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FINITE DIMENSIONAL REPRESENTATIONS OF QUANTUM GROUPS

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0. Introduction

In [2] and [4] Drinfeld and Jimbo independently noticed that there exists an algebraic object behind the theory of the quantum Yang-Baxter equation. This is the *quantum algebra*, which is a quantization of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional semisimple Lie algebra $\mathfrak{g}$. Their formulations, however, are slightly different; Drinfeld's $U_\hbar(\mathfrak{g})$ is an algebra over the formal power series ring $k[[\hbar]]$, and Jimbo's $U_q(\mathfrak{g})$ is an algebra over the rational function field $k(q^{1/2})$, where k is a field of characteristic zero. The indeterminates $\hbar$ and q are related by $q^{1/2}=e^{\hbar/4}$.

One of the purposes of this paper is to give a counter part for $U_\hbar(\mathfrak{g})$ of the results of Lusztig [6] and Rosso [7] concerning finite diemnsional $U_q(\mathfrak{g})$-modules, by fully using the advantage that $U_\hbar(\mathfrak{g})$ is a topologically free $k[[\hbar]]$-module satisfying $U_\hbar(\mathfrak{g})|_{\hbar=0}\simeq U(\mathfrak{g})$ (the indeterminate $\hbar$ can be directly specialized to 0).

Let P (resp. P^{++}) be the set of integral (resp. dominant integral) weights. As in the case for $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ we can construct a "finite dimensional highest weight module" $L(\lambda)$ for $\lambda\in P^{++}$. It is a $U_\hbar(\mathfrak{g})$-module which is free of finite rank over $R=k[[\hbar]]$ such that $L(\lambda)|_{\hbar=0}$ is isomorphic to the finite dimensional irreducible $U(\mathfrak{g})$-module with highest weight λ. Let $\mathcal{A}$ be the category of $U_\hbar(\mathfrak{g})$-modules which are free of finite rank over R. A $U_\hbar(\mathfrak{g})$-module in $\mathcal{A}$ is said to be $\mathcal{A}$-irreducible if it has no nontrivial quotients which belong to $\mathcal{A}$.

Theorem A. (i) *Any $U_\hbar(\mathfrak{g})$-module in $\mathcal{A}$ is a direct sum of $\mathcal{A}$-irreducible $U_\hbar(\mathfrak{g})$-modules.*

(ii) *A $U_\hbar(\mathfrak{g})$-module in $\mathcal{A}$ is $\mathcal{A}$-irreducible if and only if it is isomorphic to $L(\lambda)$ for some $\lambda\in P^{++}$.*

Let G be a connected split semisimple algebraic group defined over k with $\mathrm{Lie}(G)=\mathfrak{g}$. Assume that G is simply connected for simplicity. Let $A_\hbar[G]$ be the "dual" Hopf algebra of $U_\hbar(\mathfrak{g})$ (see Section 2 below). Since it can be regarded as a quantization of the coordinate algebra $k[G]$ of G, we call it the *quantum group* (see Drinfeld [3], Woronowicz [10]). We have the following analogue of the Peter-Weyl theorem:

Theorem B. *The matirx coefficients of $L(\lambda)$ for $\lambda \in P^{++}$ form a free R-basis of $A_\hbar[G]$. Hence we have an isomorphism*

$$A_\hbar[G] \simeq \oplus_{\lambda \in P^{++}}(L(\lambda) \otimes L(\lambda)^*)$$

of $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-bimodules.

Let B be a Borel subgroup of G and let $k[B]$ be its coordinate algebra. We have an induction functor Ind from the category of $k[B]$-comodules to that of $k[G]$-comodules. We can define a quantization $A_\hbar[B]$ of $k[B]$ as a quotient Hopf algebra of $A_\hbar[G]$ and the induction functor $\mathrm{Ind}_\hbar$ from the category of $A_\hbar[B]$-comodules to that of $A_\hbar[G]$-comodules is similarly defined. For $\mu \in P$ let R_μ (resp. k_μ) be the one dimensional $A_\hbar[B]$ (resp. $k[B]$)-comodule corresponding to μ. The following theorem implies that the analouge of the Borel-Weil-Bott theorem holds for quantum groups.

Theorem C. *For $\mu \in P$ the $A_\hbar[G]$-comodule $R^i \mathrm{Ind}_\hbar(R_\mu)$ is a free R-module and the $k[G]$-comodule $k \otimes_R R^i \mathrm{Ind}_\hbar(R_\mu)$ is isomorphic to $R^i \mathrm{Ind}(k_\mu)$, where R^i denotes the right derived functors.*

1. Irreducible Highest Weight Modules

1.1. Let $\mathfrak{g}$ be a finite dimensional split semisimple Lie algebra over a field k of characteristic zero. Let $A=(a_{ij})_{1 \leq i,j \leq l}$ be the Cartan matrix of $\mathfrak{g}$ and choose positive integres $d_1, \cdots, d_l$ satisfying $d_i a_{ij} = d_j a_{ji}$. The quantum algebra $U_\hbar(\mathfrak{g})$ is the associative algebra over the formal power series ring $R=k[[\hbar]]$ with 1, which is $\hbar$-adically generated by $3l$ elements t_i, e_i, f_i $(i=1, \cdots, l)$ satisfying the following fundamental relations:

$$t_i t_j = t_j t_i \qquad (i, j = 1, \cdots, l), \tag{1.1.1}$$

$$t_i e_j - e_j t_i = d_i a_{ij} e_j \qquad (i, j = 1, \cdots, l), \tag{1.1.2}$$

$$t_i f_j - f_j t_i = -d_i a_{ij} f_j \qquad (i, j = 1, \cdots, l), \tag{1.1.3}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{\sinh(\hbar t_i/2)}{\sinh(\hbar d_i/2)} \qquad (i, j = 1, \cdots, l), \tag{1.1.4}$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} e_i^{1-a_{ij}-m} e_j e_i^m = 0 \qquad (i \neq j), \tag{1.1.5}$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} f_i^{1-a_{ij}-m} f_j f_i^m = 0 \qquad (i \neq j), \tag{1.1.6}$$

where $q_i = \exp(\hbar d_i/2) \in R^*$ $(i=1, \cdots, l)$, and

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{\prod_{r=1}^n (q^r - q^{-r})}{\prod_{r=1}^m (q^r - q^{-r}) \prod_{r=1}^{n-m} (q^r - q^{-r})} \qquad (n \geqq m \geqq 0)$$

(Drinfeld [2], Jimbo [4]).

Let N^+ (resp. N^-, resp. T) be the subalgebra of $U_\hbar(\mathfrak{g})$ generated by $e_1, \cdots, e_l$ (resp. $f_1, \cdots, f_l$, resp. $t_1, \cdots, t_l$) and let $U_\hbar^f(\mathfrak{g})$ be the subalgebra generated by N^+, N^-, $\bar{T}$, where barring denotes the $\hbar$-adic closure.

Proposition 1.1.1 ([9], see also [2]). (i) *N^+ (resp. N^-) is a free R-module, and the relation* (1.1.5) (*resp.* (1.1.6)) *is a fundamental relation among the generators $e_1, \cdots, e_l$ (resp, $f_1, \cdots, f_l$) of N^+ (resp. N^-).*

(ii) *T is naturally isomorphic to the polynomial ring $R[t_1, \cdots, t_l]$, and the inclusion $T \hookrightarrow \bar{T}$ is the $\hbar$-adic completion.*

(iii) *We have the following isomorphism of R-modules:*

$$N^- \otimes \bar{T} \otimes N^+ \simeq U_\hbar^f(\mathfrak{g}) \qquad (u \otimes v \otimes w \leftrightarrow uvw)\,. \tag{1.1.7}$$

(iv) *The inclusion $U_\hbar^f(\mathfrak{g}) \hookrightarrow U_\hbar(\mathfrak{g})$ is the $\hbar$-adic completion.*

By [8] we see that the k-algebra $k \otimes_R U_\hbar(\mathfrak{g})$ is naturally isomorphic to the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, where the ring homomorphism $R \to k$ is given by $\hbar \mapsto 0$. The natural Hopf algebra structure on $U(\mathfrak{g})$ lifts to the topological Hopf algebra structure on $U_\hbar(\mathfrak{g})$ given by the following:

$$\Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i \qquad (i = 1, \cdots, l)\,, \tag{1.1.8}$$

$$\Delta(e_i) = e_i \otimes \exp(-\hbar t_i/4) + \exp(\hbar t_i/4) \otimes e_i \qquad (i = 1, \cdots, l)\,, \tag{1.1.9}$$

$$\Delta(f_i) = f_i \otimes \exp(-\hbar t_i/4) + \exp(\hbar t_i/4) \otimes f_i \qquad (i = 1, \cdots, l)\,, \tag{1.1.10}$$

$$\varepsilon(t_i) = \varepsilon(e_i) = \varepsilon(f_i) = 0 \qquad (i = 1, \cdots, l)\,, \tag{1.1.11}$$

$$S(t_i) = -t_i \qquad (i = 1, \cdots, l)\,, \tag{1.1.12}$$

$$S(e_i) = -q_i^{-1} e_i \qquad (i = 1, \cdots, l)\,, \tag{1.1.13}$$

$$S(f_i) = -q_i f_i \qquad (i = 1, \cdots, l), \tag{1.1.14}$$

where Δ, ε, S are the coproduct, the counit and the antipode, respectively (see [2], [4]).

Lemma 1.1.2. *$U_\hbar(\mathfrak{g})$ is a noetherian ring; i.e., the ascending chain conditions for left and right ideals are satisfied.*

Proof. It is known that the enveloping algebra $U(\mathfrak{g})$ is a noetherian ring and by Proposition 1.1.1 we have

$$U_\hbar(\mathfrak{g}) = \varinjlim U_\hbar(\mathfrak{g})/\hbar^n U_\hbar(\mathfrak{g})\,, \quad \hbar^n U_\hbar(\mathfrak{g})/\hbar^{n+1} U_\hbar(\mathfrak{g}) \simeq U_\hbar(\mathfrak{g})/\hbar U_\hbar(\mathfrak{g}) \simeq U(\mathfrak{g})\,.$$

Hence the assertion is proved similarly to the well known fact that the formal power series ring over a noetherian ring is noetherian. □

Let $\mathfrak{t}$ (resp. $\mathfrak{t}_0$) be the R-submodule (resp. k-subspace) of T generated by $t_1, \cdots, t_l$. By Proposition 1.1.1 $\{t_1, \cdots, t_l\}$ is a basis of the k-vector space $\mathfrak{t}_0$ and we have $\mathfrak{t}=R\otimes_k \mathfrak{t}_0$. Set $\mathfrak{t}^*=\mathrm{Hom}_R(\mathfrak{t}, R)$ and $\mathfrak{t}_0^*=\mathrm{Hom}_k(\mathfrak{t}_0, k)$. Via the ring homomorphisms $k\hookrightarrow R\to k$ we have:

$$\mathfrak{t}^*\simeq R\otimes_k \mathfrak{t}_0^* \quad \mathfrak{t}_0^*\simeq k\otimes_R \mathfrak{t}^* .$$

We will identify $\mathfrak{t}_0^*$ with a subspace of $\mathfrak{t}^*$ and the natural homomorphism $\mathfrak{t}^*\to\mathfrak{t}_0^*$ is denoted by $\lambda\mapsto\lambda^0$. We will also identify $\mathfrak{t}_0$ $(\simeq k\otimes_R \mathfrak{t})$ with a split Cartan subalgebra of $\mathfrak{g}$. We define $\alpha_i\in\mathfrak{t}_0^*$ $(i=1, \cdots, l)$ by $\alpha_i(t_j)=d_i a_{ij}$. Then $\{\alpha_1, \cdots, \alpha_l\}$ is a set of simple roots of the root system Δ of $(\mathfrak{g}, \mathfrak{t}_0)$. We denote the set of positive roots by Δ^+. Set

(1.1.15) $$Q=\oplus_{i=1}^l \boldsymbol{Z}\alpha_i ,$$

(1.1.16) $$Q^+=\oplus_{i=1}^l \boldsymbol{Z}_{\geq 0}\alpha_i ,$$

(1.1.17) $$P=\{\lambda\in\mathfrak{t}_0^* \mid \lambda(2t_i/\alpha_i(t_i))\in\boldsymbol{Z} \quad (i=1, \cdots, l)\} ,$$

(1.1.18) $$P^{++}=\{\lambda\in\mathfrak{t}_0^* \mid \lambda(2t_i/\alpha_i(t_i))\in\boldsymbol{Z}_{\geq 0} \quad (i=1, \cdots, l)\} .$$

We denote by W the Weyl group of $(\mathfrak{g}, \mathfrak{t}_0)$. It is a finite subgroup of $GL(\mathfrak{t}_0)$ generated by the reflections s_i $(i=1, \cdots, l)$ given by

$$s_i(t)=t-\frac{2\alpha_i(t)}{\alpha_i(t_i)}t_i \qquad (t\in\mathfrak{t}_0) .$$

The $\boldsymbol{Z}$-lattices P, Q in $\mathfrak{t}_0^*$ are preserved under the contragredient action of W on $\mathfrak{t}_0^*$.

1.2. Let $\mathcal{A}$ be the category of $U_\hbar(\mathfrak{g})$-modules which are free of finite rank as R-modules. This is not an abelian category but an exact category. Let M be a $U_\hbar(\mathfrak{g})$-module in $\mathcal{A}$. A $U_\hbar(\mathfrak{g})$-submodule M_1 of M is called a strict submodule if M/M_1 belongs to $\mathcal{A}$. A non-zero $U_\hbar(\mathfrak{g})$-module in $\mathcal{A}$ is said to be $\mathcal{A}$-irreducible if it does not contain non-zero proper strict submodules.

Lemma 1.2.1. *If M_1, M_2 are $U_\hbar(\mathfrak{g})$-modules in $\mathcal{A}$, we have*

$$\mathrm{Ext}^1_{U_\hbar(\mathfrak{g})}(M_1, M_2)=0$$

Proof. By the exact sequence:

$$0\to M_2\xrightarrow{\hbar} M_2\to M_2/\hbar M_2\to 0$$

of $U_\hbar(\mathfrak{g})$-modules we have:

$$\mathrm{Ext}^1_{U_\hbar(\mathfrak{g})}(M_1, M_2)\xrightarrow{\hbar}\mathrm{Ext}^1_{U_\hbar(\mathfrak{g})}(M_1, M_2)\to\mathrm{Ext}^1_{U_\hbar(\mathfrak{g})}(M_1, M_2/\hbar M_2) \quad \text{(exact)} .$$

Since $U_\hbar(\mathfrak{g})$ is a noetherian ring, we see that $\mathrm{Ext}^1_{U_\hbar(\mathfrak{g})}(M_1, M_2)$ is a finitely generated R-module. Thus it is sufficient to show $\mathrm{Ext}^1_{U_\hbar(\mathfrak{g})}(M_1, M_2/\hbar M_2)=0$. By the exact sequence:

$$0 \to U_\hbar(\mathfrak{g}) \xrightarrow{\hbar} U_\hbar(\mathfrak{g}) \to U(\mathfrak{g}) \to 0\,,$$

we have

$$\mathrm{Tor}_q^{U_\hbar(\mathfrak{g})}(U(\mathfrak{g}), M_1) = \begin{cases} M_1/\hbar M_1 & q=0 \\ 0 & q\neq 0\,. \end{cases}$$

Therefore, by the spectral sequence:

$$E_2^{pq} = \mathrm{Ext}^p_{U(\mathfrak{g})}(\mathrm{Tor}_q^{U_\hbar(\mathfrak{g})}(U(\mathfrak{g}), M_1), M_2/\hbar M_2) \Rightarrow \mathrm{Ext}^{p+q}_{U_\hbar(\mathfrak{g})}(M_1, M_2/\hbar M_2)$$

we have

$$\mathrm{Ext}^1_{U_\hbar(\mathfrak{g})}(M_1, M_2/\hbar M_2) = \mathrm{Ext}^1_{U(\mathfrak{g})}(M_1/\hbar M_1, M_2/\hbar M_2)\,.$$

The right-hand side is zero since any finite dimensional $U(\mathfrak{g})$-module is completely reducible. We are done. □

Corollary 1.2.2. *Any $U_\hbar(\mathfrak{g})$-module in $\mathcal{A}$ is a direct sum of $\mathcal{A}$-irreducible $U_\hbar(\mathfrak{g})$-modules.*

1.3. For $\lambda\in\mathfrak{t}^*$ let ξ_λ: $\bar{T}\to R$ be the unique algebra homomorphism satisfying $\xi_\lambda(t)=\lambda(t)$ for $t\in\mathfrak{t}$. For a $\bar{T}$-module M and $\lambda\in\mathfrak{t}^*$ we set

$$M_\lambda = \{m\in M \mid t\cdot m = \xi_\lambda(t)m\ (t\in\bar{T})\}\,.$$

We define an ordering on $\mathfrak{t}^*$ by

$$\lambda\geqq\mu \text{ if and only if } \lambda-\mu\in Q^+\,.$$

For $\lambda\in\mathfrak{t}^*$ we define a $U_\hbar^f(\mathfrak{g})$-module $M(\lambda)$, called the Verma module with highest weight λ, by

$$(1.3.1) \qquad M(\lambda) = U_\hbar^f(\mathfrak{g})/(\sum_{i=1}^{l} U_\hbar^f(\mathfrak{g})e_i + U_\hbar^f(\mathfrak{g})\ker\xi_\lambda) = U_\hbar^f(\mathfrak{g})m_\lambda\,,$$

where m_λ is the canonical generator corresponding to the class of 1. By Proposition 1.1.1 we have $M(\lambda)=\oplus_{\mu\leqq\lambda}M(\lambda)_\mu$ and each $M(\lambda)_\mu$ is a free R-module of finite rank. Moreover we have the character formula:

$$(1.3.2) \qquad \sum_{\mu\leqq\lambda}(\mathrm{rank}_R M(\lambda)_\mu)e^\mu = \frac{e^\lambda}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}\,.$$

Lemma 1.3.1. *If K is a $U_\hbar^f(\mathfrak{g})$-submodule of $M(\lambda)$, we have:*

$$K = \oplus_{\mu\leqq\lambda}(K\cap M(\lambda)_\mu)\,.$$

Proof. Assume that we have $m=\sum_{i=1}^{n} m_i \in K$ with $m_i \in M(\lambda)_{\mu_i}$, $\mu_i \neq \mu_j$ $(i \neq j)$. We will show that each m_i is an element of K by induction on n. The case $n=1$ being trivial, we assume that $n \geqq 2$ and the assertion holds for $n-1$. Since $\mu_1^0, \cdots, \mu_n^0$ are mutually different elements of $\mathfrak{t}_0^*$, there exists some $t \in k[t_1, \cdots, t_l] \subset \bar{T}$ satisfying $\xi_{\mu_1^0}(t)=\cdots=\xi_{\mu_{n-1}^0}(t)=0$ and $\xi_{\mu_n^0}(t)=1$. Then we have $\xi_{\mu_i}(t)=\hbar a_i$ $(i=1, \cdots, n-1)$ and $\xi_{\mu_n}(t)=1+\hbar a_n$ for some $a_1, \cdots, a_n \in R$. Hence we have

$$(1+\hbar a_n)m-t\cdot m=\sum_{i=1}^{n-1}(1+\hbar(a_n-a_i))m_i \in K\,.$$

Since $1+\hbar(a_n-a_i)$ is an invertible element of R, we have $m_i \in K$ $(i=1, \cdots, n-1)$ and hence $m_n \in K$. □

Let $K(\lambda)$ be the sum of all $U_\hbar^f(\mathfrak{g})$-submodules of $M(\lambda)$ contained in $\bigoplus_{\mu<\lambda} M(\lambda)_\mu$, and set

$$L(\lambda)=M(\lambda)/K(\lambda)\,. \tag{1.3.3}$$

Lemma 1.3.2. (i) *We have* $L(\lambda)=\bigoplus_{\mu\leqq\lambda} L(\lambda)_\mu$ *and each* $L(\lambda)_\mu$ *is a free* R*-module of finite rank.*

(ii) *If* K *is a proper* $U_\hbar^f(\mathfrak{g})$*-submodule of* $L(\lambda)$ *such that* $L(\lambda)/K$ *is a torsion free* R*-module, we have* $K=0$.

Proof. (i) Set $K'=\{m \in M(\lambda) \mid \hbar m \in K(\lambda)\}$. Then K' is a $U_\hbar^f(\mathfrak{g})$-submodule satisfying $K(\lambda) \subset K' \subset \bigoplus_{\mu<\lambda} M(\lambda)_\mu$. Hence we have $K'=K(\lambda)$ and $L(\lambda)$ is a torsion free R-module. Therefore the assertion follows from Lemma 1.3.1.

(ii) Let K_1 be a proper $U_\hbar^f(\mathfrak{g})$-submodule of $M(\lambda)$ such that $M(\lambda)/K_1$ is a torsion free R-module. By Lemma 1.3.1 we have $K_1=\bigoplus_{\mu\leqq\lambda}(M(\lambda)_\mu \cap K_1)$ and hence $M(\lambda)_\mu/M(\lambda)_\mu \cap K_1$ is a torsion free R-module for each $\mu \leqq \lambda$. Since $M(\lambda)_\lambda$ is a free R-module of rank 1, we have $M(\lambda)_\lambda \cap K_1=M(\lambda)_\lambda$ or 0. If $M(\lambda)_\lambda \cap K_1= M(\lambda)_\lambda$, then K_1 contains the generator m_λ, and hence we have $K_1=M(\lambda)$, which contradicts with the assumption. Therefore we have $M(\lambda)_\lambda \cap K_1=0$ and hence $K_1 \subset K(\lambda)$. The assertion is proved. □

We define $\rho \in \mathfrak{t}_0^*$ by $\rho(2t_i/\alpha_i(t_i))=1$ $(i=1, \cdots, l)$. For $w \in W$ set

$$l(w)=\min\{p \mid w=s_{i_1}, \cdots, s_{i_p} \text{ for some } i_1, \cdots, i_p \in [1, l]\}\,.$$

Lemma 1.3.3 ([6]). (i) $L(\lambda)$ *is finitely generated as an* R*-module if and only if* $\lambda \in P^{++}$.

(ii) *For* $\lambda \in P^{++}$, $k \otimes_R L(\lambda)$ *is an irreducible* $U(\mathfrak{g})$*-module and we have*

$$\sum_{\mu\leqq\lambda}(\operatorname{rank}_R L(\lambda)_\mu)e^\mu=\frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}\,. \tag{1.3.4}$$

(iii) *If* $\lambda \in P^{++}$, *any* $U_{\hbar}^{f}(\mathfrak{g})$*-submodule of* $L(\lambda)$ *is of the form* $\hbar^{n}L(\lambda)$ *for some non-negative integer* n.

Proof. By the arguments of [6] we see that $L(\lambda)$ is integrable (i.e., the elements e_i, f_i $(i=1, \cdots, l)$ act on $L(\lambda)$ locally nilpotently) if and only if $\lambda \in P^{++}$. If $L(\lambda)$ is finitely generated as an R-module, then it is integrable and hence we have $\lambda \in P^{++}$. If $\lambda \in P^{++}$, then $L(\lambda)$ is integrable, and hence $k \otimes_R L(\lambda)$ is an integrable highest weight module of $U(\mathfrak{g})$ with highest weight λ. Thus $k \otimes_R L(\lambda)$ is the (finite dimensional) irreducible $U(\mathfrak{g})$-module with highest weight λ. Therefore $L(\lambda)$ is finitely generated as an R-module, and Weyl's character formula implies (1.3.4). The statements (i) and (ii) are proved. Let us show (iii). Let $\lambda \in P^{++}$ and let K be a non zero $U_{\hbar}^{f}(\mathfrak{g})$-submodule of $L(\lambda)$. Take a non-negative integer n such that $K \subset \hbar^{n}L(\lambda)$ and $K \not\subset \hbar^{n+1}L(\lambda)$. Then we have $K=\hbar^{n}K_1$ for some $U_{\hbar}^{f}(\mathfrak{g})$-submodule K_1 of $L(\lambda)$. Since $(K_1+\hbar L(\lambda))/\hbar L(\lambda)$ is a non-zero $U(\mathfrak{g})$-submodule of the irreducible $U(\mathfrak{g})$-module $L(\lambda)/\hbar L(\lambda)=k \otimes_R L(\lambda)$, we have $L(\lambda)=K_1+\hbar L(\lambda)$. Since $L(\lambda)$ is a finitely generated R-module, we have $K_1=L(\lambda)$, and hence $K=\hbar^{n}L(\lambda)$. $\square$

For $\lambda \in P^{++}$ the action of $U_{\hbar}^{f}(\mathfrak{g})$ on $L(\lambda)$ uniquely lifts to that of $U_{\hbar}(\mathfrak{g})$ on $L(\lambda)$. In the following we regard $L(\lambda)$ for $\lambda \in P^{++}$ as a $U_{\hbar}(\mathfrak{g})$-module.

Corollary 1.3.4. $L(\lambda)$ *is an* $\mathcal{A}$*-irreducible* $U_{\hbar}(\mathfrak{g})$*-module for* $\lambda \in P^{++}$.

2. Quantum Groups

2.1. Define a $(U(\mathfrak{g}), U(\mathfrak{g}))$-bimodule structure on $U(\mathfrak{g})^*=\mathrm{Hom}_k(U(\mathfrak{g}), k)$ by

$$((u_1 \cdot f \cdot u_2))(u) = f(u_2 u u_1) \qquad (f \in U(\mathfrak{g})^*,\ u,\ u_1 u_2 \in U(\mathfrak{g}))\,,$$

and set

$$U(\mathfrak{g})^\circ = \{f \in U(\mathfrak{g})^* \mid \dim_k(U(\mathfrak{g})fU(\mathfrak{g}))<\infty\}\,.$$

It is an elementary fact concerning Hopf algebras that $U(\mathfrak{g})^\circ$ is also endowed with a Hopf algebra structure whose product, coproduct, unit, counit are induced by the coproduct, the product, the counit, the unit of $U(\mathfrak{g})$, respectively.

Set $U_{\hbar}(\mathfrak{g})^*=\mathrm{Hom}_R(U_{\hbar}(\mathfrak{g}), R)$. By Proposition 1.1.1. we see that $U_{\hbar}(\mathfrak{g})$ is the $\hbar$-adic completion of a free R-submodule $M=N^- \otimes T \otimes N^+$. Hence we have $U_{\hbar}(\mathfrak{g})^* \simeq \mathrm{Hom}_R(M, R)=$(a product of rank 1 free R-modules). Therefore any R-submodule of $U_{\hbar}(\mathfrak{g})^*$ is torsion free and separated. Define a $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$-bimodule structure on $U_{\hbar}(\mathfrak{g})^*$ by

$$(u_1 \cdot f \cdot u_2)(u) = f(u_2 u u_1) \qquad (f \in U_{\hbar}(\mathfrak{g})^*,\ u,\ u_1,\ u_2 \in U_{\hbar}(\mathfrak{g}))\,,$$

and set

$U_\hbar(\mathfrak{g})^\circ = \{f \in U_\hbar(\mathfrak{g})^* \mid U_\hbar(\mathfrak{g}) f U_\hbar(\mathfrak{g})$ is a finitely generated R-module$\}$.

For a $U_\hbar(\mathfrak{g})$-module V in $\mathcal{A}$ we have a natural right $U_\hbar(\mathfrak{g})$-module structure on $V^* = \mathrm{Hom}_R(V, R)$ by

$$\langle v^* \cdot u, v \rangle = \langle v^*, u \cdot v \rangle \qquad (v^* \in V^*,\ v \in V,\ u \in U_\hbar(\mathfrak{g}))\,,$$

where $\langle\ ,\ \rangle$ is the natural pairing. Define $\Phi_V\colon V \otimes V^* \to U_\hbar(\mathfrak{g})^*$ by $(\Phi_V(v \otimes v^*))(u) = \langle v^*, u \cdot v \rangle$. Then it is easily seen that Φ_V is a homomorphism of $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-bimodules and that $U_\hbar(\mathfrak{g})^\circ$ is the sum of Image(Φ_V) for $U_\hbar(\mathfrak{g})$-modules V in $\mathcal{A}$ (i.e., the R-module $U_\hbar(\mathfrak{g})^\circ$ is generated by the matrix coefficients of $U_\hbar(\mathfrak{g})$-modules in $\mathcal{A}$). Moreover the topological Hopf algebra structure on $U_\hbar(\mathfrak{g})$ defined by (1.1.8), $\cdots$, (1.1.14) induces a Hopf algebra structure on $U_\hbar(\mathfrak{g})^\circ$.

2.2. The purpose of this subsection is to prove the following:

Proposition 2.2.1. *For $\lambda \in P^{++}$ the homomorphism $\Phi_{L(\lambda)}$ is injective and we have*

$$U_\hbar(\mathfrak{g})^\circ = \oplus_{\lambda \in P^{++}} \mathrm{Image}\, \Phi_{L(\lambda)}\,.$$

Since $U_\hbar(\mathfrak{g})$ is topologically free and since $U_\hbar(\mathfrak{g})/\hbar U_\hbar(\mathfrak{g}) \simeq U(\mathfrak{g})$, we have $U_\hbar(\mathfrak{g})^*/\hbar U_\hbar(\mathfrak{g})^* \simeq U(\mathfrak{g})^*$ We denote the natural homomorphism $U_\hbar(\mathfrak{g})^* \to U(\mathfrak{g})^*$ by $f \to \bar{f}$. We first show the following:

Lemma 2.2.2. *For $\lambda \in P^{++}$ the homomorphism $\Phi_{L(\lambda)}$ is injective and we have*

$$\sum_{\lambda \in P^{++}} \mathrm{Image}\, \Phi_{L(\lambda)} = \oplus_{\lambda \in P^{++}} \mathrm{Image}\, \Phi_{L(\lambda)}\,.$$

Proof. For $\lambda \in P^{++}$ let $\{f^\lambda_{ij} \mid 1 \leqq i, j \leqq \mathrm{rank}\, L(\lambda)\}$ be the set of matrix coefficients of $L(\lambda)$ with respect to some R-basis of $L(\lambda)$. It is sufficient to show that

$$\{f^\lambda_{ij} \mid \lambda \in P^{++},\ 1 \leqq i, j \leqq \mathrm{rank}\, L(\lambda)\}$$

is linearly independent over R. The set

$$\{\overline{f^\lambda_{ij}} \mid \lambda \in P^{++},\ 1 \leqq i, j \leqq \mathrm{rank}\, L(\lambda)\}$$

is linearly independent over k, since it consists of the matrix coeficients of irreducible $U(\mathfrak{g})$-modules. Therefore the assertion follows from the fact that $U_\hbar(\mathfrak{g})^*$ is torsion free. □

Set $V(\lambda) = L(\lambda) \otimes L(\lambda)^*$ for $\lambda \in P^{++}$.

Lemma 2.2.3. *Let $\lambda_1, \cdots, \lambda_n$ be mutually different elements in P^{++} and let*

$$p_j\colon \oplus_{i=1}^n V(\lambda_i) \to V(\lambda_j)$$

be the projection. If V is a $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-submodule of $\oplus_{i=1}^n V(\lambda_i)$ such that $p_j(V)\neq 0$ for each j, there exist non-negative integers $m_1, \cdots, m_n$ satisfying $V=\oplus_{i=1}^n \hbar^{m_i} V(\lambda_i)$.

Proof. Set $U=\oplus_{i=1}^n V(\lambda_i)$. Since $p_j(V)$ is a non zero $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-submodule of $V(\lambda_j)$ and since $V(\lambda_j)/\hbar V(\lambda_j)$ is an irreducible $(U(\mathfrak{g}), U(\mathfrak{g}))$-bimodule, the argument in the proof of Lemma 1.3.3 (iii) implies that there exists some non-negative integer m_j such that $p_j(V)=\hbar^{m_j}V(\lambda_j)$. Let $F\colon U\to U$ be the $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-homomorphism defined by

$$F(\sum_{i=1}^n v_i) = \sum_{i=1}^n \hbar^{m_i} v_i \qquad (v_i\in V(\lambda_i))\,.$$

Then there exist a $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-submodule V_1 of $\oplus_{i=1}^n V(\lambda_i)$ such that $F(V_1)=V$ and $p_j(V_1)=V(\lambda_j)$. Since $V(\lambda_i)/\hbar V(\lambda_i)$ $(i=1, \cdots, n)$ are mutually non-isomorphic $(U(\mathfrak{g}), U(\mathfrak{g}))$-bimodules, we see that $(V_1+\hbar U)/\hbar U=U/\hbar U$, and hence $U=V_1+\hbar U$. Since U is a finitely generated R-module we have $V_1=U$ and hence $V=F(V_1)=\oplus_{i=1}^n \hbar^{m_i}V(\lambda_i)$. □

Set $D=\oplus_{\lambda\in P^{++}}$ Image $\Phi_{L(\lambda)}$.

Lemma 2.2.4. (i) *$U_\hbar(\mathfrak{g})^\circ=D+\hbar^n U_\hbar(\mathfrak{g})^\circ$ for any n.*
(ii) *$\hbar^n D=\hbar^n U_\hbar(\mathfrak{g})\cap D$ for any n.*

Proof. (i) Let $f\in U_\hbar(\mathfrak{g})^\circ$. Then we have $\bar{f}\in U(\mathfrak{g})^\circ$ and hence there exists some $f_1\in D$ such that $\bar{f}=\bar{f}_1$. Therefore we have $f=f_1+\hbar f_2$ for some $f_2\in U_\hbar(\mathfrak{g})^*$. Since f and f_1 are elements of $U_\hbar(\mathfrak{g})^\circ$ and since $U_\hbar(\mathfrak{g})^*$ is torsion free, we have $f_2\in U_\hbar(\mathfrak{g})^\circ$. Thus we have

$$U_\hbar(\mathfrak{g})^\circ = D+\hbar U_\hbar(\mathfrak{g})^\circ = D+\hbar(D+\hbar U_\hbar(\mathfrak{g}))^\circ = \cdots = D+\hbar^n U_\hbar(\mathfrak{g})^\circ\,.$$

(ii) Let f be an element of $U(\mathfrak{g})^\circ$ such that $\hbar^n f\in D$. Set $V=U_\hbar(\mathfrak{g})fU_\hbar(\mathfrak{g})$. Let $\{v_1, \cdots, v_p\}$ be an R-basis of V and let $\{v_1^*, \cdots, v_p^*\}$ be the dual basis of V^*. Regarding V as a left $U_\hbar(\mathfrak{g})$-module we have

$$v_i = \sum_{j=1}^p v_j(1)\Phi_V(v_i\otimes v_j^*)\in \text{Image}\,\Phi_V\,.$$

Especially we have $f\in$ Image Φ_V. Since $\hbar^n V$ is a $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-submodule of D, we see from Lemma 2.2.3 that

$$V\simeq \hbar^n V = \oplus_{i=1}^r \hbar^{m_i}(L(\lambda_i)\otimes L(\lambda_i)^*) \simeq \oplus_{i=1}^r L(\lambda_i)\otimes L(\lambda_i)^*$$

for some $\lambda_1, \cdots, \lambda_r\in P^{++}$ and $m_1, \cdots, m_r\in \boldsymbol{Z}_{\geq 0}$. Hence we have $f\in$ Image $\Phi_V=\sum_{i=1}^r$ Image $\Phi_{L(\lambda_i)}\subset D$. □

Proof of Proposition 2.2.1. By Lemma 2.2.2 it is sufficient to show $D=U_\hbar(\mathfrak{g})^\circ$. By Lemma 2.2.4 the natural R-homomorphism

$$\hat{D}(=\varprojlim D/\hbar^n D) \to (U_\hbar(\mathfrak{g})^\circ)^\wedge(=\varprojlim U_\hbar(\mathfrak{g})^\circ/\hbar^n U_\hbar(\mathfrak{g}))^\circ)$$

is an isomorphism. Therefore we can regard $U_\hbar(\mathfrak{g})^\circ$ as an R-submodule of $\hat{D}$ containing D. Since the $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-bimodule structures on D and $U_\hbar(\mathfrak{g})^\circ$ uniquely lift to the same $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-bimodule structure on $\hat{D}$, it suffices to prove that if V is a $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-submodule of $(\oplus_{\lambda\in P^{++}} V(\lambda))^\wedge$ which is finitely generated over R, then V is contained in $\oplus_{\lambda\in P^{++}} V(\lambda)$. Let q_μ: $(\oplus_{\lambda\in P^{++}} V(\lambda))^\wedge \to V(\mu)$ be the unique extention of the projection $\oplus_{\lambda\in P^{++}} V(\lambda)\to V(\mu)$. We have only to show that $q_\mu(V)=0$ except for finitely many $\mu\in P^{++}$. Assume that there exists an infinite sequence $\mu_1, \mu_2, \cdots$, of mutually different elements in P^{++} such that $q_{\mu_i}(V)\neq 0$. Let r_n: $(\oplus_{\lambda\in P^{++}} V(\lambda))^\wedge\to \oplus_{i=1}^n V(\mu_i)$ be the unique extension of the projection. By Lemma 2.2.3 we have $r_n(V)=\oplus_{i=1}^n \hbar^{m_i} V(\mu_i)$ for some non-negative integers $m_1, \cdots, m_n$. Therefore we have rank $V\geqq$ rank $r_n(V)\geqq n$ for any n. This contradicts with the assumption. We are done. □

Corollary 2.2.5. *Any $\mathcal{A}$-irreducible $U_\hbar(\mathfrak{g})$-module is isomorphic to $L(\lambda)$ for some $\lambda\in P^{++}$.*

Proof. Let V be an $\mathcal{A}$-irreducible $U_\hbar(\mathfrak{g})$-module. Take a non-zero element v^* of V^* and define F: $V\to U_\hbar(\mathfrak{g})^\circ$ by $(F(v))(u)=\langle v^*, u\cdot v\rangle$. Then F is a non-zero homomorphism of left $U_\hbar(\mathfrak{g})$-modules. By Proposition 2.3.1 the left $U_\hbar(\mathfrak{g})$-module $U_\hbar(\mathfrak{g})^\circ$ is a direct sum of $L(\lambda)$ for $\lambda\in P^{++}$. Considering the projections we see that there exists a non zero $U_\hbar(\mathfrak{g})$-homomorphism $V\to L(\lambda)$ for some $\lambda\in P^{++}$. It is seen by Lemma 1.3.3 (iii) that $L(\lambda)$ is a quotient of V. Since V is $\mathcal{A}$-irreducible, we have $V=L(\lambda)$. □

2.3. Let G be a connected split semisimple algebraic group defined over k such that the Lie algebra consisting of k-rational points of Lie (G) coincides with $\mathfrak{g}$. Then the coordinate algebra $k[G]$ is naturally endowed with a Hopf algebra structure and we have a natural injective Hopf algebra homomorphism from $k[G]\to U(\mathfrak{g})^\circ$ via the pairing

$$\langle f, u\rangle = (d_u f)(1) \qquad (f\in k[G], u\in U(\mathfrak{g}))\,,$$

where d_u is the left invariant differential operator on G corresponding to u. The image of this homomorphism is described as follows. Let L_G be the set of elements of $\mathfrak{t}_0^*$ consisting of weights of finite dimensional $U(\mathfrak{g})$-modules coming from G-modules. L_G is a $\boldsymbol{Z}$-lattice satisfying $Q\subset L_G\subset P$. Then the image of $k[G]\to U_\hbar(\mathfrak{g})^\circ$ is spanned by the matrix coefficients of finite dimensional irreducible $U(\mathfrak{g})$-modules with highest weight in $L_G\cap P^{++}$.

Set

$$A_\hbar[G] := \oplus_{\lambda\in L_G\cap P^{++}} \text{Image } \Phi_{L(\lambda)}\,. \tag{2.3.1}$$

It is easily checked that $A_\hbar[G]$ is a Hopf subalgebra of $U_\hbar(\mathfrak{g})^\circ$. We call this

Hopf algebra the *quantum group* associated to G (see [3], [10]).

Let $\mathcal{A}^\circ$ be the category of right $U_\hbar(\mathfrak{g})^\circ$-comodules which are free of finite rank over R and let $\mathcal{A}_G$ be the category of right $A_\hbar[G]$-comodules which belong to $\mathcal{A}^\circ$ as right $U_\hbar(\mathfrak{g})^\circ$-comodules. Then the natural functor $\mathcal{A}^\circ \to \mathcal{A}$ gives an equivalence of categories $\mathcal{A}^\circ \simeq \mathcal{A}$. Moreover the category $\mathcal{A}_G$ is equivalent to the full subcategory of $\mathcal{A}$ consisting of $U_\hbar(\mathfrak{g})$-modules in $\mathcal{A}$ whose $\mathcal{A}$-irreducible factors are of the form $L(\lambda)$ for some $\lambda \in L_G \cap P^{++}$.

Lemma 2.3.1. *Let V be a $U_\hbar(\mathfrak{g})$-module in $\mathcal{A}$ such that the* **Z**-*submodule of* $\mathfrak{t}^*$ *generated by the weights of V coincides with L_G. Then $A_\hbar[G]$ is generated by* Image Φ_V *as an R-algebra.*

Proof. Let H be the subalgebra of $A_\hbar[G]$ generated by Image Φ_V. We see by definition that H is a $(U_\hbar(\mathfrak{g}), U_\hbar(\mathfrak{g}))$-submodule of $A_\hbar[G]$. Hence by Lemma 2.2.3 we have

$$H = \oplus_{\lambda \in \Gamma} \hbar^{n_\lambda} \, \mathrm{Image}\, \Phi_{L(\lambda)} \tag{2.3.2}$$

for a subset Γ of P^{++} and non-negative inetgers n_λ. On the other hand we see by the assumption on V that the representation $G \to GL(V/\hbar V)$ is injective and hence the k-algebra $k[G]$ $(\simeq A_\hbar[G]/\hbar A_\hbar[G])$ is generated by the matrix coefficients of the G-module $V/\hbar V$. Therefore we have $A_\hbar[G]/\hbar A_\hbar[G] = (H + \hbar A_\hbar[G])/\hbar A_\hbar[G]$ and hence

$$H + \hbar A_\hbar[G] = A_\hbar[G]\,. \tag{2.3.3}$$

The assertion follows from (2.3.2) and (2.3.3). □

3. Borel-Weil-Bott Theorem

3.1. Let $U_\hbar^f(\mathfrak{b})$ be the subalgebra of $U_\hbar^f(\mathfrak{g})$ generated by $\bar{T}$, N^+ and let $U_\hbar(\mathfrak{b})$ be its $\hbar$-adic closure in $U_\hbar(\mathfrak{g})$. By Proposition 1.1.1 we have $U_\hbar^f(\mathfrak{b}) \simeq \bar{T} \otimes N^+$ and the inclusion $U_\hbar^f(\mathfrak{b}) \hookrightarrow U_\hbar(\mathfrak{b})$ is the $\hbar$-adic completion. Moreover we have $k \otimes_R U_\hbar(\mathfrak{b}) \simeq k \otimes_R U_\hbar^f(\mathfrak{b}) \simeq U(\mathfrak{b})$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$. Define $U_\hbar(\mathfrak{b})^\circ$ $(\subset U_\hbar(\mathfrak{b})^*)$ similarly to $U_\hbar(\mathfrak{g})^\circ$. Since $U_\hbar(\mathfrak{b})$ is a topological Hopf subalgebra of $U_\hbar(\mathfrak{g})$, we also have a natural Hopf algebra structure on $U_\hbar(\mathfrak{b})^\circ$.

Let G be a connected semisimple split algebraic group defined over k with $\mathrm{Lie}(G) = \mathfrak{g}$ and let B be the Borel subgroup of G corresponding to $\mathfrak{b}$. We denote by F: $U_\hbar(\mathfrak{g})^\circ \to U_\hbar(\mathfrak{b})^\circ$ the natural Hopf algebra homomorphism. Then $A_\hbar[B] = F(A_\hbar[G])$ is endowed with a natural Hopf algebra structure and it can be regarded as a quantization of the coordinate algebra $k[B]$ of B by the following:

Lemma 3.1.1. *$A_\hbar[B]$ is free R-module satisfying $k \otimes_R A_\hbar[B] \simeq k[B]$.*

Proof. It is easily verified using the results in Section 1 that we have

$$A_{\hbar}[B] = (\oplus_{\lambda\in L_G} R\xi_\lambda)\otimes(\oplus_{\beta\in Q^+}(N^+_\beta)^*)$$

under the identification $U^f_{\hbar}(\mathfrak{b})=\bar{T}\otimes N^+$. Hence the assertion follows from the corresponding fact for $k[B]$. □

3.2. Let $\mathrm{tirv}_{k[B]}$ (resp. $\mathrm{triv}_{A_{\hbar}[B]}$) be the right $k[B]$ (resp. $A_{\hbar}[B]$)-comodule given by the unit and let $\mathrm{triv}_{U(\mathfrak{b})}$ (resp. $\mathrm{triv}_{U_{\hbar}(\mathfrak{b})}$) be the left $U(\mathfrak{b})$ (resp. $U_{\hbar}(\mathfrak{b})$)-module given by the counit. For a right $k[B]$ (resp. $A_{\hbar}[B]$)-comodule V we set

$$\mathrm{Ind}\,(V) = \mathrm{Hom}(\mathrm{triv}_{k[B]}, k[G]\otimes_k V) \tag{3.2.1}$$

$$(\text{resp. } \mathrm{Ind}_{\hbar}(V) = \mathrm{Hom}(\mathrm{triv}_{A_{\hbar}[B]}, A_{\hbar}[G]\otimes_R V))\,. \tag{3.2.2}$$

Here $k[G]\otimes_k V$ (resp. $A_{\hbar}[G]\otimes_R V$) is endowed with a right $k[B]$ (resp. $A_{\hbar}[B]$)-comodule structure via the right $k[B]$ (resp. $A_{\hbar}[B]$)-comodule structure on $k[G]$ (resp. $A_{\hbar}[G]$) and Hom is taken in the category of right $k[B]$ (resp. $A_{\hbar}[G]$)-comodules. Then the left $k[G]$ (resp. $A_{\hbar}[G]$)-comodules structure on $k[G]$) induces a left $k[G]$ (resp. $A_{\hbar}[G]$)-comodule structure on $\mathrm{Ind}\,(V)$ (resp. $\mathrm{Ind}_{\hbar}(V)$). Hence Ind (resp. $\mathrm{Ind}_{\hbar}$) is a left exact functor from the category of right $k[B]$ (resp. $A_{\hbar}[B]$)-comodules to that of left $k[G]$ (resp. $A_{\hbar}[G]$)-comodules. We denote by $R^i\,\mathrm{Ind}$ (resp. $R^i\,\mathrm{Ind}_{\hbar}$) its right derived functors.

By the Peter-Weyl theorem for $k[G]$ and by (2.3.1) we have

$$\begin{aligned}
\mathrm{Ind}\,(V) &= \oplus_{\lambda\in L_G\cap P^{++}} \mathrm{Hom}\,(\mathrm{triv}_{k[B]}, L^0(\lambda)\otimes_k V)\otimes_k L^0(\lambda)^*\,,\\
&= \oplus_{\lambda\in L_G\cap P^{++}} \mathrm{Hom}_{U(\mathfrak{b})}(\mathrm{triv}_{U(\mathfrak{b})}, L^0(\lambda)\otimes_k V)\otimes_k L^0(\lambda)^*\,,\\
\mathrm{Ind}_{\hbar}(V) &= \oplus_{\lambda\in L_G\cap P^{++}} \mathrm{Hom}\,(\mathrm{triv}_{A_{\hbar}[B]}, L(\lambda)\otimes_R V)\otimes_R L(\lambda)^*\,,\\
&= \oplus_{\lambda\in L_G\cap P^{++}} \mathrm{Hom}_{U_{\hbar}(\mathfrak{b})}(\mathrm{triv}_{U_{\hbar}(\mathfrak{b})}, L(\lambda)\otimes_R V)\otimes_R L(\lambda)^*\,,
\end{aligned}$$

where $L^0(\lambda)=k\otimes_R L(\lambda)$. Hence we have

$$R^i\,\mathrm{Ind}\,(V) = \oplus_{\lambda\in L_G\cap P^{++}} \mathrm{Ext}^i_{U(\mathfrak{b})}(\mathrm{triv}_{U(\mathfrak{b})}, L^0(\lambda)\otimes_k V)\otimes_k L^0(\lambda)^*\,, \tag{3.2.3}$$

$$R^i\,\mathrm{Ind}_{\hbar}(V) = \oplus_{\lambda\in L_G\cap P^{++}} \mathrm{Ext}^i_{U_{\hbar}(\mathfrak{b})}(\mathrm{triv}_{U_{\hbar}(\mathfrak{b})}, L(\lambda)\otimes_R V)\otimes_R L(\lambda)^*\,. \tag{3.2.4}$$

For $\mu\in L_G$ we denote by $\hat{\xi}_\mu$: $U_{\hbar}(\mathfrak{b})\to R$ the R-algebra homomorphism given by $\hat{\xi}_\mu(t)=\xi_\mu(t)$ for $t\in\bar{T}$ and $\hat{\xi}_\mu(e_i)=0$ for $i=1,\cdots,l$. It is seen that the one dimensional left $U_{\hbar}(\mathfrak{b})$-module induced by $\hat{\xi}_\mu$ comes from a one dimensional right $A_{\hbar}[B]$-comodule R_μ. Set $k_\mu=k\otimes_R R_\mu$.

Proposition 3.2.1. *For $\mu\in L_G$ the left $A_{\hbar}[G]$-comodule $R^i\,\mathrm{Ind}_{\hbar}(R_\mu)$ is free of finite rank as an R-module and we have*

$$k\otimes_R R^i\,\mathrm{Ind}_{\hbar}(R_\mu) \simeq R^i\,\mathrm{Ind}\,(k_\mu)$$

as a left $k[G]$-comodule.

Proof. Set $V(\lambda, \mu)=L(\lambda)\otimes_R R_\mu$ and $V^0(\lambda, \mu)=L^0(\lambda)\otimes_k k_\mu$. By (3.2.3) and (3.2.4) it is sufficient to show that $\mathrm{Exit}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}_{U_\hbar(\mathfrak{b})}, V(\lambda, \mu))$ is a free R-module of rank $\dim_k(\mathrm{Ext}^i_{U(\mathfrak{b})}(\mathrm{triv}_{U(\mathfrak{b})}, V^0(\lambda, \mu))$. By the argument in the proof of Lemma 1.2.1 we have

(3.2.5) $$\mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}_{U_\hbar(\mathfrak{b})}, V^0(\lambda, \mu)) = \mathrm{Ext}^i_{U(\mathfrak{b})}(\mathrm{triv}_{U(\mathfrak{b})}, V^0(\lambda, \mu))$$

Since $U_\hbar(\mathfrak{b})$ is noetherian, the R-module $\mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}_{U_\hbar(\mathfrak{b})}, V(\lambda, \mu))$ is finitely generated, and hence the exact sequence

$$\mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}, V(\lambda, \mu)) \xrightarrow{\hbar} \mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}, V(\lambda, \mu)) \to \mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}, V^0(\lambda, \mu))$$

implies that $\mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}_{U_\hbar(\mathfrak{b})}, V(\lambda, \mu))=0$ if $\mathrm{Ext}^i_{U(\mathfrak{b})}(\mathrm{triv}_{U(\mathfrak{b})}, V^0(\lambda, \mu))=0$. Assume that $\mathrm{Ext}^i_{U(\mathfrak{b})}(\mathrm{triv}_{U(\mathfrak{b})}, V^0(\lambda, \mu))\neq 0$. By the Borel-Weil-Bott theorem for $k[G]$ there exists at most one such i for each λ, μ. Hence the natural R-homomorphism

$$\mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}, V(\lambda, \mu)) \xrightarrow{\hbar} \mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}, V(\lambda, \mu))$$

is injective and we have

$$\mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}, V(\lambda, \mu))/\hbar\, \mathrm{Ext}^i_{U_\hbar(\mathfrak{b})}(\mathrm{triv}, V(\lambda, \mu)) = \mathrm{Ext}^i_{U(\mathfrak{b})}(\mathrm{triv}, V^0(\lambda, \mu)).$$

This proves the assertion. □

Appendix

In [3] Drinfeld has given an explicit description of the quantum group $A_\hbar[SL_n]$ by generators and relations. Since [3] contains no proof, we will give a proof here.

Set $q=e^{\hbar/2}$. The quantum algebra $U_\hbar(\mathfrak{gl}_n)$ is an R-algebra $\hbar$-adically generated by the elements $h_1, \cdots, h_n, e_1, \cdots, e_{n-1}, f_1, \cdots, f_{n-1}$ satisfying the following fundamental relations:

(A.1) $$h_i h_j = h_j h_i\,,$$

(A.2) $$h_i e_i - e_i h_i = e_i\,,$$

(A.3) $$h_i e_{i-1} - e_{i-1} h_i = -e_{i-1}\,,$$

(A.4) $$h_i e_j - e_j h_i = 0 \qquad (j \neq i, j-1)\,,$$

(A.5) $$h_i f_i - f_i h_i = -f_i\,,$$

(A.6) $$h_i f_{i-1} - f_{i-1} h_i = f_{i-1}\,,$$

$$h_i f_j - f_j h_i = 0 \qquad (j \neq i, i-1), \tag{A.7}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{\sinh(\hbar(h_i - h_{i+1})/2)}{\sinh(\hbar/2)}, \tag{A.8}$$

$$e_i^2 e_j - (q+q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \qquad (|i-j| = 1), \tag{A.9}$$

$$e_i e_j = e_j e_i \qquad (|i-j| \geqq 2), \tag{A.10}$$

$$f_i^2 f_j - (q+q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \qquad (|i-j| = 1), \tag{A.11}$$

$$f_i f_j = f_j f_i \qquad (|i-j| \geqq 2). \tag{A.12}$$

Then $U_\hbar(\mathfrak{sl}_n)$ is naturally identified with the $\hbar$-adic closure of the R-subalgebra of $U_\hbar(\mathfrak{gl}_n)$ generated by $t_i = h_i - h_{i+1}$, e_i, f_i $(i=1, \cdots, n-1)$, and the topological Hopf algebra structure of $U_\hbar(\mathfrak{sl}_n)$ is extended to that of $U_\hbar(\mathfrak{gl}_n)$ by $\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i$, $\varepsilon(h_i) = 0$, $S(h_i) = -h_i$ (see [9]).

Define an R-algebra homomorphism ρ: $U_\hbar(\mathfrak{gl}_n) \to M_n(R)$ by $\rho(h_i) = E_{i,i} - E_{i+1,i+1}$, $\rho(e_i) = E_{i,i+1}$, $\rho(f_i) = E_{i+1,i}$, where $E_{i,j} \in M_n(R)$ is the matrix whose (r, s)-entry is $\delta_{ir}\delta_{js}$. Let $\hat{\rho}_{ij} \in U_\hbar(\mathfrak{gl}_n)^*$ and $\rho_{ij} \in U_\hbar(\mathfrak{sl}_n)^*$ be the matrix coefficients of ρ. They are elements of the Hopf algebras $U_\hbar(\mathfrak{gl}_n)^\circ$ and $U_\hbar(\mathfrak{sl}_n)^\circ$ ($U_\hbar(\mathfrak{gl}_n)^\circ$ is defined similarly). We see by a direct calculation that

$$\hat{\rho}_{ij} \hat{\rho}_{is} = q \hat{\rho}_{is} \hat{\rho}_{ij} \qquad (j < s), \tag{A.13}$$

$$\hat{\rho}_{ij} \hat{\rho}_{rj} = q \hat{\rho}_{rj} \hat{\rho}_{ij} \qquad (i < r), \tag{A.14}$$

$$\hat{\rho}_{ij} \hat{\rho}_{rs} = \hat{\rho}_{rs} \hat{\rho}_{ij} \qquad (i < r, j > s), \tag{A.15}$$

$$\hat{\rho}_{ij} \hat{\rho}_{rs} - \hat{\rho}_{rs} \hat{\rho}_{ij} = (q - q^{-1}) \hat{\rho}_{is} \hat{\rho}_{rj} \qquad (i < r, j < s). \tag{A.16}$$

Since ρ_{ij} is the image of $\hat{\rho}_{ij}$ under the natural algebra homomorphism $U_\hbar(\mathfrak{gl}_n)^\circ \to U_\hbar(\mathfrak{sl}_n)^\circ$, we have

$$\rho_{ij} \rho_{is} = q \rho_{is} \rho_{ij} \qquad (j < s), \tag{A.17}$$

$$\rho_{ij} \rho_{rj} = q \rho_{rj} \rho_{ij} \qquad (i < r), \tag{A.18}$$

$$\rho_{ij} \rho_{rs} = \rho_{rs} \rho_{ij} \qquad (i < r, j > s), \tag{A.19}$$

$$\rho_{ij} \rho_{rs} - \rho_{rs} \rho_{ij} = (q - q^{-1}) \rho_{is} \rho_{rj} \qquad (i < r, j < s). \tag{A.20}$$

It is also checked directly that

$$\sum_{\sigma \in \mathfrak{S}_n} (-q)^{l(\sigma)} \rho_{1\sigma(1)} \rho_{2\sigma(n)} \cdots \rho_{n\sigma(n)} = 1, \tag{A.21}$$

where $\mathfrak{S}_n$ is the symmetric group and $l(\sigma)$ for $\sigma \in \mathfrak{S}_n$ is the number of the elements of the set $\{(i, j) \mid i < j, \sigma(i) > \sigma(j)\}$.

Our purpose is to give a proof of the following:

Proposition A.1 (Drinfeld [3]). *The R-algebra $A_\hbar[SL_n]$ is generated by the elements ρ_{ij} $(i, j = 1, \cdots, n)$ satisfying the fundamental relations* (A.17), $\cdots$, (A.21).

Let C be the quotient of $R\langle x_{ij}\,|\,i, j=1, \cdots, n\rangle$ by the two-sided ideal generated by

$$\begin{aligned} &x_{ij}x_{is}-qx_{is}x_{ij} \qquad (j<s)\,,\\ &x_{ij}x_{rj}-qx_{rj}x_{ij} \qquad (i<r)\,,\\ &x_{ij}x_{rs}-x_{rs}x_{ij} \qquad (i<r, j>s)\,,\\ &x_{ij}x_{rs}-x_{rs}x_{ij}-(q-q^{-1})x_{is}x_{rj} \qquad (i<r, j<s)\,, \end{aligned}$$

where $R\langle x_{ij}\,|\,i, j=1, \cdots, n\rangle$ is the tensor algebra of the free R-module with basis $\{x_{ij}\,|\,i, j=1, \cdots, n\}$.

Lemma A.2. (i) *C is a free R-module with basis*

$$\{x_{11}^{a_{11}}x_{12}^{a_{12}}\cdots x_{1n}^{a_{1n}}x_{21}^{a_{21}}\cdots x_{2n}^{a_{2n}}\cdots x_{nn}^{a_{nn}}\,|\,a_{ij}\in \mathbf{Z}_{\geqq 0}\}\,.$$

(ii) *C is an integral domain; i.e., if f, g are elements of C satisfying $fg=0$, we have $f=0$ or $g=0$.*

Proof. (i) It is easily checked that the R-module C is generated by the elements

$$x_{11}^{a_{11}}x_{12}^{a_{12}}\cdots x_{1n}^{a_{1n}}x_{21}^{a_{21}}\cdots x_{2n}^{a_{2n}}\cdots x_{nn}^{a_{nn}} \qquad (a_{ij}\in \mathbf{Z}_{\geqq 0})\,.$$

Considering the natural algebra homomorphism $C\rightarrow U_{\hbar}(\mathfrak{gl}_n)^{\circ}$ $(x_{ij}\mapsto \rho_{ij})$, it is enough to show that the elements

$$\rho_{11}^{a_{11}}\rho_{12}^{a_{12}}\cdots \rho_{1n}^{a_{1n}}\rho_{21}^{a_{21}}\cdots \rho_{2n}^{a_{2n}}\cdots \rho_{nn}^{a_{nn}}\in U_{\hbar}(\mathfrak{gl}_n)^{*} \qquad (a_{ij}\in \mathbf{Z}_{\geqq 0})$$

are linearly independent over R. This follows from the facts that $U_{\hbar}(\mathfrak{gl}_n)^{*}$ is a torsion free R-module and that the elements

$$\rho_{11}^{a_{11}}\rho_{12}^{a_{12}}\cdots \rho_{1n}^{a_{1n}}\rho_{21}^{a_{21}}\cdots \rho_{2n}^{a_{2n}}\cdots \rho_{nn}^{a_{nn}} \bmod \hbar \qquad (a_{ij}\in \mathbf{Z}_{\geqq 0})$$

of $U_{\hbar}(\mathfrak{gl}_n)^{*}/\hbar U_{\hbar}(\mathfrak{gl}_n)^{*}=U(\mathfrak{gl}_n)^{*}$ are linearly independent over k.

(ii) This follows from (i) and the fact that $C/\hbar C$ is an integral domain. □

Set

$$\begin{aligned} \varphi &= \sum_{\sigma\in\mathfrak{S}_n}(-q)^{l(\sigma)}x_{1\sigma(1)}x_{2\sigma(2)}\cdots x_{n\sigma(n)}-1\in C\,,\\ \varphi^0 &= \sum_{\sigma\in\mathfrak{S}_n}(-1)^{l(\sigma)}y_{1\sigma(1)}y_{2\sigma(2)}\cdots y_{n\sigma(n)}-1\in k[y_{ij}]\,, \end{aligned}$$

where $k[y_{ij}]$ is the polynomial ring with variables y_{ij} $(i, j=1, \cdots, n)$. We have natural identifications $k[SL_n]=k[y_{ij}]/(\varphi^0)$ and $k\otimes_R C=k[y_{ij}]$. Let C_p (resp. $A_{\hbar}[SL_n]_p$, resp. $k[y_{ij}]_p$, resp. $k[SL_n]_p$) be the linear span in C (resp. $A_{\hbar}[SL_n]$, resp $k[y_{ij}]$, resp. $k[SL_n]$) of the mononials in x_{ij} (resp. ρ_{ij}, resp. y_{ij}, resp. y_{ij} mod φ^0) of degree $\leqq p$.

Lemma A.3. (i) C_p *is a free R-module of* rank$=$dim $k[y_{ij}]_p$.
(ii) $C_{p-n}\varphi$ *is a free R-module of* rank$=$dim $k[y_{ij}]_{p-n}$.
(iii) $\dim(k\otimes_R(C_p/C_{p-n}\varphi))=\dim k[y_{ij}]_p-\dim k[y_{ij}]_{p-n}$.
(iv) $A_\hbar[SL_n]_p$ *is a free R-dodule of* rank$\geqq\dim k[y_{ij}]_p-\dim k[y_{ij}]_{p-n}$.

Proof. The statements (i), (ii), (iii) are clear from Lemma A.2. Let us show (iv). Since $A_\hbar[SL_n]_p$ is a finitely generated R-submodule of the free R-module $A_\hbar[SL_n]$, it is a free R-module of finite rank. Hence the surjectivity of the k-linear map

$$A_\hbar[SL_n]_p/\hbar A_\hbar[SL_n]_p \to A_\hbar[SL_n]_p/(A_\hbar[SL_n]_p\cap\hbar A_\hbar[SL_n])\simeq k[SL_n]_p$$

implies that

$$\begin{aligned}\operatorname{rank} A_\hbar[SL_n]_p &= \dim(A_\hbar[SL_n]_p)/\hbar A_\hbar[SL_n]_p)\\ &\geqq \dim k[SL_n]_p\\ &= \dim k[y_{ij}]_p-\dim k[y_{ij}]_{p-n} \quad \square\end{aligned}$$

Proof of Proposition A.1. We have to show that the natural algebra homomorphism $C/C\varphi C\to A_\hbar/[SL_n]$ $(x_{ij}\mapsto\rho_{ij})$ is an isomorphism. Since this is surjective by Lemma 2.3.1, it is sufficient to show that the R-homomorphism $C_p/(C\varphi C\cap C_p)\to A_\hbar[SL_n]$ is injective for each p. Therefore it suffices to prove that the surjective R-homomorphism $C_p/C_{p-n}\varphi\to A_\hbar[SL_n]_p$ is an isomorphism. This follows from Lemma A.3. $\square$

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