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Osaka J. Math.
28 (1991), 117–127

$SK_1(Z[G])$ OF FINITE SOLVABLE GROUPS WHICH ACT LINEARLY AND FREELY ON SPHERES

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(Received February 20, 1990)

1. Introduction

Let G be a finite group, $\boldsymbol{Z}$ the ring of integers and $\boldsymbol{Q}$ the ring of rational numbers. For $R=\boldsymbol{Z}$ or $\boldsymbol{Q}$, $R[G]$ denotes the group ring of G over R. Put $GL(R[G])=\varinjlim GL_n(R[G])$ and $E(R[G])=[GL(R([G]), GL(R[G])]$ the commutator subgroup of $GL(R[G])$. Then $K_1(R[G])$ denotes the quotient group $GL(R[G])/E(R[G])$. The natural inclusion map i: $GL(\boldsymbol{Z}[G])\to GL(\boldsymbol{Q}[G])$ gives rise to a group homomorphism i_*: $K_1(\boldsymbol{Z}[G])\to K_1(\boldsymbol{Q}[G])$. Then $SK_1(\boldsymbol{Z}[G])$ is defined by setting

$$SK_1(\boldsymbol{Z}[G]) = \ker i_* .$$

In [9], C.T.C. Wall showed that $SK_1(\boldsymbol{Z}[G])$ is isomorphic the torsion subgroup of the Whitehead group $Wh(G)$ of G. Since it can be shown that

$$SK_1(\boldsymbol{Z}[G]) = \ker(\mathrm{Res}\colon Wh(G) \to \bigoplus_{C\in\mathcal{C}} Wh(C)),$$

$SK_1(\boldsymbol{Z}[G])$ gives information which cannot be obtained by restricting $Wh(G)$ to $\bigoplus_{C\in\mathcal{C}} Wh(C)$, where $\mathcal{C}$ is the class of all cyclic subgroups of G.

Incidentally, Whitehead group plays a role not only in studying simple homotopy equivalences of finite CW complexes, but also in classifying manifolds. The s-cobordism theorem says that if M and N are smooth closed n-dimensional manifolds, where $n\geq 5$, and if W is a compact $(n+1)$-dimensional manifold such that $\partial W=M\coprod N$, and such that the inclusions $M\to W$ and $N\to W$ are simple homotopy equivalences, then W is diffeomorphic to $M\times[0, 1]$ (see [5]).

For a finite group G, $SK_1(\boldsymbol{Z}[G])$ has been calculated by several authors. Let $\boldsymbol{Z}_m$ be a cyclic group of order m. At first, it was shown by Bass, Milnor, and Serre ([1]) that $SK_1(\boldsymbol{Z}[G])=0$ if G is cyclic or if $G\cong(\boldsymbol{Z}_2)^n$ for some n. Also, it was shown by T.Y. Lam ([3]) that $SK_1(\boldsymbol{Z}[G])=0$ if $G\cong\boldsymbol{Z}_{p^n}\times\boldsymbol{Z}_p$ for any prime p and any n. Later, it was shown by R. Oliver ([8]) that for a finite abelian group G, $SK_1(\boldsymbol{Z}[G])=0$ if and only if either $G\cong(\boldsymbol{Z}_2)^n$, or each Sylow subgroup of G has the form $\boldsymbol{Z}_{p^n}$ or $\boldsymbol{Z}_{p^n}\times\boldsymbol{Z}_p$. As far as non-abelian groups are concerned, it was shown in [2], [4], [6] and [7] that $SK_1(\boldsymbol{Z}[G])$ vanishies if G is a dihedral group.

The purpose of this paper is to determine $SK_1(\boldsymbol{Z}[G])$ for finite solvable groups G which act linearly and freely on spheres. As in [10, Theorem 6.1.11], there are 4 types for such kinds of groups. For the convenience of the reader, the table of these groups are cited in Appendix. In order to state our main theorem, we prepare the following notations.

Let G_1, G_2, G_3 and G_4 denote the groups of type I, II, III and IV respectively mentioned in the table in Appendix. Let $(a_1, a_2, \cdots, a_\lambda)$ denote the greatest common divisor of integers $\{a_1, a_2, \cdots, a_\lambda\}$, and let m, n, r, l, k, u, v and d be the integers appeared in the definition of G_1, G_2, G_3 and G_4. For positive integers α, β, γ and δ, put

$$\begin{aligned}
M_\beta &= (r^\beta - 1, m)\,, \\
D(\alpha) &= \{x \in \boldsymbol{N} \mid x \text{ is a divisor of } \alpha\}\,, \\
D(\alpha, \beta) &= \{x \in D(\alpha) \mid x \text{ can be divided by } \beta\}\,, \\
D(\alpha)_\gamma^\delta &= \{x \in D(\alpha) \mid x\gamma \equiv 0\,(\delta)\}\,.
\end{aligned}$$

If d is an even integer, we put $d' = d/2$, and put

$$\begin{aligned}
t(2) = \#\{(\alpha, \beta) \mid {} & \beta \in D(v)_{k-1}^v,\ \alpha \in D(M_{2^u\beta})\,, \\
& (\alpha + aM_{2^u\beta})\,(l-1,\, r^{n/4}-1) \equiv 0\,(m) \\
& \text{for some integer } a \text{ with } \quad 0 \le a < m/M_{2^u\beta}\} \\
& -\# \bigcup_{\substack{0 \le b < d \\ \lambda = 0, 1}} D(m)_{(l-1,\, r^{n/4}-1,\, l^\lambda r^b + 1)}^m\,,
\end{aligned}$$

$$\begin{aligned}
t'(2) = \#\{(\alpha, \beta) \mid {} & \beta \in D(v)_{k-1}^v,\ \alpha \in D(M_{2^u\beta})\,, \\
& (\alpha + aM_{2^u\beta})\,(l-1,\, r^{n/4}-1) \equiv 0\,(m) \quad \text{or} \\
& (\alpha + aM_{2^u\beta})\,(lr^{d'}-1,\, r^{n/4}-1) \equiv 0\,(m) \\
& \text{for some integer } a \text{ with } \quad 0 \le a < m/M_{2^u\beta}\} \\
& -\# \bigcup_{\substack{0 \le b < d \\ \lambda = 0, 1}} \left(D(m)_{(l-1,\, r^{n/4}-1,\, l^\lambda r^b + 1)}^m \cup D(m)_{(lr^{d'}-1,\, r^{n/4}-1,\, l^\lambda r^b + 1)}^m\right),
\end{aligned}$$

$$t(3) = \sum_{\beta \in D(n, 3)} \# D(M_\beta) - 1\,,$$

$$t(4) = \sum_{\beta \in D(n, 3)} \# D(M_\beta) - \sum_{\beta \in D(n, 3)_{k+1}^n} \# D(M_\beta)_{l+1}^m\,.$$

We are now ready to state our main theorem.

Theorem. (i) $SK_1(\boldsymbol{Z}[G_1]) = 0$.

(ii) $SK_1(\boldsymbol{Z}[G_2]) \cong \boldsymbol{Z}_2^{t(2)}$ *if d is an odd integer,*

$SK_1(\boldsymbol{Z}[G_2]) \cong \boldsymbol{Z}_2^{t'(2)}$ *if d is an even integer.*

(iii) $SK_1(\boldsymbol{Z}[G_3]) \cong \boldsymbol{Z}_2^{t(3)}$.

(iv) $SK_1(\boldsymbol{Z}[G_4]) \cong \boldsymbol{Z}_2^{t(4)}$.

EXAMPLE 1.1. When $d=3$, we have

(i) $SK_1(\boldsymbol{Z}[G_3])=\boldsymbol{Z}_2^{\sharp D(n,3)\cdot\sharp D(m)-1}$,

(ii) $SK_1(\boldsymbol{Z}[G_4])=\boldsymbol{Z}_2^{\sharp D(n,3)\cdot\sharp D(m)-\sharp D(n,3)^n_{k+1}\cdot\sharp D(m)^m_{l+1}}$.

EXAMPLE 1.2. For G_2, when $m=35$, $n=72$, $r=4$, $k=55$, $l=29$, we have $d=6$ and then,

$$SK_1(\boldsymbol{Z}[G_2])\cong\boldsymbol{Z}_2^8\,.$$

This paper is organized as follows: In Section 2 after proving (i) of Theorem, we state some lemmas and propositions that are necessary for the proof of (ii), (iii), (iv) of Theorem. From Section 3 to Section 5 we prove (ii), (iii), (iv) of Theorem. Section 6 presents the proofs of the lemmas in Section 2. Appendix is devoted to quoting the table of the finite solvable groups from [10] which act linearly and freely on odd dimensional spheres.

I would like to thank Professors K. Kawakubo and M. Morimoto for their many helpful suggestions.

2. Preliminaries

For every odd prime number p, since the p-Sylow subgroups of $G_i(1\leq i\leq 4)$ are cyclic, it follows from [8, Theorem 14.2] that $SK_1(\boldsymbol{Z}[G_i])_{(p)}=0$. Moreover, $Syl_2(G_1)$ the 2-Sylow subgroup of G_1 is cyclic. Hence, by [8, Theorem 14.2], we conclude that $SK_1(\boldsymbol{Z}[G_1])=0$.

For the calculation of $SK_1(\boldsymbol{Z}[G_i])$ $(2\leq i\leq 4)$, we will use the following lemmas:

Lemma 2.1. ([10, Theorem 6.1.11]). $\mathrm{Syl}_2(G_2)\cong\langle R, B^v\rangle\cong Q2^{u+1}$ $Syl_2(G_3)=\langle P, Q\rangle\cong Q8$, *and* $Syl_2(G_4)\cong\langle P, Q, R\rangle=\langle PR, P\rangle\cong Q16$, *where* $Q2^N$ *denotes the generalized quaternionic group of order* 2^N.

When H is a subgroup of G, $C_G(H)$ denotes the centralizer of H in G and $N_G(H)$ denotes the normalizer of H in G.

Lemma 2.2. ([8, Example 14.4]). *Let G be a fihite group whose 2-Sylow subgroups are dihedral, quaternionic, or semidihedral. Then*

$$SK_1(\boldsymbol{Z}[G])_{(2)}\cong\boldsymbol{Z}_2^t\,,$$

where t is the number of conjugacy classes of cyclic subgroups $\sigma\subset G$ such that (a) $|\sigma|$ is odd, (b) $C_G(\sigma)$ has a non-abelian 2-Sylow subgroup, and (c) there is no $g\in N_G(\sigma)$ with $g\,x\,g^{-1}=x^{-1}$ for all $x\in\sigma$.

By Lemma 2.1, G_2, G_3 and G_4 satisfy the assertion in Lemma 2.2. We now prepare the next lemmas for the calculation of $SK_1(\boldsymbol{Z}[G_i])$ $(i=2, 3, 4)$, whose proof will be given in the last section. For integers α and β, we put $D(\alpha)=$

$\{x \in N \mid x \text{ is a divisor of } \alpha\}$, $M_\beta = (r^\beta - 1, m)$. Then

Lemma 2.3. *For any $\beta \in D(n)$, we have $((r^n-1)/(r^\beta-1), M_\beta) = 1$.*

Lemma 2.4. *For any integer α, we have*

$$(m, r^\beta - 1, \alpha \frac{r^n-1}{r^\beta-1}) = (\alpha, M_\beta) .$$

Lemma 2.5. *Let $\langle A^\mu B^\nu \rangle$ be the cyclic group which is generated by the element of the form $A^\mu B^\nu$. We put $\beta = (n, \nu)$. Then, there exists an integer α such that $\langle A^\mu B^\nu \rangle = \langle A^\alpha B^\beta \rangle$.*

Proposition 2.6. *Let α be an integer, and β an element in $D(n)$. Put $n' = h/\beta$. Then we have*

$$|\langle A^\alpha B^\beta \rangle| = \frac{M_\beta \cdot n'}{(M_\beta, \alpha)} .$$

Proof. It is clear that $|\langle A^\alpha B^\beta \rangle|$ is divisible by n'. We have $(A^\alpha B^\beta)^{n'} = A^{\alpha(r^n-1)/(r^\beta-1)}$. Put $r^n - 1 = m \cdot s'$, $r^\beta - 1 = M_\beta \cdot s$, and $m = M_\beta \cdot t$, then we have $(r^n-1)/(r^\beta-1) = t \cdot s'/s$. Set $M_\beta = \alpha_1^{e_1} \cdots \alpha_\xi^{e_\xi}$, $t = \beta_1^{f_1} \cdots \beta_\eta^{f_\eta}$, and $s = \gamma_1^{g_1} \cdots \gamma_\iota^{g_\iota}$, where α_i, β_i and γ_i are prime numbers, and e_i, f_i, g_i are positive integers. By the fact $(t, s) = 1$ and Lemma 2.3, we have $s' = \beta_1^{f_1'} \cdots \beta_\eta^{f_\eta'} \gamma_1^{g_1'} \cdots \gamma_\iota^{g_\iota'} \delta_1^{h_1} \cdots \delta_\kappa^{h_\kappa}$ for some prime numbers $\delta_1, \cdots, \delta_\kappa$, non-negative integers $f_1', \cdots, f_\eta'$ and positive integers $g_1', \cdots, g_\iota'$, $h_1, \cdots, h_\kappa$, with $g_i' \geq g_i$ $(i = 1, \cdots \iota)$. Since

$$\frac{r^n-1}{r^\beta-1} = \frac{m \cdot s'}{M_\beta \cdot s} = \beta_1^{f_1+f_1'} \cdots \beta_\eta^{f_\eta+f_\eta'} \gamma_1^{g_1-g_1'} \cdots \gamma_\iota^{g_\iota-g_\iota'} \delta_1^{h_1} \cdots \delta_\kappa^{h_\kappa} ,$$

the smallest positive integer x satisfying that $\alpha \frac{r^n-1}{r^\beta-1} x \equiv 0(m)$ is $\frac{M_\beta}{(\alpha, M_\beta)}$. Hence we have $|\langle A^\alpha B^\beta \rangle| = \frac{M_\beta \cdot n'}{(M_\beta, \alpha)}$. □

Proposition 2.7. *Let α and α' be integers, and β and β' elements in $D(n)$. $\langle A^\alpha B^\beta \rangle$ is conjugate to $\langle A^{\alpha'} B^{\beta'} \rangle$ in G_2, G_3 and G_4 if and only if $|\langle A^\alpha B^\beta \rangle| = |\langle A^{\alpha'} B^{\beta'} \rangle|$.*

Proof. Suppose that $|\langle A^\alpha B^\beta \rangle| = |\langle A^{\alpha'} B^{\beta'} \rangle|$. By using Proposition 2.6, we obtain that $\beta = \beta'$. Since

$$A^a (A^\alpha B^\beta) A^{-a} = A^{\alpha + a(1-r^\beta)} B^\beta \quad \text{and} \quad (A^\alpha B^\beta)^{cn+\beta/\beta} = A^{\alpha(1+c(r^n-1/r^\beta-1))} B^\beta$$

for any integers a and c, by Lemma 2.4, two cyclic subgroups whose orders are same are conjugate. The converse is clear. □

As an immediate consequence of Lemma 2.5 and Proposition 2.7, we have:

Proposition 2.8. *Let μ and ν be integers. Put $\beta = (\nu, n)$, then there exists* *an element* $\alpha\in D(M_\beta)$ *such that* $\langle A^\mu B^\nu\rangle$ *is conjugate to* $\langle A^\alpha B^\beta\rangle$.

3. Proof of (ii) of Theorem

Every element in G_2 is represented by the form $A^\mu B^\nu R^e$ for some integers μ and ν, where e is either 0 or 1. We see that $|\langle A^\mu B^\nu R\rangle|$ is even, and that a generator of a cyclic subgroup of odd order is represented by the element of the form $A^\alpha B^{2^u\nu'}$ for an integer ν'. Put $\beta=(v,\nu')$. By Proposition 2.8, there exists an integer $\alpha\in D(M_{2^u\beta})$ such that $\langle A^\mu B^{2^u\nu'}\rangle$ is conjugate to $\langle A^\alpha B^{2^u\beta}\rangle$. Thus, from now on, we will consider the cyclic subgroups generated by the element of the form $A^\alpha B^{2^u\beta}$ for any $\beta\in D(v)$ and any $\alpha\in D(M_{2^u\beta})$.

At first, we state some observations on G_2.

Observation 3.1. $2v$ *is divisible by* d.

Proof. Since $r^n\equiv r^{k-1}\equiv 1\,(m)$, d is a common divisor of n and $k-1$. Since $k+1\equiv 0\ (2^u)$, $(k+1,k-1)=2$, and $u\geq 2$, $k-1$ is divisible by 2, but not divisible by 4. Since $n=2^uv$, d is a divisor of $2v$. □

When d is an even integer, we put $d'=d/2$. Then we have:

Observation 3.2. *For any integer* a,

$$\langle A^{a(1-r^{n/4})}B^{n/4},\ A^{a(1-l)}R\rangle\cong Q8\,.$$

If d *is an even integer, then for any integer* a,

$$\langle A^{a(1-r^{n/4})}B^{n/4},\ A^{a(1-lr^{d'})}B^{d'}R\rangle\cong Q8\,.$$

Lemma 3.3. *In the case that* d *is an odd integer, for any* $\beta\in D(v)$ *and any* $\alpha\in D(M_{2^u\beta})$, $C_G(\langle A^\alpha B^{2^u\beta}\rangle)$ *has a subgroup* H *which is isomorphic to* $Q8$ *if and only if* $\beta(k-1)\equiv 0\,(v)$ *and* $(\alpha+a(r^{2^u\beta}-1))\,(l-1,r^{n/4}-1)\equiv 0\,(m)$ *for some integer* a.

In the case that d *is an even integer, for any* $\beta\in D(v)$ *and any* $\alpha\in D(M_{2^u\beta})$, $C_G(\langle A^\alpha B^{2^u\beta}\rangle)$ *has a subgroup* H *which is isomorphic to* $Q8$ *if and only if* $\beta(k-1)\equiv 0(v)$ *and* $(\alpha+a(r^{2^u\beta}-1))\,(l-1,r^{n/4}-1)\equiv 0(m)$ *or* $\beta(k-1)\equiv 0(v)$ *and* $(\alpha+a(r^{2^u\beta}-1))\,(lr^{d'}-1,r^{n/4}-1)\equiv 0\,(m)$ *for some integer* a.

Proof. In the case that $\beta(k-1)\equiv 0\ (v)$ and $(\alpha+a(r^{2^u\beta}-1))\,(l-1,r^{n/4}-1)\equiv 0\,(m)$ for some integer a, we see that $C_G(\langle A^\alpha B^{2^u\beta}\rangle)\supset\langle A^{a(1-r^{n/4})}B^{n/4},A^{a(1-l)}R\rangle$. In the case that $\beta(k-1)\equiv 0\,(v)$ and $(\alpha+a(r^{2^u\beta}-1))\,(lr^{d'}-1,r^{n/4}-1)\equiv 0\,(m)$ for some integer a, we see that $C_G(\langle A^\alpha B^{2^u\beta}\rangle)\supset\langle A^{a(1-r^{n/4})}B^{n/4},A^{a(1-lr^{d'})}B^{d'}R\rangle$. Conversely, assume that $C_G(\langle A^\alpha B^{2^u\beta}\rangle)$ has *a* subgroup H which is isomorphic to $Q8$. Since $K=\langle B^v,R\rangle$ is one of the 2-Slyow subgroups of G and H is a 2-group of G, we have $g^{-1}Hg\subset K$ for some $g\in G$. Now we consider the quotient group of $K/\langle B^v\rangle$ and the projection $p\colon K\to K/\langle B^v\rangle$. Since $\ker p=\langle B^v\rangle$ and $g^{-1}Hg\cong$

$Q8$, we have $\ker (p\,|\,g^{-1}Hg)=\langle B^{n/4}\rangle$. Hence, $g^{-1}Hg=\langle B^{n/4}, B^{\tau}R\rangle$ for some integer τ which is divisible by v. Now put $g=A^a B^b R^c$ where a and b are some integers, and c is either 0 or 1. Then,

$$\begin{aligned} H &= g\langle B^{n/4}, B^{\tau}R\rangle g^{-1} \\ &= A^a B^b R^c \langle B^{n/4}, B^{\tau}R\rangle R^{-c} B^{-b} A^{-a} \\ &= A^a B^b \langle B^{n/4}, B^{\tau'}R\rangle B^{-b} A^{-a} \quad \text{(for some integer } \tau') \\ &= A^a \langle B^{n/4}, B^{\tau''}R\rangle A^{-a} \quad \text{(for some integer } \tau'') \\ &= \langle A^{a(1-r^{n/4})} B^{n/4}, A^{a(1-lr^{\tau''})} B^{t''}R\rangle . \end{aligned}$$

Since $A^{a(1-r^{n/4})} B^{n/4} \in C_G(\langle A^{\alpha} B^{2^u\beta}\rangle)$, we have

$$(r^{n/4}-1)\,\{a(r^{2^u\beta}-1)+\alpha\} \equiv 0\,(m)\,.$$

On the other hand, since $A^{a(1-lr^{\tau''})} B^{\tau''}R \in C_G(\langle A^{\alpha} B^{2^u\beta}\rangle)$, we have

$$\begin{cases} (lr^{\tau''}-1)\,\{\alpha+a(r^{2^u\beta}-1)\} \equiv 0\,(m) \\ \beta(k-1) \equiv 0\,(v)\,. \end{cases}$$

Now, since $\tau''=\tau k+b(1-k)$ if $c=1$, and $\tau''=\tau+b(1-k)$ if $c=0$, we have $r^{\tau''}=r^{\tau}$. Moreover, $r^{\tau}=1$ or $r^{d'}$ because τ is divisible by v and d is a divisor of $2v$. Thus the lemma was proved. □

As an immediate consequence of Lemma 3.3, we have:

Corollary 3.4. *In the case that d is an odd integer, for any $\beta\in D(v)$ and any $\alpha\in D(M_{2^u\beta})$, $C_G(\langle A^{\alpha} B^{2^u\beta}\rangle)$ has a subgroup H which is isomorphic to $Q8$ if and only if $\beta(k-1)\equiv 0\,(v)$ and $(\alpha+aM_{2^u\beta})\,(l-1,\ r^{n/4}-1)\equiv 0\,(m)$ for some integer a with $0\leq a<m/M_{2^u\beta}$.*

In the case that d is an even integer, for any $\beta\in D(v)$ and any $\alpha\in D(M_{2^u\beta})$, $C_G(\langle A^{\alpha} B^{2^u\beta}\rangle)$ has a subgroup H which is isomorphic to $Q8$ if and only if $\beta(k-1)\equiv 0(v)$ and $(\alpha+aM_{2^u\beta})\,(l-1,\ r^{n/4}-1)\equiv 0\,(m)$ or $\beta(k-1)\equiv 0\,(v)$ and $(\alpha+aM_{2^u\beta})\,(lr^{d'}-1,\ r^{n/4}-1)\equiv 0\,(m)$ for some integer a with $0\leq a<m/M_{2^u\beta}$. □

It is clear that $C_G(\langle A^{\alpha} B^{2^u\beta}\rangle)$ has a non-abelian 2-Sylow subgroup if and only if $C_G(\langle A^{\alpha} B^{2^u\beta}\rangle)$ has a subgroup H which is isomorphic to $Q8$. Let $\langle A^{\alpha} B^{2^u\beta}\rangle$ be a cyclic subgroup of G_2 satisfying the conditions (a) and (b). Assume that it does not satisfy the condition (c). In the case that $(A^a B^b)\,(A^{\alpha} B^{2^u\beta})\,(A^a B^b)^{-1}=(A^{\alpha} B^{2^u\beta})^{-1}$ for some integers a and b, we have

$$\begin{cases} \alpha(r^b+1) \equiv 0\,(m) \\ \beta \equiv 0(v)\,. \end{cases}$$

On the other hand, in the case that $(A^a B^b R)\,(A^{\alpha} B^{2^u\beta})\,(A^a B^b R)^{-1}=(A^{\alpha} B^{2^u\beta})^{-1}$

for some integers a and b, we have

$$\begin{cases} a+\alpha l r^b - a r^{2^u \beta k} + \alpha r^{-2^u \beta} \equiv 0\,(m) \\ \beta(k+1) \equiv 0\,(v)\,. \end{cases}$$

Since it follows from Corollary 3.4 that $\beta(k-1)\equiv 0$ (v), in this case we have

$$\begin{cases} \alpha(lr^b+1) \equiv 0\,(m) \\ \beta \equiv 0\,(v)\,. \end{cases}$$

Hence for $\alpha\in D(m)$ satisfying that $\alpha(l^\lambda r^b+1)\equiv 0$ (m) $(\lambda=0, 1)$, $\langle A^\alpha\rangle$ does not satisfy the condition (c). This completes the proof of (ii) of Theorem.

4. Proof of (iii) of Theorem

Lemma 4.1. *Let $\sigma\subset G_3$ be a cyclic subgroup of odd order. Then, there exist $\beta\in D(n)$ and $\alpha\in D(M_\beta)$ such that σ is conjugate to $\langle A^\alpha B^\beta\rangle$.*

Proof. Every element in G_3 can be represented by the form $XA^\mu B^\nu$ for some $X\in\langle P, Q\rangle$ and some integers μ and ν. We see that $\langle A^\mu B^\nu\rangle$ has odd order. In the case that $\nu\equiv 0\,(3)$, we see that $\langle XA^\mu B^\nu\rangle$ has even order. In the other cases, we see that $\langle XA^\mu B^\nu\rangle$ has even order or is conjugate to $\langle A^\mu B^\nu\rangle$. The conclusion now follows from Proposition 2.8. □

Hence from now on we will consider the cyclic subgroups generated by the element of the form $A^\alpha B^\beta$ for $\beta\in D(v)$ and $\alpha\in D(M_\beta)$. Since $\langle P, Q\rangle$ is a normal subgroup of G_3, $C_{G_3}(\langle A^\alpha B^\beta\rangle)$ has a non-abelian 2-Sylow subgroup if and only if $C_{G_3}(\langle A^\alpha B^\beta\rangle)$ includes $\langle P, Q\rangle$. And it is easy to show that $C_{G_3}(\langle A^\alpha B^\beta\rangle)$ includes $\langle P, Q\rangle$ if and only if β is an element of $D(n, 3)$. Let $\langle A^\alpha B^\beta\rangle$ be a cyclic subgroup of G_3 satisfying the conditions (a) and (b). Assume that $(A^a B^b)\,(A^\alpha B^\beta)\,(A^a B^b)^{-1}=(A^\alpha B^\beta)^{-1}$ for some integers a and b. Since n is an odd integer, we have

$$\begin{cases} \alpha(1+r^b) \equiv 0\,(m) \\ \beta \equiv 0\,(n)\,. \end{cases}$$

Since $(1+r^b,\ m)=1$ for any $b\in\mathbf{Z}$ when n is odd, we have $\langle A^\alpha B^\beta\rangle=1$. This completes the proof of (iii) of Theorem.

5. Proof of (iv) of Theorem

Lemma 5.1. *Let $\sigma\subset G_4$ be a cyclic subgroup of odd order. Then, there exist $\beta\in D(n)$ and $\alpha\in D(M_\beta)$ such that σ is conjugate to $\langle A^\alpha B^\beta\rangle$.*

Proof. Every element in G_4 can be represented by the form $XA^\mu B^\nu$ for some $X\in\langle P, Q, R\rangle$ and some integers μ and ν. We see that $\langle A^{\mu}B^{\nu}\rangle$ has odd order. And it is shown that $|\langle XA^{\mu}B^{\nu}\rangle|$ is even or $\langle XA^{\mu}B^{\nu}\rangle$ is conjugate to $\langle A^{\mu}B^{\nu}\rangle$. The conclusion now follows from Proposition 2.8. $\square$

Hence from now on we will consider the cyclic subgroups generated by the element of the form $A^{\alpha}B^{\beta}$ for $\beta\in D(v)$ and $\alpha\in D(M_{\beta})$.

Lemma 5.2. *If $C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$ has a non-abelian 2-Sylow subgroup, then $C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$ includes $\langle P\rangle$, $\langle Q\rangle$ or $\langle PQ\rangle$.*

Proof. We put $K=\langle P, Q, R\rangle=\langle PR, P\rangle$. $C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$ has a non-abelian 2-Sylow subgroup, if and only if $C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$ has a subgroup H which is isomorphic to $Q8$. Since H is a 2-group of G, we have $g^{-1}Hg\subset K$ for some $g\in G$. We note that $\langle PR\rangle$ is a cyclic subgroup of K whose order is 8. Now we consider the quotient group $K/\langle PR\rangle$, and the projection $p\colon K\to K/\langle PR\rangle$. Since $\ker p=\langle PR\rangle$ and $g^{-1}Hg\cong Q8$, we have that $\ker(p\,|\,g^{-1}Hg)$ is a cyclic subgroup of $\langle PR\rangle$ whose order is 4. Hence we have $\ker(p\,|\,g^{-1}Hg)=\langle(PR)^2\rangle=\langle Q\rangle$. Thus, we have $g^{-1}Hg=\langle Q, (PR)^{\lambda}P\rangle$ for some $\lambda\in\boldsymbol{Z}$. We note that if λ is an odd integer, then $g^{-1}Hg=\langle Q, R\rangle$, and that if λ is an even integer, then $g^{-1}Hg=\langle P, Q\rangle$. Thus, we obtain:

$$\begin{aligned}
&H=\langle P, Q\rangle \quad\text{or}\quad \langle RA^{a(l-1)}B^{b(k-1)}, Q\rangle \quad\text{if}\quad b\equiv 0\,(3)\,,\\
&H=\langle P, Q\rangle, \langle RA^{a(l-1)}B^{b(k-1)}, PQ\rangle, \langle RA^{a(l-1)}B^{b(k-1)}, P\rangle\\
&\quad\text{or}\quad \langle QRA^{a(l-1)}B^{b(k-1)}, P\rangle \quad\text{if}\quad b\equiv 1\,(3)\,,\\
&H=\langle P, Q\rangle, \langle RA^{a(l-1)}B^{b(k-1)}, P\rangle, \langle RA^{a(l-1)}B^{b(k-1)}, PQ\rangle\\
&\quad\text{or}\quad \langle RPA^{a(l-1)}B^{b(k-1)}, PQ\rangle \quad\text{if}\quad b\equiv 2\,(3)\,,
\end{aligned}$$

where a and b are integers. Hence H includes $\langle P\rangle$, $\langle Q\rangle$ or $\langle PQ\rangle$.

Lemma 5.3. *$C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$ has a non-abelian 2-Sylow subgroup if and only if $\beta\equiv 0$ (3).*

Proof. If $C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$ has a non-abelian 2-Sylow subgroup, by Lemma 5.2, we have P, Q or PQ are elements of $C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$. In the case that P or Q are elements of $C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$, we have $\beta\equiv 0\,(3)$ as in the proof of (iii) of Theorem. On the other hand it is easy to show that if PQ is an element of $C_{G_3}(\langle A^{\alpha}B^{\beta}\rangle)$, then $\beta\equiv 0\,(3)$. Conversely, if $\beta\equiv 0\,(3)$, it follows from the proof of (iii) of Theorem that $C_{G_4}(\langle A^{\alpha}B^{\beta}\rangle)$ includes $\langle P, Q\rangle$, that is a non-abelian 2-group. This completes the proof. $\square$

Now for $\beta\in D(n, 3)$ and $\alpha\in D(M_{\beta})$, we assume that $\langle A^{\alpha}B^{\beta}\rangle$ doesn't satisfy the condition (c). If $(A^{a}B^{b})(A^{\alpha}B^{\beta})(A^{a}B^{b})^{-1}=(A^{\alpha}B^{\beta})^{-1}$, then we have $A^{\alpha}B^{\beta}=1$. If $(RA^{a}B^{b})(A^{\alpha}B^{\beta})(RA^{a}B^{b})^{-1}=(A^{\alpha}B^{\beta})^{-1}$, then we have

$$\begin{cases} l(\alpha r^b + a(1-r^\beta)) + \alpha r^{-\beta} \equiv 0\ (m) \\ \beta(k+1) \equiv 0\ (n)\,. \end{cases}$$

Since d is a common divisor of n and $k-1$, we have $(k+1, d)=1$, and so β must be divisible by d. Hence we have $\alpha(1+lr^b)\equiv 0\ (m)$. Since $l^2\equiv 1\ (m)$, we have $\alpha(l+r^b)\equiv 0\ (m)$. By these equations, we have $\alpha(l+1)(r^b+1)\equiv 0\ (m)$. Since $(r^b+1, m)=1$, we have $\alpha(l+1)\equiv 0\ (m)$.

Conversely under the conditions $\beta(k+1)\equiv 0\ (m)$ and $\alpha(l+1)\equiv 0\ (m)$, we see that $R(A^\alpha B^\beta)R^{-1}=(A^\alpha B^\beta)^{-1}$, then $\langle A^\alpha B^\beta \rangle$ doesn't satisfy the condition (c). This completes the proof of (iv) of Theorem.

6. Proof of Lemmas in Section 2

Proof of Lemma 2.3. Put $n'=n/\beta$ and $r^\beta-1=M_\beta \cdot s$. Then we have

$$\begin{aligned} \frac{r^n-1}{r^\beta-1} &= \sum_{i=0}^{n'-1} r^{\beta i} \\ &= \sum_{i=0}^{n'-1} (M_\beta \cdot s+1)^i \\ &\equiv n'\ (M_\beta)\,. \end{aligned}$$

Now since $(n', M_\beta)=1$, we have $((r^n-1)/(r^\beta-1), M_\beta)=1$. □

Lemma 2.4 is an immediate consequence of Lemma 2.3.

Proof of Lemma 2.5. Since $\beta=(n, \nu)$, there exists an integer x such that $\nu x\equiv\beta(n)$. Put $n'=n/\beta$, then we see that $(x, n')=1$. We note that the order of $\langle A^\mu B^\nu \rangle$ is a divisor of mn'. If $(x, m)=1$, we have $(A^\mu B^\nu)^x=A^\alpha B^\beta$ for some integer α and $\langle (A^\mu B^\nu)^x \rangle=\langle A^\mu B^\nu \rangle$. If $(x, m)\neq 1$, since there exists an integer c such that $(x+cn', n'm)=1$, we have $(A^\mu B^\nu)^{x+cn'}=A^\alpha B^\beta$ for some integer α and $\langle (A^\mu B^\nu)^{x+cn'} \rangle=\langle A^\mu B^\nu \rangle$. This completes the proof. □

7. Appendix ([10, Theorem 6.1.11])

Let G be a finite solvable group. Then G has a fixed point free complex representation if and only if G is of type I, II, III or IV below, with the additional condition: if d is the order of r in the multiplicative group of residues modulo m, of integers prime to m, then n/d is divisible by every prime divisor of d.

TYPE I. A group of order mn that is generated by the elements of the form A and B, and that has relations:

$$A^m = B^n = 1,\ BAB^{-1} = A^r\,,$$

where m, n and r satisfy the following conditions:

$$m\geq 1,\ n\geq 1,\ (n(r-1),\ m)=1,\ r^n\equiv 1\ (m)\ .$$

Type II. A group of order $2mn$ that is generated by the elements of the form A, B and R, and that has relations:

$$R^2=B^{n/2},\ RAR^{-1}=A^l,\ RBR^{-1}=B^k$$

in addition to the relations in I, where m, n, r, l and k satisfy the following conditions:

$$l^2\equiv r^{k-1}\equiv 1\ (m),\ k\equiv -1\ (2^u)\ ,$$
$$n=2^u v\ (u\geq 2,\ (v,2)=1),\ k^2\equiv 1\ (n)$$

in addition to the conditions in I.

Type III. A group of order $8mn$ that is generated by the elements of the form A, B, P and Q, and that has relations:

$$P^2=Q^2=(PQ)^2,\ AP=PA,\ AQ=QA\ ,$$
$$BPB^{-1}=Q,\ BQB^{-1}=PQ$$

in addition to the relations in I, where m, n and r satisfy the following conditions:

$$n\equiv 1\ (2),\ n\equiv 0\ (3)$$

in addition to the conditions in I.

Type IV. A group of order $16mn$ that is generated by the elements of the form A, B, P, Q and R, and that has relations:

$$R^2=P^2,\ RPR^{-1}=QP,\ RQR^{-1}=Q^{-1}\ ,$$
$$RAR^{-1}=A^l,\ RBR^{-1}=B^k$$

in addition to the relations in III, where m, n, r, k and l satisfy the following conditions:

$$k^2\equiv 1\ (n),\ k\equiv -1\ (3),\ r^{k-1}\equiv l^2\equiv 1\ (m)$$

in addition to the conditions in III.

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