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THE AUSLANDER-REITEN QUIVER OF A RING OF RIGHT LOCAL TYPE

Dedicated to Professor Manabu Harada on his 60th birthday

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Throughout this paper, A is a right and left artinian ring and J is the Jacobson radical of A. All modules are assumed to be finitely generated over A. As well known Auslander-Reiten sequences (abbreviated AR-sequences) exist over arbitrary algebras and over representation-finite rings [6], [7], [27]. An AR-sequence of modules over a ring A and the indecomposable decomposition of the middle term of this sequence define a mesh in the Auslander-Reiten quiver (AR-quiver for short) of A, and meshes determine the AR-quiver of A. For the representation-finite rings the computation of AR-quiver gives all indecomposable modules and all non-isomorphisms between them up to isomorphisms. In the algebra case we have a general way to compute AR-sequences stopping at any non-projective modules [11], [14] although the computation of the decomposition of the middle terms is not clear. However this computation is heavily depends on the existence of the selfduality of algebras. Hence for an arbitrary artinian ring even for a representation-finite ring, this computation is not available. Over such a ring we therefore have to compute AR-sequences individually.

In this paper we will compute (1) all the AR-sequences including the indecomposable decompositions of the middle terms (Theorem I), (2) full subquivers of the AR-quiver which give the whole of the AR-quiver by gluing together and (3) all the meshes (including values) in the AR-quiver (Theorem II), over a special type of representation-finite rings, namely over a ring of right local type. As a corollary we will obtain that the AR-quiver of a ring A of right local type is a *well valued* translation quiver (i.e. the value (a, b) of any arrow of the AR-quiver maps to (b, a) by the translation between arrows associated to the AR-translation; see section 3 for detail) iff A is, in addition, a ring of left colocal type. Further for a ring A defined by a bimodule M over division rings, we will give a necessary and sufficient condition for A to be of right local type in terms of the dimension sequence of M.

In section 1 we first quote Sumioka's result on a ring of right local type, and investigate further properties of this kind of rings for the later use. Section 2 is devoted to the computation of AR-sequences over a ring of right local type. Finally in section 3 we draw the form of the AR-quiver of a ring of right local type and compute its value to obtain Theorem II. Further we give an example of a ring of right local type but not of left colocal type, which we presented in a local conference [3], and an example of a ring of left colocal type but not of right local type.

For a module M, we denote by $|M|$ and by $h(M)$ the composition length and the *height* (=the Lowey length) of M, respectively. And $\sharp I$ denotes the cardinality of a set I.

I would like to thank Dr. M. Hoshino for his useful adivce, which enabled me to remove the assumption in the original version that our ring in the title is also of left colocal type, by showing me Lemmas 2.2, 2.4 and to add the section 3 in the revised version, and Professor Sumioka for his short proof of Lemma 1.7.

1. Rings of right local type

Recall that A is said to be a ring of *right local type* in case every indecomposable right A-module is local (i.e. its top is simple), and dually A is called a ring of *left colocal type* in case every indecomposable left A-module is colocal (i.e. its socle is simple). Sumioka [23], [24] had made a precise investigation of rings of right local type and of rings of left colocal type (see also Tachikawa [25], [26] and Yoshii [28]). His result is summarized as follows:

Theorem 1.0 (Sumioka [24]). *Consider the following conditions:*

(LR) (a) *A is left serial;*

(b) *J_A is a direct sum of uniserial modules; and*

(c) *$|eJ/eJ^2|\leqq 2$ for any primitive idempotent e of A with eJ/eJ^2 not homogeneous.*

(L) $[D_1(U): D_2(U)]_r \leqq 2$ *for any uniserial left A-module U with $|U|\geqq 2$ (see [23] or [24] for the definition of $D_i(U)$).*

(R) $[D_1(U): D_2(U)]_l \leqq 2$ *for any uniserial left A-module U with $|U|\geqq 2$. Then*

(1) *If A is a ring of left colocal type, then A satisfies the conditions* (LR) *and* (L).

(2) *If A is a ring of right local type, then A satisfies the conditions* (LR) *and* (R).

(3) *If A satisfies the conditions* (LR), (L) *and* (R), *then A is a ring of right local type and of left colocal type.*

REMARK 1.1. Let A be a left serial ring. Then the following are equivalent (see [24, Remark 5] for detail):

(1) $|eJ/eJ^2|\leqq 2$ for every primitive idempotent e of A.

(2) A satisfies the conditions (LR) (c) and (L).

With this remark in mind we have the following corollary.

Corollary 1.2. *Assume that A is a ring of right local type. Then A is a ring of left colocal type iff $|eJ/eJ^2|\leqq 2$ for every primitive idempoetn e of A.*

Proposition 1.3. *If A is a ring of right local type and e is a primitive idempotent of A with $eJ^2\neq 0$, then $|eJ/eJ^2|\leqq 2$.*

Proof. See [2]. //

By Theorem 1.0 and Proposition 1.3, we can determine the form of every projective indecomposable right module over a ring of right local type as follows.

Proposition 1.4. *Let A be a rino of right iocal type and e a primitive idempotent of A. Then eA has one of the following forms:*

(a) *eA is uniserial.*

(b) *$eJ=K\oplus L$ for some non-zero uniserial modules K, L which are not isomorphic to each other.*

(c) *$eJ=K\oplus L$ for some non-zero uniserial modules K, L which are isomorphic to each other.*

(d) *eJ is a homogeneous semisimple module of length >2.*

DEFINITION 1.5. Let M be a subset of A. Then $\boldsymbol{d}(M):=\sup\{n\geqq 0\mid M\subseteqq J^n\}$ is called the *depth* of M. When $M=\{x\}$ for some $x\in A$, we simply denote $\boldsymbol{d}(\{x\})$ by $\boldsymbol{d}(x)$. By definition $\boldsymbol{d}(0)=\infty$.

NOTATION 1.6. Let $a\in A$. Then we denote by $a\cdot$ the left multiplication by a.

The following determines the form of submodules of projective indecomposable modules P with $|PJ/PJ^2|\leqq 2$ over a ring of right local type. This is taken from Harada [15]. For the benefit of the reader we give a proof which is due to Sumioka.

Lemma 1.7. *Let A be a left serial ring and e a primitive idempotent of A. If $eJ=U_1\oplus U_2$ with each U_i uniserial and $M\leqq eJ$, then there is an automorphism α of eA such that $\alpha(M)=U_1J^s\oplus U_2J^t$ for some $s, t\geqq 0$.*

Proof. Put $d:=\boldsymbol{d}(M)\geqq 1$. Then $M\leqq eJ^d=U_1J^{d-1}\oplus U_2J^{d-1}$, and there exists some $u\in M$ such that $\boldsymbol{d}(u)=d$ with $u=uf$ for some primitive idempoent f of A. Write $u=u_1+u_2$ for some $u_i\in U_i$. Then $\boldsymbol{d}(u_i)=d$ for some i, say $i=1$. This implies that $u_1A=U_1J^{d-1}$. Since Af is uniserial, we have $au=u_1$ for some

$a \in eAe \setminus eJe$. Noting that $aM \geqq u_1 A = U_1 J^{d-1}$, we have $aM = aM \cap eJ^d = U_1 J^{d-1} \oplus (aM \cap U_2 J^{d-1})$. Hence $a\cdot$ gives a desired map. //

Proposition 1.8. *Assume that A is a ring of right local type. Then every indecomposable right A-module has one of the following forms*:

(a) *$eA/(KJ^s \oplus LJ^t)$ for some primitive idempotent e of A with $eJ = K \oplus L$ where K and L are uniserial modules and $s, t \geqq 0$.*

(b) *$eA/(\bigoplus_{i=1}^{m} S_i)$ for some $0 \leqq m \leqq n$, where $eJ = \bigoplus_{i=1}^{n} S_i$, $S_1 \cong \cdots \cong S_n$ are simple and $n \geqq 3$.*

Proof. (a) follows by Lemma 1.7. (b) follows by Lemma 1.13 below. //

DEFINITION 1.9. Let $\alpha = (\alpha_i)^T \colon X \to \bigoplus_{i=1}^{n} Y_i$ be a homomorphism of right modules and let $1 \leqq j \leqq n$. Then we say that α is *j-fusible* in case $\alpha_j = \sum_{i \neq j} \beta_i \alpha_i$ for some $(\beta_j)_{i \neq j} \colon \bigoplus_{i \neq j} Y_i \to Y_j$. Further α is said to be *fusible* in case α is j-fusible for some $1 \leqq j \leqq n$, and α is said to be *infusible* if α is not fusible.

REMARK 1.10. (1) In this paper we use this definition for the case $n=2$ as follows:

Let $X_i < Y_i$ for each $i=1, 2$ and let $\alpha \colon X_1 \to X_2$ be an isomorphism. Then α is extendable to $Y_1 \to Y_2$ or α^{-1} is extendable to $Y_2 \to Y_1$ iff $(\sigma_1, \sigma_2 \alpha)^T \colon X_1 \to Y_1 \oplus Y_2$ is fusible, where σ_i are the inclusion maps $X_i \to Y_i$.

(2) In the above definition, if all Y_i are indecomposable, then α is infusible iff it is *left minimal* in the sense of Auslander-Reiten [8].

Lemma 1.11. *Let $X \xrightarrow{(\alpha_i)^T} \bigoplus_{i=1}^{n} Y_i \xrightarrow{(\beta_i)} Z \to 0$ be an exact sequence of right modules. Then*

(1) *For every $1 \leqq j \leqq n$, $(\alpha_i)^T$ is j-fusible iff β_j is a section.*

(2) *Assume that X is simple and all Y_i are local. Then $(\alpha_i)^T$ is infusible iff Z is indecomposable.*

Proof. See [1]. //

The above lemma gives a characterization of a ring of right local type, which had been obtained with the following proof before [24, Proposition 1.3].

Proposition 1.12. *The following are equivalent*:

(1) *A is of right local type.*

(2) *If S is a simple right A-module, and L_1, L_2 are local right A-modules, then any homomorphism $(\alpha_1, \alpha_2)^T \colon S \to L_1 \oplus L_2$ is fusible.*

Proof. (1)$\Rightarrow$(2). Clear from Lemma 1.11 (2).
(2)$\Rightarrow$(1). We show that any M_A is expressed as a direct sum of local modules

by induction on $|M|$. It suffices to show that M has a local direct summand. Let S be a simple submodule of M. Then by induction hypothesis $M/S=\bigoplus_{i=1}^{n} L_i/S$ for some $S\leqq L_i\leqq M$ with all L_i/S local. If $S\nleqq L_iJ$ for some i, then $L_i=S\oplus L_i'$ for some L_i', which must be a local direct summand of M. Hence we may suppose that every L_i is local, and $n>1$. Let $\sum_{i\neq 1} L_i=\bigoplus_{i=2}^{n} M$ with every M_i local. Then we have an exact sequence

$$0\to S\xrightarrow{(\alpha_i)^T} L_1\oplus M_2\oplus\cdots\oplus M_n\xrightarrow{(\beta_i)} M\to 0$$

in an obvious way. By (2), $(\alpha_1, \alpha_2)^T$ is fusible, whence so is $(\alpha_i)^T$. Thus β_i is a section for some i by Lemma 1.11 (1). //

The following is an immediate consequence of the above proposition.

Lemma 1.13. *Let A be a ring of right local type and e a primitive idempotent of A with eJ a homogeneous semisimple module of length >2. Then $eA/I\cong eA/I'$ iff $|I|=|I'|$ for any $I, I'\leqq eJ$.*

Lemma 1.14. *Let $X_i\leqq Y_i$, for each $i=1, 2$, and α: $X_1\to X_2$ an isomorphism. Put M to be the cokernel of the map $(\sigma_1, \sigma_2\alpha)^T$: $X_1\to Y_1\oplus Y_2$, where σ_i are the inclusion maps. Then* soc $M\cong$soc Y_2 *iff α is* maximal, *i.e., for any $H>X_1$ any homomorphism from H to Y_2 is not an extension of α.*

Proof. See [23]. //

Next we state the dual notions and the dual statements of the above. See [24] for the proof of Lemma 1.18. The remaining proofs are left to the reader.

DEFINITION 1.15. Let $\beta=(\beta_i)$: $\bigoplus_{i=1}^{n} Y_i\to Z$ be a homomorphism of right modules and let $1\leqq j\leqq n$. Then we say that β is *j-cofusible* in case $\beta_j=\sum_{i\neq j}\beta_i\alpha_i$ for some $(\alpha_i)_{i\neq j}^T$: $Y_j\to\bigoplus_{i\neq j} Y_i$. Further β is said to be *cofusible* in case β is j-cofusible for some $1\leqq j\leqq n$, and α is said to be *coinfusible* if α is not cofusible.

REMARK 1.16. (1) In this paper we use this definition for the case $n=2$ as follows:

Let $X_i<Y_i$ for each $i=1, 2$ and let β: $Y_1/X_1\to Y_2/X_2$ be an isomorphism. Then β is liftable to $Y_1\to Y_2$ or β^{-1} is liftable to $Y_2\to Y_1$ iff $(\beta\pi_1, \pi_2)$: $Y_1\oplus Y_2\to Y_2/X_2$ is cofusible, where π_i are the canonical epimorphisms $Y_i\to Y_i/X_i$.

(2) If in the above definition all Y_i are indecomposable, then β is coinfusible iff it is *right minimal* in the sense of [8].

Lemma 1.17. *Let $0\to X\xrightarrow{(\alpha_i)^T}\bigoplus_{i=1}^{n} Y_i\xrightarrow{(\beta_i)} Z$ be an exact sequence of right* *modules. Then*

(1) *For every* $1 \leqq j \leqq n$, (β_i) *is j-cofusible iff* α_j *is a retraction.*

(2) *Assume that* Z *is simple and all* Y_i *are colocal. Then* (β_i) *is coinfusible iff* X *is indecomposable.*

Lemma 1.18. *Let* $X_i \leqq Y_i$, *for each* $i=1, 2$, *and* β: $Y_1/X_1 \rightarrow Y_2/X_2$ *an isomorphism. Put* M *to be the kernel of the map* $(\beta\pi_1, \pi_2)$: $Y_1 \oplus Y_2 \rightarrow Y_2/X_2$, *where* π_i *are the canonical epimorphisms* $Y_i \rightarrow Y_i/X_i$. *Then* top $M \cong$ top Y_1 *iff* β *is* comaximal, *i.e., for any* $H < X_2$ *any homomorphism from* Y_1 *to* Y_2/H *is not a lift of* β.

By Lemma 1.17 (2) and Lemma 1.18, we get the following, which states that a ring of right local type has a property dual to the property [25, Theorem 5.3 II] of a ring of left colocal type.

Lemma 1.19. *Let* A *be a ring of right local type, and* Y_1, Y_2 *uniserial right A-modules such that* $|Y_1| \geqq |Y_2| > 2$. *Then any isomorphism* top $Y_1 \rightarrow$top Y_2 *is comaximal or is liftable to an epimorphism* $Y_1 \rightarrow Y_2$.

Proof. We keep the notation of Lemma 1.18. Let β: $Y_1/X_1 \rightarrow Y_2/X_3$ be an isomorphism, where $X_i := Y_i J$. If β is not comaximal, then M is not local by Lemma 1.18. Thus M is decomposable. Hence the assertion follows by Lemma 1.17(2). //

The following is a key for the computation of values of AR-quiver of a ring of right local type.

Proposition 1.20. *Let* A *be a ring of right local type and* e *a primitive idempotent of* A *such that* $eJ = K \oplus L$; K, L *are non-zero and uniserial. Then the following are equivalent*:

(1) $eA/(KJ^s \oplus LJ^{s+1}) \cong eA/(KJ^{s+1} \oplus LJ^s)$ *for some* $s \geqq 0$.

(2) $eA/(KJ^s \oplus LJ^{s+1}) \cong eA/(KJ^{s+1} \oplus LJ^s)$ *for all* $s \geqq 0$.

(3) $eA/(K \oplus LJ) \cong eA/(KJ \oplus L)$.

(4) $eA/K \cong eA/L$.

(5) $K \cong L$.

(6) top $K \cong$ top L.

Proof. The proof proceeds as follows: (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(1), (3)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(6)$\Rightarrow$(3).

(1)$\Rightarrow$(2). An isomorphism of (1) is liftable to some automorphism α of eA which gives rise to an isomorphism $KJ^s \oplus LJ^{s+1} \rightarrow KJ^{s+1} \oplus LJ^s$. This also induces an isomorphism $KJ^{s+1} \oplus LJ^{s+2} \rightarrow KJ^{s+2} \oplus LJ^{s+1}$ since the radical maps to the radical. Thus α gives $eA/(KJ^{s+1} \oplus LJ^{s+2}) \cong eA/(KJ^{s+2} \oplus LJ^{s+1})$. Conversely, taking "over socles" the last isomorphism implies the first one. Hence (1) implies (2).

(2)$\Rightarrow$(3)$\Rightarrow$(1). Trivial.
(3)$\Rightarrow$(4). We may assume that $|eA/K|\leqq|eA/L|$. Let φ: $eA/(KJ\oplus L)\to eA/(K\oplus LJ)$ be an isomorphism. This induces an automorphism β of eA/eJ. Since β is liftable to φ, Lemma 1.19 says that β is liftable to an epimorphism $eA/L\to eA/K$, which induces an isomorphism ψ: $eA/M\to eA/K$ for some $L\leqq M\leqq eJ$. Further ψ is liftable to an automorphism of eA, which gives rise to an isomorphism $M\to K$. The since $M=L\oplus(M\cap K)\cong K$ is uniserial, we have $M=L$.
(4)$\Rightarrow$(5)$\Rightarrow$(6). Trivial.
(6)$\Rightarrow$(3). Assume (6). Then there is an isomorphism between the radicals of the left and right hand sides of (3), which is extendable to a desired isomorphism by Proposition 1.12. //

Lemma 1.21. *Let A be a ring of right local type. Then A is* right coserial, *i.e., every injective indecomposable right A-module is uniserial.*

Proof. This is clear because soc^2E is local for any injective indecomposable right A-module E. //

Let A be a ring of right local type. Then by Lemma 1.21, any uniserial right A-module is quasi-injective. Now let U be a uniserial right A-module of length $n\geqq 2$. For each $1\leqq i\leqq n$, we set $D^i(U):=\mathrm{End}_A(UJ^{i-1}/UJ^i)$, which clearly is a division ring. As in Sumioka [23], we can define a ring monomorphism μ_{ij}: $D^j(U)\to D^i(U)$ for all $1\leqq i<j\leqq n$ as follows: For every $x\in D^j(U)$, x is extendable to some $y\in\mathrm{End}_A(UJ^{i-1}/UJ^j)$, whihc induces an element $\bar{y}\in D^i(U)$. Then the correspondence μ_{ij}: $x\mapsto\bar{y}$ is easily seen to be well-defined and a monomorphism. Further we have commutativity relations $\mu_{ij}=\mu_{it}\,\mu_{tj}$ for all $1\leqq i<t<j\leqq n$. Thus we can regard $D^1(U)\geqq D^2(U)\geqq\cdots\geqq D^n(U)$ by μ_{ij}'s. Under this notation we obtain the following by Lemma 1.19.

Proposition 1.22. *Let A be a ring of right local type, and U a uniserial right A-dolule of length $n>2$. Then $D^2(U)=D^3(U)=\cdots=D^n(U)$.*

Proof. We have only ot show that μ_{2n} is surjective. Let $x\in D^2(U)$. Then x is extendable to some $y\in\mathrm{End}_A(U/UJ^2)$, which induces an element $\mu_{12}(x)\in D^1(U)$. Since $\mu_{12}(x)$ is liftalbe to y, Lemma 1.19 insists that $\mu_{12}(x)$ is liftable to some $z\in\mathrm{End}_A(U)$. Put $t:=z\,|\,UJ^{n-1}$. Then $t\in D^n(U)$ and $\mu_{12}(x)=\mu_{1n}(t)=\mu_{12}\,\mu_{2n}(t)$. Thus $x=\mu_{2n}(t)$. //

DEFINITION 1.23. Let A be a ring of right local type, and S a simple right A-module with the projective cover P. Then analogous to [25], we say that S is *of 1st kind* if PJ/PJ^2 is square-free, and that S is *of 2nd kind* if it is not of 1st kind, equivalently if PJ/PJ^2 is a homogeneous semisimple module of length $\geqq 2$. Note that by Proposition 1.20, S is of 1st (resp. 2nd) kind iff P is of the from (a) or (b) (resp. (*c*) or (d)) in Proposition 1.4.

For division rings $D^1(U)$ and $D^2(U)$ we have the following, which gives another key for the computation of values of AR-quiver of a ring of right local type.

Proposition 1.24. *Let A be a ring of right local type, and U a uniserial right A-module of liength $\geqq 2$ with the projective cover P. Then*

(1) top U *is of 1st kind iff* $D^1(U)=D^2(U)$; *and*

(2) *If* top U *is of 2nd kind, then*

(a) *U is injective, and*

(b) $[D^1(U)\colon D^2(U)]_r=2$, $[D^1(U)\colon D^2(U)]_l=|PJ/PJ^2|$.

Proof. (1) ($\Rightarrow$). Assume that $D^1(U)\neq D^2(U)$, say $x\in D^1(U)\backslash D^2(U)$. Construct the kernel M of the map $(x\pi, \pi)\colon U/UJ^2\oplus U/UJ^2\to \text{top } U$, where $\pi\colon U/UJ^2 \to \text{top } U$ is the canonical epimorphism. Then by Lemma 1.18 we have top $M\cong$ top U, and $MJ/MJ^2\cong UJ/UJ^2\oplus UJ/UJ^2$ is a homogeneous semisimple module of length 2. Thus top U is of 2nd kind.

(1) ($\Leftarrow$). Assume that top U is of 2nd kind. Choose some $PJ^2\leqq I\leqq PJ$ so that $M:=P/I$ has the radical of the form $S\oplus T$ with $S\cong T$ simple modules. Then by Lemma 1.13 and Proposition 1.20 we have $M/S\cong M/T\cong U/UJ^2$. Hence we get an exact sequence of the form

$$0\to M\to U/UJ^2\oplus U/UJ^2 \xrightarrow{(\alpha,\beta)} \text{top } U\to 0\,,$$

where α, β are epimorphisms. Since M is indecomposable, we have $\bar{\alpha}\bar{\beta}^{-1}\in D^1(U)\backslash D^2(U)$ by Lemma 1.17 (2), where $\bar{\alpha}$, $\bar{\beta}$ are the maps induced from α, β, respectively. Thus $D^1(U)\neq D^2(U)$.

(2) (a). It is enough to show that $N:=U/UJ^2$ is injective. Let E be the injective hull of N, and $F:=\text{soc}^3E$. Suppose that N is not injective. Then F is a uniserial module of length 3, and $N=FJ$. Note that $D^1(N)=D^2(F)$ and $D^2(N)=D^3(F)$. Then since top N is of 2nd king, the above shows that $D^1(N)\neq D^2(N)$, which contradicts Lemma 1.22.

(2) (b). The following is easily verified:

Claim. *Let $x_1, \cdots, x_n\in D^1(U)$. Then $\{x_1, \cdots, x_n\}$ is linearly dependent in $D^1(U)_{D^2(U)}$ iff the homomorphism $(x_i\pi)\colon (U/UJ^2)^{(n)}\to$ top U is cofusible, where π: $U/UJ^2\to$ top U is the canonical epimorphism.*

If $n\geqq 3$, then Ker $(x_i\,\pi)$ is not local, whence $(x_i\,\pi)$ is cofusible by Lemma 1.17(2). Thus $[D^1(U)\colon D^2(U)]_r=2$ because $D^1(U)\neq D^2(U)$ by (1) above. For the remaining part we apply [4, Theorem and Lemma 2]. We may assume that $|U|=2$, whence that $U=P/M$ for some maximal submodule M of PJ. Note that in this case $D^1(U)=D$ and $D^2(U)=D(M)$ under the notation of [4]. Put $X^*:=\bigcap_{\alpha}\alpha^{-1}(X)$ for any $X\leqq PJ$, where α runs through $\text{End}_A P$. Then the above mentioned statements guarantee that $[D^1(U)\colon D^2(U)]_l=|PJ/M^*|$ because $PJ^2\leqq M\leqq$

PJ and P/M is quasi-injective. Since top U=top P is of 2nd kind, Proposition 1.20 and Lemma 1.13 claim that for any maximal submodule X of PJ, $P/M \cong P/X$, whence $M^*=X^*\leqq X$. Thus $PJ^2 \leqq M^* \leqq \mathrm{rad}(PJ)=PJ^2$, i.e., $M^*=PJ^2$. //

REMARK 1.25. In the setting of the proof of (2) (b), let $v \in PJ \backslash PJ^2$. Then by the above lemma, we have $[D^1(U): D^2(U)]_r=[D_1(v): D_2(v)]_l$, and $[D^1(U): D^2(U)]_l=[D_1(v): D_2(v)]_r$, where $D_i(v)$ are division rings defined in [24]. We expect that these together with Proposition 1.4, Lemma 1.21 and Theorem 1.0 will give a complete characterization of a ring of right local type.

2. *AR*-sequences over a ring of right local type

Recall that for indecomposalbe right A-modules X, Y, the bimodule ${}_{F(Y)}\mathrm{Irr}(X, Y)_{F(X)}$ of irreducible maps is, by definition, the factor module $\mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y)$ where rad is the radical of the category of finitely generated right A-modules, and $F(X):=\mathrm{End}_A(X)/\mathrm{rad}(X, X)$. The following holds for arbitrary artinian rings; see, for instance, [14] or [19] for the proof in the algebra case.

Lemma 2.1. *Let A be an aribitrary artinian ring, and $X_1, \cdots, X_n$, Y indecomposable right A-modules with $X_i \not\cong X_j$ for any $i \neq j$, and $\alpha_{ij}: X_i \to Y$ a homomorphism for every $1 \leqq i \leqq n$, $1 \leqq j \leqq d_i$. Then the homomorphism $(\alpha_{ij})_{i,j}: \bigoplus_{i=1}^{n} X_i^{(d_i)} \to Y$ is irreducible iff every α_{ij} is irreducible and $\{\bar{\alpha}_{i1}, \cdots, \bar{\alpha}_{id_i}\}$ is a linearly independent set in* $\mathrm{Irr}(X_i, Y)_{F(X_i)}$ *for every i, where $\bar{\alpha}_{ij}$ is the image of α_{ij} in* $\mathrm{Irr}(X_i, Y)$.

In the rest of this section, we assume that A is a *ring of right local type*. We now give all the irreducible maps between indecomposable right A-modules.

Lemma 2.2. *Let $\alpha: X \to Y$ be an epimorphism between local right A-modules. Then α is irreducible iff* Ker α *is simple.*

Proof. ($\Rightarrow$). This part holds for an arbitrary ring by [16, Lemma 1.3], and this is trivial in our case.

($\Leftarrow$). By assumption Y is not projective, so we have an AR-sequence (*) stopping at Y, which gives rise to a commutatiae diagram

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & \mathrm{Ker}\,\alpha & \xrightarrow{\beta} & X & \xrightarrow{\alpha} & Y & \longrightarrow 0 \\ & & & \gamma\big\downarrow & (S) & \delta\big\downarrow & & \big\| & \\ (*) & 0 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & Y & \longrightarrow 0, \end{array}$$

Consider the left sequare (S), which is a pushout and pullback diagram. Then since $(\beta, \gamma)^T$ is fusible by Proposition 1.12, δ is a section by Lemma 1.11 (1). Thus α is irreducible. //

REMARK 2.3. If A is not of right local type, then the implication ($\Leftarrow$) is not ture in general. For instance, let A^{op} be the algebra defined by the quiver $1\to 2\leftarrow 3$. Then the canonical epimorphism $(k\to k\leftarrow 0)\to(k\to 0\leftarrow 0)$ gives a counterexample.

Lemma 2.4. *Let* $0\to X \xrightarrow{(\alpha_i)^T} \bigoplus_{i=1}^{n} Y_i \xrightarrow{(\beta_i)} Z\to 0$ *be an AR-sequence with each* Y_i *indecomposable. Then* β_i *is an epimorphism for some* i, *and* $n\leqq 2$.

Proof. We may suppose that $n\geqq 2$. Since Z is local, β_i is an epimorphism for some i, say $i=1$. Then this sequence yields a pushout and pullback diagram:

$$\begin{array}{ccc} X & \xrightarrow{\alpha_1} & Y_1 \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \beta_1} \\ Y' & \xrightarrow[\beta']{} & Z \end{array}$$

where $Y' := \bigoplus_{i\neq 1} Y_i$, $\alpha' := (\alpha_i)^T_{i\neq 1}$ and $\beta' := -(\beta_i)_{i\neq 1}$. Since α' is also an epimorphism, Y' is local, whence $n\leqq 2$. //

The next gives all the irreducible monomorphisms between indecomposables, which follows by the above lemma considering "left boundaries" of Figures 2 and 3 below. Since we do not use this later and also follows by Theorem I, the precise proof is left to the reader.

Lemma 2.5. *Let* $\alpha\colon X\to Y$ *be a monomorphism between indecomposable right* A*-modules. Then* α *is irreducible iff we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \to & K & \to & U & \to & X & \to & 0 \\ & & = \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \to & K & \to & P & \to & Y & \to & 0 \end{array}$$

with exact rows such that P *is indecomposable projective and* β *is an irreducible map, i.e., an inclusion map of some direct summand of* PJ. (*Hence* P *is necessarily the projective cover of* Y.) *If this is the case,* X *is uniserial since so is* U.

The following gives all the AR-sequences over a ring of right local type.

Theorem I. *Let* A *be a ring of right local type. Then an exact sequence* *of right A-modules is an AR-sequence iff it is isomorphic to one of the following sequences*:

(I) $$0 \to eJ/eJ^{i+1} \xrightarrow{(\mu_1, -\varepsilon_1)^T} eA/eJ^{i+1} \oplus eJ/eJ^i \xrightarrow{(\varepsilon_2, \mu_2)} eA/eJ^i \to 0$$

where eA is uniserial and $i \geqq 1$.

(II) $$0 \to K/KJ^{s+1} \xrightarrow{(\mu_1, -\varepsilon_1)^T} eA/KJ^{s+1} \oplus K/KJ^s \xrightarrow{(\varepsilon_2, \mu_2)} eA/KJ^s \to 0$$

where $eJ = K \oplus L$ and K, L $(\neq 0)$ are uniserial, $s \geqq 0$, $KJ^s \neq 0$.

(III) $$0 \to eA/(KJ^{s+1} \oplus LJ^{t+1}) \xrightarrow{(\varepsilon_1', -\varepsilon_2')^T} eA/(KJ^{s+1} \oplus LJ^t) \oplus eA/(KJ^s \oplus LJ^{t+1}) \xrightarrow{(\varepsilon_1, \varepsilon_2)} eA/(KJ^s \oplus LJ^t) \to 0$$

where $eJ = K \oplus L$, and K, L are uniserial, $s, t \geqq 0$ and $KJ^s, LJ^t \neq 0$.

(IV) $$0 \to S \xrightarrow{\mu} eA \xrightarrow{\varepsilon} eA/S \to 0$$

where *$eJ = S_1 \oplus \cdots \oplus S_n$, $S := S_1 \cong \cdots \cong S_n$ are simple and $n > 2$.*

(V) $$0 \to eA/(S_2 \oplus \cdots \oplus S_{i-1}) \xrightarrow{(\varepsilon_1', -\varepsilon_2')^T} eA/(S_1 \oplus \cdots \oplus S_{i-1}) \oplus eA/(S_2 \oplus \cdots \oplus S_i) \xrightarrow{(\varepsilon_1, \varepsilon_2)} eA/(S_1 \oplus \cdots \oplus S_i) \to 0$$

where $eJ = S_1 \oplus \cdots \oplus S_n$, $S := S_1 \cong \cdots \cong S_n$ are simple and $n > 2$, $2 \leqq i \leqq n$ (for convenience we set $S_2 \oplus \cdots \oplus S_1 = 0$).

In the above, ε_*, ε_*' are the canonical epimorphisms and μ_* are the inclusion maps, and e is a primitive idempotent of A.

REMARK 2.6. In the above list of sequences, the middle term is indecomposable in the case (I), $i=1$, and in the case (II), $s=0$.

Proof of Theorem I. Let (e): $0 \to K \xrightarrow{\sigma} E \xrightarrow{\tau} C \to 0$ be an exact sequence. Then by definition (e) is an AR-sequence if

(1) (e) is non-split;
(2) K and C are indecomposable; and
(3) τ is right almost split.

Let (e) be any one of the sequences in the theorem. First it is easy to see that (e) is exact. The statements (1) and (2) are trivial because each component map of σ is non-isomorphism and K, C are local. Hence noting that all the non-projective indecomposables appear as the module C, it only remains to prove the statment (3). By definition the statement (3) is trivially equivalent to the following:

(3′) For any non-isomorphism α: $X\to C$ from any indecomposable module X, there exists a homomorphism β: $X\to E$ with $\alpha=\tau\beta$.

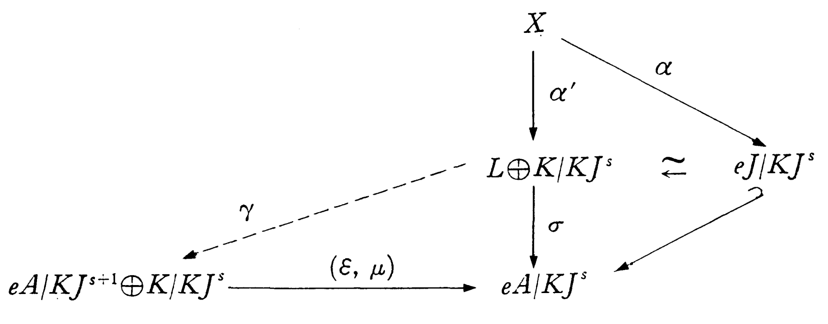
In the cases (I), (II), (IV) we directly construct the desired homomorphism β, and at the same time we construct the β for any *non-epimsrphism* α in the cases (III) and (V) for the later use. For the cases (III) and (V) we show the statemt statement (3) as follows. Let E_1 and E_2 be indecomposable modules in the middle term E so that (e) has the form $0\to K \xrightarrow{(\varepsilon_1', -\varepsilon_2')^T} E_1\oplus E_2 \xrightarrow{(\varepsilon_1, \varepsilon_2)} C\to 0$. By Lemma 2.4 it suffices to show that $(\varepsilon_1, \varepsilon_2)$ is irreducible. To this end we can apply Lemma 2.1 since by Lemma 2.2, ε_1 and ε_2 are irreducible. If $E_1\not\cong E_2$, then the assertion is trivial. Hence we have only to show that the set $\{\bar{\varepsilon}_1, \overline{\varepsilon_2\varphi}\}$ is linearly independent in $\mathrm{Irr}(E_1, C)_{F(E_1)}$ under the assumption that there is an isomorphism φ: $E_1\to E_2$.

(i) Proof of (3') for the cases (I), ⋯, (V) when α is not an epimorphism

We may asume that $X=fA/I$ for some primitive idempotent f of A and for some $I\leqq fJ$ and that $\alpha=a\cdot$ for some $a\in eJf$.

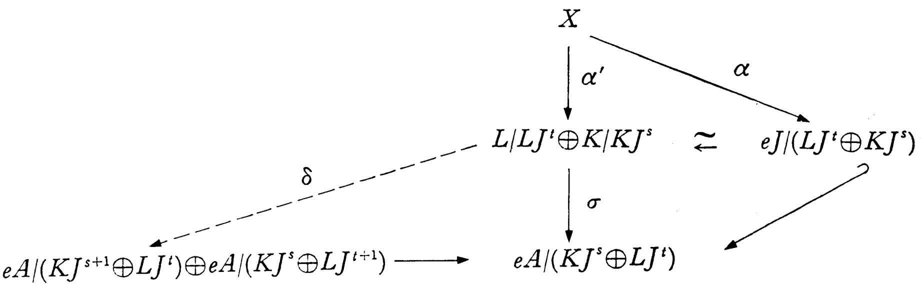
Case (I). Since $C=eA/eJ^i$ is local, Im $\alpha\leqq eJ/eJ^i$. Hence $a\cdot$: $X\to eJ/eJ^i$ is defined and is a desired map.

Case (II). Since Im $\alpha\leqq eJ/KJ^s$, we can define the maps α' and σ by the following commutative diagram



which is completed by the $\gamma:=\begin{bmatrix}\beta & 0\\ 0 & id\end{bmatrix}$ above, where id is the identity map of K/KJ^s and β is the composite $L\to K\oplus L\to(K\oplus L)/KJ^{s+1}=eJ/KJ^{s+1}\to eA/KJ^{s+1}$ of the canonical maps. Hence $\gamma\alpha'$ is a desired map.

Case (III). Im $\alpha\leqq eJ/(KJ^s\oplus LJ^t)\simeq L/LJ^t\oplus K/KJ^s$. Hence we can define the maps α' and σ by the following commutative diagram:



Define the maps $\beta: L/LJ^t \to eA/(KJ^{s+1} \oplus LJ^t)$ and $\gamma: K/KJ^s \to eA/(KJ^s \oplus LJ^{t+1})$ as the composite maps of the canonical maps: $L/LJ^t \to (K \oplus L)/LJ^t = eJ/LJ^t \to eJ/(KJ^{s+1} \oplus LJ^t) \to eA/(KJ^{s+1} \oplus LJ^t)$; and $K/KJ^s \to (K \oplus L)/KJ^s = eJ/KJ^s \to eJ/(KJ^s \oplus LJ^{t+1}) \to eA/(KJ^s \oplus LJ^{t+1})$, respectively. (Note that β, γ are non-isomorphisms.) Then as easily seen the homomorphism $\delta := \begin{bmatrix} \beta & 0 \\ 0 & \gamma \end{bmatrix}$ completes the above commutative diagram, whence the homomorphism $\delta\alpha'$ is a desired map.

Case (IV). Since eJ/S is semisimple, $0 = a(fJ) \geqq aI$. Hence $a\cdot$ gives a desired map $X \to eA$.

Case (V). Again since $eJ/(S_1 \oplus \cdots \oplus S_i)$ is semisimple, both $(a\cdot, 0)^T$ and $(0, a\cdot)^T$ gives a desired map, whose components are non-isomorphisms.

(ii) Proof of (3') for the cases (I), (II), (IV) when α is an epimorphism

Since the projective cover of X is isomorphic to eA, we may assume that $X = eA/I$ for some $I \leqq eJ$, and that $\alpha = a\cdot$ for some $a \in eAe \setminus eJe$.

Case (I). $I = eJ^j$, for some $j \geqq i+1$. Since $a(eJ^j) \leqq eJ^j \leqq eJ^{i+1}$, $a\cdot: X \to eA/eJ^{i+1}$ is defined and is a desired map.

Case (II). $aI \leqq KJ^s$. If $aI = KJ^s$, then since a is a unit, $a^{-1}KJ^s = I$, i.e., Ker $\alpha = 0$, a contradiction. Hence $aI \leqq KJ^{s+1}$. Thus $(a\cdot, 0)^T: X \to eA/KJ^{s+1} \oplus K/KJ^s$ is a desired map.

Case (IV). In this case $X = eA$ is projecitve, whence there is nothing to show.

(iii) Proof of (3) for the cases (III), (V)

Suppose that our assertion is not true. Then $\varepsilon_1 = \varepsilon_2 \eta + \theta$ for some isomorphism $\eta: E_1 \to E_2$ and for some $\theta \in \mathrm{rad}^2(E_1, C)$. Let $\theta = \gamma\beta$ with $\beta \in \mathrm{rad}(E_1, L)$, $\gamma \in \mathrm{rad}(L, C)$, where $L = \bigoplus_{i=1}^{n} L_i$ with all L_i local and $\beta = (\beta_i)^T$, $\gamma = (\gamma_i)$. Then since $|C| - |E_1| = 1$ and L_i are local, every $\gamma_i \beta_i$ is not an epimorphism, whence we have $\gamma_i\beta_i=\varepsilon_1\delta_i+\varepsilon_2\zeta_i$ for some $\delta_i\in\mathrm{rad}(E_1, E_1)$ and $\zeta_i\in\mathrm{rad}(E_1, E_2)$ by (i) above. Thus $\varepsilon_1-\varepsilon_2\eta=\varepsilon_1\delta+\varepsilon_2\zeta$, where $\delta:=\sum\delta_i$ and $\zeta:=\sum\zeta_i$. Consequently $\varepsilon_1=\varepsilon_2\lambda$ for some isomorphism λ because $1_{E_1}-\delta$ and $\eta+\zeta$ are isomorphisms. This shows that $(\varepsilon_1, \varepsilon_2)$ is cofusible, a contradiction to the fact that $\mathrm{Ker}(\varepsilon_1, \varepsilon_2)=K$ is indecomposable. //

3. The *AR*-quiver of a ring of right local type

Definition 3.1. A *translation quiver* is a pair (Q, τ) of a quiver Q and a bijection τ: $R\to L$ of subsets of the set Q_0 of vertices of Q satisfying the conditions

(a) Q has no loop $\cdot\circlearrowright$ and no multiple arrow $\cdot\rightrightarrows\cdot$; and

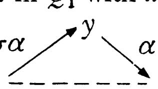
(b) For any $x\in R$, $(\tau x)^+=x^-$,

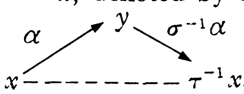
where we denote by y^+ and by y^- the set $\{z\in Q_0|$ there exists an arrow $y\to z$ in $Q\}$ and $\{z\in Q_0|$ there exists an arrow $z\to y$ in $Q\}$, respectively for every $y\in Q_0$. The full subquiver formed by $\{x, \tau x\}\cup x^-$ for any $x\in R$ is called the *mesh* starting at τx and stopping at x. For each arrow α: $y\to x$ in Q with $x\in R$ we denote by $\sigma\alpha$ the unique arrow $\tau x\to y$ in Q.

A valued translation quiver is a translation quiver (Q, τ) together with a pair (d, d') (called a *value* of (Q, τ)) of maps from Q_1(:=the set of arrows of Q) to the set of natural numbers satisfying the condition

(*c*) $d(\sigma\alpha)=d'(\alpha)$ for any α: $y\to x$ in Q_1 with $x\in R$. If further the following is satisfied, then it is called a *well valued translation quiver*:

(*d*) $d'(\sigma\alpha)=d(\alpha)$ for any α: $y\to x$ in Q_1 with $x\in R$.

For each $\alpha\in Q_1$, the diagram τx 

x, denoted by $T(\sigma\alpha, \alpha)$, is called a *triangle* stopping at α, and the diagram x 

$\tau^{-1}x$, denoted by $T(\alpha, \sigma^{-1}\alpha)$, is called a *triangle* starting at α, where we set τx, $\tau^{-1}x$, $\sigma\alpha$ or $\sigma^{-1}\alpha$ to be zero if it is not defined. A triangle $T(\alpha, \beta)$ is said to be *projective*, *injective* or *proper* if $\alpha=0$, $\beta=0$ or α, $\beta\neq0$, respectively. Proper triangles are identified with their corresponding full subquiver of Q. We define $\sigma T(\alpha, \beta):=T(\sigma\alpha, \sigma\beta)$ if $\alpha\neq0$, $\sigma^{-1}T(\alpha, \beta):=T(\sigma^{-1}\alpha, \sigma^{-1}\beta)$ if $\beta\neq0$ and $d(T(\alpha, \beta)):=\begin{cases} d(\alpha) & \text{if } \alpha\neq0 \\ d'(\beta) & \text{if } \beta\neq0\end{cases}$, which is called a *value* of $T(\alpha, \beta)$ and is well-defined by the condition (*c*). The map d: $T(\alpha, \beta)\mapsto d(T(\alpha, \beta))$ is called a *value* of triangles of (Q, τ). Note that giving a value (d, d') to a translation quiver (Q, τ) is equivalent to giving a value of triangles of (Q, τ). The condition (*d*) is equivalent to the condition

(*d'*) $d(\sigma T)=d(\sigma^{-1}T)$ for all proper triangles T.

Definition 3.2. Let A be a ring with *AR*-sequences. Then the *AR*-

quiver Γ of A is a valued translation quiver defined as follows:

(a) The set Γ_0 of vertices is the set of isomorphism classes of indecomposable right A-modules. (We identify indecomposables with their isomorphism classes.)

(b) For any X, Y in Γ_0, the number of arrows from X to Y is at most one, and there exists an arrow $X \to Y$ iff $\mathrm{Irr}(X, Y) \neq 0$.

(c) If $0 \to X \to Y \to Z \to 0$ is an AR-sequence, then translation is defined as $\tau Z := X$.

(d) The value is defined as $d(\alpha) := [\mathrm{Irr}(X, Y): F(Y)]_l$ and $d'(\alpha) := [\mathrm{Irr}(X, Y): F(X)]_r$ for each arrow $\alpha: X \to Y$.

REMARK 3.3. (1) In the above definition, the condition (*c*) in Definition 3.1 is verified by Lemma 3.4 below.

(2) If A is an artin algebra, then the AR-equiver is a well vlaued translation quiver.

The following two lemmas are well know in the algebra case, which remain valid also in our aritnian case. See for instance [20].

Lemma 3.4. *Let $0 \to X \to \bigoplus_{i=1}^{n} Y_i \to Z \to 0$ be an AR-sequence of right A-modules, where all Y_i are indecomposable. Then for any indecomposable module Y_A, we have*

$$[\mathrm{Irr}(X, Y): F(Y)]_l = \#\{i \mid Y \cong Y_i\} = [\mathrm{Irr}(Y, Z): F(Y)]_r .$$

Lemma 3.5. (1) *Let Z_A be an indecomposable projective module and $ZJ = \bigoplus_{i=1}^{n} Y_i$ with each Y_i indecomposable. Then for any indecomposable module Y_A, we have*

$$[\mathrm{Irr}(Y, Z): F(Y)]_r = \#\{i \mid Y \cong Y_i\} .$$

(2) *Let X_A be an indecomposable projective module and $X/\mathrm{soc}\, X = \bigoplus_{i=1}^{n} Y_i$ with each Y_i indecomposable. Then for any indecomposable module Y_A, we have*

$$[\mathrm{Irr}(X, Y): F(Y)]_l = \#\{i \mid Y \cong Y_i\} .$$

Theorem II. *Let A be a ring of right local type and e a primitive idempotent of A. Then*

(1) *The AR-quiver of A with the precise form of the full subquiver consisting of the meshes stopping at all the local modules thai are factor modules of eA has the follwoing form: (In the figures we write a value of a triangle inside the triangle if the value is greater than 1, and triangles without indications of values have the value*

1. *Further the symbols* [* *and* *] *indicate that* * *is projective and that* * *is injective, respectively.*)

(a) *In the case that* eA *is uniserial. Put* $h := h(eA)$. *See Fig. 1.*

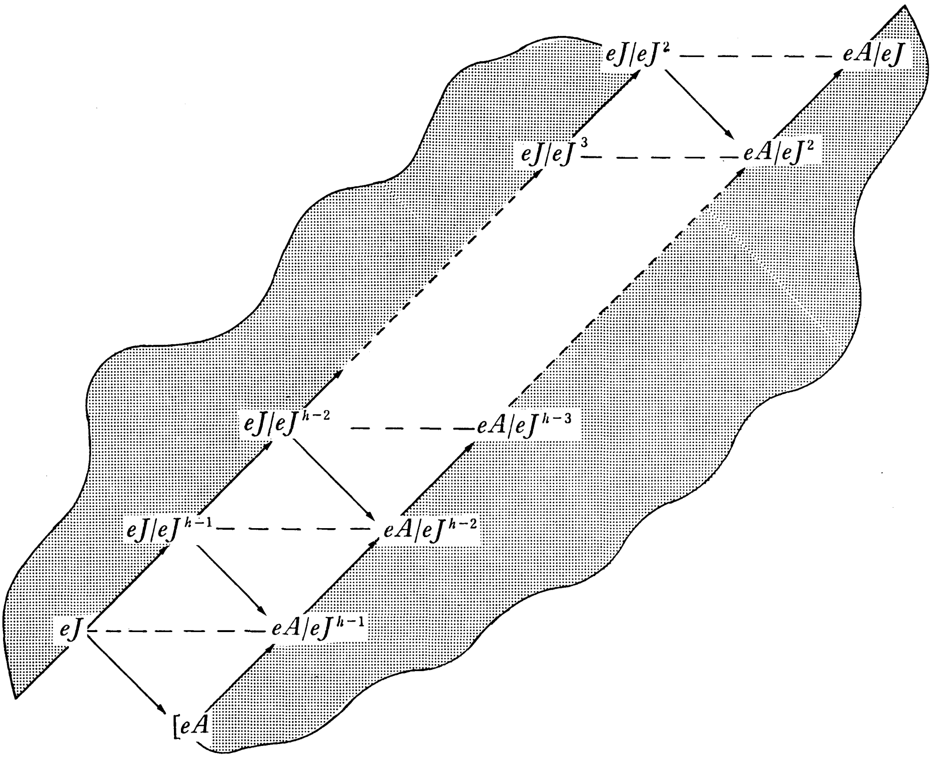


Fig. 1.

(b) *In the case that* $eJ = K \oplus L$ *for some non-zero uniserial modules* K, L *which are not isomorphic to each other. Put* $s := h(K)$, $t := h(L)$. *See Fig. 2.*

(c) *In the case that* $eJ = K \oplus L$ *for some non-zero uniserial modules* K, L *which are isomorphic to each other. Put* $h := h(K)$. *See Fig. 3.*

(d) *In the case that* $eJ = S_1 \oplus \cdots \oplus S_n$, $S_1 \simeq \cdots \simeq S_n$ *are simple and* $n > 2$. *See Fig. 4, where* $\begin{pmatrix} 0 \\ 1 \end{pmatrix} := S_1$, $\begin{pmatrix} 1 \\ i \end{pmatrix} := eA/(S_{i+1} \oplus \cdots \oplus S_n)$ $(S_{n+1} \oplus \cdots \oplus S_n := 0)$, *for all* $0 \leqq i \leqq n$.

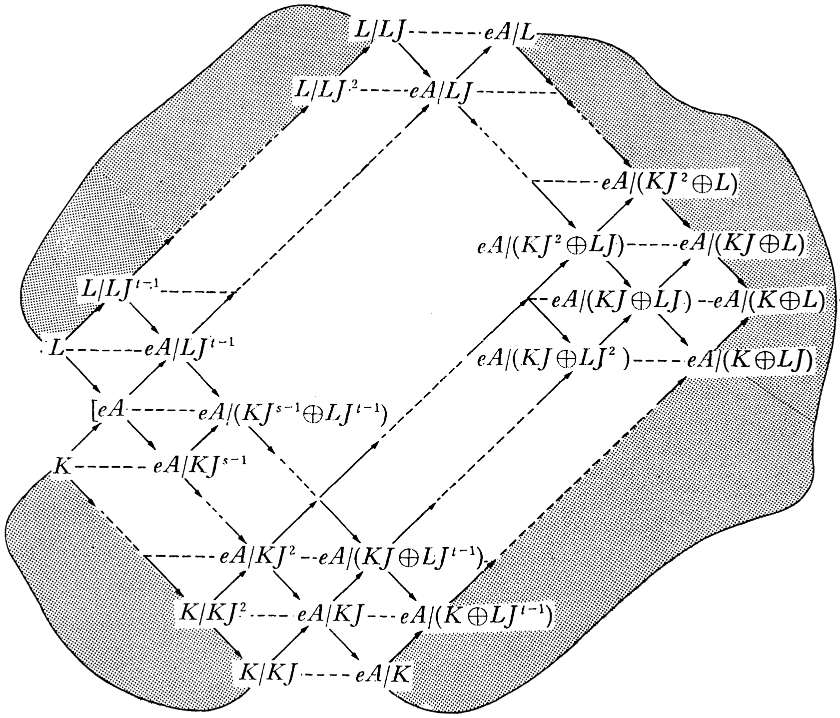
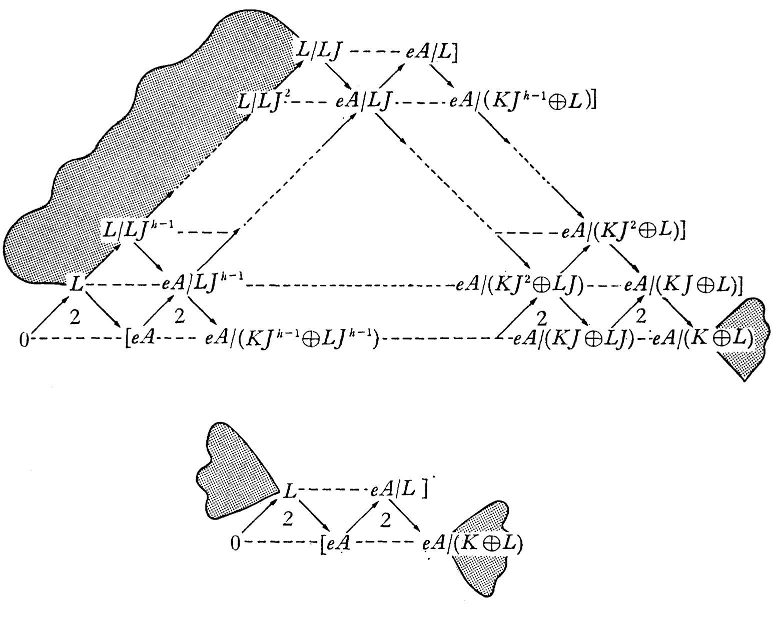


Fig. 2.



(in the case that K is simple)

Fig. 3.

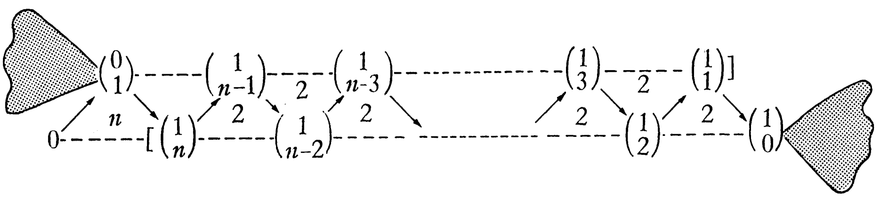
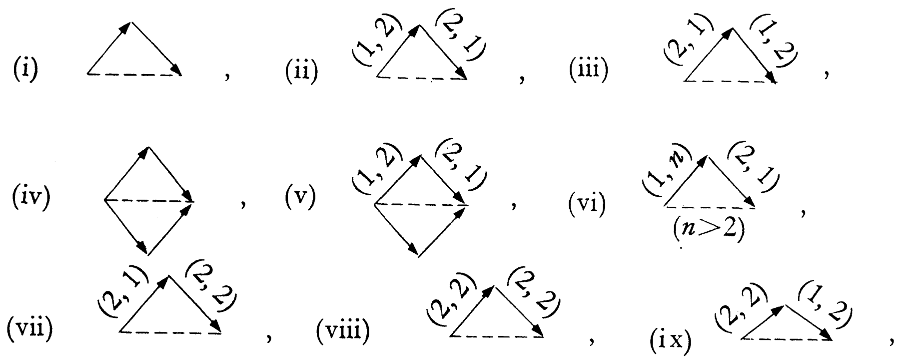


Fig. 4.

(2) *The following is the list of all the valued meshes of the AR-quiver of* A:



Here X---- Y *and* $\xrightarrow{(a,b)}$ *denote that* $X=\tau Y$, *and that this arrow* α *has the value* $(d(\alpha), d'(\alpha))=(a, b)$, *respectively, while arrows without the value notation have the value* $(1, 1)$.

(3) A *is of left colocal type iff every valued mesh has one of the forms* (*i*), $\cdots$, (*v*) *in* (2) *above.*

Proof. (1) This follows by Theorem I, Proposition 1.20 and Proposition 1.24.

(2) Let $T=T(\alpha, \beta)$, $X\overset{\alpha}{\rightarrow}Y\overset{\beta}{\rightarrow}Z$ be any proper triangle in Figures 1, $\cdots$, 4. Note that the triangles with value $\geqq 2$ appear only in the bottom of Fig. 3 and in Fig. 4. Then we see that if $d(\sigma T)\geqq 2$ or $d(\sigma^{-1}T)\geqq 2$, then $d(T)=2$ or Y is not uniserial. On the other hand we can observe that if σT or $\sigma^{-1}T$ is in the shaded part, then $d(T)=1$ and Y is uniserial, whence $d(\sigma T)=1=d(\sigma^{-1}T)$. This and Figures 1, $\cdots$, 4 give our list of all valued meshes. (Note that the condition (d') in Definition 3.1 is satisfied except for two triangles in Fig. 4.)

(3) By Corollary 1.2, A is of left colocal type iff the above case (d) does not occure. //

Notice that Theorem II makes it possible to draw the whole of the *AR*-quiver by gluing together the above figures. Now the following is an immediate consequence of Theorem II.

Theorem 3.6. *The AR-quiver of a ring A of right local type is a well valued translation quiver iff A is also a ring of left colocal type.*

This suggests us the following.

Conjecture. *If A is both of right local type and of left colocal type, then A has a selfdulity.*

Corollary 3.7. *If A is given by the form $\begin{pmatrix} F & M \\ 0 & G \end{pmatrix}$ for some division rings F, G and some F-G-bimodule M with $[M: G]_r = n \geq 1$, then A is of right local type iff the dimension sequence of M is $(n, 1, 2, 2, \cdots, 2, 1)$ ($n+2$ terms) (if $n=1$, it is $(1, 1, 1)$). Further in this case A is of left colocal type iff $n \leqq 2$.*

Proof. ($\Rightarrow$). The AR-quiver of A has just the form in Fig. 4 without the shaded parts, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is projective, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is injective, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $T_0, T_1, \cdots, T_{n+1}$ be the triangles in Fig. 4 in the same order. (Notice that T_0 is projective and T_{n+1} is injective.) Then by [12], the desired dimension sequence is given by $(d(T_0), \cdots, d(T_{n+1}))$.

($\Leftarrow$). By [12], A has exactly $n+2$ non-isomorphic indecomposable modules. Since $|eJ| = n$, there exists $n+2$ non-isomorphic local right modules, which shows that A is or right local type.

The rest follows by Corollary 1.2. //

EXAMPLE 3.8. In the algebra case, it is known that A is of right local type iff A is left serial and eJ is a direct sum of at most two uniserial modules for every primitive idempotnet e of A (note that the lateer is equivalent to saying that the conditions (LR) and (L) in Theorem 1.0 are satisfied). However this does not hold in the artinian case. We give countereaxmples of both implications, which at the same time are an example of a ring of right local type but not of left colocal type and an example of a ring of left colocal type but not of right local type. Let d be any dimension sequence of length 5. Then [21, section 13] and [12, Proposition 1] guarantee that there exists an F-G-bimodule M over division rings F, G such that the dimension sequence of M is d. Let A be a ring defined by M as in Corollary 3.7.

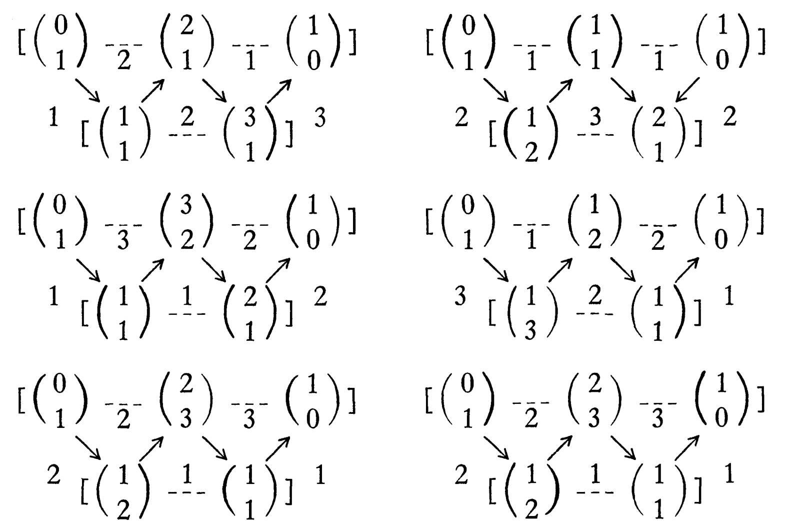
(1) (This example had been reported in Japanese [3]). If $d = (3, 1, 2, 2, 1)$, then A is of right local type but not of left colocal type by Corollary 3.7. Further A gives a counterexample of the implication ($\Rightarrow$).

(2) If $d = (2, 1, 3, 1, 2)$, then A is of left colocal type but not of right local type. In fact A^{op} is a ring defined by a bimodule over division rings having the dimension sequence $(1, 2, 2, 1, 3)$ and it has an indecomposable injective module I with $|I/\text{soc } I| = 3$, which gives 5 colocal modules. Thus A is of left colocal type, and A is not of right local type by Corollary 3.7. Further A gives

a counterexample of the implication ($\Leftarrow$).

Remark 3.9. (1) Rings defined as in Corollary 3.7 by a bimodule over division rings having dimension sequence d of length 5 is called *rings of type* $I_2(5)$ having dimension sequence d. As easily seen these rings are characterized as hereditary rings with just 2 non-isomorphic simples and just 5 non-isomorphic indecomposables. Thus if A is of type $I_2(5)$, then so are A^{op} and $D_r(A) := \mathrm{End}_A(E_1 \oplus E_2)$, where E_1 and E_2 are the injective indecomposable right A-modules. More precisely, if A has dimension sequence $(a_1, \cdots, a_5)$, then A^{op} has $(a_2, a_1, \cdots)$, and $D_r(A)$ has $(a_5, \cdots, a_1)$. It should be noted that a dimension sequence of length 5 is completely determined by the first two terms.

(2) The AR-quiver of a ring of type $I_2(5)$ is as follows:



In the above, modules X are presented by the dimension type $([Xe_1: F]_r, [Xe_2: G]_r)^T$ where $e_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $e_2=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. (Notice that for non-simple indecomposables X, $Xe_1 = \text{top } X$ and $Xe_2 = XJ = \text{soc } X$.) Further the dimension sequences are just the sequence of values of triangles, and dimension sequences $(a, b, \cdots)$ and $(b, a, \cdots)$ are placed horizontally to express left modules and right modules. The last two quivers are the same. The AR-quiver of a ring A and that of $D_r(A)$ are placed diagonally (left-up to right-down).

(3) As well known a ring is serial iff it is of (left and right) local type. By Lemma 1.21, rings of local type are coserial. Hence serial rings are coserial. But the converse is not true as the above AR-sequence of a ring of type $I_2(5)$ having dimension sequence (2, 2, 1, 3, 1) shows.

(4) A ring is called a right Tachikawa ring if every indecomposable right module is local or colocal, and is called a Tachikawa ring if both A and A^{op} are

right Tachikawa rings. A ring of right local and left colocal type is an exrmple of Tachikawa rings. However the AR-sequence of a ring of type $I_2(5)$ having dimension sequence (3, 1, 2, 2, 1) and that of its oppsite ring tell us that a ring of right local type does not need to be a Tachikawa ring.

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