V *the radical of V*. The bilinear form g is non-degenerate if and only if the radical $V^\perp$ is trivial: $V^\perp=(0)$.

The orthogonal complements have usual properties. For examples, the inclusion relationship of submodules are reversed, namely, $V^\perp \subset W^\perp \subset U^\perp$ whenever $U \subset W \subset V$, the orthogonal complement $(W^\perp)^\perp$ of $W^\perp$ includes W and coincides with W whenever g is non-degenerate, and so forth.

To define the regularity of the submodule W, consider the restriction of the quadratic form g to W:

$$g|_W \colon W \times W \to \Lambda . \tag{10.5}$$

Then the duality map for $g|_W$, which we denote by δ_W, makes the following diagram commutative:

$$\begin{array}{ccc} W & \xrightarrow{\delta_g} & V^* \\ & \searrow^{\delta_W} & \downarrow \\ & & W^* , \end{array}$$

where the vertical arrow is the restriction map. We call a free submodule W *regular* if δ_W is an isomorphism. A regular submodule is nothing but a free submodule to which the restriction of a quadratic form g is non-degenerate. A free submodule W is regualar if and only if $W \cap W^\perp=(0)$.

Now let us find the normal form of the quadratic form g. To do this, for the sake of simplicity, we consider only regular quadratic forms.

EXAMPLES. First, we give examples of regular quadratic forms on the $m|n$-dimensional standard free Λ-module $\Lambda^{m|n}$, which illustrates normal forms of quadratic forms.

We regard each element $v \in \Lambda^{m|n}$ as a column vector:

$$v = \begin{pmatrix} v^1 \\ \vdots \\ v^{m+n} \end{pmatrix}$$

and its supertranspose by a row vector:

$$v^{st} = (v^1, \cdots, v^m, (-1)^{|v|} v^{m+1}, \cdots, (-1)^{|v|} v^{m+n}).$$

For an arbitrary element λ of the supernumber algebra Λ, we define $\bar{\lambda} \in \Lambda$ by

$$\bar{\lambda} = \lambda_0 - \lambda_1 \qquad (\lambda = \lambda_0 + \lambda_1,\ \lambda_0 \in \Lambda_0,\ \lambda_1 \in \Lambda_1).$$

Then the supertranspose of v can be rewritten as

$$v^{st} = (v^1, \cdots, v^m, -\overline{v^{m+1}}, \cdots, -\overline{v^{m+n}}).$$

The standard basis $e_1, \cdots, e_{m+n}$ of $\Lambda^{m|n}$ are

$$e_\mu = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ <\mu \\ \\ \\ \\ \end{matrix} \qquad (\mu = 1, \cdots, m+n)$$

and their supertransposes are

$$e_\mu{}^{st} = (0, \cdots, 0, (-1)^{|\mu|}, 0, \cdots, 0),$$

where $|\mu|$ is defined by

$$|\mu| = \begin{cases} 0 & (1 \leqq \mu \leqq m) \\ 1 & (m+1 \leqq \mu \leqq m+n). \end{cases} \tag{10.6}$$

Let p, q and r are non-negative integers satisfying $p+q \leqq m$ and $2r \leqq n$. We define a square matrix Q by

$$Q = (Q^\mu{}_\nu) = \underbrace{\begin{pmatrix} I_p & O & O \\ O & -I_q & O \\ O & O & O \\ O & O & O \\ O & O & O \\ O & O & O \end{pmatrix}}_{m}\!\!\underbrace{\begin{matrix} O & O & O \\ O & O & O \\ O & O & O \\ O & -I_r & O \\ I_r & O & O \\ O & O & O \end{matrix}}_{n} \begin{matrix} \}\, p \\ \}\, q \\ \}\, m-(p+q) \\ \}\, r \\ \}\, r \\ \}\, n-2r \end{matrix}, \tag{10.7}$$

where I_p denotes the unit matrix of size p. Then we define a bilinear form g on $\Lambda^{m|n}$ by

$$g(v, w) = v^{st}Qw \qquad (v, w \in \Lambda^{m|n}). \tag{10.8}$$

It is clear that g is even and supersymmetric. Moreover, since the matirx Q represents not only the bilinear form g but also the duality map δ_g for g, it can be shown that g is regular. Hence we obtain a regular quadratic form on $\Lambda^{m|n}$. For the quadratic form g to be non-degenerate, it is necessary and sufficient that $p+q=m$ and $2r=n$. Consequently, in this case, the odd dimension n must be an even number.

By using the components of v and w, (10.8) can be expressed as

$$\begin{aligned} g(v, w) &= \sum_{\mu,\nu=1}^{m+n} (-1)^{|\nu|\cdot|\mu|} v^{\mu}\cdot Q^{\mu}{}_{\nu}\cdot w^{\nu} \\ &= v^1\cdot w^1 + \cdots + v^p\cdot w^p - v^{p+1}\cdot w^{p+1} - \cdots v^{p+q}\cdot w^{p+q} \\ &\quad + (\overline{v^{m+1}}\cdot w^{m+r+1} + \cdots + \overline{v^{m+r}}\cdot w^{m+2r} - \overline{v^{m+r+1}}\cdot w^{m+1} - \cdots - \overline{v^{m+2r}}\cdot w^{m+r}) \end{aligned} \tag{10.9}$$

To express this more briefly, we denote the odd elements $e_{m+1}, \cdots, e_{m+h}$ of the standard basis by $f_1, \cdots, f_n$. Furthermore, we use Latin indices $i, j, \cdots$ and Greak ones $\alpha, \beta, \cdots$, respectively for even and odd elements of the standard basis:

$$\begin{aligned} &e_i \quad (1 \leqq i \leqq m) \text{ (even basis)} \\ &f_\alpha \quad (1 \leqq \alpha \leqq n) \text{ (odd basis)}. \end{aligned}$$

According to this, we express v and w as

$$v = \begin{pmatrix} x^i \\ \xi^\alpha \end{pmatrix}, \quad w = \begin{pmatrix} y^i \\ \eta^\alpha \end{pmatrix}.$$

Then (10.9) can be rewritten as

$$g(v, w) = \sum_{i=1}^{p} x^i\cdot y^i - \sum_{j=1}^{q} x^{p+j}\cdot y^{p+j} + \sum_{a=1}^{r} (\overline{\xi^\alpha}\cdot \eta^{r+\alpha} - \overline{\xi^{r+\alpha}}\cdot \eta^\alpha). \tag{10.10}$$

From this expression, we might roughly say that g is a symmetric inner product in the usual sence with the signature (p, q) in variables v's and w's and a symplectic form in variables ξ's rnd η's.

We note that if v is even then, since ξ'ss are odd elements of Λ, we have

$$g(v, w) = \sum_{i=1}^{p} x^i\cdot y^i - \sum_{j=1}^{q} x^{p+j}\cdot y^{p+j} - \sum_{a=1}^{r} (\xi^\alpha\cdot \eta^{r+\alpha} - \xi^{r+\alpha}\cdot \eta^\alpha).$$

It follows from (10.10) that the standard basis $e_1, \cdots, e_m, f_1, \cdots, f_n$ satisfies

$$g(e_i, e_j) = \begin{cases} 1 & (i=j \text{ and } 1 \leqq i \leqq p) \\ -1 & (i=j \text{ and } p+1 \leqq i \leqq p+q) \\ 0 & (\text{otherwise}) \end{cases}, \tag{10.11}$$

$$(10.12)\qquad g(f_\alpha, f_\beta) = \begin{cases} 1 & (1 \leqq \alpha \leqq r \text{ and } \beta = \alpha + r) \\ -1 & (r+1 \leqq \alpha \leqq 2r \text{ and } \beta = \alpha - r) \\ 0 & (\text{otherwise}) \end{cases},$$

$$(10.13)\qquad g(e_i, f_\alpha) = 0 \quad (1 \leqq i \leqq m,\ 1 \leqq \alpha \leqq n).$$

10.2. Normal forms. Now we go back to the problem of finding the normal forms of quadratic forms.

In general, for a regular quadratic form g on a free Λ-module V, a pure basis $e_1, \cdots, e_m$ $f_1, \cdots, f_n$ of V satisfying (10.11), (10.12) and (10.13) is called *an orthosymplectic basis.*

The normal form of a regular quadratic form is given by finding an orthosymplectic basis. This will be done, by reducing the quadratic form restricted to proper regular submodules.

The first step of reduction is to eliminate the radical of the quadratic form. That is, by means of the following proposition, our problem can be reduced to the one in the case when quadratic forms are non-degenerate:

Proposition 34. *Let g be a regular quadratic form on a free Λ-module V. Then*

$$(0) \to V^{\perp} \to V \xrightarrow{\delta_g} V^*$$

is an exact sequnce of free Λ-modules. In particular, an arbitrary complement W to $V^{\perp}$ in V is a regular submodule and the restriction $g|_W$ of g to W is non-degenerate.

Hence, we may assume that the quadratic forms are non-degenerate. The second step is to characterize the simplest regular submodules. Since for the existence of a non-degenerate quadratic form, the odd dimension must be an even number, the dimension of a non-trivial minimal regular submodule differs according to the parity of generators:

Lemma 35. *Let g be a non-degenerate quadratic form on V.*
i) *For an even element $v \in V_0$, the submodule $W = v\Lambda$ generated by v is regular if and only if $g(v, v)$ is an invertible element in Λ. In this case, W is a free submodule of dimension $1|0$.*
ii) *For odd elements $v, w \in V_1$, the submodule $W = v\Lambda + w\Lambda$ generated by v and w is regular if and only if $g(v, w)$ is an invertible element in Λ. In this case, W is a free submodule of dimension $0|2$.*

We note that an element $\lambda \in \Lambda$ is invetible if and only if $\varepsilon(\lambda) \in R$ is not zero, where

$$\varepsilon\colon \Lambda \ni \lambda \to \lambda_\phi \in R$$

is the augmentation.

By our assumption that the quadratic form g is non-degenerate, the duality map δ_g is an isomorphism of V onto V^*. It follows immediately from this that if the even dimension m of V is not zero then there exists an even element $v \in V_0$ such that $g(v, v)$ is invetible. Similarly, if the odd dimension n of V is not zero, then there exist odd elements $v, w \in V_1$ such that $g(v, w)$ is invertible. Moreover, the values of g can be normalized by using the following lemma:

Lemma 36. *Let $\lambda \in \Lambda_0$ be an invetible element. If $\varepsilon(\lambda)>0$, then there exists $\mu \in \Lambda_0$ such that $\lambda=\mu^2$. Similarly, if $\varepsilon(\lambda)<0$, then there exist $\mu \in \Lambda_0$ such that $\lambda=-\mu^2$.*

Summarizing the above argument, we have:

Proposition 37. *Let g be a non-degenerate quadratic form on a free Λ-module V of dimension $m|n$. If the even dimension m is not zero, then there exists an even element $v \in V_0$ such that*

$$g(v, v) = 1 \text{ or } -1 .$$

If the odd dimension n is not zero, then there exist odd elements $v, w \in V_1$ such that

$$g(v, w) = 1 .$$

Hence by induction on the dimension of V, we can conclude that:

Theorem 38. *For an arbitrary regular quadratic form g on V, there exists an orthosymplectic basis of V. In particular, by using the components of each element in V with respect to an orthosymplectic basis, the quadratic form g can be expressed as*

$$g(v, w) = \sum_{i=1}^{p} x^i \cdot y^i - \sum_{j=1}^{q} x^{p+j} \cdot y^{p+j} + \sum_{\alpha=1}^{r} (\xi^\alpha \cdot \eta^{r+\alpha} - \xi^{r+\alpha} \cdot \eta^\alpha) \qquad (v, w \in V) .$$

By virtue of this theorem, each qudratic form can be identified with the one on $\Lambda^{m|n}$ which is given by (10.8). Then it can be shown that the integers p, q and r which appear in the matrix Q of (10.7) are independent of the choice of an orthosymplectic basis. The pair (p, q) is called *the signature of g* and the integer *r the rank of g.*

10.3 Orthosymplectic group and its Lie superalgebra. By using the normal form, the orthosymplectic group of each quadratic form can be identified with a linear Lie supergroup. For the sake of simplicity, we consider non-degenerate qudratic forms.

Let g be the qudratic form on $\Lambda^{m|n}$ defined by (10.8). By the assumption of non-degeneracy, the signature (p, q) and the rank r must satisfy $p+q=m$ and $2r=n$, respectively. In this case, the matrix Q is

$$Q = \begin{bmatrix} I_p & O & O & O \\ O & -I_q & O & O \\ O & O & O & -I_r \\ O & O & I_r & O \end{bmatrix}.$$

Then the orthosymplectic group of g can be identified with the linear Lie supergroup

$$OSp(p, q|2r) = \{a \in GL(m|n; \Lambda) : a^{st}Qa = Q\}$$

and its Lie superalgebra can be identified with the linear Lie superalgebra

$$\mathfrak{osp}(p, q|2r) = \{X \in \mathfrak{gl}(m|n; \Lambda) : X^{st}Q + QX = 0\} ,$$

where the supertranspose of a matrix is defined by

$$X^{st} = \begin{pmatrix} A^t & (-1)^{|X|}\,{}^tC \\ (-1)^{|X|+1}\,{}^tB & {}^tD \end{pmatrix}$$

for

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

From these matrix expression, it follows that the orthosymplectic group and its Lie superalgebra are $m'|n'$-dimensional, where

$$m' = \tfrac{1}{2}(m(m-1)+n(n+1)) , \qquad n' = mn .$$

References

[1] M. Fujio: *G-structures on supermanifolds*, Master thesis, Osaka Univ. (1986), (in Japanese).

[2] M. Fujio: *Differential invariants of superwebs*, Proc. Japan Acad. **63** Ser A (1987), 330–333.

[3] M. Fujio: *Foundations of super differential geometry (A categorical approach)*, (1989), (preprint).

[4] S. Kobayashi: *Transformation groups in differential geometry*, Springer, Berlin, 1972.

[5] B. Kostant: *Graded manifolds, graded Lie theory and prequantization.* Lecture Notes in Math. 570, Springer, 1977, 177–306.

[6] A.A. Rosly and A.S. Schwarz: *Geometry of $N=1$ supergravity*, Commun. Math. Phys. **95** (1984), 161–184.

[7] A.A. Rosly and A.S. Schwarz: *Geometry of $N=1$ supergravity (II)*, Commun. Math. Phys. **96** (1984), 285–309.

[8] I.M. Singer and S. Sternberg: *The infinite groups of Lie and Cartan I*, Ann. Inst. Fourier **15** (1965), 1–114.

[9] S. Sternberg: *Lectures on differential geometry*, Prentice-Hall Inc., New Jersey, 1964.

[10] A.S. Schwarz: *Supergravity, complex geometry and G-structures*, Commun. Math. Phys. **87** (1982), 37–63.

[11] B.S. De Witt: *Supermanifolds*, Cambridge Univ. Press, Cambridge, 1984.

Department of Mathematics
Faculty of Science
Osaka University
Toyonaka, Osaka 560
Japan

Present address
Department 4
Information System Research
and Development Center
Corporate Research and Development Group
SHARP CORPORATION
2613–1, Ichinomoto, Tenri, Nara 632
Japan