Ikegami, T.
Osaka J. Math.
28 (1991), 93–116

DUALITY ON HARMONIC SPACES

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(Received March 13, 1990)

Introduction

Our consideration of duality on harmonic spaces originates from the following problem: given a Green function $k(x, y)$, i.e. a system of extremal potentials $\{k_y(x); y \in X\}$ on a harmonic space $(X, \mathcal{U})$ in the sense of Constantinescu-Cornea with additional properties, can we construct a harmonic structure $(X, \mathcal{U}^*)$ such that $\{k_x^*(y)=k(x, y); x \in X\}$ is a system of Green functions on the second space? (we call this harmonic space $(X, \mathcal{U}^*)$ is the dual of $(X, \mathcal{U})$). This problem was first considered by R-M. Hervé [9] in Brelot's axiomatic setting and then by J. Taylor [19] in the framework of Bauer. Later Taylor gave an affirmative answer to the problem for general harmonic spaces in terms of the probability theory.

The purpose of this paper is to give a purely potential theoretic proof of the problem. We start from assuming the existence of a Green function apparently less restricted to that of K. Janssen [10] though it turnes out to be equivalent in the last section. Next we form a sweeping system $\Omega^*=(\{\mu_y^{*V}; y \in V\})_V$, then define $\mathcal{U}^*$ to be the sheaf of Ω^*-hyperharmonic functions. In proving axioms of harmonic spaces, the axiom of convergence and the axiom of resolutivity are crucial. In the former axiom, the Mokobodzki measure plays an important role and in the second we use the idea of R-M. Hervé. In the course of the proof it is revealed that $(X, \mathcal{U}^*)$ satisfies the Doob convergence property and later we can find it is also same for $(X, \mathcal{U})$ as was pointed out by U. Schirmeire [15].

The following two sections are devoted to establish the duality theorem for the Green function and the measure representation of potentials by Green functions in the dual space. Then it is concluded that the original space $(X, \mathcal{U})$ is the dual of $(X, \mathcal{U}^*)$.

In the final section we make several remarks on duality and the equivalence of measure representation of potentials by Green functions.

An important problem on duality of resolvents and the relation with the duality theorem of H-cones will be discussed under a more general setting elesewhere.

The author wishes to express his hearty thanks to Ursula Schirmeier who kindly read the manuscript and pointed out some error.

1. Preliminaries

Throughout this article, we use notations in [7] without any reference. Let $(X, \mathcal{U})$ be a P-harmonic space of Constantinescu-Cornea with countable base, where $\mathcal{U}$ denotes a sheaf of cones of hyperharmonic functions. We assume that X admits a Green function $k(x, y)$ possessing the following properties:

1) $(x, y) \to k(x, y)$: $X \times X \to \overline{\boldsymbol{R}}_+$ is lower semicontinuous and finite continuous if $x \neq y$,
2) $x \to k(x) = k(x, y)$ is a potential such that $S(k_y) = \{y\}$ for every $y \in X$,
3) for every potential p on X with compact $S(p)$ there exists a unique measure μ such that $p(x) = \int k_y(x)\, d\mu(y)$.

We write $k\mu(x) = \int k_y(x) d\mu(y)$ and $k_x^*(y) = k_y(x)$.

REMARKS 1.1.

(1) K. Janssen [10] proved that if $(X, \mathcal{U})$ satisfies
 i) the Doob convergence property,
 ii) (A): every point $x \in X$ has a neighborhood $V(x)$ such that $A_{X \setminus V(x)} = X$,
 iii) (P): the proportionality axiom

then there is a Green function which represents every potential on X by a measure and the mapping $y \to k_y$ is continuous with respect to a topology on the cone of positive superharmonic functions on X.

(2) If $k\mu = k\nu$ is a potential then $\mu = \nu$. For, let K be a compact subset of X and f be a continuous function on X with compact support such that $f = 1$ on K, then the specific multiplication $f \cdot k\mu = k(f\mu)$ is a potential and $S(f \cdot k\mu)$ is compact. Thus $f\mu = f\nu$, i.e., $\mu = \nu$ on K.

(3) $p(x) = \int k_y(x) d\mu(y)$ implies that the support of μ is $S(p)$.

(4) $k\mu$ is a potential on X whenever μ has a compact support.

(5) if $k(x, y)$ is a Green function then for evrey finite continuous function g on X which is strictly positive $k(x, y)/g(y)$ is also a Green function.

(6) if $(X, \mathcal{U})$ admits a Green function, then X has no isolated point.

We list up the notations used continuously in the following.

$M^+(X)(M_K^+(X))$: the set of positive Radon measures on X (with compact support)

$\mathcal{U}_+(\mathcal{S}_+)$: the set of non-negative hyperharmonic (superharmonic) functions on X.

$\mathcal{C}(E)$: the set of finite continuous functions on E.

$\mathcal{C}_K^+(X)(\mathcal{C}_b^+(X))$: the set of non-negative continuous functions on X with compact support (bounded).

∂A: the boundary of A.

We use the notations $\mathcal{C}A = X \backslash A$, $\mathrm{R}^A u = \mathrm{R}_u^A$ and $\hat{\mathrm{R}}^A u = \hat{\mathrm{R}}_u^A$.

2. The measure ε_x^φ.

In [14], G. Mokobodzki considered the smoothing of balayaged measures and it revealed to be usefull for the consideration of the topology on $\mathcal{S}_+$ ([1], [14], [17], [19]).

Let $\varphi \in \mathcal{C}_b^+(X)$. For every $v \in \mathcal{U}_+$ we define

$$v_\varphi(x) = \int_0^{\sup \varphi} \mathrm{R}_v^{\{\varphi > \alpha\}}(x) d\alpha .$$

It is easily checked that $v_\varphi \in \mathcal{U}_+$, $v_\varphi \leq (\sup \varphi) v$, thus v_φ is superharmonic (resp. potential) if v is superharmonic (resp. potential). For $x \notin \operatorname{supp} \varphi$ there is $\varepsilon_x^\varphi \in M^+(X)$ such that $\varepsilon_x^\varphi(v) = \int v d\varepsilon_x^\varphi = v_\varphi(x)$ for every $v \in \mathcal{U}_+$ and $\operatorname{supp} \varepsilon_x^\varphi \subset \operatorname{supp} \varphi$.

REMARK 2.1. In the same way the following result is proved: let φ be a finite continuous function on X such that $0 \leq \varphi \leq 1$ and $\{\varphi < 1\}$ is relatively compact, and let $\mu \in M_K^+(X)$ be such that $\operatorname{supp} \mu \subset$(the interior of $\{\varphi = 0\}$), then we have $\mu^\varphi \in M^+(X)$ satisfying

1) $\int v_\varphi(x) d\mu(x) = \int v(x) d\mu^\varphi(x) \qquad \forall v \in \mathcal{U}_+$,

2) $\operatorname{supp} \mu^\varphi \subset \overline{\{0 < \varphi < 1\}}$.

We call (φ, x) is a *pair* when $\varphi \in \mathcal{C}_K^+(X)$ and $x \notin \operatorname{supp} \varphi$. The family of measures $\mathcal{M} = \{\varepsilon_x^\varphi; (\varphi, x) \text{ is a pair}\}$ plays an important role in the article. For $\varepsilon_x^\varphi \in \mathcal{M}$, let consider the function of y: $\varepsilon_x^\varphi(k_y) = \int k_y(z) d\varepsilon_x^\varphi(z) = \int k_z^*(y) d\varepsilon_x^\varphi(z) = k^* \varepsilon_x^\varphi(y)$.

Proposition 2.1. *$k^*\varepsilon_x^\varphi$ is continuous on X for every $\varepsilon_x^\varphi \in \mathcal{M}$.*

Proof. Let $y_n \to y_0$. We claim that $\lim_{n\to\infty} \mathrm{R}^{\{\varphi>\alpha\}} k_{y_n}(x) = \mathrm{R}^{\{\varphi>\alpha\}} k_{y_0}(x)$ if $\alpha \neq \varphi(y_0)$. For, (1) if $\alpha < \varphi(y_0)$ then $y_n \in \{\varphi > \alpha\}$ for sufficiently large n and $\mathrm{R}^{\{\varphi>\alpha\}} k_{y_n}(x) = k_{y_n}(x) \to k_{y_0}(x) = \mathrm{R}^{\{\varphi>\alpha\}} k_{y_0}(x)$, since $x \neq y_0$. (2) if $\alpha > \varphi(y_0)$ then $\mathrm{R}^{\{\varphi>\alpha\}} k_{y_n}(x) = \int k_{y_n}(z) d\varepsilon_x^{\{\varphi>\alpha\}}(z) \to \int k_{y_0}(z) d\varepsilon_x^{\{\varphi>\alpha\}}(z) = \mathrm{R}^{\{\varphi>\alpha\}} k_{y_0}(x)$, since $\operatorname{supp} \varepsilon_x^{\{\varphi>\alpha\}}$ is compact and disjoint with a neighborhood of y_0. Also, we can prove that $\{\mathrm{R}^{\{\varphi>\alpha\}} k_{y_n}(x)\}$ is uniformly bounded in $[0, \sup \varphi]$, thus the proposition is proved by Lebesgue's bounded convergence theorem.

REMARK 2.2. Let φ be a function in Remark 2.1. Then $k^*\varepsilon_x^\varphi$ is continuous.

Proposition 2.2. *For every compact subset K of X, the family $\{k^*\varepsilon_x^\varphi|_K; \varepsilon_x^\varphi \in \mathcal{M}\}$ is total in $C(K)$, where $\cdot|_K$ denotes the restrictionon on K.*

Proof. We prove that for positive Radon measure λ, μ on K, if $\lambda(k^*\varepsilon_x^\varphi)=\mu(k^*\varepsilon_x^\varphi)$ for every $\varepsilon_x^\varphi \in \mathcal{M}$ then $\lambda=\mu$. Since $\lambda(k^*\varepsilon_x^\varphi)=\int k\lambda(z)d\varepsilon_x^\varphi(z)=\int_0^{\sup\varphi} \mathrm{R}_{k\lambda}^{\{\varphi>\alpha\}}(x)d\alpha$, we have $\int_0^{\sup\varphi} \mathrm{R}_{k\lambda}^{\{\varphi>\alpha\}}(x)d\alpha=\int_0^{\sup\varphi} \mathrm{R}_{k\mu}^{\{\varphi>\alpha\}}(x)d\alpha$. Let $\psi=\inf(\varphi, t)$ with $t<\sup\varphi$, then

$$\int_0^{\sup\psi} \mathrm{R}_{k\lambda}^{\{\psi>\alpha\}}(x)d\alpha = \int_0^t \mathrm{R}_{k\lambda}^{\{\varphi>\alpha\}}(x)d\alpha\,.$$

Hence, $\mathrm{R}_{k\lambda}^{\{\varphi>\alpha\}}(x)=\mathrm{R}_{k\mu}^{\{\varphi>\alpha\}}(x)d\alpha$-a.e., and thus $\mathrm{R}^{CU}k\lambda(x)=\mathrm{R}^{CU}k\mu(x)$ for a fundamental system $\{U\}$ of neighborhoods of x, which implies $k\lambda(x)=k\mu(x)$ and finally $\lambda=\mu$, since $k\lambda$ and $k\mu$ are potentials on X such that $S(k\lambda)=S(k\mu)\subset K$.

Let $\{f_n\}\subset C_K(X)$ be a countable family which is uniformly dense in $C_K(X)$. Each f_n^+ and f_n^- can be approximated uniformly on $\operatorname{supp} f_n$ by affine combinations of $k^*\varepsilon_x^\varphi$, i.e., $\sum_{i=1}^m c_i k^*\varepsilon_{x_i}^{\varphi_i}$ $(c_i>0,\ \varepsilon_{x_i}^{\varphi_i}\in\mathcal{M})$. We have thus a countable family of measures $\mathcal{M}_0$ such that

Proposition 2.3.

(1) $\mathcal{M}_0=\{\mu_n\}\subset M_K^+(X)$,

(2) $\forall f\in C_K(X)\forall\varepsilon>0\exists\mu,\ \mu'\in\mathcal{M}_0 \,|\, f-(k^*\mu-k^*\mu')|<\varepsilon$ *on* $\operatorname{supp} f$.

The following property of ε_x^φ is of fundamental importance in [19], and is derived from the topology on $\mathcal{S}_+$. We prove it without using T-topology.

Proposition 2.4. *Every measure ε_x^φ of $\mathcal{M}$ does not charge on every semipolar set.*

Proof. It is sufficient to prove $\varepsilon_x^\varphi(A)=0$ if A is compact and totally thin. $A=\{\hat{\mathrm{R}}^A p<\mathrm{R}^A p\}$ for a continuous strict potential p on X. Let $\{O_n\}$ be a decreasing sequence of relatively compact open sets with $\bigcap_{n=1}^\infty O_n=A$ and $s_n=R^{O^n}p$. Then from the continuity of p it is easily seen that $s_n=\mathrm{R}^A p=\inf s_n$. Since the decreasing function of α $\mathrm{R}^{\{\varphi>\alpha\}}p(x)$ admits only countable discontinuities, $\varepsilon_x^{\{\varphi>\alpha\}}=\varepsilon_x^{\{\varphi\geq\alpha\}}=\mu_x^{\{\varphi<\alpha\}}d\alpha$-a.e., where $\mu^{\{\varphi<\alpha\}}$ is the harmonic measure of $\{\varphi<\alpha\}$. On the other hand, since $\mathrm{H}_h^{\{\varphi<\alpha'\}}=h$, where $h=\mathrm{H}_s^{\{\varphi<\alpha\}}$ and $\alpha'<\alpha$, we have

$\mu_x^{\{\varphi<\alpha'\}}(\hat{s})\geq\mu_x^{\{\varphi<\alpha\}}(s)$ whenever $\alpha'<\alpha$. From these we conclude that, for $\eta>0$,

$$\int_\eta^{\sup\varphi}\Big(\int s\,d\varepsilon_x^{\{\varphi>\alpha\}}\Big)d\alpha\leq\int_0^{\sup\varphi}\Big(\int\hat{s}\,d\varepsilon_x^{\{\varphi>\alpha\}}\Big)d\alpha=\varepsilon_x^\varphi(\hat{s})$$

and letting $\eta\to0$

$$\varepsilon_x^\varphi(\hat{s})\geq\int_0^{\sup\varphi}\Big(\int s\,d\varepsilon_x^{\{\varphi>\alpha\}}\Big)d\alpha=\lim_{u\to\infty}\varepsilon_x^\varphi(s_n)=\varepsilon_x^\varphi(s)\,,$$

which proves the proposition.

3. The dual sweeping system

Let V be a relatively compact open set. By the property (3) of a Green function, there exists $\mu_y^{*V}\in M^+(X)$ such that

$$\hat{\mathrm{R}}^{CV}k_y(x)=\int k_z(x)d\mu_y^{*V}(z)\,.$$

We note that μ_y^{*V} concentrates in ∂V (Remark 1.1 (3))

We form a sweeping system on X, $\Omega^*:=\{(\mu_y^{*V})_{y\in V};\ V$ is a relatively compact open subset of $X\}$ and define, for every open set U

$$\mathcal{U}^*(U)=\{u^*;\ \text{locally }\Omega^*\text{-hyperharmonic functions on }U\},$$

i.e., $u^*\in\mathcal{U}^*(U)$ if and only if u^* is lower semicontinuous and lower finite on U, and for every relatively compact open set V such that $\bar{V}\subset U$ we have $\mu_y^{*V}(u^*)\leq u^*(y)$ for every $y\in V$.

In the following we shall prove that $\mathcal{U}^*$: $U\to\mathcal{U}^*(U)$ defines a sheaf of cones of functions and that $(X, \mathcal{U}^*)$ is a P-harmonic space of Constantinescu-Cornea, which we call the dual of $(X, \mathcal{U})$.

We note that

Proposition 3.1. *For every x and for every subset A of X, the function of y, $k_x^*(y)$ and $\hat{\mathrm{R}}^A\,k_y(x)$ are non-negative and belong to $\mathcal{U}^*(X)$. Further, for every compact set K and $y_0\in X$ there is $u^*\in\mathcal{U}^*(X)$ such that $u^*(y_0)<\infty$ and $\inf_K u^*>0$.*

The first assertion is easily seen from the definition of the sweeping system. For the second, by the properties (1), (2) of a Green function for every $y\in K$ there is $x=x(y)\neq y, y_0$ with $k_y(x)=k_x^*(y)>0$, hence there is a neighborhood $U(y)$ of y such that $k_x^*>0$ on $U(y)$. Now the compactness argument implies the existence of a finite system $\{x_i=x(y_i);\ i=1, 2, \cdots, n\}$ and $u^*(y)=\sum_{i=1}^n k_{x_i}^*(y)$ satisfies the requirement.

Proposition 3.2. *$\mathcal{U}^*(X)$ separates points of X.*

Proof. For $y_1, y_2 \in X$, $y_1 \neq y_2$, let V be a relatively compact open set with $y_1 \in V$, $y_2 \notin \bar{V}$. Then there is $x \in V$ such that $k_{y_1}(x) \neq \hat{R}^{CV} k_{y_1}(x)$. Setting $u^*(y) = k_x^*(y)$ and $v^*(y) = \hat{R}^{CV} k_y(x)$ we have

$$u^*(y_1) = k_x^*(y_1) = k_{y_1}(x) \neq \hat{R}^{CV} k_{y_1}(x) = v^*(y_1),$$
$$u^*(y_2) = k_{y_2}(x) = \hat{R}^{CV} k_{y_2}(x) = v^*(y_2)$$

and thus, $u^*(y_1)v^*(y_2) \neq u^*(y_2)v(y_1)$ if $k_{y_2}(x) \neq 0$. The other case is much simpler.

Lemma 3.1. $\sup_{y \in V} \int \mu_y^{*V} < \infty$ *for every relatively compact open set V.*

Proof. From the same argument as in the proof of Prop. 3.1, we have

$$\sup \{ \int d\mu_y^{*V} ; y \in K \} < \infty \text{ for every compact subset } K \text{ of } V.$$

If the lemma is not valid, we have $\{y_n\} \subset V$ such that $\mu_{y_n}^{*V}(1) \to \infty$. Above consideration implies $\{y_n\}$ has no limit point in V, and we may assume that $y_n \to y_0 \in \partial V$. But this is absurd since in the proof of Prop. 3.1 we can take $x(z) \notin \{y_n; n=0, 1, 2, \cdots\}$ for each $z \in \partial V$.

Lemma 3.2. *Let V be a relatively compact open set and $f \in C(\partial V)$. Then $g(y) = \mu_y^{*V}(f)$ is bounded and *harmonic on V, i.e., g is continuous on V and $\mu^{*V_1}(g) = g$ for every open set V_1 with $\bar{V}_1 \subset V$.*

Proof. First, we consider the case where $f = k^* \varepsilon_x^\varphi |_{\partial V}$ for a pair (φ, x) i.e., the restriction of $k^* \varepsilon_x^\varphi$ on ∂V. Then, $g(y) = \mu_y^{*V}(f) = \int \hat{R}^{CV} k_y(z) d\varepsilon_x^\varphi(z) = \int k_y(z) d(\varepsilon_x^\varphi)^{CV}(z)$. Since $\operatorname{supp} (\varepsilon_x^\varphi)^{CV} \subset \partial V \cup (\operatorname{supp} \varepsilon_x^\varphi \backslash V)$, $g(y)$ is continuous on V. Letting W be open satisfying $y_0 \in W \subset \bar{W} \subset V$,

$$\int g(y) d\mu_{y_0}^{*W}(y) = \int \hat{R}^{CW} k_{y_0}(z) d(\varepsilon_x^\varphi)^{CV}(z) = \int k_{y_0}(z) d(\varepsilon_x^\varphi)^{CV}(z) = g(y_0),$$

which means that g is *harmonic on V.

Next, we proceed to the general case. Given $\varepsilon > 0$, by Prop. 2.3, there exist $\mu_1, \mu_2 \in \mathcal{M}_0$ such that $|f - (k^*\mu_1 - k^*\mu_2) < \varepsilon$ on ∂V. By the previous observation $\mu^{*V}(g')$ is *harmonic on V, where $g' = k^*\mu_1|_{\partial V} - k^*\mu_2|_{\partial V}$ and $g(y) = \mu_y^{*V}(f)$ is *harmonic since it is the uniform limit of *harmonic functions. The boundedness of g is an immediate consequence of Lemma 3.1.

For a relatively compact open set V and $x, y \in V$ we define

$$g(x, y) = k(x, y) - \hat{R}^{CV}k_y(x)\,.$$

The function of x, $g(x, y)$ is a potential on V for every $y \in V$. The following lemma admits the existence of *Evans functions.

Lemma 3.3. *Let V be a relatively compact open set and $y_0 \in V$. Then, for every $x \in V$, there exists a finite continuous $w_x^* \in \mathcal{U}^*(V)$ such that*

1) $w_x^*(y_0) \leq 1$,

2) if $\{y_l\} \subset V$, $y_l \to z \in \partial V$, $\lim_{l\to\infty} g(x, y_l) > 0$ then $\lim_{l\to\infty} w_x^*(y_l) = \infty$.

Proof. Letting $\{V_n\}$ be a sequenc of open sets satisfying $\bar{V}_n \subset V_{n-1}$, $V = \bigcup_{n=1}^{\infty} V_n$ and $x \in V_1$, we consider $\{\varphi_n\} \subset C(X)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n = 0$ on $\bar{V}_n$, $\varphi_n = 1$ on CV_{n+1}, and define $u_n^* = k^*\varepsilon_x^{\varphi_n}$. Then, u_n^* is continuous (Remark 2.2) and from $\hat{R}^{CV}k_y(x) = \mu_y^{*V}(k_x^*)$, $w_n^*(y) = u_n^*(y) - \hat{R}^{CV}k_y(x)$ is finite continuous and $w_n^* \in \mathcal{U}^*(V)$.

$$w_n^*(y) = \int_0^1 R^{\{\varphi_n > \alpha\}}k_y(x)d\alpha - \hat{R}^{CV}k_y(x) \leq \hat{R}^{CV_n}k_y(x) - \hat{R}^{CV}k_y(x)$$

implies $\lim_{u\to\infty} w_n^*(y) = 0$ and the convergence is locally uniform in V. Thus, there is a sequence of integers $\{m(n)\}$ such that $w_{m(n)}^* \leq 2^{-n}$ on V_n. The function $w_x^*(y) = \sum_{n=1}^{\infty} w_{m(n)}^*(y)$ is clearly finite continuous and belongs to $\mathcal{U}^*(V)$. Let $\{y_l\} \subset V$ be such that $y_l \to z \in \partial V$, $\lim_{l\to\infty} g(x, y_l) > 0$ and let $\lim_{l\to\infty} w_x^*(y_l) < \infty$. Then, for sufficiently large l, $g(x, y_l) \geq \alpha > 0$ and $y_l \notin V_{m(n_0+1)}$ where n_0 is an integer such that $n_0\alpha > \sup_l w_x^*(y_l)$. Now a contradiction $w_x^*(y_l) \geq n_0\alpha$ is derived since $g(x, y_l) = k(x, y_l) - \hat{R}^{CV}k_{y_l}(x) = w_{m(n)}^*(y_l)$ for every $n \leq n_0$. We may find $c > 0$ so that cw_x^* satisfies 1).

Given a relatively compact open set V, let consider a countable dense subset $\{x_j\}$ of V and define $w^*(y) = \sum_{j=1}^{\infty} (1/2^j) w_{x_j}^*(y)$, where $w_{x_j}^*(y)$ is the function in the previous lemma. We have therefore $w^* \in \mathcal{U}^*(V)$, $w^* \geq 0$ and $w^*(y_0) < \infty$. In view of this function we have

Lemma 3.4. *Let $f \in C(\partial V)$ and $\varepsilon > 0$. Then the function $\bar{v}^*(y) = \mu_y^{*V}(f) + \varepsilon w^*(y)$ satisfies $\liminf_{y\to z} \bar{v}^*(y) \geq f(z)$ for every $z \in \partial V$.*

Proof. Fix $z \in \partial V$ and consider $\{y_l\} \subset V$ such that $y_l \to z$. First suppose that $g(x, y_l) \to 0$ for every $x \in V$ and prove $\mu_{y_l}^{*V} \to \varepsilon_z$ vaguely. To see that it is sufficient to show $\mu_{y_l}^*(k^*\varepsilon_x^\varphi) \to k^*\varepsilon_x^\varphi(z)$ for every pair (φ, x) such that $z \notin \operatorname{supp} \varphi$ since the family $\{k^*\varepsilon_x^\varphi|_{\partial V};\ \varepsilon_x^\varphi \in \mathcal{M}_0$ and $z \notin \operatorname{supp} \varphi\}$ is total in $C(\partial V)$. In this situation it is proved $\lim_{l\to\infty} \hat{R}^{CV}k_{y_l}(\xi) = k_z(\xi)$ for $d\varepsilon_x^\varphi$-almost all ξ. For, if $\xi \notin \bar{V}$

then $\hat{R}^{CV}k_{y_l}(\xi)=k_{y_l}(\xi)$ and $\hat{R}^{CV}k_{y_l}(\xi)=k_{y_l}(\xi)-g(\xi, y_l)$ for $\xi\in V$. Thus we have $\lim_{l\in\infty} R^{CV}k_{y_l}(\xi)=k_z(\xi)$ for $\xi\in V\cup C\bar{V}$. For $\xi\in\partial V$, let $A=\bigcup_{l=1}^{\infty}\{\hat{R}^{CV}k_{y_l}<R^{CV}k_{y_l}\}$. The set A is semipolar, then by Prop. 2.4 with our assumption $\varepsilon_x^{\varphi}(A\cup\{z\})=0$, and $\hat{R}^{CV}k_{y_l}(\xi)=k_{y_l}(\xi)$ whenever $\xi\in\partial V\backslash(A\cup\{z\})$. As the function $\hat{R}^{CV}k_{y_l}(\xi)$ of ξ is uniformly bounded on supp ε_x^{φ} for sufficiently large l, $\mu_{y_l}^{*V}(k^*\varepsilon_x^{\varphi})=(\hat{R}^{CV}k_{y_l})\to\varepsilon_x^{\varphi}(k_z)=k^*\varepsilon_x^{\varphi}(z)$.

Next, suppose that $\lim_{l\to\infty} g(x, y_l)>0$ for some $x\in V$. Then there is $x_j\in V$ such that $\lim_{l\to\infty} g(x_j, y_l)>0$. For if $\lim_{l\to\infty} g(x_j, y_l)=0$ for every x_j, then since the family $\{g(x, y_l);\ l=1, 2, \cdots\}$ is locally uniformly bounded in V for sufficiently large l it is equi-continuous, which implies that $\lim_{l\to\infty} g(x, y_l)=0$ on V. Hence, by Lemma 3.3, we have for suitable M, $\liminf_{l\to\infty} \bar{v}^*(y_l)\geq -M+\varepsilon\liminf_{l\to\infty} w^*(y_l)\geq -M+\varepsilon(1/2^j)\liminf_{l\to\infty} w_{x_j}^*(y_l)=\infty$. Hence, we have $\liminf_{y\to z} \bar{v}^*(y)\geq f(z)$ in any case.

We can now prove that $\mathcal{U}^*(U)$ possesses the sheaf property, that is

Lemma 3.5. *Let U_1, U_2 be open and $u|_{U_i}\in\mathcal{U}^*(U_i)$ $(i=1, 2)$ then $\mu^{*V}(u)\leq u$ for every relatively compact open set V with $\bar{V}\subset U_1\cup U_2$.*

For, let $f\in C(\partial V)$ with $f\leq u$ and w^* be the function in Lemma 3.4, i.e., $w^*\in\mathcal{U}^*(V)$, $w^*\geq 0$ and $w^*(y_0)<\infty$, where y_0 is a fixed point of V. Consider $\bar{u}(y)=u(y)-[\mu_y^{*V}(f)-\varepsilon w^*(y)]$. For every $y\in V$, we may find a relatively compact open neighborhood W of y such that $\bar{W}\subset V\cap U_1$ or $\bar{W}\subset V\cap U_2$. In both cases $\mu_y^{*W}(\bar{u})\leq\bar{u}(y)$ and $\liminf_{y\to z}\bar{u}(y)\geq\liminf_{y\to z} u(y)-\limsup_{y\to z}[\mu_y^{*V}(f)-\varepsilon w^*(y)]\geq u(z)-f(z)\geq 0$ for every $z\in\partial V$. Hence, by Prop 3.1, Prop. 3.2 and [7] Th. 1.3.1, we have $\bar{u}\geq 0$ on V. Especially, $u(y_0)+\varepsilon w^*(y_0)\geq\mu_{y_0}^{*V}(f)$ this implies $u(y_0)\geq\mu_{y_0}^{*V}(f)$ since ε is arbitray and $w^*(y_0)<\infty$, and finally we have $\mu_{v_0}^{*V}(u)\leq u(y_0)$.

Here, it is clear that $\mathcal{U}^*$: $U\to\mathcal{U}^*(U)$ defines a hyperharmonic sheaf and every relatively compact open set is an *MP*-set with respect to $\mathcal{U}^*$. Further we have

Proposition 3.3. *Every relatively compact open set V is resolutive with respect to $\mathcal{U}^*$. The *harmohic measure of V at y is μ_y^{*V}.*

In order to obtain the convergence property of *harmonic functions, we follow the idea of R-M. Hervé [9], Lemma 29.1 and Lemma 29.2

Throughout the following lemmas, let U be an open set, $y_0\in U$ and V, V_1 be relatively compact open sets such that $y_0\in V_1\subset\bar{V}_1\subset V\subset\bar{V}\subset U$. We assume that V is connected.

Lemma 3.6. *For every $\varepsilon>0$ there exists a neighborhood Y of y_0 such that $\bar{Y}\subset V_1$ and* $\sup\{|[k_y(x)-\hat{R}^{CV}k_y(x)]-[k_{y_0}(x)-\hat{R}^{CV}k_{y_0}(x)]|\,;\,x\in\partial V_1\}\leq\varepsilon$ *for $y\in Y$.*

Proof. For $x\in\partial V_1$ there correspond a neighborhood $U(x)$ of x and a neighborhood $U(y_0)$ of y_0 such that $x'\in U(x)$ and $y\in U(y_0)$ implies $|k_y(x')-k_{y_0}(x')|<\varepsilon/2$ and $|\hat{R}^{CV}k_y(x')-\hat{R}^{CV}k_{y_0}(x')|<\varepsilon/2$. For the latter fact, we note that the family $\{\hat{R}^{CV}k_y\colon y\in U(y_0)\}$ is equi-continuous on a relatively compact open set W such that $U(y_0)\cap\bar{W}=\emptyset$. The compactness argument derives the result.

Now, let Q be a dense subset of U. Then for each $x\in\partial V_1$ there is $y\in(V\setminus\{x\})\cap Q$ such that $k_y(x)\neq\hat{R}^{CV}k_y(x)$, thus we have a potential $q=\sum_{i=1}^{d}k_{y_i}$, where $y_i\in(V\setminus\{x\})\cap Q$, fins $q>\hat{R}_q^{CV}$ on ∂V_1. Select $c>0$ so that $c\inf\{q(x)-\hat{R}_q^{CV}(x);\,x\in\partial V_1\}\geq\varepsilon$ and put $p=cq$. By the above lemma,

$$|[k_y-\hat{R}^{CV}k_y]-[k_{y_0}-\hat{R}^{CV}k_{y_0}]|\leq p-\hat{R}_p^{CV}\qquad\text{on } \partial V_1 \text{ for every } y\in Y\,.$$

We define:

$$v_y=R^{CV}k_{y_0}-R^{CV}k_y+\hat{R}_p^{CV}\qquad\text{for every } y\in Y\,.$$

Lemma 3.7. $v_y\leq k_{y_0}-k_y+p$ *on CV_1 for every $y\in Y$.*

This is an immediate consequence of boundary minimum principle (cf. [10] Lemma 2.2), since $s=k_{y_0}-k_y+p-v_y$ is superharmonic on $V\setminus\bar{V}_1$, dominates $-(k_y-\hat{R}^{CV}k_y)$ on V where $k_y-\hat{R}^{CV}k_y$ is a potential on V and $\liminf s\geq 0$ at every point of ∂V_1.

For every $x\in X$, let $\mathcal{W}_x$ be a family of relatively compact open neighborhood W of x such that (1) $\bar{W}\subset V$ if $x\in V$ (2) $\bar{W}\subset C\bar{V}$ if $x\notin\bar{V}$ (3) $\bar{W}\cap\bar{V}_1=\emptyset$ if $x\in\partial V$, and let $\mathcal{W}=\{\mathcal{W}_x;\,x\in X\}$. Then

Lemma 3.8. *v_y is $\mathcal{W}$-nearly hyperharmohic for every $y\in Y$, i.e.,* $\bar{H}_{v_y}^{W}=\mu^{W}(v_y)\leq v_y$ *on W for every $W\in\mathcal{W}$. Thus $\hat{v}_y\in\mathcal{U}(X)$.*

Proof. It is clear that v_y is harmonic or superharmonic on W in the case of $\bar{W}\subset V$ or $\bar{W}\subset C\bar{V}$ respectively. For a delicate case where $W\in\mathcal{W}_x$ and $x\in\partial V$ we follow the idea of [9] lemma 29.1. Since by Lemma 3.7, $v_y\leq k_{y_0}-k_y+p$ on ∂W we have $\mu_z^W(v_y)\leq v_y(z)$ whenever $z\in W\setminus V$. On the other hand, when $z\in W\cap V$ we define

$$v'=\begin{cases} v_y & \text{on}\quad \partial(W\cap V)\cap\partial W\\ \bar{H}_{v_y}^{W} & \text{on}\quad \partial(W\cap V)\cap W\end{cases}$$

Then $\bar{H}_{v'}^{W\cap V}=\bar{H}_{v_y}^{W}$ ([7]. Prop. 2.4.4) and, since $v'\leq v_y$ by the above observation, $\mu_z^W(v_y)=\bar{H}_{v_y}^W(z)\leq\bar{H}_{v_y}^{W\cap V}(z)=\bar{H}_{(k_{y_0}-k_y+p)}^{V}(z)=v_y(z)$.

Lemma 3.9. *There exists a measure $\nu \in M_K^+(X)$ satisfying*

$$\mu_y^{*V} \leq \mu_{y_0}^{*V} + \nu \qquad \text{for every } y \in Y .$$

Proof. $v_y + \mathrm{R}^{CV} k_y = \mathrm{R}^{CV} k_{y_0} + \mathrm{R}^{CV} p$ by definition, which implies also $\hat{v}_y + \hat{\mathrm{R}}^{CV} k_y = \hat{\mathrm{R}}^{CV} k_{y_0} + \hat{\mathrm{R}}^{CV} p$. Thus $\hat{v}_y$ is a potential and $S(\hat{v}_y) \subset \partial V$, hence $\hat{v}_y = k\lambda_y$ for some $\lambda_y \in M^+(X)$. Since $p = \sum_{i=1}^{d} ck_{y_i}$ $(y_i \in V)$ $\hat{\mathrm{R}}^{CV} p(x) = k\nu(x)$, where $\nu = \sum_{i=1}^{d} c\mu_{y_i}^{*V}$. Therefore we may conclude the assertion from the equality $\lambda_y + \mu_y^{*V} = \mu_{y_0}^{*V} + \nu$.

Proposition 3.4. *Let U be an open subset of X and let $\{h_n^*\}$ be an increasing sequence of functions in $\mathcal{H}^*(U) = \mathcal{U}^*(U) \cap [-\mathcal{U}^*(U)]$ such that $g^* = \sup_n h_n^*$ is finite on a dense subset Q of U. Then $g^* \in \mathcal{H}^*(U)$, i.e., the harmonic sheaf $\mathcal{H}^*$ possesses the Doob convergence property.*

Proof. Let $y_0 \in U \cap Q$ and let V be a relatively compact open neighborhood of y_0 such that $\bar{V} \subset U$. By Lemma 3.9, there exist a neighborhood Y of y_0, a finite set $\{y_i : i = 1, 2, \cdots, d\} \subset Q \cap V$ and $c > 0$ such that $\mu_y^{*V} \leq \mu_{y_0}^{*V} + c \sum \mu_{y_i}^{*V}$ for every $y \in Y$. For $\varepsilon > 0$ there is an integer n_0 such that $0 \leq h_m^*(y_i) - h_n^*(y_i) < \varepsilon$ for $i = 0, 1, 2, \cdots, d$ whenever $m > n \geq n_0$. Then, if $y \in Y$ and if $m > n \geq n_0$ we have

$$0 \leq h_m^*(y) - h_n^*(y) = \int (h_m^* - h_n^*) d\mu_y^{*V} \leq \int (h_m^* - h_n^*) d\mu_{y_0}^{*V} + c \sum_{i=1}^{d} \int (h_m^* - h_n^*) d\mu_{y_i}^{*V}$$
$$< (1 + cd)\varepsilon .$$

This implies that g^* is the locally uniform limit of $\{h_n^*\}$.

As a consequence of the above consideration, we conclude:

Theorem 3.1. *$(X, \mathcal{U}^*)$ is a P-harmonic space in the sense of Constantinescu-Cornea which satisfies the Doob convergence property.*

Proof. We shall check the axioms of the harmonic space.

Axiom of positivity: this is derived from the following fact; for every $y \in X$ and for every neighborhood $U(y)$ of y there is a point $x \in U(y) \setminus \{y\}$ such that $k_x^*(y) > 0$, since k_x^* is *harmonic on $X \setminus \{x\}$.

Axiom of Convergence and *Axiom of Resolutivity*: they are a consequence of Prop. 3.4 and Prop. 3.3 respectively.

The definition of dual sheaf $\mathcal{U}^*$ implies *Axiom of Completeness*. Hence, it is enough to show that k_x^* is a potential on X. Let $\{X_n\}$ be an exhaustion of X, i.e., an increasing sequence of relatively compact open sets such that $\bar{X}_n \subset X_{n+1}$ and $\bigcup_{n=1}^{\infty} X_n = X$. Then for every $y \in X$, $\mu_y^{*X_n}(k_x^*) = \hat{\mathrm{R}}^{CX_n} k_y(x) \to 0$ as $n \to \infty$.

REMARKS 3.1.

(1) if we denote by $(X, \bar{\mathcal{U}}^*)$ the dual harmonic space with respect to the Green function $\bar{k}(x, y)=k(x, y)/g(y)$, where g is continuous and strictly positive on X, then $\bar{\mathcal{U}}^*=\mathcal{U}_g^*$ ([7], Excercises 1.2.3). For, by

$$\int \bar{k}_z(x)d\bar{\mu}_y^{*V}(z) = \hat{\mathrm{R}}^{CV}\bar{k}_y(x) = (\hat{\mathrm{R}}^{CV}k_y(x))/g(y) = \int \bar{k}_z(x)b(z)/g(y)d\mu_y^{*V}(z)$$

we obtain $d\bar{\mu}_y^{*V}(z)=g(z)/g(y)d\mu_y^{*V}(z)$.

(2) later (§ 6), we shall show that $(X, \mathcal{U})$ possesses the Doob convergence property if X has a Green function ([15]), and if $(X, \mathcal{U})$ is a Brelot space then so its dual $(X, \mathcal{U}^*)$ is also a Brelot space

4. Duality of Green functions and semipolarity

In this section, we establish a fundamental relation of the duality of Green functions: for each subset E of X and for every $x, y\in X$

$$\hat{\mathrm{R}}^E k_y(x) = \hat{\mathrm{R}}^{*E} k_x^*(y)\,.$$

First we consider a special case where E is open in the above relation.

Proposition 4.1. *Let G be an open subset of X and $x, y\in X$. Then*

$$\mathrm{R}^G k_y(x) = \mathrm{R}^{*G} k_x^*(y)\,.$$

Proof. Let x be fixed and consider the function of y: $\mathrm{R}^G k_y(x)$. It is non-negative and *hyperharmonic on X (Prop. 3.1). Further, by the property of Green functions, $\mathrm{R}^G k_y(x)=k_y(x)$ if $y\in G$. Thus, $\mathrm{R}^G k_y(x)=k_x^*(y)$ for every $y\in G$, which implies $\mathrm{R}^{*G}k_x^*(y)\leq \mathrm{R}^G k_y(x)$. Similarly we have $\mathrm{R}^G k_y(x)\leq \mathrm{R}^{*G}k_x^*(y)$.

REMARK 4.1. For $\mu\in M_K^+(X)$ we have

$$\mathrm{R}^G k\mu(x) = k\mu^{*G}(x)\,,$$
$$\mathrm{R}^{*G} k^*\mu(y) = k^*\mu^G(y)\,.$$

In fact,

$$\mathrm{R}^G k\mu(x) = \int [\int k_y(z)d\mu(y)]d\varepsilon_x^G(z) = \int \mathrm{R}^G k_y(x)d\mu(y)$$
$$= \int \mathrm{R}^{*G}k_x^*(y)d\mu(y) = \int k_x^*(y)d\mu^{*G}(y) = \int k_y(x)d\mu^{*G}(y) = k\mu^{*G}(x)\,.$$

Similarly we can derive the second equality.

Here we introduce a countable family $\mathcal{L}_0$ of measures: let

$$\mathcal{L} = \{\mu\in M_K^+(X);\ k\mu \text{ is finite and continuous}\}$$

and let $\mathcal{L}_0$ be a countable subfamily of $\mathcal{L}$ such that

i) $S(k\lambda)$ is sompact $\forall\lambda\in\mathcal{L}_0$,

ii) $\forall f\in C_K(X)\forall\varepsilon>0\exists\lambda', \lambda''\in\mathcal{L}_0 | f-(k\lambda'-k\lambda'')|<\varepsilon$ on X.

The existence of $\mathcal{L}_0$ is assured by [7], Th. 2.3.1.

Lemma 4.1. *For every relatively compact E of X there is a decreasing sequence $\{G_n\}$ of relatively compact open sets containing E and such that*

$$\hat{R}^E k\lambda = \widehat{\inf_n R^{G_n} k\lambda} \qquad \forall\lambda\in\mathcal{L}_0\,, \tag{4.1}$$

$$\sup_n \lambda^{*G_n}(X)<\infty \qquad \forall\lambda\in\mathcal{L}_0\,. \tag{4.2}$$

Proof. Let $\mathcal{L}_0=\{\lambda_j; j=1, 2, \cdots\}$. If $k\lambda$ is finite and continuous

$$R^E k\lambda = \inf\{R^G k\lambda;\ G\supset E \text{ open}\}$$

and using Choquet's lemma ([7], p. 169) there is a sequence $\{U_n\}$ of open sets such that $E\subset U_n$ and $\hat{R}^E k\lambda=\widehat{\inf_n R^{U_n}k\lambda}$. We may assume that $\{U_n\}$ is a decreasing sequence of relatively compact open sets. Thus we have $\{G_i^j; i, j=1, 2, \cdots\}$ such that

$$G_1^j\supset G_2^j\supset\cdots\supset G_i^j\supset\cdots \qquad j=1, 2, \cdots\,,$$

$$G_i^j\supset E\,,\quad \text{relatively compact},\ i, j=1, 2, \cdots\,,$$

$$\hat{R}^E k\lambda_j = \widehat{\inf_i R^{G_i^j} k\lambda_j} \qquad j=1, 2, \cdots\,.$$

Clearly the set $G_n=\bigcap_{i,j=1}^{n} G_i^j$ satisfies (4.1). For (4.2), we consider a finite system $\{x_i; i=1, 2, \cdots, N\}$ such that

$$\sum_{i=1}^{N} k_{x_i}^* \geq c>0 \qquad \text{on} \quad \bar{G}_1$$

Since supp $\lambda^{*G_n}\subset\bar{G}_1$ for every $\lambda\in\mathcal{L}_0$, by Prop. 4.1,

$$c\lambda^{*G_n}(X)\leq\int\sum_{i=1}^{N} k_{x_i}^*(y)d\lambda^{*G_n}(y) = \sum_{i=1}^{N}\int R^{*G_n}k_{x_i}^*(y)d\lambda(y)$$

$$= \sum_{i=1}^{N}\int R^{G_n}k_y(x_i)d\lambda(y)\leq\sum_{i=1}^{N} R^{G_n}k\lambda(x_i)\leq\sum_{i=1}^{N} k\lambda(x_i)<\infty\,.$$

Lemma 4.2. *Let $\{\nu_n; n=1, 2, \cdots\}$ be a sequence of measures in $M_K^+(X)$ such that*

i) *supp $\nu_n\subset K$ for all n, where K is a compact subset of X,*

ii) *$\{k\nu_n; n=1, 2, \cdots\}$ is decreasing,*

iii) *$\nu_n\to\nu$ vaguely.*

Then $k\nu=\widehat{\inf_n k\nu_n}$.

Proof. Let X_∞ be the one-point compactification of X and $f\in C^+(X_\infty)$. It is readily seen ([7], Th. 8.1.1.1)

$$f\cdot k\nu_n(x) = \int f(y)k(x, y)d\nu_n(y)$$

and $f\cdot k\nu_n$ is non-negative and harmonic on $X\backslash\operatorname{supp} f$. Since, for a compact set L with $L\cap \operatorname{Supp} f=\emptyset$,

$$\sup \{\int f(y)k(x, y)d\nu_n(y);\ x\in L\}$$
$$\leq (\sup \{k(x, y);\ x\in L, y\in(\operatorname{supp} f)\cap K\}) \int f(y)d\nu_n(y) < \infty$$

$\{f\cdot k\nu_n;\ n=1, 2, \cdots\}$ is equi-continuous on $X\backslash\operatorname{supp} f$. From iii) we deduce $f\cdot k\nu_n \to f\cdot k\nu$ on $X\backslash\operatorname{supp} f$, therefore the convergence is locally uniform on $X\backslash\operatorname{supp} f$, which implies that $T\text{-}\lim_{n\to\infty} k\nu_n = k\nu$ ([7], p. 288) and finally $k\nu=\widehat{\inf_n k\nu_n}$ ([7], Prop. 11.2.8).

REMARK 4.2. After proving Theorem 4.1, we may conclude, under the assumption of Lemma 4.1, that

$$\lambda^{*G_n} \leftarrow \lambda^{*E} \quad \text{vaguely for every } \lambda\in\mathcal{L}_0\,.$$

For, using the same notations as in Lemma 4.1, it is clear that $\{\lambda^{*G_n}\}$ satisfies the hypothesis i) of Lemma 4.2. Now suppose that $\lambda^{*G_n} \to \nu$ vaguely. Since $k\lambda^{*G_n}(x) = \mathrm{R}^{G_n}k\lambda(x)$, $\{k\lambda^{*G_n}\}$ is decreasing. Hence $k\nu=\widehat{\inf_n \mathrm{R}^{G_n}k\lambda} = \hat{\mathrm{R}}^E k\lambda = k\lambda^{*E}$, which implies $\nu=\lambda^{*E}$ by the property of Green functions.

Now we prove

Theorem 4.1. *Let E be an arbitrary subset of X and $x, y\in X$. Then*

$$\hat{\mathrm{R}}^E k_y(x) = \hat{\mathrm{R}}^{*E} k_x^*(y)\,.$$

Proof. From a property of balayage functions it is sufficient to prove the case where E is relatively compact. Let $\{G_n\}$ be a sequence in Lemma 4.1. Then the set

$$A = \cup\, \{\widehat{\inf_n \mathrm{R}^{G_n}k\lambda} < \inf_n \mathrm{R}^{G_n}k\lambda;\ \lambda\in\mathcal{L}_0\}$$

is semipolar. We assert that $\varepsilon_x^{G_n} \to \varepsilon_x^E$ vaguely for every $x\in X\backslash A$. In fact, if $x\notin A$ and $\lambda\in\mathcal{L}_0$ then $\inf_n \mathrm{R}^{G_n}k\lambda(x) = \widehat{\inf_n \mathrm{R}^{G_n}k\lambda(x)} = \hat{\mathrm{R}}^E k\lambda(x)$ by Lemma 4.1. Thus $\int k\lambda d\varepsilon_x^{G_n} \to \int k\lambda d\varepsilon_x^E$ and the assertion is derived,since $\{k\lambda;\ \lambda\in\mathcal{L}_0\}$ is total in $C_K(X)$. Applying Lemma 4.2 to the dual harmonic space $(X, \mathcal{U}^*)$, $k^*\varepsilon_x^E(y) = \liminf_{y'\to y} [\lim_{n\to\infty} k^*\varepsilon_x^{G_n}(y')]$ for each $x\notin A$, that is $\hat{\mathrm{R}}^E k_y(x) = \liminf_{y'\to y} [\inf_n \mathrm{R}^{G_n}k_{y'}(x)]$ for every $x\in X\backslash A$ and $y\in X$. In the same way, letting $A' = \cup\, \{\widehat{\inf_n \mathrm{R}^{*G_n}k^*\mu} <$

$\inf_n \hat{R}^{*G_n}k^*\mu$; $\mu\in\mathcal{M}_0\}$, where $\mathcal{M}_0$ is the family of measurex measures defined in § 2.

$\hat{R}^{*E}k_x^*(y)=\liminf_{x'\to x}[\inf_n R^{*G_n}k_{x'}^*(y)]$ for every $y\in X\backslash A'$ and $x\in X$. Thus, for $x\notin A$, $y\in X$ we have $\hat{R}^E k_y(x)=\liminf_{y'\to y}[\inf_n R^{*G_n}k_x^*(y')]=\widehat{\inf_n R^{*G_n}k_x^*(y)}\geq \hat{R}^{*E}k_x^*(y)$. And, similarly, for $y\notin A'$, $x\in X$ we have $\hat{R}^{*E}k_x^*(y)\geq\hat{R}^E k_y(x)$. Therefore $\hat{R}^E k_y(x)=\hat{R}^{*E}k_x^*(y)$ whenever $x\in X\backslash A$ and $y\in X\backslash A'$, i.e., the theorem is proved except semipolar sets.

Next, we shall remove the restriction. Let, for arbitrary x, $y\in X$, $u_y(x)=\hat{R}^{*E}k_x^*(y)$ and $v_y(x)=\hat{R}^E k_y(x)$. The function u_y and v_y are hyperhar-harmoinc since $u_y=k\varepsilon_y^{*E}$, and $u_y(x)=v_y(x)$ if $x\notin A$ and $y\notin A'$. Now, fix $y\in X\backslash A'$ u_y coincides with v_y except the polar set A, hence $\hat{R}^E k_y(x)=\hat{R}^{*E}k_x^*(y)$ for every $x\in X$ and $y\in X\backslash A'$. In the same way, $u_x^*(y)=\hat{R}^{*E}k_x^*(y)=u_y(x)$ and $v_x^*(y)=\hat{R}^E k_y(x)=v_y(x)$ are *hyperharmonic and $u_x^*=v_x^*$ except the *polar set A' thus $u_x^*(y)=v_x^*(y)$ for all y, which proves the theorem.

We conclude this section with the theorem that the sets are semipolar if and only if they are *semipolar.

Before proceed the problem we remark that the polarity and the *polarity are same. This is a direct consequence of Th. 4.1.

To prove the theorem, we introduce some notations and lemmas that were considered by Talyor [18]. For a set E and $u\in\mathcal{U}_+(X)$, let $(\hat{R}^E)^0u=u$, $(\hat{R}^E)^1u=\hat{R}^E u$ and $(R^E)^{j+1}u=\hat{R}^E[(\hat{R}^E)^j u]$ for $j\geq 1$. Then:

Lemma 4.3. ([18] Prop. 4.7.) *Let B be a relatively compact and totally thin. Then, for every $u\in\mathcal{U}_+(X)$ which is bounded on B there exists $\{B_n; n=1, 2, \cdots\}$ such* m*that*

$$B=\bigcup_{u=1}^{\infty}B_n \text{ and } \lim_{j\to\infty}(\hat{R}^{B_n})^j u=0 \text{ for every } n\,.$$

Proof. Let p be a continuous strict potential on X and let $w=\hat{R}^B u+p$. The function w is finite and $\hat{R}^B u\leq w$, thus $(\hat{R}^B)^2 u\leq\hat{R}^B w$. On the other hand, $\hat{R}^B w\leq\hat{R}^B u+\hat{R}^B p<\hat{R}^B u+p=w$, since B is totally thin and p is strict. Now let $B_n=\{x\in B;\ \hat{R}^B w(x)\leq(1-1/n)w(x)\}$. We have clearly $B=\bigcup_{n=1}^{\infty}B_n$. By induction, $(\hat{R}^{B_n}\hat{R}^B)^j w\leq(1-1/n)^j w$, which implies $(\hat{R}^{B_n})^{2j+1}u\leq(\hat{R}^{B_n}\hat{R}^B)^{j-1}\hat{R}^{B_n}(\hat{R}^B)^2 u\leq(\hat{R}^{B_n}\hat{R}^B)^j w\leq(1-1/n)^j w$. This completes the proof since $\{(\hat{R}^{B_n})^j u; j=1, 2, \cdots\}$ is decreasing.

For the simplicity of notations, we write $\langle\lambda, f\rangle$ instead of $\int f d\lambda$. By Th. 4.1 and by induction, we can easily prove:

Lemma 4.4. For $\lambda\in M_K^+(X)$, $A\subset X$ and for every integer n we have

$$\langle\lambda, (\hat{R}^A)^n k\mu\rangle = \langle\mu, (\hat{R}^{*A})^n k^*\lambda\rangle .$$

Theorem 4.2. *Semipolarity coincides with *semipolarity.*

Proof. We prove that a semipolar set A is *semipolar. The converse is similar. We may assume that A is relatively compact and totally thin. Let $\{a_i;\ i=1, 2, \cdots\}$ be dense in X, $\{W_j; j=1, 2, \cdots\}$ be a base of relatively compact open sets for the topology of X. We consider pairs such that $a_i \in W_j$. They form a countable family of pairs $\{(z_n, V_n);\ n=1, 2, \cdots\}$. Fix a relatively compact open set G with $\bar{A}\subset G$. Since $k\varepsilon_{z_n}^{*CV_n} = \hat{R}^{CV_n}k_{z_n}$ is bounded on $\bar{G}$ for each pair (z_n, V_n), we may find a sequence $\{c_n\}$ of positive numbers such that

$$\sum_{n=1}^{\infty} c_n \varepsilon_{z_n}^{*CV_n}(X) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} c_n k\varepsilon_{z_n}^{*CV_n} \leq M < \infty \qquad \text{on} \quad \bar{G} .$$

Letting $\mu^* = \sum_{n=1}^{\infty} c_n \varepsilon_{z_n}^{*CV_n}$, we have $k\mu^* \in \mathcal{U}_+(X)$, and by Lemma 4.3, there is $\{A_n\}$ satisfying $A = \bigcup_{n=1}^{\infty} A_n$ and $\lim_{j\to\infty} (\hat{R}^{A_n})^j k\mu^* = 0$ for all n. The proof will be completed if we show that A_n is *semipolar. From now on we fix A_n. It is easily seen that there exists $\lambda \in M_K^+(X)$ such that $\langle\lambda, k\mu^*\rangle < \infty$ and $k^*\lambda > 0$ on A. Putting $w_j^* = (\hat{R}^{*A_n})^j k^*\lambda$ and define $w_\lambda^* = \inf_j w_j^* = \lim_{j\to\infty} w_j^*$. We claim that $\hat{w}_\lambda^* = 0$; for w_λ^* is nearly *hyperharmonic and, by Lemma 4.4,

$$\langle\mu^*, w_\lambda^*\rangle = \lim_{j\to\infty} \langle\mu^*, w_j^*\rangle = \lim_{j\to\infty} \langle\lambda, (\hat{R}^{A_n})^j k\mu^*\rangle = 0 ,$$

that is, $\langle\varepsilon_{z_n}^{*CV_n}, w_\lambda^*\rangle = 0$ for every n. If we consider a subsequence $\{(z_{n(m)}, V_{n(m)});\ m=1, 2, \cdots\}$ of $\{(z_n, V_n)\}$ so that $z_{n(m)} = y$ and $V_{n(m)}$ tends to y, we have $\hat{w}_\lambda^*(y) = \lim_{m\to\infty} \langle\varepsilon_y^{*CV_{n(m)}}, w_\lambda^*\rangle$. Thus we obtain $\hat{w}_\lambda^* = 0$ on a dense subset of X, hence $\hat{w}_\lambda^* = 0$. The fact that $\{w_\lambda^* > 0\} = \{w_\lambda^* > \hat{w}_\lambda^*\}$ is *semipolar implies $A_n \cap \{w_\lambda^* > 0\}$ is *semipolar. On the other hand, since we have $w_j^* = R^{*A_n} w_j^*$ on A_n and $w_{j+1}^* = \hat{R}^{*A_n} w_j^*$, $B_j = A_n \cap \{w_\lambda^* = 0\} \cap \{w_{j+1}^* < w_j^*\}$ is *semipolar. On account of $A_n \cap \{w_\lambda^* = 0\} = \bigcup_{j=0}^{\infty} B_j$, $A_n = [\{w_\lambda^* > 0\} \cap A_n] \cap \bigcup_{n=0}^{\infty} B_n$ is *semipolar.

5. The bidual

In the previous sections we have developed the duality theory of harmonic spaces, that is, given a Green function $k(x, y)$ on a harmonic space $(X, \mathcal{U})$, we can construct a dual harmonic space $(X, \mathcal{U}^*)$ such that the function $k(x, y) = k_x^*(y)$ of y is a *potential with carrier $\{x\}$ for every $x \in X$. In this section we prove that k_x^* satisfies the condition 3) of Green functions. Thus we may consider the dual of $(X, \mathcal{U}^*)$, and it is revealed that this bidual coincides with $(X, \mathcal{U})$.

To this end, we follow the idea of [19] to make *potentials more restrictive, which corresponds to consider a quotient sheaf of the original $\mathcal{U}^*$.

Proposition 5.1. ([19], Prop. 1.5) *There exists a strict positive $g_0 \in C(X)$ such that $\bar{k}(x, y)=k(x, y)/g_0(y)$ satisfies, besides 1)~3) in the definition of Green functions,*

4) *there is a measure $\eta \in M^+(X)$ with $\bar{k}^*\eta=1$,*
5) *$\bar{k}^*\mu \in C_0(X)$ for every $\mu \in \mathcal{M}_0$.*

The proof is carried out as in [19]: so we only sketch the outline. Since, by Prop. 2.3, every continuous function with compact support is approximated by the difference of *potentials on the support, there exists $\sigma \in \mathcal{M}_0$ such that $k^*\sigma>0$ on a given compact set. Hence, letting $\{X_n\}$ be a compact exhaustion of X, we may obtain $c_n>0$ and $\sigma_n \in \mathcal{M}_0$ satisfying $c_n\sigma_n(X)>1/2^n$ and $0<c_n k^*\sigma_n <1/2^n$ on $\bar{X}_n$. Let $\mu_0=\sum_{n=1}^{\infty} c_n\sigma_n$. On the other hand, for each $\mu_j \in \mathcal{M}_0$ there is $\nu_j \in M^+(X)$ such that $k^*\nu_j \in C(X)$ and $k^*\mu_j \in o(k^*\nu_j)$. We select $b_j^i>0$ so that $\sum_{j=1}^{\infty} b_j^i(\sup \{k^*\nu_j; \bar{X}_j\})<\infty$, $\sum_{j=1}^{\infty} b_j^i\nu_j(X_i)<\infty$ and $b_j^{i+1}\leq b_j^i$; and then let $\nu_0=\sum_{j=1}^{\infty} b_j^j\nu_j$. The measure $\eta=\mu_0+\nu_0$ and the function $g_0=k^*\eta$ are the required.

Corollary 5.1. *For each compact K there is $\mu \in \mathcal{M}_0$ such that $\bar{k}^*\mu \in C_0(X)$ and $\bar{k}^*\mu>0$ on K.*

In the following, we assume that Green functions $k(x, y)$ satisfy

1) *lower semicontinuous on $X \times X$ and continuous off the diagonal,*
2) *$x \to k(x, y)$ is a potential on X with carrier $\{y\}$ for every $y \in X$,*
3) *every potential p possessing a compact carrier can be represented uniquely by $\mu \in M^+(X)$, i.e., $p(x)=\int k(x, y)d\mu(y)$,*
4) *there exists $\eta \in M^+(X)$ with $k^*\eta(y)=\int k(x, y)d\eta(x)=1$,*
5) *$k^*\mu \in C_0(X)$ for each $\mu \in \mathcal{M}_0$.*

We note that 1 is *superharmonic.

Lemma 5.1. *Let $\mu \in M^+(X)$ with $\mu(X)<\infty$, then $k\mu$ is a potential on X.*

Proof. We prove first that $\hat{R}^{CU}k\mu$ is continuous on U for every relatively compact open set U. Let $x_0 \in U$, $x_j \to x_0$ and let W, V be open such that $x_0 \in W \subset \bar{W} \subset V \subset \bar{V} \subset U$.

$$\hat{R}^{CU}k\mu(x)=\int \hat{R}^{CU}k_y(x)d\mu(y)=\int_V \hat{R}^{CU}k_y(x)d\mu(y)+\int_{CV}\hat{R}^{CU}k_y(x)d\mu(y)\,.$$

Let $f_j(y)=\hat{R}^{CU}k_y(x_j)$, $j=1, 2, \cdots$, and let each term of the last integrals be $u(x)$

and $v(x)$ respectively. It is trivially seen that $f_j(y) \to f_0(y)$ for every y. It is also easily seen that $\sup_j (\sup \{f_j(y); y \in V\}) < \infty$. As the remaining case, $\sup_j (\sup \{f_j(y); y \in CV\}) \leq \sup_j (\sup \{k_y(x_j); y \in CV\}) = \sup_j (\sup \{k^*_{x_j}(y); y \in CV\})$, since $R^{*W} k^*_{x_j} = k^*_{x_j}$ and $\sup_j (\sup \{k^*_{x_j}(y); y \in \partial(C\bar{W})\}) < \infty$ in view of $1 \in \mathcal{U}^*(X)$. Thus the Lebesgue convergence theorem is applied to obtain $v(x_j) \to v(x_0)$ and $u(x_j) \to u(x_0)$. The remainder of the proof is routin, i.e., $\hat{R}^{CV} k\mu$ is harmonic in U and $\hat{R}^{CK} k\mu \to 0$ when the compact sets K tend to X increasingly.

REMARK 5.1. Above proof shows also that $k\mu$ is a potential if it is finite and superharmonic, and under the assumption that 1 is superharmonic, if $\mu(X) < \infty$ then $k^*\mu$ is a *potential.

Let $\mathcal{F} = \{k^* \varepsilon^\varphi_x; \varepsilon^\varphi_x \in \mathcal{M}\}$ and $\mathcal{A}$ be the family of finite linear combinations of $\mathcal{F}$. Suggested by the proof of [19], Th. 2.5, we have

Lemma 5.2. *$\mathcal{A}$ is uniformly dense in $C_0(X)$.*

Proof. Let $\varepsilon^\varphi_x \in \mathcal{M}$. Then $K = \operatorname{supp} \varepsilon^\varphi_x$ is compact and $k^* \varepsilon^\varphi_x$ is finite continuous on X. By Cor. 5.1, there is a measure σ such that $k^*\sigma > k^* \varepsilon^\varphi_x$ on K and $k^*\sigma \in C_0(X)$. Further $k^*\sigma \geq k^* \varepsilon^\varphi_x$ since $k^* \varepsilon^\varphi_x$ is a *potential which is *harmonic on CK and $k^*\sigma \in \mathcal{U}^*_+(X)$. Hence $k^* \varepsilon^\varphi_x \in C_0(X)$. To prove the assertion of the lemma, let X_∞ be the one-point compactification of X and let $H = \{u^* + \alpha; u^* \in \mathcal{A}, \alpha \text{ is real}\}$, where all functions of $\mathcal{A}$ are defined to be 0 at the infinity ∞. Suppose $\mu_1, \mu_2 \in M^+(X_\infty)$ and $\mu_1 = \mu_2$ on H and let $\mu'_i = \mu_i | X$, $\mu''_i = \mu_i |_{\{\infty\}}$ $(i = 1, 2)$. Then $0 = \mu''_1(u^*) = \mu''_2(u^*)$ for every $u^* \in \mathcal{A}$. Thus, for $u^* \in \mathcal{A}$, $\mu'_1(u^*) = \mu'_2(u^*)$. We have, in particular, $\langle \varepsilon^\varphi_x, k\mu'_1 \rangle = \mu'_1(k^* \varepsilon^\varphi_x) = \mu'_2(k^* \varepsilon^\varphi_x) = \langle \varepsilon^\varphi_x, k\mu^\varphi_2 \rangle$ for every $\varepsilon^\varphi_x \in \mathcal{M}$. It follows that $k\mu'_1 = k\mu'_2$, thus we may conclude that $\mu'_1 = \mu'_2$ on account of Remark 1.1 (2), since, by Lemma 5.1, $k\mu'_1$ and $k\mu'_2$ are potentials on X, $\mu'_1(\alpha) + \mu''_1(\alpha) = \mu_1(\alpha) = \mu_2(\alpha) = \mu'_2(\alpha) + \mu''_2(\alpha)$ for real α implies $\mu''_1 = \mu''_2$. Thus $\mu_1 = \mu_2$, which is just to say that the set H is uniformly dense in $C(X_\infty)$. In particular, for $f \in C_0(X)$ and for $\varepsilon > 0$ there is $f^* \in H$ such that $|f - f^*| < \varepsilon$ on X_∞. Since f^* is of the form $\sum_{i=1}^{n} c_i u^*_i + \alpha$, where $u^*_i \in \mathcal{F}$, and $f(\infty) = \sum_{i=1}^{\infty} c_i u^*_i(\infty) = 0$, this means $|\alpha| < \varepsilon$ and $|f - \sum_{i=1}^{\infty} c_i u^*_i| < 2\varepsilon$ on X.

Approximating functions of a uniformly dense countable subfamily of $C_0(x)$ by functions in $\mathcal{A}$, we can obtain a countable family of positive measures $\{\sum_{i=1}^{n} c_i \varepsilon^{\varphi_i}_{x_i}\}$, which is denoted by $\mathcal{M}_0$ again since there is no serious confusion. Let $\mathcal{M}_0 = \{\mu_n\}$ and, select $c_n > 0$ so that $\sum_{n=1}^{\infty} c_n \mu_n(X) < \infty$ and $\sum_{n=1}^{\infty} c_n \sup (k^* \mu_n) < \infty$, we define $\mu = \sum_{n=1}^{\infty} c_n \mu_n$ and $V^*(y, dx) = k^*_x(y) d\mu(x)$.

Proposition 5.2. *$p^* = V^* 1$ is a bounded continuous strict *potential.*

There exists a sub-Markov resolvent $\boldsymbol{V}^*=(V_\alpha^*)$ *such that* $V_0^*=V^*$ *and* $E_{V^*}=\mathcal{U}_+^*(X)$, *where* E_{V^*} *is the set of* $\boldsymbol{V}^*$*-excesseive functions. Further* $V^*f\in C_0(X)$ *if* $f\in\mathcal{B}_b(X)$ (*i.e., f is a bounded Borel function on X*).

The only essential part to be proved is that p^* is a strict *potential. The other assertion is a consequence of the theory of balayage (cf. [3]), II. Th,. 7.8). It is clear that $p^*=k^*\mu\in C_0(X)$ and p^* is a *potential. To show that p^* is strict, let, λ, $\nu\in M^+(X)$ such that $\lambda(u^*)\leq\nu(u^*)$ for every $u^*\in\mathcal{U}_+^*(X)$ and $\lambda(p^*)=\nu(p^*)<\infty$. From these relations it follows that $\langle\lambda, k^*\mu_n\rangle=\langle\nu, k^*\mu_n\rangle$ for all n. In fact, $\lambda(p^*)=\int[\int k_x^*(y)d\mu(x)]d\lambda(y)=\int[\int k_y(x)d\lambda(y)]d\mu(x)=\langle\mu, k\lambda\rangle=\sum_{n=1}^{\infty}c_n\langle\mu_n, k\lambda\rangle=\sum_{n=1}^{\infty}c_n\langle\lambda, k^*\mu_n\rangle\leq\sum_{n=1}^{\infty}c_n\langle\nu, k^*\mu_n\rangle=\nu(p^*)=\lambda(p^*)$. Thus we have $\langle\lambda, \varphi\rangle=\langle\nu, \varphi\rangle$ for every $\varphi\in C_K(X)$, i.e., $\lambda=\nu$.

Lemma 5.3. *Let* $\{\sigma_n\}$ *be a sequence of measures in* $M^+(X)$ *converging to* σ_0 *vaguely. If there exists* $c>0$ *and a finite system* $\{x_i;\ i=1, 2, \cdots, N\}$ *of points such that* $k^*\sigma_n\leq ck^*(\sum_{i=1}^{N}\varepsilon_{x_i})$ *for all n, then* $\limsup_{n\to\infty}\int pd\sigma_n\leq\int pd\sigma_0$ *for every finite continuous potential p on X.*

Proof. Let p be a finite continuous potential. Then there is a finite continuous potential p_1 with $p\in o(p_1)$ ([7], Prop. 2.2.4). Thus, given $\varepsilon>0$ there is a compact set K such that $p\leq\varepsilon p_1$ on $X\backslash K$. Consider $\varphi\in C_K(X)$ such that $0\leq\varphi\leq 1$ and $\varphi=1$ on K. It is trivially seen that

$$\lim_{n\to\infty}\int\varphi pd\sigma_n=\int\varphi pd\sigma_0\leq\int pd\sigma_0\,.$$

On the other hand, $\int(1-\varphi)pd\sigma_n\leq\int_{\{\varphi<1\}}pd\sigma_n\leq\varepsilon\int p_1d\sigma_n\leq\varepsilon c\sum_{i=1}^{N}p_1(x_i)$; for, let $\{q_m\}$ be a sequence of potentials with compact carrier and tends to p_1 increasingly (e.g., $q_m=\mathrm{R}^{X_m}p_1$ for a compact exhaustion $\{X_m\}$ of X), and let $q_m=k\nu_m$. Then $\langle\sigma_n, q_m\rangle=\langle\nu_m, k^*\sigma_n\rangle\leq c\langle\nu_m, k^*(\sum_{i=1}^{N}\varepsilon_{x_i})\rangle=c\sum_{i=1}^{N}q_m(x_i)\leq c\sum_{i=1}^{N}p_1(x_i)$. Thus, we have

$$\limsup_{n\to\infty}\int pd\sigma_n\leq\int pd\sigma_0+\varepsilon c\sum_{i=1}^{N}p_1(x_i)\,,$$

and, since ε being arbitrary, we completes th proof.

Proposition 5.3. *If* q^* *is a *potential with compact carrier K, then there exists a unique measure* $\tau\in M^+(X)$ *such that* $q^*=k^*\tau$, *i.e.,* $k_x^*(y)=k(x, y)$ *is a Green function of* $(X, \mathcal{U}^*)$.

Proof. First, we restrict c ourseleves to the case where q^* is bounded. Let G be a relatively compact open set with $K\subset G$. Then it is clear that $\mathrm{R}^{*G}q^*=q^*$.

Since $q^*\in E_{V^*}$, $q^*=\sup_n k^*\sigma_n=\lim_{n\to\infty} k^*\sigma_n$, where $\sigma_n=f_n\mu$ and $f_n=nq^*-n^2V_n^*q^*$. We may suppose that $\text{supp}\sigma_n\subset\bar{G}$, for otherwise since $\sigma_n\in\Lambda$ consider σ_n^G instead of σ_n. There exists a finite system $\{x_i; i=1, 2, \cdots, N\}$ of points such that $k^*\delta>0$ on $\bar{G}$, where $\delta=\sum_{i=1}^N \varepsilon_{x_i}$. We then select $c>0$ so that $ck^*\delta>\sup q^*$ on $\bar{G}$. Let E be a relatively compact open set with $\bar{G}\cup\{x_i; i=1, 2, \cdots, N\}\subset E$ and let $k\lambda=\hat{R}^E 1$, then

$$\sigma_n(\bar{G})\leq\langle\sigma_n, k\lambda\rangle=\langle\lambda, k^*\sigma_n\rangle\leq\langle\lambda, q^*\rangle\leq c\langle\lambda, k^*\delta\rangle=c\langle\delta, k\lambda\rangle$$
$$=c\sum_{i=1}^N k\lambda(x_i)<\infty, \qquad n=1, 2, \cdots.$$

Therefore, there exists a subsequence of $\{\sigma_n\}$ which is convergent vaguely to σ_0. We denote this subsequence by $\{\sigma_n\}$ for simplicity. Then it follows from the lower semicontinuity of Green functions

$$k^*\sigma_0\leq\liminf_{n\to\infty} k^*\sigma_n=q^*.$$

On the other hand, by Lemma 5.3,

$$\limsup_{n\to\infty}\int p\,d\sigma_n\leq\int p\,d\sigma_0 \text{ for every finite continuous potential } p.$$

Fix $y\in X$, and consider seqeunce $\{p_j\}$ of finite continuous potentials with compact carrier, tending to k_y increasingly. Then, since $\int p_j d\sigma_n=\langle\lambda_j, k^*\sigma_n\rangle$ is increasing, where $p_j=k\lambda_j$

$$q^*(y)=\lim_{n\to\infty}\int k_y(x)d\sigma_n(x)=\sup_n\int k_y(x)d\sigma_n(x)=\sup_n\sup_j\int p_j d\sigma_n$$
$$=\sup_j\sup_n\int p_j d\sigma_n\leq\sup_j\int p_j d\sigma_0=k^*\sigma_0(y).$$

Hence, we have $q^*=k^*\sigma_0$ with $\text{supp}\,\sigma_0\subset S(q^*)$.

For a general case, let G be as above. The relation $q^*=\hat{R}^{*\bar{G}}q^*=H_{q^*}^{*C\bar{G}}$ on $C\bar{G}$ with the fact that $1\in\mathcal{U}_+^*(X)$ implies q^* is bounded on $X\setminus\bar{G}$. Letting $M=\sup\{q^*(y); y\notin\bar{G}\}$, for $n\geq M$ we define $q_n^*=\inf(q^*, n)$. q_n^* is a bounded *potential with $S(q_n^*)\subset\bar{G}$. Thus, by the previous consideration, there is $\tau_n\in M^+(X)$ such that $q_n^*=k^*\tau_n$ and $\text{supp}\,\tau_n\subset\bar{G}$. It is trivial that $q_n^*\uparrow q^*$ and $A=\{q^*=\infty\}$ is a *polar subset of K. As usual, we may find a finite system $\{y_1, \cdots, y_N\}\subset X\setminus A$ of points and $c>0$ satisfying $c\sum_{i=1}^N k_x^*(y_i)\geq M$ for every $x\in\bar{G}$. The inequalities

$$M\tau_n(X)\leq\int c\sum_{i=1}^N k_x^*(y_i)d\tau_n(x)=c\sum_{i=1}^N k^*\tau_n(y_i)\leq c\sum_{i=1}^N q^*(y_i)<\infty$$

imply that the total mass of τ_n is bounded. Then we may select a subsequence $\{\tau'_n\}$ of $\{\tau_n\}$ tending to τ vaguely. It is evident

$$k^*\tau \leq \liminf_{n\to\infty} k^*{}'_n = q^* .$$

Let $\{p_j\}$ be as above and let $\varphi\in C_K(X)$, $0\leq\varphi\leq 1$, $\varphi=1$ on $\bar{G}$. Since

$$\langle\tau'_n, p_j\rangle = \langle\tau'_u, p_j\varphi\rangle + \langle\tau'_n, p_j(1-\varphi)\rangle \to \langle\tau, p_j\varphi\rangle \leq \langle\tau, p_j\rangle$$

we have as in the previous case $\lim_{n\to\infty} k^*\tau'_n \leq k^*\tau$.

Finally we shall prove the unicity of τ. Suppose that $q^*=k^*\tau=k^*\tau'$ for τ, $\tau'\in M^+(X)$ with compact support. It follows then

$$\langle\tau, k\lambda\rangle = \langle\lambda, k^*\tau\rangle = \langle\lambda, k^*\tau'\rangle = \langle\tau', k\lambda\rangle \qquad \forall\lambda\in M^+(X) .$$

Let $\varphi\in C_K(X)$. Since for each $\varepsilon>0$ there exist λ, $\lambda\in M^+_K(X)$ such that $0\leq k\lambda-k\lambda'\leq\varphi\leq k\lambda-k\lambda'+\varepsilon$ we have

$$\begin{aligned}|\langle\tau, \varphi\rangle - \langle\tau', \varphi\rangle| &= |\langle\tau, k\lambda-k\lambda'+\psi\rangle - \langle\tau', k\lambda-k\lambda'+\psi\rangle| \\ &\leq \varepsilon[\tau(\mathrm{X})+\tau'(X)] ,\end{aligned}$$

where $\psi=\varphi-(k\lambda-k\lambda')$. This concludes $\tau=\tau'$.

Coming back to the original difinition of Green functions (§ 1), in view of Remark 3.1 (1), we have proved that $k^*(x, y)=k(y, x)$ is a Green function of $(X, \mathcal{U}^*)$, i.e.,

1*) $(x, y)\to k^*(x, y)$ is lower semicontinuous and continuous if $x\neq y$,

2*) $y\to k^*_x(y)=k^*(y, x)$ is a *potential on X with carrier $\{x\}$ for every $x\in X$,

3*) for every *potential q^* with compact carrier there exists a unique $\tau\in M^+(X)$ such that $q^*=k^*\tau$.

Hence, we can construct the dual $(X, \mathcal{U}^{**})$ of $(X, \mathcal{U}^*)$, that is the bidual of $(X, \mathcal{U})$. In view of this bidual we obtain

Theorem 5.1. $(X, \mathcal{U}^{**})=(X, \mathcal{U})$.

Proof. The hyperharmonic sheaf $\mathcal{U}^{**}$ is defined by a family of sweepings $\{\mu^{**V}_x;\ V$ is a relatively compact open set, $x\in V\}$. The balaged function can be expressed by

$$\hat{\mathrm{R}}^{*CV}k^*_x(y) = \int k^*_x(y)d\mu^{**V}_x(z) = \hat{\mathrm{R}}^{CV}k_y(x) = \int k_y(z)d\mu^V_x(z) = \int k^*_z(y)d\mu^V_x(z) ,$$

which implies that $\mu^{**V}_x=\mu^V_x$ by 3*).

6. Consequences

In this section we give several supplementary remarks. Some of them contain the known results.

Proposition 6.1. *Suppose that there is a function* $k(x, y)$ *satisfying* 1), 2), *of the Green function in* § 1. *Then* 3) *is equivalent to the following:*

3') *for every potential* p *on* X *there exists a unique measure* μ *such that* $p(x) = k\mu(x)$,

3'') *for every continuous potential* p *on* X *there exists a unique measure* μ *such that* $p(x) = k\mu(x)$.

Proof. First we consider the measure representation.

3)$\Rightarrow$3'): let $\{U_n\}$ be a locally finite open covering of x consisting of relatively compact open sets and $\{f_n\}$ be the decomposition of the unity corresponding to $\{U_n\}$, i.e.,

$$f_n \in C_K(X),\ 0 \leq f_n \leq 1,\ \operatorname{supp} f_n \subset U_n,\ \sum_{n=1}^{\infty} f_n = 1\,.$$

Now since the specific mutiplication $f_n \cdot p$ is a potential with compact carrier, there is $\mu_n \in M^+(X)$ such that $f_n \cdot p = k\mu_n$ and supp $\mu_n = S(f_n \cdot p) \subset \operatorname{supp} f_n \subset U_n$. We define $\mu = \sum_{n=1}^{\infty} \mu_n \in M^+(X)$. Then $p = k\mu$. For, let $\bar{\mu}_n = \sum_{i=1}^{n} \mu_i$ and $\bar{f}_n = \sum_{i=1}^{n} f_i$. We have $k\mu = \sup_n k\bar{\mu}_n$ and $\bar{f}_n \cdot p = (\sum_{i=1}^{n} f_i) \cdot p = \sum_{i=1}^{n} (f_i \cdot p) = \sum_{i=1}^{n} k\mu_i = k\bar{\mu}_n$. Hence, $\bigvee_n k\bar{\mu}_n = 1 \cdot p = p$ since $\sup_n \bar{f}_n = 1$ ([7], Th. 8.1.3) and $p < k\mu$. On the other hand, $k\bar{\mu}_n < p$ implies $k\mu = p$.

3')$\Rightarrow$3'') is trivial.

3'')$\Rightarrow$3): let p be a potential with $S(p) = K$ is compact, U be a relatively compact open neighborhood of K and let $\{f_n\} \subset C_K^+(X)$ such that f_n increasingly tends to $p1_U$. Then it is easily seen that Rf_n is a continuous potential with $S(\mathrm{R}f_n) \subset \bar{U}$ and R$f_n \uparrow p$. Thus, by the assumption, there is $\mu_n \in M^+(X)$ such that R$f_n = k\mu_n$ and supp $\mu_n \subset \bar{U}$. The next step is routin: we may find $c > 0$ and a finite system $\{x_i;\ i = 1, 2, \cdots, N\}$ of points such that $p(x_i) < \infty$ and $c \sum_{i=1}^{N} k^*_{x_i} \geq 1$ on $\bar{U}$. Since $\mu_n(X) = \mu_n(\bar{U}) \leq c \sum_{i=1}^{N} p(x_i) < \infty$ for all n, $\{\mu_n\}$ has a subsequence convergents to μ vaguely. We suppose $\mu_n \rightarrow \mu$ for simplicity. Then supp $\mu \subset \bar{U}$. Recall that $k^* \varepsilon_x^{\varphi}$ is finite continuous on X, where ε_x^{φ} is the measure defined in § 2. Then,

$$\langle \varepsilon_x^{\varphi}, p \rangle = \lim_{n \to \infty} \langle \varepsilon_x^{\varphi}, k\mu_n \rangle = \lim_{n \to \infty} \langle \mu_n, k^* \varepsilon_x^{\varphi} \rangle = \langle \mu, k^* \varepsilon_x^{\varphi} \rangle = \langle \varepsilon_x^{\varphi}, k\mu \rangle$$

for every pair (φ, x) implies $p = k\mu$.

For the unicity of measures we follow [8], [4], [11]. The crucial point is the following: if $k\lambda = 0$ for $\lambda \in M^+(X)$ then $\lambda = 0$. This is a consequence of a property of Green functions: for every $y \in X$ there is $x \in X$ such that $k_y(x) > 0$.

REMARK 6.1. (1) when 1 is superharmonic, under the assumptions 1), 2) of Green functions $k(x, y)$, the property 3) is equivalent to the following ([11], [15]):

there exists a sub-Markov resolvent $\boldsymbol{V}=(V_\alpha)_{\alpha\geq 0}$ and a measure $\lambda\in M^+(X)$ such that

(i) V_0 is proper,

(ii) the family of $\boldsymbol{V}$-excessive functions coincides with $\mathcal{U}_+(X)$,

(iii) $V_0 f(x)=\int f(y)k_y(x)d\lambda(y) \qquad \forall f\in\mathcal{B}^+(X)$.

(2) in [12], F-Y. Maeda investigated the Dirichlet integral and energy of potentials on harmonic spaces with adjoint structure. In the starting point he assumed five properties of Green functions (G0), (G1), (G2), (G*1) and (G*2) but we can observe that (G*1) and (G*2) are superfluous.

As a consequence of the observation in § 3 and § 4, we have an important result which was obtained by Schirmeier in a different context [15].

Proposition 6.2. *If a harmonic space $(X, \mathcal{U})$ has a Green function, then $(X, \mathcal{U})$ has the Doob convergence property.*

Hence, when $(X, \mathcal{U})$ possesses a Green function, the topoloyg T defined in the vector space $[\mathcal{S}]$ formed by differences of two positive superharmonic functions coincides with the topology of graph convergence and also with the Mokobodzki topology T_M, i.e., for $\{s_n\}\subset\mathcal{S}_+$, $s_n\to s(T_M)$ is defined $\langle\varepsilon_x^\varphi, s_n\rangle\to\langle\varepsilon_x^\varphi, s\rangle$ for every $\varepsilon_x^\varphi\in\mathcal{M}$ [16]. As an easy consequence, we have

Corollary 6.1. *k_{y_n} converges to k_{y_0} with respect to the topology T if and only if $y_n\to y_0$.*

For, $k_{y_n}\to k_{y_0}$ (T) is equivalent to $k^*\varepsilon_x^\varphi(y_n)=\langle\varepsilon_x^\varphi, k_{y_n}\rangle\to\langle\varepsilon_x^\varphi, k_{y_0}\rangle=k_x^\varphi(y_0)$, this is equivalent to $y_n\to y_0$, since $k^*\varepsilon_x^\varphi$ are continuous and separate points of X.

Now we consider the case where $(X, \mathcal{U})$ is elliptic ([7], p. 66).

Proposition 6.3. *If $(X, \mathcal{U})$ is elliptic then the dual $(X, \mathcal{U}^*)$ is also elliptic.*

Proof. X has a base of connected, relatively compact open sets ([7], Th. 1.1.1). For every open connected set V, $V_1=\mathring{\bar{V}}$ is also open connected and $\mathring{\bar{V}}_1=V_1$, which implies $\partial\bar{V}_1=\partial V_1$. Thus we have a base $\mathcal{D}=\{V\}$ of connected, reltively compact open sets with $\partial V=\partial\bar{V}$. The sweeping $((\mu_y^{*V})_{y\in V})_{V\in\mathcal{D}}=((\varepsilon_y^{*CV})_{y\in V})_{V\in\mathcal{D}}$ is an elliptic swepping system on $(X, \mathcal{U}^*)$. For, suppose, on the contrary, that there exist $V_1\in\mathcal{D}$ contained in a connected set and $y\in V_1$ such that supp $\mu_y^{*V_1}\not\supset\partial\bar{V}_1=\partial V_1$. Let $A=\partial V_1\backslash$supp μ_y^{*V} and $x_0\in A$. As a function of x, $\hat{\mathrm{R}}^{CV_1}k_y(x)=\int k_x^*(z)d\mu_y^{*V_1}(z)$ is continuous at x_0. This is an immediate consequence of Lebesgue's convergence theorem, since $x_0 \notin \operatorname{supp} \mu_y^{*V_1}$. We may find a connected open neighborhood W of x_0 so that $\hat{R}^{CV_1}k_y$ is continuous on W. Then, $W \cap C\bar{V}_1 \neq \emptyset$ and $\hat{R}^{CV}k_y = k_y$ on $C\bar{V}_1$. The function $k_y - \hat{R}^{CV}k_y$ is a non-negative superharmonic function on a connected open set $U = W \cup V_1$ and $k_y - \hat{R}^{CV_1}k_y = 0$ on $W \cap C\bar{V}_1$. This derives a contradiction that $k_y = \hat{R}^{CV_1}k_y$ on V_1 ([7], Prop. 3.1.4).

Collorary 6.2. *If an elliptic harmonic space $(X, \mathcal{U})$ has a Green function, then $(X, \mathcal{U})$ is a Brelot space.*

In fact, as is pointed out in Remark 1.1, X has no isolated point. By a theorem of Constantinescu-Cornea ([7], Prop. 3.1.4), an elliptic space has a base of regular sets. The Brelot convergence property is derived as in [2], Th. 1.5.6.

Collorary 6.3. *If a Brelot space $(X, \mathcal{U})$ has a Green function then its dual $(X, \mathcal{U}^*)$ is also a Brelot space.*

For, every Brelot space is elliptic ([7], Th. 3.1.2) and $(X, \mathcal{U}^*)$ is also elliptic.

Finally we consider the case $(X, \mathcal{U}^*) = (X, \mathcal{U})$. The following result was obtained by Schirmeier by using the fine topology [15].

Proposition 6.4. *If the Green function is symmetric, i.e., $k(x, y) = k(y, x)$, then $(X, \mathcal{U})$ is a Brelot space.*

Proof. We prove that $(X, \mathcal{U})$ is elliptic. Suppose, on the contrary, there exists $y_0 \in X$ and a base of connected neighborhood $\{V\}$ of y_0 such that $\partial V = \partial \bar{V}$ and $\operatorname{supp} \varepsilon_{y_0}^{CV} \neq \partial V$. Fix one of them, say V, and $x_0 \in \partial V \setminus \operatorname{supp} \varepsilon_{y_0}^{CV}$. As in the proof of Prop. 6.3, we may find a connected open neighborhood $U(x_0)$ of x_0 so that $p = k_{y_0} - \hat{R}^{CV}k_{y_0}$ is non-negative superharmonic on $U = V \cup U(x_0)$. The set $F = U \cap [p = 0]$ is an absorbent set of the space U and $\emptyset \neq F \neq U$. Thus, there is a point $z_0 \in \partial F \cap U$. Now let W be an open connected neighborhood of z_0 with $\bar{W} \subset U$. The set $W \cap F$ is an absorbent set of the space W. For $y \in W \setminus F$ and for any neighborhood $V(y)$ of y such that $\overline{V(y)} \subset W \setminus F$, we have ${}^W\hat{R}^{\overline{V(y)}} p_y = 0$ on $W \cap F$, where $p_y = k_y - \hat{R}^{CV}k_y$. On the other hand, since p_y is a potential on W, by the minimum principle, $p_y = {}^W\hat{R}^{\overline{V(y)}} p_y$. Thus $p_y = 0$ on $W \cap F$ for every $y \in W \setminus F$, i.e., $p_y(z) = 0$ for every $z \in W \cap F$ and for every $y \in W \setminus F$. The lower semicontinuity of p_y implies $p_y(z) = 0$ for every $z \in W \cap F$ and $y \in \overline{W \setminus F} \cap W$. In particular, we have $p_y(z_0) = p_{z_0}(y) = 0$ for every $y \in \overline{W \setminus F} \cap W$. While, for $z \in W \cap F$, $p_{z_0}(z) = p_z(z_0) = 0$ since $z_0 \in \overline{W \setminus F} \cap W$. Hence, we have a contradiction $p_{z_0} = 0$ on W.

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