Kono, A. and Ōshima, H.
Osaka J. Math.
28 (1991), 129–139

CODEGREE OF SIMPLE LIE GROUPS-II

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(Received December 19, 1989)

0. Introduction

In [19] the n-th codegree (number) $\mathrm{cdg}(X, n)\in \boldsymbol{Z}$ and its stable version ${}^s\mathrm{cdg}(X, n)\in \boldsymbol{Z}$ were defined for every pair of a path-connected space X and a positive integer n. In [18], ${}^s\mathrm{cdg}_p(G, 3)$, the exponent of a prime p in ${}^s\mathrm{cdg}(G, 3)$, was determined for some simply connected simple Lie groups G. The purpose of this paper is to continue computing ${}^{(s)}\mathrm{cdg}(G, n)$ for some (G, n). We use notations in [19] and [18]. Our results are the following.

Theorem 1. *If* $r\geqq 3$, *then*

$$r\leqq {}^s\mathrm{cdg}_2(Spin(n), 3)\leqq r+1 \quad \textit{for} \quad 2^r\leqq n\leqq 2^r+6\,,$$
$${}^s\mathrm{cdg}_2(Spin(n), 3) = r+1 \quad \textit{for} \quad 2^r+7\leqq n\leqq 2^{r+1}-1\,.$$

Theorem 2. ${}^s\mathrm{cdg}_3(E_6, 3)={}^s\mathrm{cdg}_3(F_4, 3)=2$.

Theorem 3. (1) $\mathrm{cdg}(SU(3), 3)=2^2$ *and* $\mathrm{cdg}(SU(3), 5)={}^s\mathrm{cdg}(SU(3), n)=2$ *for* $n=3, 5$; $[SU(3), S^3]=\boldsymbol{Z}\oplus\boldsymbol{Z}_2$; $\{SU(3), S^3\}=\boldsymbol{Z}$; $[SU(3), S^5]=\{SU(3), S^5\}=\boldsymbol{Z}$.

(2) ${}^s\mathrm{cdg}(SU(4), 3)=2^2\cdot 3\,|\,\mathrm{cdg}(SU(4), 3)\,|\,2^5\cdot 3^2$; $\mathrm{cdg}(SU(4), 5)={}^s\mathrm{cdg}(SU(4), 5)=2$; $\mathrm{cdg}(SU(4), 7)={}^s\mathrm{cdg}(SU(4), 7)=2\cdot 3$.

(3) $\mathrm{cdg}(G_2, 11)={}^s\mathrm{cdg}(G_2, 11)={}^s\mathrm{cdg}(G_2, 3)=2^3\cdot 3\cdot 5\,|\,\mathrm{cdg}(G_2, 3)\,|\,2^5\cdot 3^2\cdot 5$.

(4) $\mathrm{cdg}(Spin(n), 11)=\mathrm{cdg}(SO(n), 11)=2^3\cdot 3\cdot 5$ *for* $n=7, 8$; $\mathrm{cdg}(SO(7)/SO(5), 11)=2^3$.

(5) $2^5\cdot 3\,|\,{}^s\mathrm{cdg}(Sp(3), 7)\,|\,\mathrm{cdg}(Sp(3), 7)\,|\,2^8\cdot 3$.

Proposition 4.

$$\begin{aligned}
&2^3\cdot 3^2\cdot 5\,|\,\mathrm{cdg}(Spin(9), 7)\,,\\
&2^4\cdot 3^2\cdot 5\,|\,\mathrm{cdg}(Spin(9), 11)\,,\\
&2^2\cdot 3^2\cdot 5\cdot 7\,|\,\mathrm{cdg}(Spin(9), 15)\,,\\
&2^3\cdot 3\,|\,\mathrm{cdg}(SU(5), 5)\,,\\
&2^2\cdot 3\,|\,\mathrm{cdg}(SU(5), 7)\,,\\
&2^7\cdot 3\cdot 5\cdot 7\,|\,\mathrm{cdg}(F_4, n) \quad \textit{for} \quad n = 11, 15\,,\\
&2^7\cdot 3^2\cdot 5\cdot 7\cdot 11\,|\,\mathrm{cdg}(F_4, 23)\,.
\end{aligned}$$

1. Proof of Theorem 1

Let $g\colon V_{2n-1}=SO(2n+1)/SO(2)\times SO(2n-1)\to\Omega\mathrm{Spin}(2n+1)$ be the generating map for $\mathrm{Spin}(2n+1)$ ($n\geqq 3$) (see [2]). Let $g'\colon \Sigma\Omega\mathrm{Spin}(2n+1)\to\mathrm{Spin}(2n+1)$ be the canonical map. Then $(g'\circ\Sigma g)_*\colon \pi_3(\Sigma V_{2n-1})\cong\pi_3(\mathrm{Spin}(2n+1))$, hence

$$\begin{aligned} &{}^s\mathrm{cdg}(V_{2n-1}, 2)\,|\,{}^s\mathrm{cdg}(\mathrm{Spin}(2n+1), 3)\,, \\ &{}^s\mathrm{cdg}^K(V_{2n-1}, 2)\,|\,{}^s\mathrm{cdg}^K(\mathrm{Spin}(2n+1), 3)\,. \end{aligned} \tag{1.1}$$

We will calculate the 2-components of these numbers.

The inclusions $U(n)\subset SO(2n)=SO(2n)\times I_1\subset SO(2n+1)$, $SO(2n+1)=SO(2n+1)\times I_2\subset SO(2n+3)$, and $U(n)=U(n)\times I_1\subset U(n+1)$ induce maps:

$$\begin{aligned} \sigma_n &\colon P(\boldsymbol{C}^n) = U(n)/U(1)\times U(n-1)\to V_{2n-1}\,, \\ \tau_n &\colon V_{2n-1}\to V_{2n+1}\,, \\ \tau'_n &\colon P(\boldsymbol{C}^n)\to P(\boldsymbol{C}^{n+1}) \end{aligned}$$

such that $\tau_n\circ\sigma_n=\sigma_n\circ\tau'_n$. Let L_n be the canonical complex line bundle over the complex projective $(n-1)$-space $P(\boldsymbol{C}^n)$, and let $a_n\in H^2(P(\boldsymbol{C}^n);\boldsymbol{Z})$ be the first Chern class of L_n. Then

$$\tau'^*_n(a_{n+1}) = a_n\,.$$

As is easily seen (e.g., [2]), we have

$$\begin{aligned} H^*(V_{2n-1};\boldsymbol{Z}) &= \boldsymbol{Z}[x_n, y_n]/(x_n^n-2y_n, y_n^2)\,, \\ \dim(x_n) = 2\,, &\quad \dim(y_n) = 2n\,, \\ \sigma_n^*(x_n) &= a_n\,, \\ \tau_n^*(x_{n+1}) &= x_n\,. \end{aligned}$$

Hence

$$\begin{aligned} \sigma_n^*&\colon H^i(V_{2n-1};\boldsymbol{Z})\cong H^i(P(\boldsymbol{C}^n);\boldsymbol{Z}) \qquad \text{for}\quad i\leqq 2n-2\,, \\ \tau_n^*&\colon H^i(V_{2n+1};\boldsymbol{Z})\cong H^i(V_{2n-1};\boldsymbol{Z}) \qquad \text{for}\quad i\leqq 2n-2\,, \end{aligned} \tag{1.2}$$

$$H^*(V_{2n-1};\boldsymbol{Q}) = \boldsymbol{Q}[x_n]/(x_n^{2n})\,,$$

$$\tau_n^*\colon H^i(V_{2n+1};\boldsymbol{Q})\cong H^i(V_{2n-1};\boldsymbol{Q}) \qquad \text{for}\quad i\leqq 4n-2\,. \tag{1.3}$$

Recall from Clarke [4] that

$$K(V_{2n-1}) = \boldsymbol{Z}[X_n, Y_n]/(X_n^n-2Y_n-X_nY_n, Y_n^2)\,. \tag{1.4}$$

Hence

$$K(V_{2n-1})\otimes\boldsymbol{Q} = \boldsymbol{Q}[X_n]/(X_n^{2n})\,.$$

By the construction of X_n ([4]), we have

$$\sigma_n^*(X_n) = L_n - 1 ,$$
$$\tau_n^*(X_{n+1}) = X_n .$$

The Chern character of X_n is given by

Lemma 1.5. $ch(X_n) = \exp(x_n) - 1$.

Proof. We have

$$\begin{aligned}\sigma_{2n}^*(ch(X_{2n})) &= ch(\sigma_{2n}^*(X_{2n})) \\ &= ch(L_{2n}-1) = \exp(a_{2n})-1 = \sigma_{2n}^*(\exp(x_{2n})-1) .\end{aligned}$$

Hence $ch(X_{2n}) \equiv \exp(x_{2n})-1 \bmod x_{2n}^{2n}$ by (1.2), thus $ch(X_n)=\exp(x_n)-1$ by (1.3). This proves 1.5.

Proposition 1.6. (1) ${}^s\mathrm{cdg}^K{}_2(V_{2n-1}, 2) = r$ *if* $2^{r-1} < n \leqq 2^r$.

(2) ${}^s\mathrm{cdg}_2(V_{2n-1}, 2) = r$ *if* $2^{r-1} < n < 2^r$,
$r \leqq {}^s\mathrm{cdg}_2(V_{2n-1}, 2) \leqq r+1$ *if* $n = 2^r$.

Proof. Put $D = {}^s\mathrm{cdg}(V_{2n-1}, 2)$. Let f: $V_{2n-1} \to S^2$ be a stable map such that the induced homomorphism f_*: ${}^s\pi_2(V_{2n-1}) = \boldsymbol{Z} \to {}^s\pi_2(S^2) = \boldsymbol{Z}$ is multiplication by D. Let $\beta \in \tilde{K}(S^2) = \boldsymbol{Z}$ be a generator. For simplicity, we set $X = X_n$, $Y = Y_n$ and $x = x_n$. Set

$$f^*(\beta) = \textstyle\sum_{1\leq i<n} a_i X^i + Y \cdot \sum_{0\leq i<n} b_i X^i = \sum_{1\leq i<2n} a_i X^i$$

in $\tilde{K}(V_{2n-1}) \otimes \boldsymbol{Q}$, where $a_i \in \boldsymbol{Z}(1 \leqq i < n)$, $b_i \in \boldsymbol{Z}(0 \leqq i < n)$, and $a_i \in \boldsymbol{Q}(n \leqq i < 2n)$. Then

$$\begin{aligned}D \cdot \textstyle\sum_{i\geq 1}((-1)^{i-1}/i)\,(e^x-1)^i &= D \cdot \log(e^x - 1 + 1) \\ = D \cdot x = f^* ch(\beta) &= ch(f^*(\beta)) = \textstyle\sum_{i\geq 1} a_i (e^x-1)^i .\end{aligned}$$

Hence $a_i = D \cdot (-1)^{i-1}/i$ $(1 \leqq i < 2n)$. We then have

$$\begin{aligned}f^*(\beta) &= \textstyle\sum_{1\leq i<n} a_i X^i + (2Y + XY) \sum_{n\leq i<2n} a_i X^{i-n} \\ &= \textstyle\sum_{1\leq i<n} (D(-1)^{i-1}/i)\, X^i + Y \cdot 2D(-1)^{n-1}/n \\ &\quad + Y \cdot \textstyle\sum_{n\leq i\leq 2n-2} \{D(-1)^{i-1}/i + 2D(-1)^i/(i+1)\}\, X^{i-n+1} .\end{aligned}$$

Thus

(1.7) D/i $(1 \leqq i < n)$, $2D/n$, and $D/i - 2D/(i+1)$ $(n \leqq i \leqq 2n-2)$ are in $\boldsymbol{Z}$.

Let $r \geqq 1$ be an integer such that $2^{r-1} < n \leqq 2^r$. Then the relation $D/2^r - 2D/(2^r+1) \in \boldsymbol{Z}$ implies that $2^r \mid D$. Conversely, if $2^r \mid D$, then (1.7) with $\boldsymbol{Z}$ replaced by its 2-localized ring $\boldsymbol{Z}_{(2)}$ holds. Therefore ${}^s\mathrm{cdg}^K{}_2(V_{2n-1}, 2) = r$. This proves (1).

A map $V_{2n-1}\to K(\boldsymbol{Z}, 2)$ which represents x_n factorizes as $V_{2n-1}\to P(\boldsymbol{C}^{2n})\subset K(\boldsymbol{Z}, 2)$. Hence ${}^s\mathrm{cdg}(V_{2n-1}, 2)\,|\,{}^s\mathrm{cdg}(P(\boldsymbol{C}^{2n}), 2)$ so that $r={}^s\mathrm{cdg}^K{}_2(V_{2n-1}, 2)\leqq{}^s\mathrm{cdg}_2(V_{2n-1}, 2)\leqq{}^s\mathrm{cdg}_2(P(\boldsymbol{C}^{2n}), 2)$. By [18], we have ${}^s\mathrm{cdg}_2(P(\boldsymbol{C}^{2n}), 2)={}^s\mathrm{cdg}_2(SU(2n), 3)$ which is $r+1$ or r according as $n=2^r$ or $2^{r-1}<n<2^r$. Hence we have (2).

Corollary 1.8. *${}^s\mathrm{cdg}^K{}_2(Spin(2n+1), 3)\geqq r$ if $2^{r-1}<n\leqq 2^r$.*

Proof. This follows from 1.1 and 1.6.

Proof of Theorem 1. The complexification induces isomorphisms of representation rings:

$$RO(Spin(m)) \cong R(Spin(m)) \quad \text{if} \quad m\equiv 0,1,7 \bmod 8$$

(see [7, p. 193]). By the proof of [18, 4.4], we then have

$${}^s\mathrm{cdg}^{KO}(Spin(m), 3) = 2\cdot{}^s\mathrm{cdg}^K(Spin(m), 3) \quad \text{if} \quad m\equiv 0,1,7 \bmod 8\,.$$

Thus, by 1.8, we have

$${}^s\mathrm{cdg}_2(Spin\,(2n+1), 3)\geqq r+1 \quad \text{if} \quad n\equiv 0,3 \bmod 4 \text{ and } 2^{r-1}<n\leqq 2^r\,.$$

On the other hand, if $n\geqq 2$, then the canonical homomorphism

$$\boldsymbol{Z} = \pi_3(Spin(2n+1)) \to \pi_3(SO(2n+1)) \to \pi_3(SU(2n+1)) = \boldsymbol{Z}$$

is multipliaction by 2, so that

$${}^s\mathrm{cdg}_2(Spin(2n+1), 3)\leqq 1+{}^s\mathrm{cdg}_2(SU(2n+1), 3)\,.$$

The latter number is $r+2$ or $r+1$ according as $n=2^r$ or $2^{r-1}<n<2^r$, by [18]. Hence

$${}^s\mathrm{cdg}_2(Spin(2n+1), 3) = r+1 \quad \text{if} \quad n\equiv 0,\,3 \bmod 4 \text{ and } 2^{r-1}<n<2^r\,.$$

In particular, if $r\geqq 3$, then ${}^s\mathrm{cdg}_2(Spin(2^r-1), 3)=r$ and ${}^s\mathrm{cdg}_2(Spin(2^r+7), 3)={}^s\mathrm{cdg}_2(Spin(2^{r+1}-1), 3)=r+1$. Hence, if $r\geqq 3$, then

$$\begin{aligned} &r\leqq{}^s\mathrm{cdg}_2(Spin(n), 3)\leqq r+1 \quad \text{for} \quad 2^r\leqq n\leqq 2^r+6\,,\\ &{}^s\mathrm{cdg}_2(Spin(n), 3) = r+1 \quad \text{for} \quad 2^r+7\leqq n\leqq 2^{r+1}-1\,. \end{aligned}$$

This proves Theorem 1.

2. Proof of Theorem 2

The relations ${}^s\mathrm{cdg}_3(E_6, 3)={}^s\mathrm{cdg}_3(F_4, 3)\geqq 2$ were proved in [18]. We will prove ${}^s\mathrm{cdg}_3(F_4, 3)\leqq 2$. By [6] and [12], there exist a mod 3 H-space X of dimension 26 and a mod 3 homotopy equivalence

$$F_4 \simeq_3 X \times B_5(3)$$

where $B_5(3)$ is the total space of an S^{11}-bundle over S^{15} [15]. It follows from [3] that the top cell of the localized space $X_{(3)}$ splits off stably, that is, $X \simeq_3 X^{(23)} \vee S^{26}$ (stably), where $X^{(23)}$ is the 23-skeleton of X, and it follows from [5] that $X^{(23)}{}_{(3)}$ is stably homotopy equivalent to $X_1 \vee X_2$ where X_2 is 17-connected and $H^*(X_1; \boldsymbol{Z}_3) = \boldsymbol{Z}_3\{1, x_3, x_7, x_8, x_{18}, x_{19}, x_{23}\}$ such that $\dim(x_i) = i$, $\mathcal{P}^1 x_3 = x_7$, $\beta x_7 = x_8$, $\beta x_{18} = x_{19}$, and $\mathcal{P}^1 x_{19} = x_{23}$.

Lemma 2.1. $X_1 = S^3{}_{(3)} \cup e^7{}_{(3)} \cup e^8{}_{(3)} \cup e^{18}{}_{(3)} \cup e^{19}{}_{(3)} \cup e^{23}{}_{(3)}$.

In the rest of this section we work in the stable homotopy category of mod 3 local spaces. For simplicity we omit the notation "(3)".

Proof of 2.1. Let

$$S^3 \to X_1 \to Y_1 \to \Sigma S^3 \to \Sigma X_1$$

be a cofibre sequence such that Y_1 is 6-connected. Then $X_1 = S^3 \cup C\Sigma^{-1} Y_1$. Inductively we have cofibre sequences

$$\begin{aligned}
S^7 &\to Y_1 \to Y_2 \to \Sigma S^7 \to \Sigma Y_1\,, \\
S^8 &\to Y_2 \to Y_3 \to \Sigma S^8 \to \Sigma Y_2\,, \\
S^{18} &\to Y_3 \to Y_4 \to \Sigma S^{18} \to \Sigma Y_3\,, \\
S^{19} &\to Y_4 \to Y_5 \to \Sigma S^{19} \to \Sigma Y_4
\end{aligned}$$

and

$$\begin{aligned}
Y_1 &= S^7 \cup C\Sigma^{-1} Y_2\,, \\
Y_2 &= S^8 \cup C\Sigma^{-1} Y_3\,, \\
Y_3 &= S^{18} \cup C\Sigma^{-1} Y_4\,, \\
Y_4 &= S^{19} \cup C\Sigma^{-1} Y_5 = S^{19} \cup e^{23}
\end{aligned}$$

where the last equality follows from the fact that $Y_5 = S^{23}$. Therefore we have

$$X_1 = S^3 \cup C(S^6 \cup C(S^6 \cup C(S^{15} \cup C(S^{15} \cup e^{19})))) \,.$$

This proves 2.1.

Proof of Theorem 2. Put $Y = S^3 \cup e^7 \cup e^8 \cup e^{18} \cup e^{19} \cup e^{23}$. Then ${}^s\mathrm{cdg}_3(F_4, 3) = {}^s\mathrm{cdg}_3(X, 3) = {}^s\mathrm{cdg}_3(Y, 3)$. Let $\alpha_i \in {}^s\pi_{4i-1}(S^0)$ $(1 \leqq i \leqq 5)$ be the element of order 3 defined in [20, p. 178]. Let $\alpha_i'\colon S^{4i-1} \cup_3 e^{4i} \to S^0$ and $\alpha_1''\colon S^4 \to S^0 \cup_3 e^1$ be an extension of α_i and a coextension of α_1 respectively. The 8-skeleton $Y^{(8)}$ of Y is equivalent to the mapping cone $C(\Sigma^3 \alpha_1') = S^3 \cup C(S^6 \cup_3 e^7)$ and $Y/Y^{(8)}$ is equivalent to $C(\Sigma^{15} \alpha_1'') = S^{18} \cup_3 e^{19} \cup e^{23}$. Hence we have a cofibre sequence

$$C(\Sigma^3 \alpha_1') \to Y \to C(\Sigma^{15} \alpha_1'') \xrightarrow{h} C(\Sigma^4 \alpha_1') \xrightarrow{k} \Sigma Y .$$

Let g: $C(\Sigma^4 \alpha_1')\to S^4$ be an extension of 3: $S^4\to S^4$. As is easily seen, we have an exact sequence

$$\begin{aligned} {}^s\pi^3(S^{18} \cup_3 e^{19}) = \boldsymbol{Z}_3\{\alpha_4'\} &\xrightarrow{\alpha_1''^*} {}^s\pi^3(S^{22}) = \boldsymbol{Z}_3\{\alpha_5\} \\ &\to {}^s\pi^4(S^{18}\cup_3 e^{19} \cup e^{23}) \to {}^s\pi^4(S^{18}\cup_3 e^{19}) = \boldsymbol{Z}_3 \to 0 . \end{aligned}$$

Since $\alpha_1''^*(\alpha_4')\in\langle\alpha_4, 3, \alpha_1\rangle=\alpha_5$, it follows that ${}^s\pi^4(S^{18}\cup_3 e^{19}\cup e^{23})=\boldsymbol{Z}_3$ and $3g\circ h=0$. Hence there exists a map r: $\Sigma Y\to S^4$ such that $r\circ k=3g$, so that r has degree 9 on the bottom sphere S^4 and ${}^s\mathrm{cdg}_3(Y)\leqq 2$. Hence ${}^s\mathrm{cdg}(F_4, 3)=2$ as desired.

3. Proof of Theorem 3

Lemma 3.1. *Given an integer $n\geqq 2$ and a connected finite CW-complex X such that*

X and its $(n-1)$-skeleton Y are simply connected;

$\pi_{n-1}(X) = 0$;

$\pi_{n-1}(Y) = \boldsymbol{Z}_m$;

$\mathrm{rank}(\pi_n(X/Y)) = \mathrm{rank}(\pi_n(X)) = 1$,

then we have $\mathrm{Cdg}(X, n)\subset m\cdot\mathrm{Hom}(\pi_n(X), \pi_n(S^n))$. *If moreover* Cdg *is surjective for $(X/Y, n)$, then* $\mathrm{Cdg}(X, n)=m\cdot\mathrm{Hom}(\pi_n(X), \pi_n(S^n))$.

Proof. By a theorem of Blakers-Massey, the collapsing map induces an isomorphism c_*: $\pi_n(X, Y)\cong\pi_n(X/Y)$. From the assumptions and the homotopy exact sequence of the pair (X, Y), it follows that c_*: $\pi_n(X)/\mathrm{Tor}=\boldsymbol{Z}\to\pi_n(X/Y)=\boldsymbol{Z}$ is multiplication by m. Hence the assertion follows from the commutative diagram

$$\begin{array}{lcl} [X/Y, S^n] & \to & \mathrm{Hom}(\pi_n(X/Y), \pi_n(S^n)) = \boldsymbol{Z} \\ c^* \downarrow & & \downarrow c_*{}^* = m \\ [X, S^n] & \to & \mathrm{Hom}(\pi_n(X), \pi_n(S^n)) = \boldsymbol{Z} \end{array}$$

since c^* is surjective.

Proof of Theorem 3(1). We shall use notations and results in [20]. The group $SU(3)$ has a cell structure: $SU(3)=S^3\cup_{\eta_3} e^5\cup_f e^8$. As noticed in [14, p. 475], we have $\Sigma f=j_*(\nu_4\circ\eta_7)$ from [9, 3.1], where j: $S^4\subset S^4\cup e^6$. Let h: $S^3\cup e^5\to S^3$ be a map having degree 2 on S^3. We have

$$\begin{aligned} 2\iota_4\circ\nu_4 &= 2\nu_4+[\iota_4, \iota_4], \quad \text{by [22, XI]} , \\ &= 4\nu_4-\Sigma\nu', \quad \text{by [20, p. 43]} . \end{aligned}$$

Hence $\Sigma(h\circ f) = 2\iota_4\circ\nu_4\circ\eta_7=-\Sigma\nu'\circ\eta_7=\Sigma(\nu'\circ\eta_6)$, so $h\circ f=\nu'\circ\eta_6$, since Σ is injective on $\pi_7(S^3)=\boldsymbol{Z}_2\{\nu'\circ\eta_6\}$. It follows that $2\iota_3\circ h$ can be extended to $SU(3)$ and that $\mathrm{cdg}(SU(3), 3)=2^2$. By the same method as [18, 4.3(1)], we can prove that $\mathrm{cdg}(SU(3), 5)={}^s\mathrm{cdg}(SU(3), n)=2$ for $n=3, 5$. By applying the functor $[\ , S^3]$ to the cofibre sequence $S^7\to S^3\cup e^5\subset SU(3)$, we have $[SU(3), S^3]=\boldsymbol{Z}\oplus\boldsymbol{Z}_2$. Other assertions of (1) can be proved easily.

Proof of Theorem 3(2). By [13] there is a stable homotopy equivalence $SU(4)\simeq\Sigma P(\boldsymbol{C}^4)\vee Y$, where Y is a 7-connected finite CW-complex. Hence, for $n=3, 5, 7$, we have ${}^s\mathrm{cdg}(SU(4), n)={}^s\mathrm{cdg}(P(\boldsymbol{C}^4), n-1)$ which can be easily determined by using the cell structure of $P(\boldsymbol{C}^4)$ (see [17, 1.15]).

By using a cell structure of $SU(4)$ and known structures of $\pi_*(S^3)$ and ${}^s\pi_{14}(S^3)$ (see [20]), we can construct a map $SU(4)\to S^3$ which has degree $2^5\cdot 3^2$ on S^3, hence $\mathrm{cdg}(SU(4), 3)\,|\,2^5\cdot 3^2$. On the other hand we have $2^2\cdot 3={}^s\mathrm{cdg}(SU(4), 3)\,|\,\mathrm{cdg}(SU(4), 3)$ by [18, 3.3] and [19, 3.4(3)].

Taking $(X, n)=(SU(4), 5)$ in 3.1, we have $\mathrm{Cdg}(SU(4), 5)\subset 2\boldsymbol{Z}$. By the homotopy exact sequence of the principal Sp(2)-bundle $q\colon SU(4)\to S^5$, we have $\mathrm{Cdg}(q)=2$, so $\mathrm{cdg}(SU(4), 5)=2$. By [19, 4.2(1)], $\mathrm{cdg}(SU(4), 7)=6$. This completes the proof of (2).

Proof of Theorem 3(3). Using a stable duality map $S^{14}\to G_{2+}\wedge G_{2+}$ (see [3]), we can prove ${}^s\mathrm{cdg}(G_2, 11)={}^s\mathrm{cdg}(G_2, 3)$ which is $2^3\cdot 3\cdot 5$ as proved in [18]. The group G_2 has a cell structure: $G_2=S^3\cup e^5\cup e^6\cup e^8\cup e^9\cup e^{11}\cup e^{14}$. Hence $G_2/Y=S^{11}\vee S^{14}$, where Y is the 9-skeleton of G_2, so Cdg is surjective for $(G_2/Y, 11)$. Consider the exact sequence:

$$\pi_{11}(G_2)=\boldsymbol{Z}\oplus\boldsymbol{Z}_2\to\pi_{11}(G_2, Y)=\boldsymbol{Z}\to\pi_{10}(Y)\to\pi_{10}(G_2)=0\,.$$

By [16, 4.2], $\pi_{10}(Y)=\boldsymbol{Z}_{120}$. Taking $(X, n)=(G_2, 11)$ in 3.1, we have $m=2^3\cdot 3\cdot 5$ and $\mathrm{cdg}(G_2, 11)=2^3\cdot 3\cdot 5$. Since Sq^2 is trivial on $H^6(G_2; \boldsymbol{Z}_2)$ by [1], the attaching map of the 8-dimensional cell of G_2 factorizes as $S^7\to S^3\cup e^5\subset S^3\cup e^5\cup e^6$. Using this fact and the additive structures of $\pi_*(S^3)$ and ${}^s\pi_{13}(S^3)$ (see [20]), we can construct a map $G_2\to S^3$ which has degree $2^5\cdot 3^2\cdot 5$ on S^3, hence $\mathrm{cdg}(G_2, 3)\,|\,2^5\cdot 3^2\cdot 5$. This proves (3).

Proof of Theorem 3(4). Applying $\pi_*(\ \ )$ to the following commutative diagram

$$\begin{array}{ccccccc} & & SO(5) & \subset & SO(6) & & \\ & & \downarrow & & \downarrow & & \\ & & SO(7) & = & SO(7) & & \\ & & p\downarrow & & \downarrow & & \\ S^5 & \subset & V_{7,2} & = & SO(7)/SO(5) & \to & S^6 \end{array}$$

we have $p_*=15\colon \pi_{11}(SO(7))/\mathrm{Tor}=\boldsymbol{Z}\to\pi_{11}(V_{7,2})/\mathrm{Tor}=\boldsymbol{Z}$. Hence

$$\mathrm{cdg}(SO(7), 11) \mid 3\cdot 5\cdot \mathrm{cdg}(V_{7,2}, 11).$$

The space $V_{7,2}$ has a cell structure: $V_{7,2}=S^5\cup e^6\cup e^{11}$, Let q: $V_{7,2}\to S^{11}$ be the collapsing map. By [8, 2.8 and 2.9], we have an exact sequence $\pi_9(S^5)=\boldsymbol{Z}_2\to \pi_{10}(S^5\cup e^6, S^5)\to\pi_5(S^5\cup e^6)=\boldsymbol{Z}_2\to 0$, hence the order of $\pi_{10}(S^5\cup e^6, S^5)$ is at most 4, so that the order of $\pi_{10}(S^5\cup e^6)$ is at most 8 because $\pi_{10}(S^6)=\boldsymbol{Z}_2$ by [20]. Since $\pi_{10}(V_{7,2})=0$, it then follows from the homotopy exact sequence of the pair $(V_{7,2}, S^5\cup e^6)$ that $\mathrm{Coker}[q_*\colon \pi_{11}(V_{7,2})\to\pi_{11}(S^{11})]\cong\pi_{10}(S^5\cup e^6)$, so that the last group is cyclic. Hence by the commutative diagram

$$\begin{array}{ccc}
[S^{11}, S^{11}] & \to & \mathrm{Hom}(\pi_{11}(S^{11}), \pi_{11}(S^{11})) = \boldsymbol{Z} \\
q^*\downarrow & & \downarrow q_*^* \\
[V_{7,2}, S^{11}] & \to & \mathrm{Hom}(\pi_{11}(V_{7,2}), \pi_{11}(S^{11})) = \boldsymbol{Z}
\end{array}$$

$\mathrm{cdg}(V_{7,2}, 11)$ is the order of $\pi_{10}(S^5\cup e^6)$, since q^* is surjective. Thus $\mathrm{cdg}(V_{7,2}, 11) \mid 2^3$. On the other hand, $\mathrm{cdg}(G_2, 11)\mid\mathrm{cdg}(Spin(7), 11)$ by use of the principal G_2-bundle $Spin(7)\to S^7$. Hence $2^3\cdot 3\cdot 5=\mathrm{cdg}(G_2, 11)\mid\mathrm{cdg}(Spin(7), 11)\mid\mathrm{cdg}(SO(7), 11)\mid 3\cdot 5\cdot\mathrm{cdg}(V_{7,2}, 11)\mid 2^3\cdot 3\cdot 5$, therefore these numbers are equal and $\mathrm{cdg}(V_{7,2}, 11)=2^3$. Then [19, 3.7(2) and 4.4] complete the proof of (4).

Proof of Theorem 3(5). The groups Sp(2) and Sp(3) have cell structures: $Sp(2)=S^3\cup_g e^7\cup e^{10}$ and $Sp(3)=S^3\cup_g e^7\cup e^{10}\cup e^{11}\cup e^{14}\cup e^{18}\cup e^{21}$. They contain quasi-projective spaces $Q_2=S^3\cup_g e^7$ and $Q_3=S^3\cup_g e^7\cup e^{11}$ respectively. The inclusions induce isomorphisms $\pi_7(Q_2)\cong\pi_7(Q_3)\cong\pi_7(Sp(3))$. In $Q_{3,2}=Q_3/S^3=S^7\cup_h e^{11}$, h has order 8 (see [10, p. 38]). Hence the homomorphisms $\{Q_{3,2}, S^7\}=\boldsymbol{Z}\to\{S^7, S^7\}=\boldsymbol{Z}$ and $\{Q_3, S^7\}=\boldsymbol{Z}\to\{Q_2, S^7\}=\boldsymbol{Z}$ induced by the inclusions are multiplications by 8. Let t be the order of the cokernel of the stabilization $\pi_7(Sp(3))=\boldsymbol{Z}\to{}^s\pi_7(Sp(3))=\boldsymbol{Z}$. Consider the following commutative diagram.

$$\begin{array}{ccc}
[Sp(3), S^7] & \to & \mathrm{Hom}(\pi_7(Sp(3)), \pi_7(S^7)) = \boldsymbol{Z} \\
\downarrow & & \uparrow t \\
\{Sp(3), S^7\} & \to & \mathrm{Hom}({}^s\pi_7(Sp(3)), {}^s\pi_7(S^7)) = \boldsymbol{Z} \\
\downarrow & & \downarrow\cong \\
\{Q_3, S^7\} = \boldsymbol{Z} & \to & \mathrm{Hom}({}^s\pi_7(Q_3), {}^s\pi_7(S^7)) = \boldsymbol{Z} \\
2^3\downarrow & & \downarrow\cong \\
\{Q_2, S^7\} = \boldsymbol{Z} & \to & \mathrm{Hom}({}^s\pi_7(Q_2), {}^s\pi_7(S^7)) = \boldsymbol{Z}
\end{array}$$

By the proof of [19, 4.3(1)], the bottom homomorphism is multiplication by $2^2\cdot 3$. Hence

$$t\cdot 2^5\cdot 3\mid t\cdot{}^s\mathrm{cdg}(Sp(3), 7)\mid\mathrm{cdg}(Sp(3), 7).$$

Let p: $Sp(3)\to X_{3,2}=Sp(3)/Sp(1)$ be the canonical fibration. Since $\mathrm{Coker}[p_*\colon \pi_7(Sp(3))\to\pi_7(X_{3,2})]\cong\pi_6(S^3)=\boldsymbol{Z}_{12}$, we have $\mathrm{cdg}(Sp(3), 7)\mid 2^2\cdot 3\cdot\mathrm{cdg}(X_{3,2}, 7)$. The space $X_{3,2}$ has a cell structure: $X_{3,2}=S^7\cup_h e^{11}\cup_v e^{18}$. Since the order of h

is 8, there is a map $u: Q_{3,2}\to S^7$ which has degree 8 on S^7. Since $u\circ v\in\pi_{17}(S^7)=\boldsymbol{Z}_{24}\oplus\boldsymbol{Z}_2$ (see [20]), $2^3\cdot 3u$ can be extended to a map $X_{3,2}\to S^7$ which has degree $2^6\cdot 3$ on S^7. Therefore $\mathrm{cdg}(X_{3,2}, 7)\,|\,2^6\cdot 3$ and so $\mathrm{cdg}(Sp(3), 7)\,|\,2^8\cdot 3^2$. On the other hand, $\mathrm{cdg}_3(Sp(3), 7)=\mathrm{cdg}_3(\mathrm{Spin}(7), 7)=1$ by [19, 4.2(2) and 4.3(2)]. Hence $\mathrm{cdg}(Sp(3), 7)\,|\,2^8\cdot 3$. This proves (5) and completes the proof of Theorem 3.

4. Proof of Proposition 4

By [19, 2.4], we have

Proposition 4.1 (cf., [19, 3.15(2)]). *If n is odd, $\pi_n(X)=\boldsymbol{Z}\{s\}\oplus\mathrm{Tor}$ and $H^n(X;\boldsymbol{Z})=\boldsymbol{Z}\{x_n\}\oplus\mathrm{Tor}$, then there exists a map $f\colon X\to S^n$ such that $\mathrm{cdg}(X, n)=\deg(f_* s)=|AB|$, where integers A and B are defined by $s^*(x_n)=A[S^n]$ and $f^*[S^n]\equiv Bx_n \bmod \mathrm{Tor}$ respectively. (Here $[S^n]$ is a generator of $H^n(S^n;\boldsymbol{Z})$.) Stable version also holds.*

Hence the next result proves Proposition 4.

Lemma 4.2. *In 4.1, $(|A|, |B|)$ is equal to*

$$\begin{array}{lll}
(3, 2^3\cdot 3\cdot 5\cdot y_1) & \textit{for} & (Spin(9), 7)\,, \\
(2^3\cdot 3\cdot 5, 2\cdot 3\cdot y_2) & \textit{for} & (Spin(9), 11)\,, \\
(2^2\cdot 3^2\cdot 5\cdot 7, y_3) & \textit{for} & (Spin(9), 15)\,, \\
(2, 2^2\cdot 3\cdot y_4) & \textit{for} & (SU(5), 5)\,, \\
(2\cdot 3, 2\cdot y_5) & \textit{for} & (SU(5), 7)\,, \\
(2^3\cdot 5, 2^4\cdot 3\cdot 7\cdot y_6) & \textit{for} & (F_4, 11)\,, \\
(2^3\cdot 3\cdot 7, 2^4\cdot 5\cdot y_7) & \textit{for} & (F_4, 15)\,, \\
(2^7\cdot 3^2\cdot 5\cdot 7\cdot 11, y_8) & \textit{for} & (F_4, 23)
\end{array}$$

for some integers y_i. In these cases, $B/y_i\,|\,{}^s\mathrm{cdg}(G, n)$.

Proof. We prove the assertion only for $(F_4, 11)$, because others can be proved similarly. Assertion about A has been known (for example, see [11]). Recall that the type of F_4 is (3, 11, 15, 23). Let $x_n\in H^n(F_4;\boldsymbol{Z})$ be a generator of the free part for $n\in\{3, 11, 15, 23\}$ (see [1]). Consider the commutative diagram:

$$\begin{array}{ccc}
K^1(S^{11}) \overset{ch}{\cong} H^{11}(S^{11};\boldsymbol{Z}) & \subset & H^*(S^{11};\boldsymbol{Q}) \\
f^*\downarrow & & \downarrow f^* \\
K^1(F_4) & \xrightarrow[ch]{\qquad\qquad} & H^*(F_4;\boldsymbol{Q})
\end{array}$$

As is well-known, $K^*(F_4)$ is an exterior algebra generated by some elements

$\beta_1, \cdots, \beta_4 \in K^1(F_4)$ whose Chern characters were determined in [21]. Let g be a generator of $K^1(S^{11})$. Let ${}^K\mathrm{Dec}$ and ${}^H\mathrm{Dec}$ be the groups of decomposable elements with respect to $\{\beta_1, \cdots, \beta_4\}$ and $\{x_3, x_{11}, x_{15}, x_{23}\}$, respectively. Express $f^*(g) \equiv \Sigma a_i \beta_i \bmod {}^K\mathrm{Dec}$ and $\Sigma a_i\, ch(\beta_i) = P_3 x_3 + P_{11} x_{11} + P_{15} x_{15} + P_{23} x_{23}$, where P_i is a polynomial of $a_1, \cdots, a_4$ with rational coeffiients. Then $ch(f^*(g)) \equiv \Sigma P_i x_i \bmod {}^H\mathrm{Dec}$. On the other hand, $ch(f^*(g)) = f^* ch(g) = \pm f^*[S^{11}] \in H^{11}(F_4; \boldsymbol{Z})$. Hence $P_{11} = \pm B$ and $P_3 = P_{15} = P_{23} = 0$, then $B = |P_{11}| = 2^4 \cdot 3 \cdot 7 \cdot |a_3|$ by elementary calculation. This argument can also be applied to stable case. By setting $y_6 = |a_3|$, we have the desired result.

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