Hirashima, Y.
Osaka J. Math.
28 (1991), 141–152

MULTIPLICATIVE STRUCTURES OF MONOIDS OF SELF HOMOTOPY EQUIVALENCES OF $K(G, 1)$-SPACES

Dedicated to Professor S. Araki on his 60th birthday

YASUMASA HIRASHIMA

(Received January 16, 1990)

Introduction

In the present work we study multiplicative structures of monoids of self homotopy equivalences of $K(G, 1)$-spaces. Historicaly, in order to study $K(G, 1)$-fibrations, the multiplicative structures were not used and many authors still obtained certain results without them. However, we would like to show that the multiplicative structures are determined easily. This enables us to make use of several techniques on bundle theory in our subjects and some results by predecessors are obtained in a rather conceptual and easy way.

Our main result is Theorem 1 which states the explicit formula of multiplication of the (simplicial) monoid $E\bar{W}G$. This theorem implies the famous result by Gottlieb [4] about the homotopy groups of universal $K(G, 1)$-fibration, as a corollary (Corollary 2). Further applications are Hill's result on the homotpoy type of the classifying space $\bar{W}E\bar{W}G$ (Theorem 3) and the existence of a principal refinement of every nilpotent fibration (Thearem 4 and the continued Remark).

1. Main Theorem and applications

We first fix our notations and definitions. Almost all of them are found in [11], [3] or [2], some of the rest are the following. Let EX denote the simplicial monoid of self weak equivalences of a simplicial set X which is a sub simplicial monoid of $\hom(X, X)$. The set of invertible simplices of EX forms its maximal subgroup AX. If X is minimal, then $AX=EX$ is proved by making use of minimality of every fibration $\Delta[n]\times X\to\Delta[n]$.

Let G be a simplicial group and X a G-space. We define a simplicial set $W(G; X)$ whose set of n-simplices $W_n(G; X)$ is $G_0\times G_1\times\cdots\times G_{n-1}\times X_n$ by adopting the formulae of the faces and degeneracies as

$$\partial_i(g_0, g_1, \cdots, g_{n-1}; x_n) =$$

$$(g_0, \cdots, g_{i-2}, g_{i-1}\,\partial_i g_i, \partial_i g_{i+1}, \cdots, \partial_i g_{n-1}; \partial_i x_n) \quad \text{if} \quad i<n\,,$$
$$\partial_n(g_0, g_1, \cdots, g_{n-1}; x_n) = (g_0, \cdots, g_{n-2}; g_{n-1}\,\partial_n x_n)$$

and

$$s_i(g_0, g_1, \cdots, g_{n-1}; x_n) = (g_0, \cdots, g_{i-1}, 1, s_i g_i, \cdots, s_i g_{n-1}; s_i x_n)\,.$$

The universal G-bundle $G \to WG \to \bar{W}G$ is defined to be $G \to W(G; G) \to W(G; *)$ and $\bar{W}G$ is called the $\bar{W}$-construction. The canonical twisting function $t(G) = (t(G)_n\colon \bar{W}_n G \to G_{n-1})_{n\geq 1}$ is defined by $t(G)_n(g_0, g_1, \cdots, g_{n-1}) = g_{n-1}$. A twisting function $t = (t_n\colon B_n \to G_{n-1})_{n\geq 1}$ is defined to be the composition $t(G)\,\theta(t)$ with some simplicial map $\theta(t)\colon B \to \bar{W}G$, which is uniquely determined by t. The twisted cartesian product (T.C.P.) with base B, G-space X and twisting function $t\colon B \to G$ is the simplicial set $B \times_t X$ whose set of n-simplices is $B_n \times X_n$, with faces and degeneracies given by $\partial_n(b, x) = (\partial_n b, t_n(b)\,\partial_n x)$, $\partial_i(b, x) = (\partial_i b, \partial_i x)$, $0 \leqq i \leqq n-1$, $s_i(b, x) = (s_i b, s_i x)$, $0 \leqq i \leqq n$. The T.C.P. $\bar{W}G \times_{t(G)} X$ is identified with $W(G; X)$. In this paper the nerve functor ([9]) is also denoted by $\bar{W}$.

In order to state our main theorem we fix some groups and group homomorphisms. Let G be a group, Aut$G$ the group of automorphisms of G and Inn: $G \to$ Aut$G$ the homomorphism which sends an element g to its inner automorphism $\mathrm{Inn}(g)(\ \ ) = g(\ \ )g^{-1}$. These groups have natural Aut G-actions, evaluation of G and conjugation of Aut$G$ respectively, and we find the homomorphism Inn to be Aut$G$-equivariant. The kernel of Inn is the center ZG of G and the cokernel of Inn the group of outer auotmorphisms Out$G$. So we have the long exact sequence of Aut$G$-groups $1 \to ZG \to G \xrightarrow{\mathrm{Inn}} \mathrm{Aut}G \to \mathrm{Out}G \to 1$.

Theorem 1. *$E\bar{W}G$, the simplicial group of self weak equivalences of $K(G, 1)$-space $\bar{W}G$, is isomorphic to $W(G;$ Aut$G)$ where G acts on* Aut$G$ *through the homomorphism* Inn *and the multiplication is given by the formula*

$$(x_0, x_1, \cdots, x_{n-1}; \alpha)\,(y_0, y_1, \cdots, y_{n-1}; \beta)$$
$$= (z_0, z_1, \cdots, z_{n-1}; \alpha\beta),\ z_i = x_i\ \mathrm{Inn}(x_{i+1}\cdots x_{n-1})\,(\alpha(y_i))\,.$$

Corollary 2. ([4], [5]). *The homotopy groups of the classifying space $\bar{W}E\bar{W}G$, $\pi_n \bar{W}E\bar{W}G$ are isomorphic to ZG if $n=2$,* Out$G$ *if $n=1$ and $\{1\}$ in the other cases, and moreover the $\pi_1 W(E\bar{W}G; \bar{W}G)$-equivariant homotopy sequence of the universal $K(G, 1)$-fibration $\bar{W}G \to W(E\bar{W}G; \bar{W}G) \to \bar{W}E\bar{W}G$ is isomorphic to the* Aut$G$-*exact sequence* $1 \to ZG \to G \to \mathrm{Aut}\,G \to \mathrm{Out}\,G \to 1$.

It was Hill, Jr.[5] who determined the k-invariant of the two stage Postnikov space $\bar{W}E\bar{W}G$. The k-cocycle is defined as follows (see [10]). Let s be a section of the quotient $\mathrm{Aut}\,G \to \mathrm{Out}\,G$ satisfying $s(1)=1$. The difference between $s(x)\,s(y)$ and $s(xy)$ is measured by an element $f(x, y) \in \mathrm{Inn}\,G$ and the equation $s(x)\,s(y) = s(xy)\,f(x, y)$, so we have the function $f\colon \mathrm{Out}\,G^2 \to \mathrm{Inn}\,G$.

Further we have a lifting g: $\mathrm{Out}\,G^2 \to G$, $\mathrm{Inn}\ g=f$ with the property $g(x, 1)=1=g(1, x)$. The associative law $(s(x)\,s(y))\,s(z) = s(x)\,(s(y)\,s(z))$ provides that there exists a function u: $\mathrm{Out}\,G^3 \to ZG$ such that $u(x, y, z)\,g(xy, z)\,s(z)^{-1}(g(x, y)) =g(x, yz)\,g(y, z)$. The function u: $\bar{W}_3\mathrm{Out}\,G \to ZG$ is a 3-cochain of the space $\bar{W}\mathrm{Out}\,G$ which is twisted by the canonical twisting function $t(\mathrm{Out}\,G)$ and by the group homomorphism $s=s|_{ZG}$: $\mathrm{Out}\,G \to \mathrm{Aut}\,ZG$, the composition of the set function s and the restriction $\mathrm{Aut}\,G \to \mathrm{Aut}\,ZG$. The cochain is a cocycle (see [10]).

REMARK. Contrary to [10] we adopt right action to define local coefficient cohomology theory. Under the isomorphism $B_*(\mathrm{Out}\,G, \mathrm{Out}\,G, *) \cong B_*(*, \mathrm{Out}G, \mathrm{Out}\,G)=W\,\mathrm{Out}\,G$, $(x; x_0, x_1, \cdots, x_{n-1}) \to (x_0, x_1, \cdots, x_{n-1}; (xx_0 \cdots x_{n-1})^{-1})$, our cocycle corresponds to that of MacLane.

Thdorem 3 (Hill, Jr.[5]). *The classifying space $\bar{W}E\bar{W}G$ of the $K(G, 1)$-fibrations has a strong deformation retract which is the two stage Postnikov space with the k-cocycle u^{-1}.*

Finally, we will show another type of application of Theorem 1.

Theorem 4. *Let p: $E \to B$ be a (based) nilpotent $K(G, 1)$-fibration (see [7] or [2]) where E, B are connected. Then p is decomposed into a finite tower of principal fibrations with abelian $K(A, 1)$-fibres.*

REMARK. Theorem 4 and its $K(A, n)$ version induce the more general theorem: the Moore-Postnikov decomposition of a (based) nilpotent fibration $F \to E \to B$, where F, E and B are all connected, admits a principal refinement (see [7; Thm. 2.14] or [2; 4.7 Prop.]).

2. Proof of Theorem 1

A group G is regarded as a small category with one object $*$. A functor α: $G \to G$ is only a group homomorphism, and a natural transformation $\alpha \to \beta$ only a relation $\beta=\mathrm{Inn}(x)\,\alpha$, $x \in G$. So the functor category (see [9]) G^G has the following structure maps: S, T: $MG^G=G \times \mathrm{End}\,G \to OG^G=\mathrm{End}\,G$, $S(x, \alpha)=\alpha$, $T(x, \alpha)=\mathrm{Inn}(x)\,\alpha$, I: $\mathrm{End}\,G \to G \times \mathrm{End}\,G$, $I(\alpha)=(1, \alpha)$ and m: $G \times G \times \mathrm{End}\,G \to G \times \mathrm{End}\,G$, $m(x, y, \beta)=(xy, \beta)$.

Proposition 5. *$W(G; \mathrm{End}\,G)$ is isomorphic to* hom$(\bar{W}G, \bar{W}G)$.

Proof. The correspondence, $(x; \alpha)=(x_0, x_1, \cdots, x_{n-1}; \alpha_n) \to ((x_0, \alpha_1), (x_1, \alpha_2), \cdots, (x_{n-1}, \alpha_n))$, $\alpha_i=\mathrm{Inn}(x_i \cdots x_{n-1})\,\alpha$, makes $W(G; \mathrm{End}\,G)$ and $\bar{W}G^G$ isomorphic. The n-simplex $(x; \alpha)$, considered to be an n-simplex of $\bar{W}G^G$, is usually regarded as the functor $\boldsymbol{n} \to G^G$, $(i, j) \to (x_i \cdots x_{j-1}, \alpha_j)$ where $\boldsymbol{n}=0 \leftarrow 1 \leftarrow \cdots \leftarrow n$ is the category with $n+1$ objeats 0, 1, $\cdots$, n and one morphicm (i, j) from j to i if $i \leqq j$. By adjointness, $\mathrm{Cat}(\boldsymbol{n}, G^G) \cong \mathrm{Cat}(\boldsymbol{n} \times G, G)$, the n-simplex $(x; \alpha)$ is also regarded as the functor $\boldsymbol{n}\times G\to G$, $((i,j),g)\to((x_i\cdots x_{j-1},\alpha_j),g)\to x_i\cdots x_{j-1}\alpha_j(g)$. Further, applying the nerve $\bar{W}$ to the functor $\boldsymbol{n}\times G\to G$, we have an n-simplex $\Delta[n]\times\bar{W}G=\bar{W}\boldsymbol{n}\times\bar{W}G\cong\bar{W}(\boldsymbol{n}\times G)\to\bar{W}G$ of $\hom(\bar{W}G,\bar{W}G)$. Since the nerve functor is fully-faithful, the correspondence $\mathrm{Cat}(\boldsymbol{n}\times G, G)\to\boldsymbol{S}(\bar{W}(\boldsymbol{n}\times G),\bar{W}G)$ is bijeative. Naturalities of these correspondenses make the composition $W(G;\mathrm{End}\,G)\to\hom(\bar{W}G,\bar{W}G)$ simplicial, and the proposition is proved.

An n-simplex or a functor, correspoding to an n-simplex $(x;\alpha)\in W_n(G;\mathrm{End}\,G)$, is also called $(x;\alpha)$.

Proposition 6. *Let* $(x;\alpha)=(x_0,\cdots,x_{n-1};\alpha)$, $(y;\beta)=(y_0,\cdots,y_{n-1};\beta)$ *be n-simplices of* $\hom(\bar{W}G,\bar{W}G)$, *and let* $(u,g)=((u(o),u(1),\cdots,u(k)),(g_0,g_1,\cdots,g_{k-1}))$ *be a k-simplex of* $\Delta[n]\times\bar{W}G$, *then we have (i)* $(x;\alpha)(u,g)=(z_0,z_1,\cdots,z_{k-1})$, $z_i=x_{u(i)}\,\mathrm{Inn}(x_{u(i)+1}\cdots x_{u(i+1)-1})\,\alpha_{u(i+1)}(g_i)$, *and (ii)* $(x;\alpha)(y;\beta)=((x;\alpha)((0,1,\cdots,n),y);\alpha\beta)$.

Proof. The k-simplex (u,g) is regarded as the following sequence of morphisms in $\boldsymbol{n}\times G$ $(((u(0),u(1)),g_0),((u(1),u(2)),g_1),\cdots,((u(k-1),u(k)),g_{k-1}))$, where $((u(i),u(i+1),g_i)\colon(u(i),*)\leftarrow(u(i+1),*)$. Applying the functor $(x;\alpha)\colon\boldsymbol{n}\times G\to G$, $((i,j),g)\to x_i\cdots x_{j-1}\alpha_j(g)$, to the above sequence, we have a sequence of morphisms in G which is the k-simplex $(x,\alpha)(u,g)$. This proves (i). Put $(z;\gamma)=(x;\alpha)(y;\beta)$, and apply a 1-simplex $((i,i+1),g)$ to both sides, then we get the following equations, $x_i\,\mathrm{Inn}(x_{i+1}\cdots x_{n-1})(\alpha(y_i\,\mathrm{Inn}(y_{i+1}\cdots y_{n-1})(\beta(g))))=z_i\,\mathrm{Inn}(z_{i+1}\cdots z_{n-1})(\gamma(g))$, $0\leqq i\leqq n-1\cdots(1)$. Substituting 1 for g in the equations (1), we find, $z_i=x_i\,\mathrm{Inn}(x_{i+1}\cdots x_{n-1})(\alpha(y_i))\cdots(2)$. Comparing (1), (2) in the case $i=n-1$, we have $\gamma=\alpha\beta$, and (ii) is completed.

REMARK. The monoid $\hom(\bar{W}G,\bar{W}G)$ acts on $\bar{W}G$ by the formula $(x;\alpha)y=(x;\alpha)((0,1,\cdots,n),y)$.

Since it is easy for the reader to see $(x;\alpha)$ is invertible iff $\alpha\in\mathrm{Aut}\,G$, Theorem 1 is proved.

Next we prove Corollary 2. Fundamental maps related to $E\bar{W}G\cong W(G;\mathrm{Aut}\,G)$ are the $\mathrm{Aut}\,G$-principal bundle $\mathrm{Aut}\,G\xrightarrow{i}W(G;\mathrm{Aut}\,G)\xrightarrow{p}\bar{W}G$, the short exact sequence of simplicial groups $\bar{W}ZG\xrightarrow{j}W(G;\mathrm{Aut}\,G)\xrightarrow{q}W(\mathrm{Inn}\,G;\mathrm{Aut}\,G)$, and the canonical epimorphism of simplicial groups with the contractible kernel $W\,\mathrm{Inn}\,G$, $r\colon W(\mathrm{Inn}\,G;\mathrm{Aut}\,G)\to\mathrm{Out}\,G$. All maps except p have their deloopings, $\bar{W}i$, $\bar{W}j$, $\bar{W}q$ and $\bar{W}r$.

Proposition 7. $\bar{W}q\colon\bar{W}W(G;\mathrm{Aut}\,G)\to\bar{W}W(\mathrm{Inn}\,G;\mathrm{Aut}\,G)$ *is a minimal fibration.*

Since $\bar{W}W(G;\mathrm{Aut}\,G)$ is fibrant, the proof can be reduced to the following

Lemma 8. *Let x, y be elements of $\bar{W}_n W(\mathrm{Inn}\,G; \mathrm{Aut}\,G)$. If $n \geqq 3$, then $\partial_i x = \partial_i y$, for all $i \neq k$ imply $x = y$.*

Proof. Put $x = (\alpha_0, (x(1); \alpha_1), \cdots, (x(n-1); \alpha_{n-1}))$, $x(i) = (x(i)_0, x(i)_1, \cdots, x(i)_{i-1})$ and $y = (\beta_0, (y(1); \beta_1), \cdots, (y(n-1); \beta_{n-1}))$, $y(i) = (y(i)_0, y(i)_1, \cdots, y(i)_{i-1})$. In case of $0 \leqq k \leqq n-2$, $\partial_n x = \partial_n y$ proves $\alpha_0 = \beta_0$ and $(x(i); \alpha_i) = (y(i); \beta_i)$ for $0 \leqq i \leqq n-2$, further $\partial_{n-1} x = \partial_{n-1} y$ implies $(x(n-1)_0, \cdots, x(n-1)_{n-3}) = (y(n-1)_0, \cdots, y(n-1)_{n-3})$ and the rest, $x(n-1)_{n-2} = y(n-1)_{n-2}$, $\alpha_{n-1} = \beta_{n-1}$, are proved by $\partial_j x = \partial_j y$, $0 \leqq j \leqq n-2$, $j \neq k$. Such a j exists by the condition $n \geqq 3$. Therfore we have $x = y$. Similar procedure, taking $\partial_0 x = \partial_0 y$, $\partial_1 x = \partial_1 y$ and $\partial_j x = \partial_j y$ for $2 \leqq j \leqq n$, $j \neq k$, implies $x = y$ in the case $2 \leqq k \leqq n$. This completes the lemma.

Examining the homotopy long exact sequence of the fibration $\bar{W}q$, we find that the first half of Corollary 2 is proved.

The homomorphism i: $\mathrm{Aut}\,G \to W(G; \mathrm{Aut}\,G)$ induces the map of contractible free $\mathrm{Aut}\,G$-spaces, Wi: $W\mathrm{Aut}\,G \to WW(G; \mathrm{Aut}\,G)$, and therefore induces the homotopy equivalence, $I = Wi/\mathrm{Aut}\,G$: $\bar{W}\mathrm{Aut}\,G \to WW(G; \mathrm{Aut}\,G)/\mathrm{Aut}\,G = W(W(G; \mathrm{Aut}\,G); \bar{W}G)$, $I(\alpha_0, \alpha_1, \cdots, \alpha_{n-1}) = (\alpha_0, (1; \alpha_1), \cdots, (1^{n-1}; \alpha_{n-1}); 1^n)$, $1^k = (1, 1, \cdots, 1)$. It enables us to fix the identification $\pi_1 I$: $\mathrm{Aut}\,G = \pi_1 \bar{W}\mathrm{Aut}\,G \to \pi_1 W(W(G; \mathrm{Aut}\,G); \bar{W}G)$. For the second half we need more identifications, $\pi_2 \bar{W}j$: $ZG = \pi_2 \bar{W}^2 ZG \to \pi_2 \bar{W}W(G; \mathrm{Aut}\,G)$, $\pi_1 \bar{W}rq$: $\pi_1 \bar{W}W(G; \mathrm{Aut}\,G) \to \pi_1 \bar{W}\mathrm{Out}\,G = \mathrm{Out}\,G$. For the universal $K(G; 1)$-fibration, $\bar{W}G \xrightarrow{k} W(W(G; \mathrm{Aut}\,G); \bar{W}G) \xrightarrow{p_r} \bar{W}W(G; \mathrm{Aut}\,G)$, we examine $\pi_1 k$, $\pi_1 p_r$ and the connecting homomorphism ∂: $\pi_2 \bar{W}W(G; \mathrm{Aut}\,G) \to \pi_1 \bar{W}G = G$ under the above identifications. It is easy to see that $rqp_r I$ is only the canonical projection, $\bar{W}\mathrm{Aut}\,G \to \bar{W}\mathrm{Out}\,G$. So $\pi_1 p_r$ is identified with the projection $\mathrm{Aut}\,G \to \mathrm{Out}\,G$. As for $\pi_1 k$ we introduce a left inverse f of I. Define f_1: $W_1(W(G; \mathrm{Aut}\,G); \bar{W}G) \to \bar{W}_1 \mathrm{Aut}\,G$ by $f_1(\alpha; x) = \alpha\,\mathrm{Inn}(x)$. Once f_1 is thus given, f_2 and other f_n's are determined through the routine computations such that f becomes simplicial. It is easy to see $fI = 1_{\mathrm{Aut}\,G}$ and $fk = \bar{W}\mathrm{Inn}$. In order to examine the connecting homomorphism it is enough to compare the homotopy sequences of two fibratiobs:

$$\begin{array}{ccccc} \bar{W}ZG & \to & W\bar{W}ZG & \to & \bar{W}^2 ZG \\ \downarrow \subset & & \downarrow W(j; \subset) & & \downarrow \bar{W}j \\ \bar{W}G & \to & W(W(G; \mathrm{Aut}\,G); \bar{W}G) & \to & \bar{W}W(G; \mathrm{Aut}\,G). \end{array}$$

We notice that $1_{ZG} = \partial: \pi_2 \bar{W}^2 ZG \to \pi_1 \bar{W}ZG$. This completes examinations. With respect to $\mathrm{Aut}\,G$-actions, the well known fact that π_1 acts on itself by conjugation and the following proposition imply the second half of Corollary 2.

Proposition 9. *In the universal $K(G; 1)$-fibration, $\mathrm{Aut}\,G$ acts on $\pi_1 \bar{W}G = G$ by evaluation through the isomorphism $\pi_1 I$.*

Proof. Let α be an element of $\bar{W}_1 \mathrm{Aut}\, G = \mathrm{Aut}\, G$. Then we have $I(\alpha) = (\alpha; 1)$ and $p_r I(\alpha) = \alpha \in \bar{W}_1 W(G; \mathrm{Aut}\, G)$. These elements define paths $\bar{I}(\alpha)$: $\Delta[1] \to W(W(G; \mathrm{Aut}\, G); \bar{W}G)$, $\bar{\alpha}$: $\Delta[1] \to \bar{W}W(G; \mathrm{Aut}\, G)$. By making use of the canonical twisting function $t = t(W(G; \mathrm{Aut}\, G))$: $\bar{W}W(G; \mathrm{Aut}\, G) \to W(G; \mathrm{Aut}\, G)$ we define a simplicial map $\tilde{\alpha}$: $\Delta[1] \times \bar{W}G \to W(W(G; \mathrm{Aut}\, G); \bar{W}G)$, $\tilde{\alpha}(u, x) = (u^*\alpha; t((u, 1)^*\alpha)x)$ for $(u, x) \in \Delta[1]_k \times \bar{W}_k G$. Here u^* denotes the map $\bar{W}_1 W(G; \mathrm{Aut}\, G) \to \bar{W}_k W(G; \mathrm{Aut}\, G)$ induced by $u \in \Delta[1]_k = \Delta([k], [1])$. Substituting $1^{k+1} = (1, 1, \cdots, 1)$ for u, we find that the restriction $\tilde{\alpha}|_{(1)\times \bar{W}G}$ is identified with the canonical inclusion k. Since $\tilde{\alpha}((0, 1), 1) = (\alpha; s_1 1) = (\alpha; 1)$, $\tilde{\alpha}|_{\Delta[1]\times 1}$ is identified with the path $\bar{I}(\alpha)$. These show that the homotopy class of $I(\alpha)$ operates on $\pi_1 \bar{W}G = G$ as the homomorphism $\pi_1(k^{-1}\tilde{\alpha}|_{(0)\times \bar{W}G})$: $\pi_1 \bar{W}G \to \pi_1 \bar{W}G$. Since $\tilde{\alpha}((0, 0), x) = (1_G; (1; \alpha)x) = (1_G; (1; \alpha)((0, 1), x)) = (1_G; \alpha(x))$, the homomorphism is equal to α. This proves Proposition 9.

3. Proof of Theorem 3

In section 1 we have fixed a section s: $\mathrm{Out}\, G \to \mathrm{Aut}\, G$ and a function f: $\mathrm{Out}\, G^2 \to \mathrm{Inn}\, G$ which satisfies $s(x)s(y) = s(xy)f(x, y)$. By making use of them we construct a right inverse R: $\bar{W}\mathrm{Out}\, G \to \bar{W}W(\mathrm{Inn}\, G; \mathrm{Aut}\, G)$ to $\bar{W}r$. Put $h(x, y) = s(y)f(x, y)^{-1}s(y)^{-1}$ and define R by the formulae

$$R_n(b) = (R_{n,0}(b), R_{n,1}(b), \cdots, R_{n,n-1}(b)), \quad R_{n,i}(b) = (c_0, c_1, \cdots, c_{i-1}; s(b_i)),$$
$$c_j = h(b_j \cdots b_{i-1}, b_i)\, h(b_{j+1} \cdots b_{i-1}, b_i)^{-1} \quad \text{for} \quad b = (b_0, b_1, \cdots, b_{n-1}) \in \bar{W}_n \mathrm{Out}\, G.$$

Long but routine calculations make us find R to be simplicial. We find immediately it to be a right inverse to $\bar{W}r$.

The proof of Theorem 3 is organized as follows.

(i) We obtain a simplicial map over $\bar{W}\mathrm{Out}\, G$ U^{-1}: $\bar{W}\mathrm{Out}\, G \to \bar{W}(\bar{W}^2 ZG \times_s \mathrm{Out}\, G)$ corresponding to the cocylce $u^{-1} \in Z_t^3(\bar{W}\mathrm{Out}\, G; ZG)$. Here $\bar{W}^2 ZG \times_s \mathrm{Out}\, G$ is the semi-direct product of simplicial groups $\bar{W}^2 ZG$, $\mathrm{Out}\, G$ with the action s: $\mathrm{Out}\, G \times \bar{W}^2 ZG \to \bar{W}^2 ZG$ which is the canonical extension of the group aation $s_{|ZG}$: $\mathrm{Out}\, G \to \mathrm{Aut}\, ZG$ defined in section 1. The map U^{-1} defines a twisting function $t' = t(\bar{W}^2 ZG \times_s \mathrm{Out}\, G)U^{-1}$ and a fibration $\bar{W}^2 ZG \to \bar{W}\mathrm{Out}\, G \times_{t'} \bar{W}^2 ZG \to \bar{W}\mathrm{Out}\, G$.

(ii) We can lift the map R to a cofibration $\tilde{R}$: $\bar{W}\mathrm{Out}\, G \times_{t'} \bar{W}^2 ZG \to \bar{W}W(G; \mathrm{Aut}\, G)$ which makes the diagram

$$\begin{array}{ccccc} \bar{W}^2 ZG & \to & \bar{W}\mathrm{Out}\, G x_{t'} \bar{W}^2 ZG & \to & \bar{W}\mathrm{Out}\, G \\ \downarrow = & & \downarrow \tilde{R} & & \downarrow R \\ \bar{W}^2 ZG & \to & \bar{W}W(G; \mathrm{Aut}\, G) & \to & \bar{W}W(\mathrm{Inn}\, G; \mathrm{Aut}\, G) \end{array}$$

commutative.

After these procedures, Theorem 3 would be proved because $\tilde{R}$ is a trivial cofibration and all the spaces in the diagram are fibrant. Covering homotopy property and some other techniques, concerning closed model categories, provide moreover the fact that $(\tilde{R}, R)$ has a left inverse which is a strong deformation retraction of these fibrations.

We begin with procedure (i). Let $C_t^n(B, A)$ be the set of A-valued twisted normalized n-cochains of a simplicial set B with a twisting function $t=t(\Gamma)\theta(t)$: $B\to\Gamma$ and a group action $\phi: \Gamma\to\mathrm{Aut}A$, where Γ is a group and A a (multiplicative) commutative group. The differentials $\delta f(b)=\prod_{i=0}^{n} f(\partial_i b)^{\varepsilon(i)}\, t(b)^{-1}(f(\partial_{n+1} b))$, $\varepsilon(i)=(-1)^{n+1+i}$, $b\in B_{n+1}$, $f\in C_t^n(B, A)$, make $(C_t^*(B, A), \delta)$ a cochain complex. Applying the normalized cochain complex functor to the cosimplicial simplicial set $\Delta[*]$ we have a cochain complex $\boldsymbol{S}_{\bar{W}\Gamma}(B, \bar{W}\Gamma\times_t C^*(\Delta[*]; A))$ where the set of n-cochains $\boldsymbol{S}_{\bar{W}\Gamma}(B, \bar{W}\Gamma\times_t C^n(\Delta[*]; A))$ is the set of simplicial maps over $\bar{W}\Gamma$ (see [8]).

Proposition 10. *These cochain complexes are isomorphic by the correspondence* $\mu: C_t^n(B; A)\to\boldsymbol{S}_{\bar{W}\Gamma}(B, \bar{W}\Gamma\times_t C^n(\Delta[*]; A))$, $\mu(f)(b)=(\theta(t)(b), [a\to t((a, k)*b)^{-1} f(a*b)])$ *where* $f\in C_t^n(B; A)$, $b\in B_k$ *and* $a\in\Delta[k]_n$.

Proof. The inverse function ν is defined as $\nu(\theta(t), g)(b)=g(b)(0, 1, \cdots, n)$, $b\in B_n$. Details are left to the readers (see [8]).

The shot exact sequence $1\to Z^n(\Delta[*]; A)\to C^n(\Delta[*]; A)\xrightarrow{\delta} Z^{n+1}(\Delta[*]; A)\to 1$ is a model of the universal fibration $\bar{W}^n A\to W\bar{W}^n A\to\bar{W}^{n+1}A$. We construct an isomorphism $Z^n(\Delta[*]; A)\to\bar{W}^n A$ as follows. A twisting function t_k^{n+1}: $Z^{n+1}(\Delta[k]; A)\to Z^n(\Delta[k-1]; A)$ is defined as $t_k^{n+1}(f)=f(\ \ , k-1)^{-1} f(\ \ , k)$ (see [11; §23]). We have the isomorphism of Eilenberg-MacLane spaces $\theta(t^{n+1})$: $Z^{n+1}(\Delta[*]; A)\to\bar{W}Z^n(\Delta[*]; A)$, $\theta(t^{n+1})_k(f)=(t_1^{n+1}(\partial_2\partial_3\cdots\partial_k f), t_2^{n+1}(\partial_3\partial_4\cdots\partial_k f), \cdots, t_k^{n+1}(f))$, and the isomorphism $\eta: Z^0(\Delta[*]; A)\to A\otimes\Delta[0]$, $\eta(f)=f(k)$ for $f\in Z^0(\Delta[k]; A)$. The composition $\eta^n=\bar{W}^n\eta\bar{W}^{n-1}\theta(t^1)\,\bar{W}^{n-2}\theta(t^2)\cdots\theta(t^n))$: $Z^n(\Delta[*]; A)\to\bar{W}^n A$, is the isomorphism which we need.

Lemma 11. *We have the explicit formula*

$$\eta_3^3(f)=f(0, 1, 2, 3).$$

We need one more isomorphism ξ: $W(\Gamma; \bar{W}^n A)\to\bar{W}(\bar{W}^{n-1}A\times_\phi\Gamma)$ which is defined by $\xi(y; a)=((y_0\cdots y_{k-1}(a_0), y_0), (y_1\cdots y_{k-1}(a_1), y_1), \cdots, (y_{k-1}(a_{k-1}), y_{k-1}))$ for $(y; a)=(y_0, y_1, \cdots, y_{k-1}; a_0, a_1, \cdots, a_{k-1})\in W_k(\Gamma; \bar{W}^n A)$. Mixing up above isomorphisms we have the isormorphism $Z_t^n(B; A)\to\boldsymbol{S}_{\bar{W}\Gamma}(B, \bar{W}(\bar{W}^{n-1}A\times_\phi\Gamma))$. We conclude that the cocycle u^{-1} determines the simplicial map over $\bar{W}\mathrm{Out}G$ U^{-1}: $\bar{W}\mathrm{Out}G\to\bar{W}(\bar{W}^2 ZG\times_s\mathrm{Out}G)$, the twisting function t': $\bar{W}\mathrm{Out}G\to\bar{W}^2 ZG\times_s\mathrm{Out}\,G$ and the fibration $\bar{W}^2 ZG\to\bar{W}\mathrm{Out}G\times_{t'}\bar{W}^2 ZG\to\bar{W}\mathrm{Out}G$. We have moreover explicit formulae $t_1'(b_0)=s(b_0)$, $t_2'(b_0, b_1)=s(b_1)$ and $t_3'(b_0, b_1, b_2)=(s(b_2)(u(b_0, b_1,$

$b_2)^{-1}), b_2)$.

We turn to procedure (ii) at once. We define the following injective maps $\tilde{R}_0: 1\to 1$, $\tilde{R}_1: \mathrm{Out}\,G\times 1\to \mathrm{Aut}\,G$, $\tilde{R}_2: \mathrm{Out}\,G^2\times 1\times ZG\to \mathrm{Aut}\,G\times(G\times \mathrm{Aut}\,G)$ and $\tilde{R}_3: \mathrm{Out}\,G^3\times 1\times ZG\times ZG^2\to \mathrm{Aut}\,G\times(G\times \mathrm{Aut}\,G)\times(G^2\times \mathrm{Aut}\,G)$ by $\tilde{R}_0(1)=1$, $\tilde{R}_1(b_0, 1)=s(b_0)$, $\tilde{R}_2((b_0, b_1), (1, a_1))=(s(b_0), (s(b_1)(g(b_0, b_1)^{-1}a_1); s(b_1))$ and $\tilde{R}_3((b_0, b_1, b_2), (1, a_1, (a_{2,0}, a_{2,1})))=(s(b_0), (s(b_1)(g(b_0, b_1)^{-1})\, s(b_1b_2)\,(u(b_0, b_1, b_2)^{-1}a_1); s(b_1)), (s(b_2)(g(b_0b_1, b_2)^{-1}g(b_1, b_2)\,a_{2,0}), s(b_2)(g(b_1, b_2)^{-1}a_{2,1}); s(b_2)))$ respeatively. By making use of the above explicit formulae of t' and the relation $u(b_0, b_1, b_2)\,g(b_0b_1, b_2)\,s(b_2)^{-1}(g(b_0, b_1))=g(b_0, b_1b_2)\,g(b_1, b_2)$ it turns out that $(\tilde{R}_0, \tilde{R}_1, \tilde{R}_2, \tilde{R}_3)$ is a simplicial map truncated at level 3 or equivalently a simplicial map $sk^3(\bar{W}\mathrm{Out}\,G\times_{t'}\bar{W}^2ZG)\to \bar{W}W(G; \mathrm{Aut}\,G)$ (see [1]). We find further that the following diagram truncated at level 3

$$\begin{array}{ccccc}
\bar{W}^2ZG & \to & \bar{W}\mathrm{Out}\,G\times_{t'}\bar{W}^2ZG & \to & \bar{W}\mathrm{Out}\,G \\
\downarrow (1,1,1,1) & & \downarrow (\tilde{R}_0,\tilde{R}_1,\tilde{R}_2,\tilde{R}_3) & & \downarrow R \\
\bar{W}^2ZG & \to & \bar{W}W(G; \mathrm{Aut}\,G) & \to & \bar{W}W(\mathrm{Inn}\,G; \mathrm{Aut}\,G)
\end{array}$$

or equivalently the following diagram of simplicial maps

$$\begin{array}{ccccc}
\mathrm{sk}^3\bar{W}^2ZG & \to & \mathrm{sk}^3(\bar{W}\mathrm{Out}\,G\times_{t'}\bar{W}^2ZG) & \to & \bar{W}\mathrm{Out}\,G \\
\cap & & \downarrow & & \downarrow R \\
\bar{W}^2ZG & \to & \bar{W}W(G; \mathrm{Aut}\,G) & \to & \bar{W}W(\mathrm{Inn}\,G; \mathrm{Aut}\,G)
\end{array}$$

is commutative.

Proposition 12. *For any commutative diagram*

$$\begin{array}{ccc}
\mathrm{sk}^3X & \subset & X \\
\downarrow & & \downarrow \\
\bar{W}W(G; \mathrm{Aut}\,G) & \to & \bar{W}W(\mathrm{Inn}\,G; \mathrm{Aut}\,G)
\end{array}$$

the filler $X\to \bar{W}W(G; \mathrm{Aut}\,G)$ *exists uniquely.*

Proof. The filtration $\mathrm{sk}^3X\subset \mathrm{sk}^4X\subset\cdots\subset X$ and the push out diagram

$$\begin{array}{ccc}
\coprod\dot{\Delta}[n+1] & \subset & \coprod\Delta[n+1] \\
\downarrow & & \downarrow \\
\mathrm{sk}^nX & \subset & \mathrm{sk}^{n+1}X
\end{array}$$

reduce the proposition to the following

Lemma 13. *When* $n\geqq 3$, *for any commutative diagram*

$$\begin{array}{ccc}
\dot{\Delta}[n+1] & \subset & \Delta[n+1] \\
\downarrow z & & \downarrow \\
\bar{W}W(G; \mathrm{Aut}\,G) & \to & \bar{W}W(\mathrm{Inn}\,G; \mathrm{Aut}\,G)
\end{array}$$

the filler $\Delta[n+1]\to\bar{W}W(G;\mathrm{Aut}\,G)$ *exists uniquely.*

Proof. The equivalent condition to give a simplicial map z is to give n-simplices $z_0, z_1, \cdots, z_{n+1}\in\bar{W}_nW(G;\mathrm{Aut}G)$ such that $\partial_iz_j=\partial_{j-1}z_i$ for $i<j$. The diagram restricted to the horn $\Lambda^{n+1}[n+1]\subset\dot{\Delta}[n+1]$ has a filler $\bar{w}$ because $\bar{W}q$ is a fibration. The $n+1$-simplex w satisfies equation $\partial_iw=z_i$ for any $i\neq n+1$. Since $\partial_i\partial_{n+1}w=\partial_n\partial_iw=\partial_nz_i=\partial_iz_{n+1}$ for $i=0, 1, \cdots, n$ analogous arguments in Lemma 8 imply $\partial_{n+1}w=z_{n+1}$. Therefore $\bar{w}$ is a filler without any restriction. Similar arguments show the uniquencess of $\bar{w}$ and the lemma is proved.

We are given a commutative diagram

$$\begin{array}{ccccc} \bar{W}^2ZG & \to & \bar{W}\mathrm{Out}\,G\times_{t'}\bar{W}^2ZG & \to & W\mathrm{Out}\,G \\ \downarrow & & \downarrow\tilde{R} & & \downarrow R \\ W^2ZG & \to & \bar{W}W(G;\mathrm{Aut}\,G) & \to & \bar{W}W(\mathrm{Inn}\,G;\mathrm{Aut}G)\,. \end{array}$$

The restriction $\tilde{R}_{|\bar{W}^2ZG}$: $\bar{W}^2ZG\to\bar{W}^2ZG$ is equal to the identity of $\bar{W}^2ZG$ because $\bar{W}^2ZG=\mathrm{cosk}^3\bar{W}^2ZG$. Since $\tilde{R}_0, \cdots, \tilde{R}_3$ are injeative, injectivity of $\tilde{R}$ is proved by induction. Hence completing Procedure (ii), Theorem 3 is proved.

4. Proof of Theorem 4

Let us consider a fibration $\bar{W}A\to E\to B$, where A is an abelian group, B an one vertexed fibrant simplicial set and E a T.C.P. (twisted cartesian product) $B\times_t\bar{W}A$ with t: $B\to W(A;\mathrm{Aut}A)$. The fibration is classified as the following commutative diagram

$$\begin{array}{ccccc} \bar{W}A & \to & E & \xrightarrow{p_1} & B \\ \downarrow = & & \downarrow\tilde{\theta}(t) & & \downarrow\theta(t) \\ \bar{W}A & \to & W(W(A;\mathrm{Aut}A);\bar{W}A) & \xrightarrow{p_r} & \bar{W}W(A;\mathrm{Aut}A)\,. \end{array}$$

Lemma 14. *For the above fibration the* π_1E*-action on* $\pi_1\bar{W}A=A$ *is determined by the homomorphism* $\pi_1\theta(t)\,\pi_1p_1$: $\pi_1E\to\pi_1B\to\mathrm{Aut}A$, *and the action is trivial iff the fibration is principal.*

Proof. By naturality π_1E acts on A through $\pi_1\tilde{\theta}(t)$. For the universal fibration it is proved in Proposition 9 that $\mathrm{Aut}A\cong\pi_1W(W(A;\mathrm{Aut}A);\bar{W}A)$ acts on A by evaluation. In this abelian case π_1p_r is identified with $1_{\mathrm{Aut}\,A}$ under the specific identification π_1I: $\mathrm{Aut}A\to\pi_1W(W(A;\mathrm{Aut}A);\bar{W}A)$. Therefore we have $\pi_1I^{-1}\pi_1\tilde{\theta}(t)=\pi_1\theta(t)\,\pi_1p_1$ and the first half of the lemma is concluded. Let $t(B)$: $B\to\pi_1B$ be the twisting function defined by $t(B)_n(b)$=the homotopy class of $\partial_0^{n-1}b$. Since $t(\mathrm{Aut}A)_n(a_0, a_1, \cdots, a_{n-1})=a_{n-1}=\partial_0^{n-1}(a_0, \cdots, a_{n-1})$, we have $t(\mathrm{Aut}A)\,\bar{W}q\theta(t)=\pi_1\bar{W}q\pi_1\theta(t)\,t(B)$ by naturality, furthermore since $\pi_1\bar{W}q=1_{\mathrm{Aut}A}$,

$t(\mathrm{Aut}\,A)\,\bar{W}q\theta(t)=qt$, we have $qt=\pi_1\theta(t)\,t(B)$. By making use of the last equation, surjectivity of $t(B)$ and $\pi_1 p_1$, and exactness of $1\to\bar{W}A\to W(A;\mathrm{Aut}\,A)\to\mathrm{Aut}\,A\to 1$, the rest of the lemma is proved.

REMARK (1) When the fibre $\bar{W}A$ is replaced by $\bar{W}^n A$, similar lemma can be proved by similar arguments (see [8]).

REMARK (2). After Lemma 14 and Remark (1), a $K(A, n)$-fibration is called principal when it satisfies the following equivalent conditions; (1) it has the untwisted k-invariant, (2) the value of its twisted function is reduced to the sub simplicial group $K(A, n)$ (i.e. $\bar{W}^n A\subset\bar{W}^n A\times_{ev}\mathrm{Aut}\,A\cong EK(A, n)$), (3) it is fibre homotopy equivalent to a principal $K(A, n)$-fibration (bundle), (4) the fundamental group of its base space acts trivially on A.

Let $\bar{W}G\to E\to B$, $E=B\times_t\bar{W}G$, be a nilpotent fibration ([7]), which is classified as

$$\begin{array}{ccccc} \bar{W}G & \to & E & \xrightarrow{p_1} & B \\ \downarrow = & & \downarrow \tilde{\theta}(t) & & \downarrow \theta(t) \\ \bar{W}G & \to & W(W(G;\mathrm{Aut}\,G);\bar{W}G) & \to & \bar{W}W(G;\mathrm{Aut}\,G)\,. \end{array}$$

Comparing the homotopy long exact sequences of the diagram with the following commutative diagram which is analogous to that appeared in the proof of Lemma 14,

$$\begin{array}{ccc} B & \xrightarrow{t} & W(G;\mathrm{Aut}\,G) \\ \downarrow t(B) & & \downarrow rq \\ \pi_1 B & \xrightarrow{\pi_1\theta(t)} & \mathrm{Out}\,G\,, \end{array}$$

we see that the structure group is reduced to the subgroup $W(G;\Gamma)\subset W(G;\mathrm{Aut}\,G)$, $\Gamma=\mathrm{Im}\,\pi_1\tilde{\theta}(t)$. Then we have the reduced diagram

$$\begin{array}{ccccc} \bar{W}G & \to & E & \to & B \\ \downarrow = & & \downarrow \tilde{\theta}(t) & & \downarrow \theta(t) \\ \bar{W}G & \to & W(W(G;\Gamma);\bar{W}G) & \to & \bar{W}W(G;\Gamma)\,, \end{array}$$

and the homotopy long exact sequence of the lower fibration in the diagram becomes $1\to ZG\to G\to\Gamma\to\Gamma/\mathrm{Inn}\,G\to 1$. Naturality implies that this new "universal" fibration is nilpotent. If it admits a principal refinement our original fibration has the induced refinement and that would prove Theorem 4.

Let $G=G_0\supset G_1\supset\cdots\supset G_q\supset\cdots\supset G_N=\{1\}$ be a lower central Γ-series of the nilpotent Γ(or π_1E)-group $G=\pi_1\bar{W}G$, which satisfies that (i) G_q is normal and Γ-invariant, (ii) G_q/G_{q+1} is contained in $Z(G/G_{q+1})$ and (iii) the induced Γ-action on G_q/G_{q+1} is trivial. The series associates the following series of subgroups $W(G_0;\Gamma)\supset W(G_1;\Gamma)\supset\cdots\supset W(G_q;\Gamma)\supset\cdots\supset W(G_N;\Gamma)=\Gamma$.

Lemma 15. *$W(G_{q+1};\Gamma)$ is normal in $W(G_q;\Gamma)$ and there exists a natural isomorphism $W(G_q;\Gamma)/W(G_{q+1};\Gamma)\cong\bar{W}(G_q/G_{q+1})$.*

Proof. Define a simplicial map $W(G_q;\Gamma)\to\bar{W}(G_q/G_{q+1})$ by $(x;\alpha)=(x_0,x_1,\cdots,x_{n-1};\alpha)\to\bar{x}=(\bar{x}_0,\bar{x}_1,\cdots,\bar{x}_{n-1})$. If $(x;\alpha)(y;\beta)=(z_0,z_1,\cdots,z_{n-1};\alpha\beta)$ we have $\bar{z}_i=\bar{x}_i\,\bar{x}_{i+1}\cdots\bar{x}_{n-1}\,\alpha(\bar{y}_i)\,\bar{x}_{n-1}^{-1}\cdots\bar{x}_{i+1}^{-1}=\bar{x}_i\,\bar{y}_i$ by Theorem 1. Hence we find that the map is a homomorphism of simplicial groups, is epic and its kernel is equal to $W(G_{q+1};\Gamma)$.

Successive factorizations of the contractible space $WW(G;\Gamma)$ with these subgroups decompose the fibration $W(W(G;\Gamma);\bar{W}G)\cong WW(G;\Gamma)/\Gamma\to\bar{W}W(G;\Gamma)\cong WW(G;\Gamma)/W(G;\Gamma)$ into a series of simplicial sets $WW(G_0;\Gamma)/\Gamma\to WW(G_0;\Gamma)/W(G_{N-1};\Gamma)\to\cdots\to WW(G_0;\Gamma)/W(G_0;\Gamma)$.

Lemma 16. *Let H be a simplicial group, K a sub simplicial group of H and L a normal sub simplicial group of K. Then the canonical projection $WH/L\to WH/K$ is a principal K/L-fibration.*

Proof. The canonical projection $WH/L\to WH/K$ is identified with $W(H;H/L)\to W(H;H/K)$. The canonical right action $H/L\times K/L\to H/L$ and the isomorphism of orbit spaces $(H/L)/(K/L)\cong H/K$ induce the action $W(H;H/L)\times K/L\to W(H;H/L)$ and the isomorphism $W(H;H/L)/(K/L)\cong W(H;H/K)$.

By making use of these lemmas we find that every $\bar{W}(G_q/G_{q+1})\to WW(G;\Gamma)/W(G_{q+1};\Gamma)\to WW(G;\Gamma)/W(G_q;\Gamma)$ is a principal $\bar{W}(G_q/G_{q+1})$-fibration (bundle), and therefore is principal (see Remark (2) after Lemma 14). We obtained a principal refinement of $W(W(G;\Gamma);\bar{W}G)\to\bar{W}W(G;\Gamma)$, and the proof of Theorem 4 is completed.

References

[1] M. Artin and B. Mazur: Etale homotopy, Lecture Notes in Math. 100, Springer, Berlin, 1969.

[3] A.K. Bousfield and D.M. Kan: Homotopy limits, completions and localizations, Lecutre Notes in Math. 304, Springer, Berlin, 1972.

[3] P. Gabriel and M. Zisman: Calculus of fractions and homotopy theory, Springer, Berlin, 1967.

[4] D.H. Gottlieb: *On fibre spaces and the evaluation map*, Ann. of Math. (2) **87** (1968), 42–55.

[5] R.O. Hil, Jr.: *A geometric interpretation of a group cohomology obstruction*, Proc. Amer. Math. Soc. **54** (1976), 405–412.

[6] P. Hilton: Nilpotent actions on nilpotent groups, Lecture Notes in Math. 450, Springer, Berlin, 1975.
[7] P. Hilton, G. Mislin and J. Roitberg: Localization of nilpotnet groups and spaces, Math. Studies 15, North Holland, Amsterdam, 1975.
[8] Y. Hirashima: *A note on cohomology with local coefficients,* Osaka J. Math. **16** (1979), 219–231.
[9] D.M. Latch, R.W. Thomason, and W.S. Wilson: *Simplicial sets from categories,* Math. Z. **164** (1979), 195–214.
[10] S. MacLane: Homology, Die Grundlehren der Math. Wissenschaften 114, Springer, Berlin, 1963.
[11] J.P. May: Simplicial objects in algeblaic topology, Van Nostrand, Toront, 1967.
[12] J.P. May: Classifying spaces and fibrations, Mem. Amer. Math. Soc. No. 155, 1975.

Faculty of Engineering
Kansai University
Suita, Osaka 564, Japan