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Q-HOMOLOGY PLANES WITH C^{**}-FIBRATIONS

Dedicated to Professor Heisuke Hironaka on his sixtieth birthday

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Introduction. Let X be a nonsingular algebraic surface defined over the complex number field $\boldsymbol{C}$. We call X a *homology plane* (resp. a $\boldsymbol{Q}$*-homology plane*) if the homology groups $H_i(X; \boldsymbol{Z})$ (resp. $H_i(X; \boldsymbol{Q})$) vanish for $1 \leq i \leq 4$. We can also define a *logarithmic homology plane* X as a normal affine surface which has only quotient singularities and $H_i(X; \boldsymbol{Z})=0$ for all $i>0$.

In our previous paper [7], $\boldsymbol{Q}$-homology planes with Kodaira dimension less than 2 are classified and it is shown that there are many $\boldsymbol{Q}$-homology planes which have non-trivial automorphisms of finite order. A structure theorem is given on logarithmic homology planes of Kodaira dimension $-\infty$ and 1. In particular, it is proved that a logarithmic homology plane of Kodaira dimension $-\infty$ is isomorphic to one of the following surfaces:

(1) $\boldsymbol{C}^2$;

(2) $\boldsymbol{C}^2/G$, where G is a small finite subgroup of $GL(2, \boldsymbol{C})$;

(3) A surface X with an $\boldsymbol{A}^1$-fibration $\rho: X \rightarrow \boldsymbol{A}^1$ such that every fiber is irreducible and that there are N multiple fibres $H_1, \cdots, H_N$ with respective multiplicities $d_1, \cdots, d_N$, each of them carrying a cyclic quotient singular point of type d_i/e_i, where N is an arbitrary positive integer.

Similarly, logarithmic homology planes of Kodaira dimension 1 are studied by making use of $\boldsymbol{C}^*$-fibrations.

In the present paper we are interested in homology planes with $\kappa=2$. An example of a contractible algebraic surface with $\kappa=2$, which is a special case of a homology plane, was first given by C.P. Ramanujam [9] and many examples were recently found by Gurjar-Miyanishi [2], Miyanishi-Sugie [6] and Petrie-tom Dieck [11, 12]. We constructed in [6] homology planes by the blowing-up method from the configurations of two curves on the projective plane $\boldsymbol{P}^2$ and Petrie-tom Dieck [11] from the line arrangements on $\boldsymbol{P}^2$. In order to construct further examples, we propose to think of algebraic surfaces with fibrations of curves. As a natural extension of the $\boldsymbol{C}$-fibrations and the $\boldsymbol{C}^*$-fibrations which are so effective in the cases of $\kappa=-\infty$ and 1, we shall look into a surface with a $\boldsymbol{C}^{**}$-fibration, where $\boldsymbol{C}^{**}$ is the affine line with two points deleted off. We can consider a $\boldsymbol{C}^{**}$-fibration as an analogy of a fibration by curves of genus 2 in the complete case.

Besides, in the study of structures of homology planes, it is an intresting problem to verify or negate the following:

Homology Plane Conjecture. *Let X be a homology plane admitting a non-trivial automorphism of finite order. Then X is isomorphic to $\boldsymbol{C}^2$.*

In §1 of this paper, we classify singular fibers of $\boldsymbol{C}^{**}$-fibrations, and in §2 we classify $\boldsymbol{Q}$-homology planes which have $\boldsymbol{C}^{**}$-fibrations. In §3 we calculate the homology groups $H_i(X; \boldsymbol{Z})$ and the Kodaira dimension $\kappa=\kappa(X)$ of certain surfaces listed in §2. Thus, we obtain infinitely many homology planes and logarithmic homology planes of $\kappa=2$ and some of these surfaces have, indeed, nontrivial automorphisms of finite order, which negates the above homology plane conjecture. We give an explicit description of those examples in §4.

By the way, there seems to be a misunderstanding about the difference between homology planes and contractible surfaces in the case of $\kappa=1$. Petrie proved that contracttble surfaces of $\kappa=1$ have no non-trivial automorphisms. There exist, however, homology planes with $\kappa=1$ which have non-trivial automorphisms. We also include these examples in §5.

NOTATIONS: We denote by $\boldsymbol{C}^{(N*)}$ a rational curve $\boldsymbol{C}$-{N points}. In particular, $\boldsymbol{C}^{*}$ is a curve $\boldsymbol{C}$-{1 point} and $\boldsymbol{C}^{**}$ is a curve $\boldsymbol{C}$-{2 points}. A (-1) curve means an exceptional curve of the first kind. We refer to Miyanishi [5] for the definition of Kodaira dimension κ and relevant results. We employ also the notations and results in [7].

1. Singular fibers of $\boldsymbol{C}^{**}$-fibrations

Let X be a normal affine surface defined over the complex number field $\boldsymbol{C}$ with a $\boldsymbol{C}^{(N*)}$-fibration $\pi: X \to B$, where B is a smooth algebraic curve. Let V be a normal projective surface which contains X as an open subset and is smooth along $D: =V-X$. Moreover, we assume that D is an effective divisor with simple normal crossings and that the fibration π is extended to a $\boldsymbol{P}^1$-fibration $p: V \to C$, where C is a smooth complete curve. Let $f: W \to V$ be a minimal resolution of singularities of V. Then $q=p\cdot f: W \to C$ is a $\boldsymbol{P}^1$-fibration on a smooth projective surface W and if we set $Y=f^{-1}(X)$, $\rho: =p|_Y: Y \to B$ defines a $\boldsymbol{C}^{(N*)}$-fibration on Y. We identify the divisor D on V with the divisor $f^{-1}(D)$ on W.

The following property of a $\boldsymbol{P}^1$-fibration is well-known. We shall make use of it freely.

Lemma 1.1. *Let W be a smooth pıojective surface with a $\boldsymbol{P}^1$-fibration q:*

$W \to C$ and let F be a singular fiber of q. Write $F = \sum_{i=1}^{n} m_i F_i$ as a sum of irreducible components. Then the following hold:

(1) *Three irreducible components of F do not meet in one point;*

(2) Supp (F) *does ont contain a loop;*

(3) *If $m_k = 1$ and $n \geq 2$ there exists a (-1) curve in F other than E_k.*

We consider first the case where D contains $N+1$ different cross-sections and denote these sections by $S_1, \cdots, S_{N+1}$. If two or more of S_i's meet in one point, we blow up these intersection points until the proper transforms of S_i's are disjoint from each other, and we include the resulting exceptional curves into the boundary divisor D. We may thus assume that $S_1, S_2, \cdots, S_{N+1}$ do not meet. In this case we call a $\boldsymbol{C}^{(N*)}$-fibration $\pi: X \to C$ *untwisted.* If not all of S_i's are cross-sections, we call a $\boldsymbol{C}^{(N*)}$-fibratopm *twisted.* Since singular fibers of $\boldsymbol{C}^{(N*)}$-fibrations on normal surfaces can be obtained easily from the smooth case, we consider first a smooth affine surface with a $\boldsymbol{C}^{(N*)}$-fibration. We call a fiber $\pi^{-1}(P)$ a *singular fiber* if it is not isomorphic to $\boldsymbol{C}^{(N*)}$ as a subscheme.

Let A be a singular fiber of π, let $A_1, \cdots, A_k$ be all connected components of A and let A_{ij} be irreducible components of A_i. Let $T := p^{-1}(\pi(A))$ be the fiber of p containing A. We denote by T_i the connected component of $T \cap D$ which intersects S_i. We may assume that $T_i \neq \phi$ for every i. Indeed, if $T_i = \phi$, blow up the point $T \cap S_i$ and include the exceptional curve into the boundary divisor D. Note that T_i and T_j might coincide with each other for different indices i and j. Set

$$a_i + 1 := \text{the number of points of } (\bar{A}_i - A_i) \quad \text{and} \quad a := \sum_i a_i\,,$$

where $\bar{A}_i$ is the closure of A_i in V. Then we have

Lemma 1.2. *$a \leq N$.*

Proof. We have only to consider the connected components A_i for which $a_i \geq 1$. We assume that $a_i \geq 1$ for $1 \leq i \leq m$ and $a_i = 0$ for $m < i \leq k$. First, there are $a_1 + 1$ connected components $T_1, \cdots, T_{a_1+1}$ of $T \cap D$ which intersect $\bar{A}_1$. Secondly, there are $a_2 + 1$ connected components $T_{a_1+2}, \cdots, T_{a_1+a_2+2}$ of $T \cap D$ which intersect $\bar{A}_2$, at most one of which can be taken from $T_1, \cdots, T_{a_1+1}$ since a fiber of a $\boldsymbol{P}^1$-fibration contains no loops. Let α_2 be the number of different connected components in $\{T_1, \cdots, T_{a_1+1}, T_{a_1+2}, \cdots, T_{a_1+a_2+2}\}$ and let $\beta_2 = a_1 + a_2 + 2 - \alpha_2$. Then $\alpha_2 \geq a_1 + a_2 + 1$ and $2 - \beta_2$ equals to the number of connected components of the support of $\bar{A}_1 + \bar{A}_2 + \sum_{i=1}^{a_1+a_2+2} T_i$. In the third step, we need $a_3 + 1$ connected components $T_{a_1+a_2+3}, \cdots, T_{a_1+a_2+a_3+3}$ which intersect $\bar{A}_3$ and at most $2 - \beta_2$ can be taken from $T_1, \cdots, T_{a_1+1}, T_{a_1+2}, \cdots, T_{a_1+a_2+2}$. Let α_3 be the number of different connected components in $\{T_1, \cdots, T_{a_1+a_2+a_3+3}\}$ and let $\beta_3 = a_1 + a_2 + a_3 + 3 - \alpha_3$. Then $\alpha_3 \geq \alpha_2 + \{a_3 + 1 - (2 - \beta_2)\} = a_1 + a_2 + a_3 + 1$. Continuing this way to the

m-th step, we see $\alpha_m \geq \sum_{i=1}^{m} a_i + 1$. Since $\alpha_m \leq N+1$, we have the stated inequality. Q.E.D.

We have also the following lemma.

Lemma 1.3. *Assume $N \geq 2$. If A_{red} is isomorphic to $\boldsymbol{C}^{(N*)}$ then $A = A_{\mathrm{red}}$ and A itself is isomorphic to $\boldsymbol{C}^{(N*)}$.*

Proof. In this case, the connected components $T_1, \cdots, T_{N+1}$ must be all different. If T contains a (-1) curve, we contract it. Then the images of T_i's are all different, though one of T_i's might become the empty set and the image of A_{red} is still isomorphic to $\boldsymbol{C}^{(N*)}$. We can thus assume from the beginning that $T_1, \cdots, T_{N+1}$ do not contain (-1) curves. Then $\bar{A}_{\mathrm{red}}$ must be a unique (-1) curve in T. Contract then $\bar{A}_{\mathrm{red}}$ and let σ be the contraction morphism. If $T_i \neq \phi$ for every i, then more than three of $\sigma(T_i)$'s intersect in one point, which contradicts the property of a singular fiber of a $\boldsymbol{P}^1$-fibration. Therefore at least one component of T_i's is the empty set, say, $T_1 = \phi$. Then $\bar{A}_{\mathrm{red}}$ meets the section S_1 and there must be a (-1) curve in the fiber T other than $\bar{A}$ unless $\bar{A} = T$. From the assumption that either $T_i = \phi$ or T_i contains no (-1) curves, we conclude that $T = \bar{A} = \boldsymbol{P}^1$ and $A = \boldsymbol{C}^{(N*)}$ as a subscheme.

Q.E.D.

Lemma 1.3 states a property particular to the case $N \geq 2$. For example, a $\boldsymbol{C}^*$ fibration has a singular fiber of the form $m\boldsymbol{C}^*$ $(m \geq 2)$. The following result is easy to verify if one takes into account that X is affine.

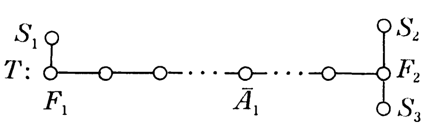
Lemma 1.4. *If $a_i = 0$, then A_i is irreducible and isomorphic to $\boldsymbol{C}$.*

From now on we restrict ourselves to the case where $N=2$. Let Γ be the union of A_i's for which $a_i \geq 1$ and let Δ be the union of A_i's for which $a_i = 0$. Then $A = \Gamma + \Delta$ and Δ is a disjoint union of curves which are isomorphic to $\boldsymbol{C}$. With the above notations we have the following:

Lemma 1.5. *Let X be a smooth affine surface with an untwisted $\boldsymbol{C}^{**}$-fibration $\rho: X \to C$ and employ the notations A, T, Γ, Δ, T_i's, etc. as above. Assume $\Delta = \phi$. Then Γ and the dual graph of $T + S_1 + S_2 + S_3$ are described as one of the following:*

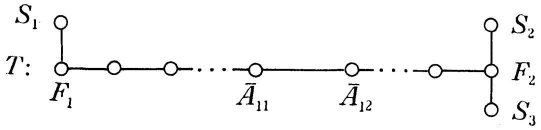
(0) $\Gamma = \phi$.

(I_1) $\Gamma = A_1 = \boldsymbol{C}^*$, *$\bar{A}_1$ is a (-1) curve and* $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_2) = 1$:



where T_1 migth be empty.

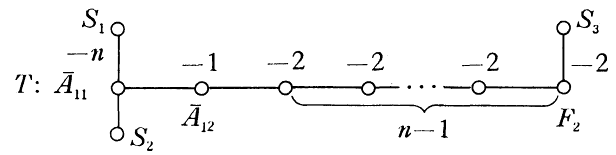
(I_2) $\Gamma = A_1 = A_{11} + A_{12}$, *where* $A_{11} \simeq A_{12} \simeq \boldsymbol{C}$ *and either* $\bar{A}_{11}$ *or* $\bar{A}_{12}$ *is a* (-1) *curve, and* $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_2) = 1$:



where T_1 *may be empty.*

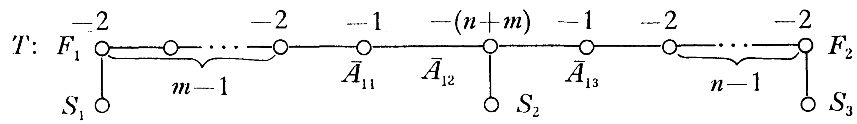
(II_1) $\Gamma = A_1 = \boldsymbol{C}^{**}$ (*This case occurs if* A *is not a smooth fiber, i.e.,* $\Delta \neq \phi$).

(II_2) $\Gamma = A_1 = A_{11} + A_{12}$, $A_{11} \simeq \boldsymbol{C}^*$, $A_{12} \simeq \boldsymbol{C}$, $\bar{A}_{12}$ *is a* (-1) *curve and* $(S_1 \cdot \bar{A}_{11}) = (S_2 \cdot \bar{A}_{11}) = (S_3 \cdot F_2) = 1$:



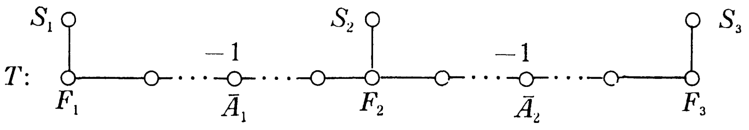
where T_3 *might be empty.*

(II_3) $\Gamma = A_1 = A_{11} + A_{12} + A_{13}$, $A_{11} \simeq A_{12} \simeq A_{13} \simeq \boldsymbol{C}$, $\bar{A}_{11}$ *and* $\bar{A}_{13}$ *are* (-1) *curves and* $(S_1 \cdot F_1) = (S_2 \cdot \bar{A}_{12}) = (S_3 \cdot F_2) = 1$:



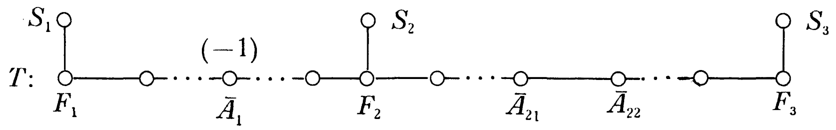
where T_1 *and* T_3 *might be empty.*

(III_1) $\Gamma = A_1 \coprod A_2$, $A_1 = A_2 = \boldsymbol{C}^*$, $\bar{A}_1$ *and* $\bar{A}_2$ *are* (-1) *curves, and* $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_3) = 1$:



where T_1 *and* T_3 *might be empty.*

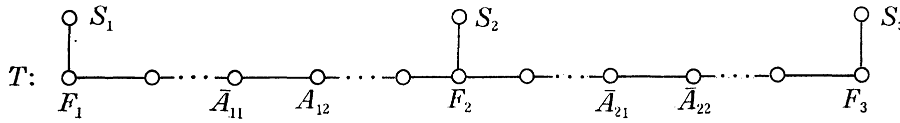
(III_2) $\Gamma = A_1 \coprod A_2$, $A_1 = \boldsymbol{C}^*$, $A_2 = A_{21} + A_{22}$, $A_{21} \simeq A_{22} \cong \boldsymbol{C}$, $\bar{A}_1$ *is a* (-1) *curve, either* $\bar{A}_{21}$ *or* $\bar{A}_{22}$ *is a* (-1) *curve and* $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_3) = 1$:



where T_1 *and* T_3 *might be empty.*

(III_3) $\Gamma = A_1 \coprod A_2$, $A_1 = A_{11} + A_{12}$, $A_2 = A_{21} + A_{22}$, $A_{11} \simeq A_{12} \simeq A_{21} \simeq A_{22} \simeq \boldsymbol{C}$, *either* $\bar{A}_{11}$ *or* $\bar{A}_{12}$ *is a* (-1) *curve, either* $\bar{A}_{21}$ *or* $\bar{A}_{22}$ *is a* (-1) *curve and* $(S_1 \cdot F_1) =$

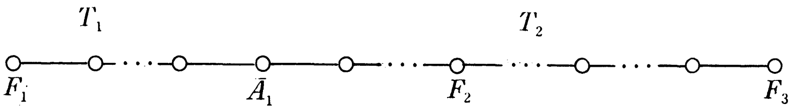
$(S_2 \cdot F_2) = (S_3 \cdot F_3) = 1$:



where T_1 and T_3 might be empty.

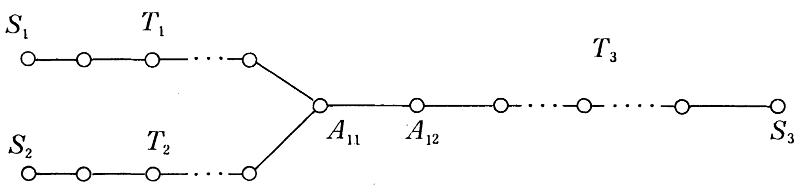
Proof. We use the notations set forth before. We can assume that T_1, T_2 and T_3 do not contain (-1) curves other than those which intersect at least two sections.

(1) **Case $a=1$.** In this case $\Gamma = A_1$.
(1–1) Consider the case where A_1 is irreducible. Then $A_1 \simeq \boldsymbol{C}^*$. We may assume that $T_2 = T_3$ and $\bar{A}_1$ intersects T_1 and T_2. Then T must contain curves whose dual graph is given as follows:



where $(S_1 \cdot F_1) = (S_2 \cdot F_2) = (S_3 \cdot F_3) = 1$. From the assumption, we see that T does not contain other components than those included in the above dual graph. Since there exists a composite of blowing-downs τ such that $\tau(T) = \tau(F_2) \simeq \boldsymbol{P}^1$, the above dual graph must contain a (-1) curve in the branch on the right hand side of F_2 of the graph unless $F_2 = F_3$. Thus we have $F_2 = F_3$ and $\bar{A}_1$ must be a (-1) curve. This case corresponds to (I_1) in the statement of the lemma.
(1–2) If $\Gamma = A_1$ is reducible, we have $A_1 = A_{11} + A_{12}$ and $A_{11} = A_{12} = \boldsymbol{C}$. We can show the statement in (I_2) be a similar argument as in (1–1).

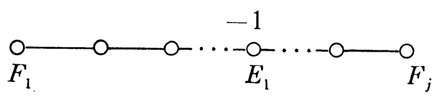
2) **Case $a = a_1 = 2$.** In this case $\Gamma = A_1$.
(2–1) If A_1 is irreducible, Lemma 1.3 shows that A is a smooth fiber.
(2–2) If A_1 is reducible and consists of two components, we have $A_1 = A_{11} + A_{12}$ and $A_{11} \simeq \boldsymbol{C}^*$ and $A_{12} \simeq \boldsymbol{C}$. We may assume here that T_1 and T_2 intersect A_{11} and T_3 intersects with A_{12}. Then, by these assumptions we know that the dual graph of $T + S_1 + S_2 + S_3$ is given as follows:



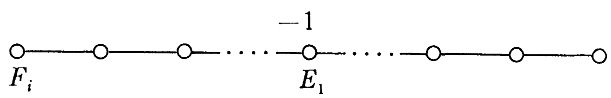
If A_{11} is a (-1) curve, by contracting A_{11}, we can easily show $T_1=T_2=\phi$. Then since A_{11} intersects the sections S_1 and S_2, there exists another (-1) curve, which must be A_{12} and we have $T_3=\phi$. If A_{12} is a (-1) curve, first contract A_{12} and continue the contractions of the curves $F_1, \cdots, F_{n-1}$ contained in T_3 and assume that after contracting F_{n-1}, A_{11} becomes a (-1) curve. At this step, we get to a situation similar to the above. This means that $A_{11}, A_{12}, F_1, \cdots, F_{n-1}$ exhaust all curves in the fiber T. The statement in (II_2) is now easy to prove.
(2–3) If A_1 is reducible and consists of three components, $A_1=A_{11}+A_{12}+A_{13}$ and $A_{11}\simeq A_{12}\simeq A_{13}\simeq \boldsymbol{C}$. Arguing as in (2,2), we can prove the statement in (II_3).

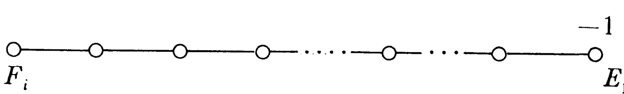
(3) **Case $a=2$, $a_1=a_2=1$.** In this case $\Gamma=A_1\coprod A_2$. We divide it into the following three cases:
(3–1) A_1 and A_2 are irreducible.
(3–2) A_1 is irreducible and A_2 is reducible.
(3.2) A_1 and A_2 are reducible.
In each case we argue as in the cases (1) and (2) and obtain accordingly the statements from (III_1) to (III_3). Q.E.D.

The method of obtaining a singular fiber in the case where $\Delta\neq\phi$ from one of the above singular fibers is explained in [7]. Namely, starting from an initial point P_0 in $T\cap D$, we obtain a component of Δ as follows:
(a) If P_0 is an intersection point $P_0=F_i\cap F_j$ of two components F_i and F_j of T, let σ_1: $Z_1\rightarrow V$ be an oscilating sequence of blowing-ups with initial point P_0. The dual graph of the configuration of curves $F_1+\sigma_1^{-1}(P_0)+F_j$ is given as follows:

L_1:


We say that this oscilating sequence of blowing-ups is of type (a).
(b) If P_0 belongs to only one component F_i of T, let σ_1: $Z_1\rightarrow V$ be an oscilating sequence of blowing-ups with initial point P_0. The dual graph of curves $F_i+\sigma_1^{-1}(P_0)$ is given by one of the following:

L_2:


L_3:


We say that an oscilating sequence producing the dual graph L_2 or L_3 is of type (b–1) or (b–2), respectively.

Next we choose a second initial point P_1 from E_1. If P_1 is an intersection point of E_1 with other component of the fiber, we perform an oscilating sequence of blowing-ups of type (a). If P_1 does not belong to other components of the fiber, perform an oscilating sequence of blowing-ups of type (b–1).

We proceed this way several times and at the last step we perform an oscilating sequence of blowing-ups of type (b–2). Then we obtain a surface $\tilde{Z}$ and a (-1) curve $\tilde{E}$ which is an end component of the dual graph of the curves contained in the fiber corresponding to T. We include all exceptional curves obtained by the above sequence of blowing-ups except for $\tilde{E}$ into the boundary divisor $\tilde{D}$. Then $\tilde{E}-\tilde{D}\simeq \boldsymbol{C}$ is a component of Δ. Every component of Δ is given in this fashion. We denote by $(\mathrm{II}_2)_{(k)}$, for example, a singular fiber which is obtained by adding k components of Δ to (II_2).

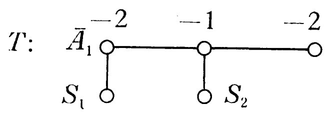
Next, we consider the case where affine surfaces have twisted $\boldsymbol{C}^{**}$-fibrations. We use here the notations similar to those employed above. Let $\pi\colon X\to C$ be a twisted $\boldsymbol{C}^{**}$-fibration. There are two cases to consider, that is, the case where p has two sections S_1 and S_2 contained in D such that $\deg p|_{S_1}=1$ and $\deg p|_{S_2}=2$, and the case where p has one section contained in D such that $\deg p|_S=3$. We call the first the 2-section case and the second the 3-section case, respectively.

Lemma 1.6. *Let X be a smooth affine surface which has a twisted $\boldsymbol{C}^{**}$-fibration $\rho\colon X\to C$.*

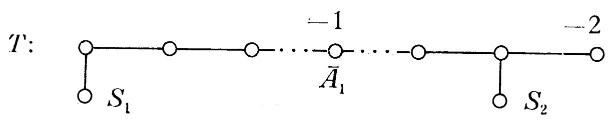
(A) **The 2-section case.** *Assume $\Delta=\phi$. Then Γ and the dual graph of $T+S_1+S_2$ are exhausted by one of the following graphes* (IV-0) *to* (IV–3) *which correspond to a fiber containing a branch point of $p|_{S_2}\colon S_2\to C$ and by one of the modifications of the graphes listed as* (I_1)-(II_3) *in Lemma* 1.5, *where S_2 meets a fiber in two points and two branches of the* 2*-section S_2 are identified suitably with two of three sections S_1, S_2 and S_3.*

(IV_0) $\Gamma=\phi$.

(IV_1) $\Gamma=A_1=\boldsymbol{C}^*$:



(IV_2) $\Gamma=A_1=\boldsymbol{C}^*$ *and $\bar{A}_1$ is a (-1) curve*:



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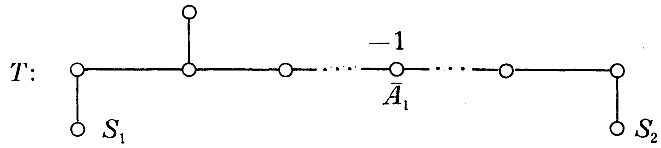
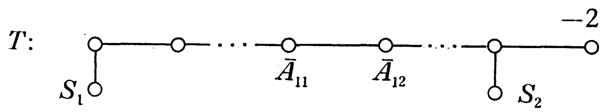
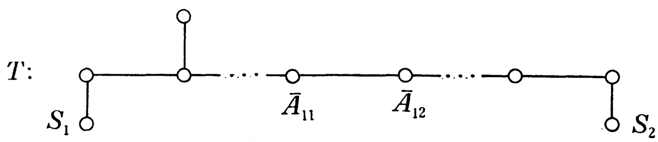


(IV$_3$) $\Gamma=A_1=A_{11}+A_{12}$, $A_{11}\simeq A_{12}\simeq \boldsymbol{C}$, *either* A_{11} *or* A_{12} *is a* (-1) *curve*:



or



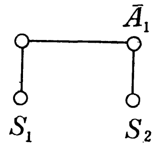
(B) **The 3-section case.** *Assume* $\Delta=\phi$. *Then* Γ *and the dual graph of* $T+S$ *are exhausted by the modifications of the graphes* (I$_1$)-(IV$_3$) *in Lemmas* 1.5 *and* 1.6(A), *where the* 3*-section meets a fiber in either* 3 *points or* 2 *points and we identify accordingiy branches of the* 3*-section* S *either with sections* S_1, S_2 *and* S_3 *or with the section* S_1 *and the* 2*-section* S_2, *and the following extra case*:
(V) $\Gamma=\phi$, *which corresponds to a fiber containing a totally ramified point of* $p|_S$.

Proof. Note that we have $a\leq 1$ in the case (A). The proof in this case is similar to the one in Lemma 1.5. We omit the details of the proof. Q.E.D.

A singular fiber with $\Delta\neq\phi$ is obtained by the same way as explained after Lemma 1.5. We use the notaions like $(\mathrm{IV}_1)_{(3)}$, for example, to signify a singular fiber obtained from (IV) by adding three affine lines to A.

Next we consider the case where a surface X has quotient singularities. Then we have

Lemma 1.7. *Let* X *be a logarithmic affine surface with a* $\boldsymbol{C}^{**}$*-fibration* $\pi\colon Y\to B$. *We use the same notations as above. Then we have:*
(1) $A=\Gamma+\Delta$, *and* Γ_{red} *together with* T *is given by one of* (I) *to* (V) *listed above and the following* (IV$_4$):
(IV$_4$) $\Gamma=A_1=\boldsymbol{C}^*$ *and* $\bar{A}_1$ *has one singularity of type* A_1:



(2) *Each component of* Δ_{red} *has at most one cyclic quotient singular point.*
(3) *The point (s) indicated below can be a cyclic quotient singular point*:

$(\mathrm{I})_2$ $A_{11}\cap A_{12}$,
$(\mathrm{II})_2$ $A_{11}\cap A_{12}$,
$(\mathrm{II})_3$ $A_{11}\cap A_{12}$ *and* $A_{12}\cap A_{13}$,
$(\mathrm{III})_2$ $A_{21}\cap A_{22}$,
$(\mathrm{III})_3$ $A_{11}\cap A_{12}$ *and* $A_{21}\cap A_{22}$,
$(\mathrm{IV})_3$ $A_{11}\cap A_{12}$.

If one of the above points is a singular point, the statement in Lemmas 1.5 *and* 1.6 *concerning the curves* A_{ij} *being a* (-1) *curve need not be true.*

It is easy to prove the above statement. We omit the details. In order to indicate a singular fiber of a logarithmic surface as listed above, we refer to it by the same notations as in the smooth case corresponding to it.

2. Logarithmic $\boldsymbol{Q}$-homology planes with $\boldsymbol{C}^{**}$-fibrations

Let X be a logarithmic $\boldsymbol{Q}$-homology plane with a $\boldsymbol{C}^{**}$-fibration $\pi: X\rightarrow B$. Let $A^{(1)}, \cdots, A^{(l)}$ be all singular fibers of π. We define $A_j^{(i)}$, $A_{j,k}^{(i)}$ and $a_j^{(i)}$ etc. for a singular fiber $A=A^{(i)}$ as in §1. First we cite the following result from [MS2].

Lemma 2.1. (1) *Let* X *be a rational logarithmic* $\boldsymbol{Q}$*-homology plane. Then the boundary divisor* D *is simply connected,* $\Gamma(X, \mathcal{O}_X)^*=\boldsymbol{C}^*$ *and* $\mathrm{Pic}(X)$ *is a finite group.*
(2) *Assume that* X *is a smooth rational surface such that the boundary divisor* D *is connected and simply connected,* $\mathrm{Pic}(X)$ *is a finite group and* $H^2(V;\boldsymbol{Q})\rightarrow H^2(D;\boldsymbol{Q})$ *is an isomorphism. Then* X *is a* $\boldsymbol{Q}$*-homology plane and we have the following isomorphisms*:

$$\mathrm{Pic}(X)\simeq H_1(X;\boldsymbol{Z})\simeq \mathrm{Coker}(H^2(V;\boldsymbol{Z})\rightarrow H^2(D;\boldsymbol{Z}))\,.$$

Moreover, if $H_1(X;\boldsymbol{Z})=0$ *then* X *is a homology plane.*
(3) *Assume that* X *is a rational logarithmic surface such that* D *is connected and simply connected,* $H^2(W;\boldsymbol{Q})\rightarrow H^2(D\cup\Theta;\boldsymbol{Q})$ *is an isomorphism, where* Σ *is the singular locus of* X *and* Θ *is* $f^{-1}(\Sigma)$ *(cf. the notaions in* §1*). Then* X *is a logarithmic* $\boldsymbol{Q}$*-homology plane and we have the following exact sequence and isomorphisms*:

$$0\rightarrow H_1(\partial T;\boldsymbol{Z})\rightarrow H_1(X^0;\boldsymbol{Z})\rightarrow H_1(X;\boldsymbol{Z})\rightarrow 0$$
$$H_1(X^0;\boldsymbol{Z})\simeq \mathrm{Pic}(W-D-\Theta)\simeq \mathrm{Coker}(H_2(D\cup\Theta;\boldsymbol{Z})\rightarrow H_2(W;\boldsymbol{Z}))\,,$$

where $X^0:=X-\Sigma$ *and* ∂T *is a disjoint union of the boundaries of closed neighbourhoods of singular points. In particular, if* $H_1(X;\boldsymbol{Z})=0$ *then* X *is a logarithmic homology plane.*

In this section, we determine the singular fibers of a logarithmic $\boldsymbol{Q}$-homology plane with a $\boldsymbol{C}^{**}$-fibration $\pi: X \rightarrow B$. First, note that D can contain at most one complete fiber of $p: V \rightarrow C$ since D is simply connected. Therefore $B \simeq \boldsymbol{P}^1$ or $\boldsymbol{C}$. From Lemma 2.1. (1)., we also have the following:

Lemma 2.2. *Assume that $p: V \rightarrow C$ is untwisted. Then we have:*

$$\begin{cases} \sum_{i=1}^{l} a^{(i)} = 2(l-1) & \text{if } B \simeq \boldsymbol{P}^1 \\ \sum_{i=1}^{l} a^{(i)} = 2l & \text{if } B \simeq \boldsymbol{C}. \end{cases}$$

Proof. Since D is connected and simply connected, we have:

$$\sum_{i=1}^{l} (2-a^{(i)}) = \begin{cases} 2 & \text{if } B \simeq \boldsymbol{P}^1 \\ 0 & \text{if } B \simeq \boldsymbol{C}. \end{cases}$$

From this follow the above inequalities. Q.E.D.

Let $b^{(i)}$ be the number of irreducible components in $A^{(i)}$. Since $\mathrm{Pic}(X)$ is a finite group and $\Gamma(X, \mathcal{O}_X^*) = \boldsymbol{C}^*$, rank $(\mathrm{Pic}(V))$ is equal to the number of irreducible components in D. Hence we have the following:

Lemma 2.3. *We have:*

(1) *The untwisted case:*

$$\sum_{i=1}^{l} (b^{(i)}-1) = \begin{cases} 1 & \text{if } B \simeq \boldsymbol{P}^1 \\ 2 & \text{if } B \simeq \boldsymbol{C}. \end{cases}$$

(2) *The twisted case with a s 2-section:*

$$\sum_{i=1}^{l} (b^{(i)}-1) = \begin{cases} 0 & \text{if } B \simeq \boldsymbol{P}^1 \\ 1 & \text{if } B \simeq \boldsymbol{C}. \end{cases}$$

(3) *The twisted case with a 3-section: $b^{(i)}=1$ for all i if $B \simeq \boldsymbol{C}$. The case $B \simeq \boldsymbol{P}^1$ does not occur.*

Now we shall determine the structure of logarithmic $\boldsymbol{Q}$-homology planes with $\boldsymbol{C}^{**}$-fibrations.

Lemma 2.4. *Let X be a logarithmic $\boldsymbol{Q}$-homology plane with a $\boldsymbol{C}^{**}$-fibration $\pi: X \rightarrow B$. Then the set of all singular fibers of π is given by one of the following:*

(1) *The case where π is untwisted and $B \simeq \boldsymbol{P}^1$;*

(UP_1) *π has only one singular fiber; $\mathrm{A}^{(1)}$: type $(0)_{(2)}$;*

(UP_2) *π has two singular fibers;*

(UP_{2-1}) $\mathrm{A}^{(1)}$: *type* $(\mathrm{I}_1)_{(1)}$, $A^{(2)}$: *type* (I_1);

(UP_{2-2}) $\mathrm{A}^{(1)}$: *type* (I_2), $\mathrm{A}^{(2)}$: *type* (I_1);

(UP_{2-3}) $A^{(1)}$: *type* $(II_1)_{(1)}$, $A^{(2)}$: *type* $(0)_{(1)}$;

(UP_{2-4}) $A^{(1)}$: *type* (II_2), $A^{(2)}$: *type* $(0)_{(1)}$;

(UP_3) *π has three singular fibers*;

(UP_{3-1}) $A^{(1)}$: *type* (III_1), $A^{(2)}$: *type* (I_1), $A^{(3)}$: *type* (I_1);

(UP_{3-2}) $A^{(1)}$: *type* (II_2), $A^{(2)}$: *type* (I_1), $A^{(3)}$: *type* (I_1);

(UP_{3-3}) $A^{(1)}$: *type* $(II_1)_{(1)}$, $A^{(2)}$: *type* (I_1), $A^{(3)}$: *type* (I_1);

(2) *The case where π is untwisted and $B \simeq \boldsymbol{C}$.*

(UC_1) *π has one singular fibers*;

(UC_{1-1}) $A^{(1)}$: *type* $(III_1)_{(1)}$;

(UC_{1-2}) $A^{(1)}$: *type* (III_2);

(UC_{1-3}) $A^{(1)}$: *type* $(II_1)_{(2)}$;

(UC_{1-4}) $A^{(1)}$: *type* $(II_2)_{(1)}$;

(UC_{1-5}) $A^{(1)}$: *type* (II_3);

(UC_2) *π has two singular fibers*;

(UC_{2-1}) $A^{(1)}$: *type* (III_1), $A^{(2)}$: *type* (III_1);

(UC_{2-2}) $A^{(1)}$: *type* (III_1), $A^{(2)}$: *type* $(II_1)_{(1)}$;

(UC_{2-3}) $A^{(1)}$: *type* (III_1), $A^{(2)}$: *type* (II_2);

(UC_{2-4}) $A^{(1)}$: *type* $(II_1)_{(1)}$, $A^{(2)}$: *type* $(II_1)_{(1)}$;

(UC_{2-5}) $A^{(1)}$: *type* $(II_1)_{(1)}$, $A^{(2)}$: *type* (II_2);

(UC_{2-6}) $A^{(1)}$: *type* (II_2), $A^{(2)}$: *type* (II_2);

(3) *The case where π is twisted with a 2-section and $B \simeq \boldsymbol{P}^1$*;

(TP_1) *π has two singular fibers,* $A^{(1)}$: *type* $(IV_0)_{(1)}$,
$A^{(2)}$: *type* (IV_1) *or* (IV_2) *or* (IV_4);

(TP_2) *π has three singular fibers,* $A^{(1)}$: *type* (IV_1) *or* (IV_2) *or* (IV_4),
$A^{(2)}$: *type* (IV_1) *or* (IV_2) *or* (IV_4), $A^{(3)}$: *type* (I_1);

(4) *The case where π is twisted with a 2-section and $B \simeq \boldsymbol{C}$*;

(TC_1) *π has one singular fiber*;

(TC_{1-1}) $A^{(1)}$: *type* $(IV_1)_{(1)}$ *or type* $(IV_2)_{(1)}$ *or type* $(IV_4)_{(1)}$;

(TC_{1-2}) $A^{(1)}$: *type* (IV_3);

(TC_2) *π has two singular fibers*;

(TC_{2-1}) $A^{(1)}$: *type* (IV_1) *or* (IV_2) *or* (IV_4), $A^{(2)}$: *type* (III_1);

(TC_{2-2}) $A^{(1)}$: *type* (IV_1) *or* (IV_2) *or* (IV_4), $A^{(2)}$: *type* (II_2);

(5) *The case where π is twisted with a 3-section and $B \simeq \boldsymbol{C}$.*

($T3C_1$) *π has one singular fiber,* $A^{(1)}$: *type* $(V_0)_{(1)}$;

($T3C_2$) *π has two singular fibers,* $A^{(1)}$: *type* (IV_1) *or* (IV_2) *or* (IV_4)
$A^{(2)}$: *type* (IV_1) *or* (IV_2) *or* (IV_4);

The case $B \simeq \boldsymbol{P}^1$ does not occur.

Proof. We make use of the inequalities in Lemmas 2.2 and 2.3. Consider, for example, the case where π is untwisted and $B \simeq \boldsymbol{P}^1$. We have

$$2(l-1) = \sum_{i=1}^{l} a^{(1)} \quad \text{and} \quad \sum_{i=1}^{l} (b^{(1)}-1) = 1\,,$$

where l is the number of singular fibers of π. If $l=1$, we have $a^{(1)}=0$ and $b^{(1)}=2$. If $l=2$, we may assume $b^{(1)}=2$, $b^{(2)}=1$ and then have either $a^{(1)}=1$, $a^{(2)}=1$ or $a^{(1)}=2$, $a^{(2)}=0$. If $l=3$, we may assume $b^{(1)}=2$, $b^{()}=1$, $b^{(3)}=1$ and then have $a^{(1)}=2$, $a^{(2)}=1$, $a^{(3)}=1$. This case divides into two cases: $a_1^{(1)}=2$ or $a_1^{(1)}=a_2^{(1)}=1$. If $l\geq 4$, we may assume $b^{(1)}=2$ and $b^{(i)}=1$ for $i\geq 2$, but $a^{(i)}=2$ for at least two i's with $i\geq 2$. This contradicts Lemma 1.3. Therefore the number of singular fibers is at most three. We argue in a similar fashion in the other cases. Q.E.D.

We can exhibit the configurations of the divisor D on V or $D\cup\Theta$ on W and also the configurations of the image curves of D on the relatively minimal model which is obtained from W by the contraction of curves in the fibers. We remark that there exists a contraction $\alpha\colon V\to\Sigma_a$ from V to a Hirzebruch surface Σ_a of degree a such that the images of horizontal components of D are disjoint and smooth. This means that $a=0$ if p is untwisted or if p is twisted and has a 3-section, that $a=1$ if p is twisted and has a 2-section. Conversely, starting from Σ_0 or Σ_1, we can construct a $\boldsymbol{Q}$-homology plane with a $\boldsymbol{C}^{**}$-fibration which has singular fibers as described in Lemma 2.4. It is rather easy to show, by means of criteria given in [5] or by looking into the configuration of the boundary curves, that $\boldsymbol{Q}$-homology planes with $\boldsymbol{C}^{**}$-fibrations have also $\boldsymbol{C}$-fibrations or $\boldsymbol{C}^*$-fibrations except for the following cases: Type(UP_{3-1}), (UC_{2-1}), $(\mathrm{UC}_{2-1})'$ (see below), (TP_2) and (TC_{2-1}).

3. $H_i(X; Z)$ and $\kappa(X)$

In this section we compute the homology groups and Kodaira dimensions of $\boldsymbol{Q}$-homology planes given in Lemma 2.4. Some of these surfaces have $\boldsymbol{C}$-fibrations or $\boldsymbol{C}^*$-fibrations and the homology groups and Kodaira dimensions are computed for them in the previous paper [7]. So, we omit the computation for them and restrict ourselves to the cases listed at the end of §2.

Type (UP_{3-1}). The configuration of singular fibers and sections S_1, S_2, S_3 of p is given as follows:

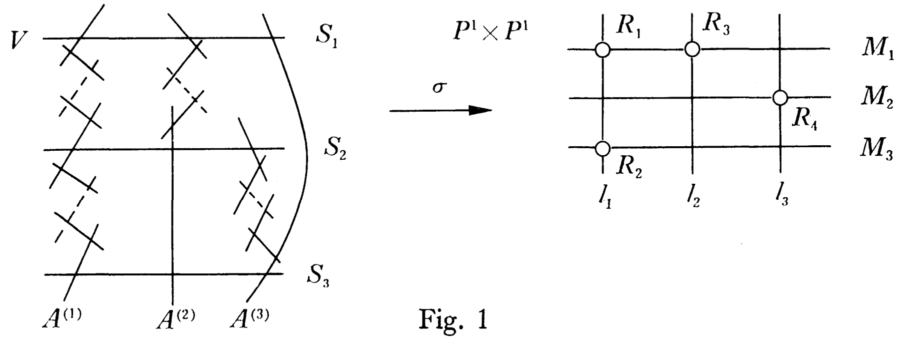


Fig. 1

We obtain the above configuration starting from a configuration as given in Fig. 1 consisting of curves on $\boldsymbol{P}^1\times\boldsymbol{P}^1$ by oscilating sequences of blowing-ups σ: $V\to\boldsymbol{P}^1\times\boldsymbol{P}^1$ with initial points R_1, R_2, R_3 and R_4. We represent by l_1, l_2 and l_3 the fibers of the first projection p_1: $\boldsymbol{P}^1\times\boldsymbol{P}^1\to\boldsymbol{P}^1$ and by M_1, M_2 and M_3 the fibers of the second projection p_2: $\boldsymbol{P}^1\times\boldsymbol{P}^1\to\boldsymbol{P}^1$. The projection p_1 induces a $\boldsymbol{P}^1$-fibration on V. Let E_i $(1\leq i\leq 4)$ be a unique (-1) curve contained in $\sigma^{-1}(R_i)$. For example, E_1 is $\bar{A}_1^{(1)}$ and E_2 is $\bar{A}_2^{(1)}$ in the notations of Lemma 1.4. The total transforms of l_i's and M_i's are written as follows:

$$\begin{aligned}
&\sigma^*(l_1)\sim u_1E_1+u_2E_2+(\text{fiber components of } D)\\
&\sigma^*(l_2)\sim u_3E_3+(\text{fiber components of } D)\\
&\sigma^*(l_3)\sim u_4E_4+(\text{fiber components of } D)\\
&\sigma^*(M_1)\sim v_1E_1+v_3E_3+(\text{other components of } D)\\
&\sigma^*(M_2)\sim v_4E_4+(\text{other components of } D)\\
&\sigma^*(M_3)\sim v_2E_2+(\text{other components of } D).
\end{aligned}$$

Since $l_1\sim l_2\sim l_3$ and $M_1\sim M_2\sim M_3$ on $\boldsymbol{P}^1\times\boldsymbol{P}^1$, $H_1(X;\boldsymbol{Z})$ has generators $\xi_i:=[E_i]$ and relations

$$\begin{aligned}
u_1\xi_1+u_2\xi_2-u_3\xi_3 \qquad\qquad &= 0\\
u_3\xi_3-u_4\xi_4 &= 0\\
v_1\xi_1 \qquad +v_3\xi_3-v_4\xi_4 &= 0\\
v_2\xi_2 \qquad\quad -v_4\xi_4 &= 0\,.
\end{aligned}$$

Therefore the order of $H_1(X;\boldsymbol{Z})$ is equal to $|d|$, where

$$d=\begin{vmatrix} u_1 & u_2 & -u_3 & 0\\ 0 & 0 & u_3 & -u_4\\ v_1 & 0 & v_3 & -v_4\\ 0 & v_2 & 0 & -v_4\end{vmatrix}=u_3u_4v_1v_2+u_1u_4v_2v_3-u_2u_3v_1v_4-u_1u_3v_2v_4\,.$$

The equation $d=\pm 1$ has infinitely many solutions of positive integers for u_i and v_i. The following is a solution of the equation $d=\pm 1$:

$$u_1=u_2=u_3=u_4=1,\ v_1=m,\ v_2=mn-m\pm 1,\ v_3=m-n,\ v_4=n-1\,,$$

where m and n are positive integers such that $n>m$ and $n\geq 2$. The homology planes obtained this way are isomorphic to those which we constructed in our paper [6].

Next we compute the Kodaira dimensions. Write the canonical divisor of V as $K_V\sim\sigma^*(K_{\Sigma_0})+G$, where G is supported on the exceptional curves of σ. Starting with Σ_0 and comparing the multiplicity of the newly obtained exceptional curve in G and $\sigma^*(l_i)$'s and $\sigma^*(M_i)$'s inductively at each step, it is easy to verify the following:

$$\sigma^*(l_1)+\sigma^*(l_2)+\sigma^*(l_3)+\sigma^*(M_1)+\sigma^*(M_2)+\sigma^*(M_3)$$
$$\sim l_1'+l_2'+l_3'+S_1+S_2+S_3+G+H\,,$$

where l_i' is the proper transform of l_i and H is the sum of all exceptional curves of σ with $H_{\mathrm{red}}=H$. Since $K_{\Sigma_0}\sim -l_2-l_3-M_2-M_3$, we have the following expression for K_V+D:

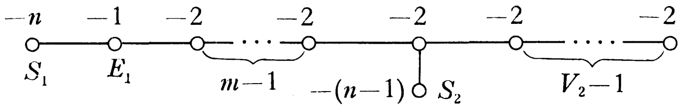
$$K_V+D\sim\sigma^*(l_1)+\sigma^*(M_1)-(E_1+E_2+E_3+E_4)\,.$$

There exist many examples of homology planes of Kodaira dimension two. For example, we will show that $\kappa=2$ for almost all values of u_i's and v_i's given earlier as a solution of $d=\pm 1$.

(a) case $n\geq m+2$, $v_2=mn-m+1$ or $n\geq m+2$, $m\geq 2$, $v_2=mn-m-1$.
In this case, we have

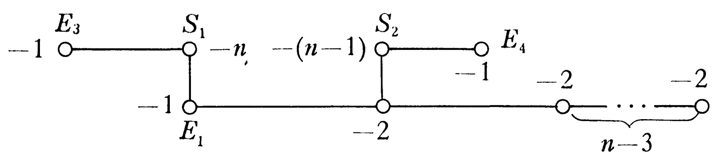
$$2(K_V+D)\sim\sigma^*(M_1)+\sigma^*(M_2)+2\sigma^*(l_1)-2(E_1+E_2+E_3+E_4)$$
$$\sim S_1+S_2+(v_1E_1+v_3E_3)+v_4E_4+2(u_1E_1+u_2E_2)-2(E_1+E_2+E_3+E_4)$$
$$+(\text{fiber components of } D)$$
$$\sim S_1+S_2+mE_1+(n-m-2)\,E_3+(n-3)\,E_4+(\text{fiber components of } D)\,.$$

Therefore $2(K_V+D)$ is effective and there exists an effective member A of $|2(K_V+D)|$ whose support contains the curves, the configuration of which is given as follows:



Note that $-n+1+(m-1)+1+(v_2-1)=n(m-1)\pm 1\geq 1$. Therefore if we contract all curves contained in the above graphs other than S_1 and S_2, the proper transform of S_1 becomes a nonsingular rational curve of positive self-intersection number. This shows that Kodaira dimension of $X:=V-D$ is equal to two.

(b) case $m=1$, $n\geq 4$ and $v_2=n-2$.
In this case also $2(K_V+D)$ is effective and $|2(K_V+D)|$ contains an effective member whose support contains the union of curves given by the following dual graph.



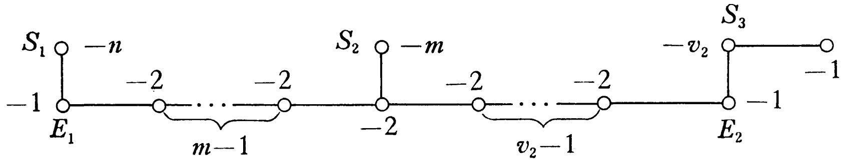
If we contract all curves contained in the above graph except S_1, S_2, E_3 and E_4, then S_1 and S_2 become nonsingular rational curves whose self-intersection numbers are (-1) and the proper transform of S_1 and that of S_2 intersect at one point with multiplicity $n-2$. From this it is easy to show that $\kappa(X)=2$.

(c) case $n=m+1$, $m\geq 3$ and $v_2=m^2\pm 1$.

In this case we have the following expression for $3(K_V+D)$.

$$\begin{aligned}3(K_V+D) &\sim \sigma^*(M_1)+\sigma^*(M_2)+\sigma^*(M_3)+\sigma^*(l_1)+2\sigma^*(l_2)-3(E_1+E_2+E_3+E_4)\\ &\sim S_1+S_2+S_3+(v_1E_1+v_3E_3)+v_4E_4+v_2E_2+(u_1E_1+u_2E_2)+2u_3E_3\\ &\quad +(\text{fiber components of } D)-3(E_1+E_2+E_3+E_4)\\ &\sim S_1+S_2+S_3+(m-2)\,E_1+(v_2-2)\,E_2+(m-3)\,E_4+(\text{fiber components of } D)\end{aligned}$$

Therefore $3(K_V+D)$ is effective and there exists an effective member of $|3(K_V+D)|$ whose support contains the union of curves given by the following dual graph.



The similar argument shows that $\kappa(X)=2$.

Type (UC_{2-1}). The configuration of singular fibers and sections S_1, S_2, S_3 of p is given as follows:

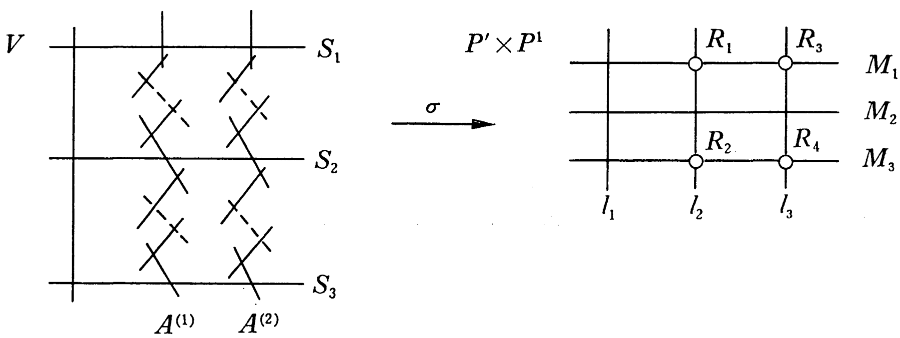


Fig. 2

We obtain the above configuration starting from a configuration of curves as given in Fig. 2 on $\boldsymbol{P}^1\times\boldsymbol{P}^1$ by oscilating sequences of blowing-ups σ: $V\rightarrow\boldsymbol{P}^1\times\boldsymbol{P}^1$ with initial points R_1, R_2, R_3 and R_4. We use the notatonos l_i, M_i and E_i in the same way as in the previous acse. Write the total transforms of l_i's and M_i's as follows:

$$\sigma^*(l_2) \sim u_1 E_1 + u_2 E_2 + (\text{fiber components of } D)$$
$$\sigma^*(l_3) \sim u_3 E_3 + u_4 E_4 + (\text{fiber components of } D)$$
$$\sigma^*(M_1) \sim v_1 E_1 + v_3 E_3 + (\text{other components of } D)$$
$$\sigma^*(M_3) \sim v_2 E_2 + v_4 E_4 + (\text{other components of } D)\,.$$

$H_1(X; \boldsymbol{Z})$ has generators $\xi_1 := [E_i]$ and relations:

$$\begin{aligned} u_1\xi_1 + u_2\xi_3 \qquad\qquad &= 0 \\ u_3\xi_3 + u_4\xi_4 &= 0 \\ v_1\xi_1 \qquad + v_3\xi_3 \qquad &= 0 \\ v_2\xi_2 \qquad + v_4\xi_4 &= 0\,. \end{aligned}$$

The order of $H_1(X; \boldsymbol{Z})$ is equal to $|d|$, where $d = u_2u_3v_1v_4 - u_1u_4v_2v_3$ and the equation $d = \pm 1$ should take positive integer solutions for u_i's and v_i's.

On the other hand, a computation shows that we have the same expression as above for $K_V + D$ and there are many examples of homology planes of $\kappa = 2$ of this type.

Type $(UC_{2-1})'$. The configuration of singular fibers and sections S_1, S_2, S_4 of p is given as follows:

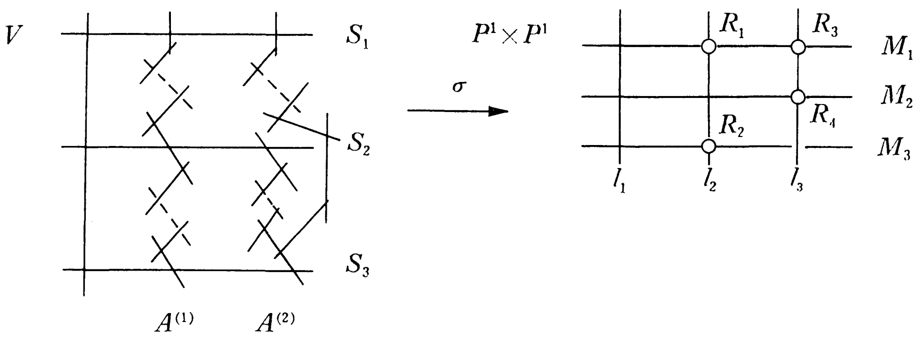


Fig. 3

We obtain the above configuration starting from a configuration of curves on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ as given in Fig. 3 by oscilating sequences of blowing-ups $\sigma: V \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ with initial points R_1, R_2, R_3 and R_4. We use the notations l_i, M_i and E_i in the same way as in the previous case. Write the total transforms of l_i's and M_i's as follows.

$$\sigma^*(l_2) \sim u_1 E_1 + u_2 E_2 + (\text{fiber components of } D)$$
$$\sigma^*(l_3) \sim u_3 E_3 + u_4 E_4 + (\text{fiber components of } D)$$
$$\sigma^*(M_1) \sim v_1 E_1 + v_3 E_3 + (\text{other components of } D)$$
$$\sigma^*(M_2) \sim v_4 E_4 + (\text{other components of } D)$$
$$\sigma^*(M_3) \sim v_2 E_2 + (\text{other components of } D)\,.$$

$H_1(X; \boldsymbol{Z})$ has generators $\xi_i := [E_i]$ and relations:

$$\begin{aligned} u_1\xi_1 + u_2\xi_2 \qquad\qquad\qquad &= 0 \\ u_3\xi_3 + u_4\xi_4 &= 0 \\ v_1\xi_1 \qquad + v_3\xi_3 - v_4\xi_4 &= 0 \\ -v_2\xi_2 \qquad + v_4\xi_4 &= 0\,. \end{aligned}$$

The order of $H_1(X; \boldsymbol{Z})$ is equal to $d = u_1u_3v_2v_4 + u_1u_4v_2v_3 + u_2u_3v_1v_4$. Since u_i's and v_j's are positive integers, there are no solutions for $d = \pm 1$. Hence there exist no homology planes of this type. The divisor $K_V + D$ is also given by the same expression as in the previous cases and there are many examples of $\boldsymbol{Q}$-homology planes of $\kappa = 2$ of this type. We remark that the surface corresponding to the values $u_1 = u_2 = u_3 = u_4 = v_1 = v_3 = 1$ and $v_2 = v_4$ has $\kappa = 0$ and isomorphic to $Y\{2, 4, 4\}$ according to Fuijta [F].

Type $(\boldsymbol{TP}_2)$. The configuration of singular fibers and sections S_1, S_2 of p is given as follows:

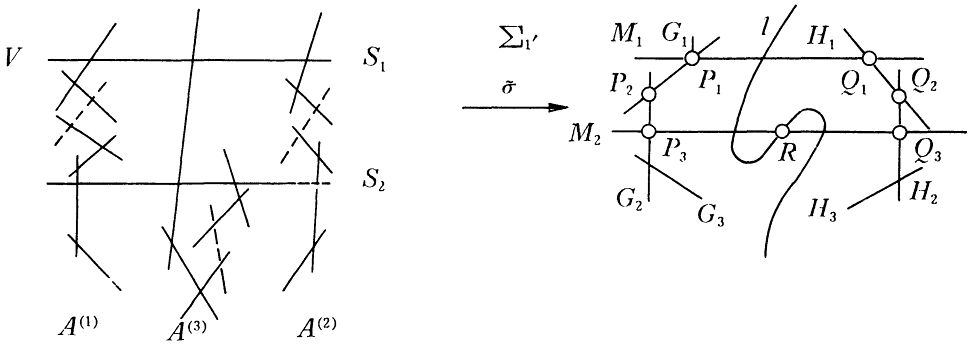


Fig. 4

We obtain this configuration starting from a configuration of curves Σ_1 as given in Fig. 4. Let M_1 be the minimal section of Σ_1 and let p_1 be the morphism from Σ_1 to $\boldsymbol{P}^1$ which gives the natural $\boldsymbol{P}^1$-bundle structure on Σ_1. Let C be a 2-section of Σ_1 disjoint from M_1 and let l_1 and l_2 be fibers of p_1 containing ramification points of $p_1|_C$ and let l_3 be a third fiber of p_1. Consider a divisor $D_0 = M_1 + C + l_1 + l_2 + l_3$ on Σ_1. First we blow up $l_1 \cap C$, $l_2 \cap C$ and their infinitely near points over C in order to get a simple normal crossing divisor. We call this surface Σ_1' and let $\tau\colon \Sigma_1' \to \Sigma_1$ be the composition of these four blowing-ups. The configuration of curves on Σ_1' is given in Fig. 4, where M_2 is the proper transform of C. Next we perform oscilating sequences of blowing-ups $\tilde{\sigma}\colon V \to \Sigma_1'$. We choose an initial point R_1 on $\tau^{-1}(l_1)$ to be one of P_1, P_2 and P_3 if $A^{(1)}$ has type (IV_2). Similarly, we choose an initial point R_2 on $\tau^{-1}(l_2)$. Let R_3 be an initial point on $\tau^{-1}(l_3)$. Let $\sigma = \tau \cdot \tilde{\sigma}$ and let $E_i (1 \leq i \leq 3)$ be a unique (-1) curve contained in $\sigma^{-1}(R_i)$. E_1 is $\bar{A}^{(1)}$ and E_2 is $\bar{A}^{(2)}$ in the notations of Lemma 1.4. We choose G_1 as E_1 (resp. G_2 as E_1) if $A^{(1)}$ has type (IV_1) (resp. (IV_4)). The same remark applies also to $A^{(2)}$. Write the total transforms of l_i's and M_i's as follows:

$$\begin{aligned}
&\sigma^*(l_1) \sim u_1E_1 + (\text{fiber components of } D)\\
&\sigma^*(l_2) \sim u_2E_2 + (\text{fiber components of } D)\\
&\sigma^*(l_3) \sim u_3E_3 + (\text{fiber components of } D)\\
&\sigma^*(M_1) \sim v_1E_1 + v_2E_2 + (\text{other boundary components})\\
&\sigma^*(C) \sim w_1E_1 + w_2E_2 + w_3E_3 + (\text{other boundary components})\,.
\end{aligned}$$

Note that if the initial point R_1 is P_1, w_1 is 0 and if R_1 is P_2 or P_3, v_1 is 0. If $A^{(1)}$ has type (IV_4), $u_1=w_1=2$ and $v_1=0$. The same remark applies also to $A^{(2)}$. Since $l_1 \sim l_2 \sim l_3$ and $2M_1+2l_3 \sim C$, $H_1(X; \boldsymbol{Z})$ has generators $\xi_i := [E_i]$ and relations:

$$\begin{aligned}
u_1\xi_1 \quad - \quad u_2\xi_2 \qquad\qquad &= 0\\
u_2\xi_2 \quad - \quad u_3\xi_3 &= 0\\
(2v_1-w_1)\,\xi_1+(2v_2-w_2)\,\xi_2+(2u_3-w_3)\,\xi_3 &= 0\,,
\end{aligned}$$

and the order of $H_1(X; \boldsymbol{Z})$ is equal to $|d|$, where

$$d = u_1u_2(2u_3-w_3)+u_2u_3(2v_1-w_1)+u_1u_3(2v_2-w_2)\,.$$

The equation $d=\pm 1$ has infinitely many solutions of positive integers for u_i, v_i and w_1. The following is a solution of the equation $d=\pm 1$.

$$w_i = w_2 = 0,\ u_1 = u_2 = 1,\ u_3 = m,\ v_1 = v_2 = n,\ w_3 = 2m+4mn\pm 1\,,$$

where m, n are positive integers. In the next section we prove that homology planes corresponding to the above values have involutions.

We compute the Kodaira dimensions of the above examples, Write the canonical divisor of V as follows:

$$K_V \sim \sigma^*(K_{\Sigma_1})+2G_2'+G_3'+2H_2'+H_3'+Z_1+Z_2$$

where $\mathrm{Supp}(Z_1)=\tau^{-1}(P_1)\cup\tau^{-1}(\mathbf{Q}_1)$, $\mathrm{Supp}(Z_2)=\tau^{-1}(R)$ and we denote the proper transform of the curve using'. Starting with Σ_1 and checking inductively at each step, it is easy to prove the following:

$$\begin{aligned}
&\sigma^*(l_1)+\sigma^*(l_2)+\sigma^*(l_3)+\sigma^*(M_1)+\sigma^*(C)\\
&\quad \sim l_1'+l_2'+l_3'+S_1+S_2+4G_2'+2G_3'+4H_2'+2H_3'+Z_1+U_1+Z_2+U_2
\end{aligned}$$

where U_1 is the sum of all exceptional curves contained in $\tau^{-1}(P_1)\cup\tau^{-1}(\mathbf{Q}_1)$ with $(U_1)_{\mathrm{red}}=U_1$ and U_2 is the sum of all exceptional curves contained in $\tau^{-1}(R)$ with

$(U_2)_{\rm red}=U_2$. Since $K_{\Sigma_1}\sim -C-l_3$, we have the following expression for K_V+D:

$$\begin{aligned}K_V+D &\sim K_V+l_1'+l_2'+l_3'+S_1+S_2+G_2'+G_3'+H_2'+H_3'+U_1+U_2\\ &\quad -(E_1+E_2+E_3)\\ &\sim -\sigma^*(C)-\sigma^*(l_3)+Z_1+Z_2+2G_2'+G_3'+2H_2'+H_3'\\ &\quad +\sigma^*(l_1)+\sigma^*(l_2)+\sigma^*(l_3)+\sigma^*(M_1)+\sigma^*(C)-3G_2'-G_3'-3H_2'-H_3'\\ &\quad -Z_1-Z_2-(E_1+E_2+E_3)\\ &\sim \sigma^*(l_1)+\sigma^*(l_2)+\sigma^*(M_1)-G_2'-H_2'-(E_1+E_2+E_3)\,.\end{aligned}$$

Since

$$4\{\sigma^*(l_1)+\sigma^*(l_2)+\sigma^*(M_1)\}\sim\sigma^*(C)+2\sigma^*(M_1)+3\sigma^*(l_1)+3\sigma^*(l_2)\,,$$

we have

$$\begin{aligned}4(K_V+D) &\sim 2(v_1E_1+v_2E_2)+(w_3E_3+2G_2'+2H_2')+3(E_1+2G_2')+3(E_2+2H_2')\\ &\quad -4(E_1+E_2+E_3+G_2'+H_2')+2S_1+S_2+(\text{fiber components of } D)\\ &\sim (2v_1-1)E_1+(2v_2-1)E_2+(W_3-4)E_3+4G_2'+4H_2'\\ &\quad +2S_1+S_2+(\text{fiber components of } D)\end{aligned}$$

and $4(K_V+D)$ is effective. From this expression, it is easy to see that $\kappa(X)=2$.

If $A^{(1)}$ has type (IV_4), the above number $|d|$ is the order of $H_1(X-\Sigma;\boldsymbol{Z})$. The surface obtained this way has a unique A_1 singular point. We have $d=\pm 2$ for the following values:

$$u_1=2,\ v_1=0,\ w_1=2,\ u_2=1,\ v_2=m,\ w_2=0,\ u_3=n,\ w_3=2mn+n\pm 1\,.$$

Thus we have examples of logarithmic homology planes, each of which has a unique A_1 singular point.

Type (TC_{2-1}). We use the same notations as in the previous cases. We obtain a surface V starting from Σ_1' by oscilating sequences of blowing-ups with initial points R_1, R_2 and R_3.

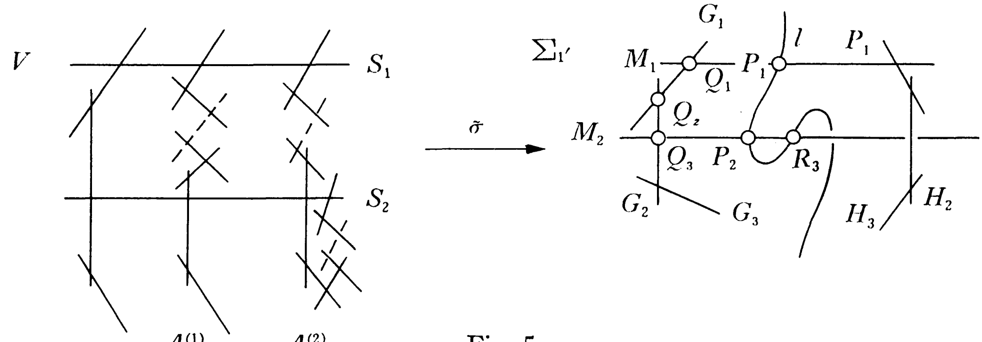


Fig. 5

We choose an initial point R_1 from $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ if $A^{(1)}$ has type (IV_2) and we choose an initial point R_2 from P_1 and P_2. Let E_i be the unique exceptional curve contained in $\sigma^{-1}(R_i)$. E_1 is the proper transform of G_1 (resp. G_2) if $A^{(1)}$ has type (IV_1) (resp. (IV_4)).

Write the total transform of l_i's and M_i's as follows:

$$\sigma^*(l_1) \sim u_1 E_1 + (\text{fiber components of } D)$$
$$\sigma^*(l_2) \sim u_2 E_2 + u_3 E_3 + (\text{fiber components of } D)$$
$$\sigma^*(M_1) \sim v_1 E_1 + v_2 E_2 + (\text{other boundary components})$$
$$\sigma^*(C) \sim w_1 E_1 + w_2 E_2 + w_3 E_3 + (\text{other boundary components}).$$

The order of $H_1(X; \boldsymbol{Z})$ (or $H_1(X-\Sigma; \boldsymbol{Z})$ if $A^{(1)}$ has type (IV_4)) is equal to $|d|$ where $d = 2u_1u_3v_2 + u_1u_2w_3 - u_1u_3w_2$. We note that $v_2w_2 = 0$. Therefore we have homology planes when $v_2 = 0$, $u_1 = 1$ and $u_2w_3 - u_3v_2 = \pm 1$ and logarithmic homology planes when $v_2 = 0$, $u_2 = 2$ and $u_2w_3 - u_3v_2 = \pm 1$. The computation of the Kodaira dimension is similar to the previous case.

Type ($\boldsymbol{T3C_2}$). The configuration of singular fibers and sections of p is given as follows:

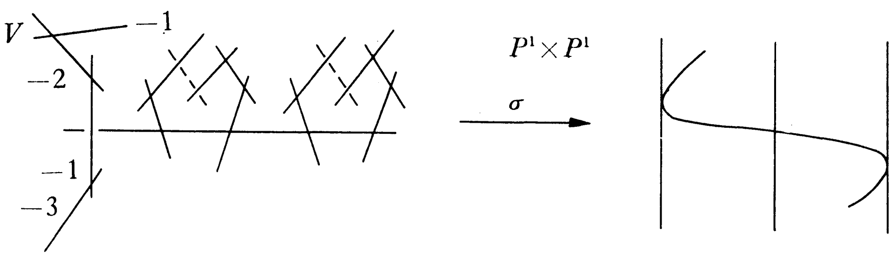


Fig. 6

We start from a nonsingular member C of $|3M+l|$ on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ which totally ramifies at one point and has two other ramification points when it is considered a covering of $\boldsymbol{P}^1$ through the first projection p_1. Here M is a section and l is a fiber of p_1. We can prove that such a curve exists. The calculation of $H_1(X; \boldsymbol{Z})$ is same as before. In this case, there exist no homology planes. Also we can show that there exist $\boldsymbol{Q}$-homology planes of $\kappa = 2$ of this type. We omit the details.

Summarizing the results of this section we obtain the following theorem:

Theorem 1. *There exist infinitely many homology planes of* $\kappa = 2$ *with* $\boldsymbol{C}^{**}$-*fibrations of type* (UP_{3-1}), (UC_{2-1}), (TP_2) *and* (TC_{2-1}) *and there exist infinitely many logarithmic homology planes of* $\kappa = 2$ *with* $\boldsymbol{C}^{**}$-*fibrations of type* (TP_2) *and* (TC_{2-1}). *Conversely, if* X *is a homology plane or a logarithmic homology plane of* $\kappa = 2$ *with a* $\boldsymbol{C}^{**}$-*fibration,* X *belongs to one of the above classes.*

4. Homology planes which have automorphisms

In this section we give examples of homology planes which admit non-trivial automorphisms of finite order. First we show that homology planes of type (TP_2) with values of u_i, v_i and w_i assigned in the previous section have involutions. These examples are constructed by tom Dieck and Petrie. We found out them by differenc approach.

We can start from the following configuration of curves on $\boldsymbol{P}^2$ to obtain the homology plane of type (TP_2):

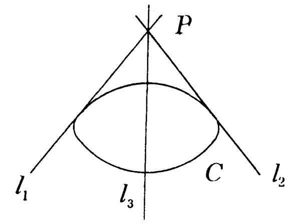


Fig. 7

Let $(X_0: X_1: X_2)$ be the homogeneous coordinates of $\boldsymbol{P}^2$ and l_1, l_2 and l_3 be lines on $\boldsymbol{P}^2$ and let C be a conic on $\boldsymbol{P}^2$ whose equations are given respectively as follows:

$$l_1\colon X_1+X_0=0\,, \qquad l_2\colon X_1-X_0=0\,,$$
$$l_3\colon X_1=0\,, \qquad C\colon X_1^2+X_2^2=X_0^2\,.$$

Let $i\colon \boldsymbol{P}^2 \to \boldsymbol{P}^2$ be an involution defined by $(X_0: X_1: X_2) \to (X_0: -X_1: X_2)$. Then we have $i(C)=C$, $i(l_1)=l_2$ and l_3 is pointwise fixed. Moreover, i has an isolated fixed point $(0, 1, 0)$. To obtain a homology plane, we first perform the blowing-ups with centers P, $l_1 \cap C$ plus its infinitely near point over C, and $l_2 \cap C$ plus its infinitely near point over C. Let $\sigma\colon \Sigma_1' \to \boldsymbol{P}^1$ be the composition of these blowing-ups. Then we have the following configuration of curves on Σ_1':

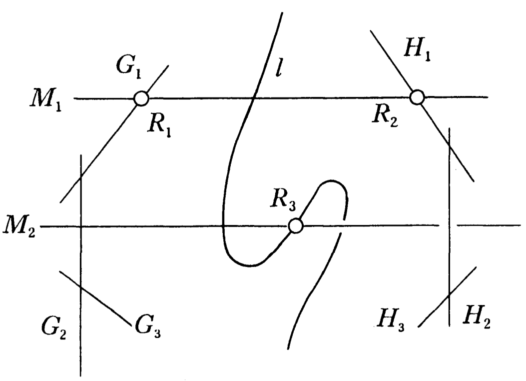
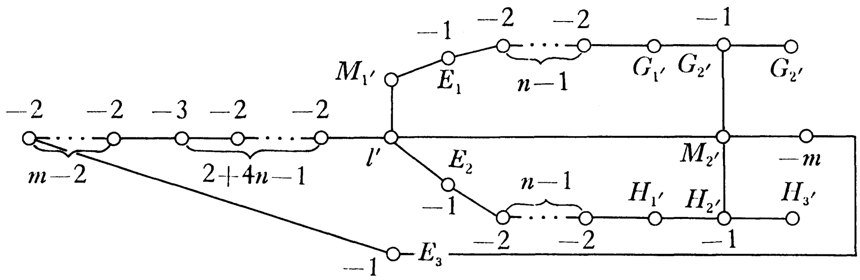


Fig. 8

Next we perform the blowing-ups with centers the point R_1, its $(n-1)$ infinitely near points consecutively lying on M_1, the point R_2 and its $(n-1)$ infinitely near points consecutively lying on M_1, and perform an oscilating sequence of blowing-ups with initial point R_3 such that, if we set $\sigma: V \to \Sigma_1'$ the composition of the above blowing-ups, we have

$$\sigma^*(M_2) \sim (2m+4mn\pm 1) E_3 + (\text{other boundary components}) \quad \text{and}$$
$$\sigma^*(l) \sim m\, E_3 + (\text{other boundary components}),$$

where E_i represents a unique (-1) curve contained in $\sigma^{-1}(R_i)$. Set $D = \sigma^{-1}(M_1+M_2+G_1+G_2+G_3+H_1+H_2+H_3+l)_{\text{red}} - (E_1+E_2+E_3)$. We give the dual graph of divisor $D+E_1+E_2+E_3$ on V in the case where $w_3 = 2m+4mn-1\,(m\geq 2)$:

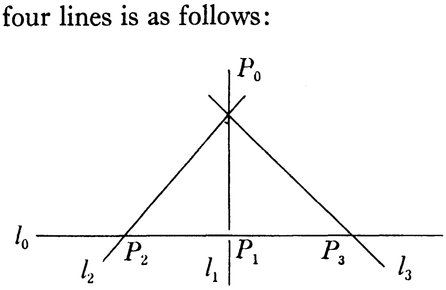


It is shown in §3 that the affine surface $X := V - D$ is a homology plane. Since we perform the blowing-ups at the intersection points of pairwise stable curves or at the fixed points of the involution i or the involutions induced by i, the involution i is liftable to an involution $\tilde{i}$ on V such that $\tilde{i}(D) = D$. Hence $\tilde{i}$ induces an involution $\tilde{i}$ on X, which has a unique fixed point inside X. Thus we obtain homology planes which have involutions. The quotient surface $X/\tilde{i}$ is a logarithmic homology plane with a cyclic quotient singular point of Dynkin type A_1 and belongs to the class (TP_2).

Next we show that there exist homology planes of $\kappa=1$ which have non-trivial automorphisms. A method of constructnig homology planes with $\kappa=1$ is given in [2]. Let $(X_0: X_1: X_2)$ be the homogeneous coordinates of $\boldsymbol{P}^2$. We take four lines on $\boldsymbol{P}^2$ as follows:

$$l_0: X_2 = 0,\ l_1: X_1 = 0,\ l_2: X_1 = -X_2,\ l_3: X_1 = X_2\,.$$

The configuration of four lines is as follows:



First blow-up $P_0=(0:0:1)$ and then perform the oscilating sequences of blowing-ups of type (a) (cf. §1) with initial points P_2 and P_3. Finally we perform an oscilating sequence of blowing-ups with initial point P_1 to produce a singular fiber isomorphic to $\boldsymbol{C}$ on $X:=V-D$. Let $\sigma: V\to \boldsymbol{P}^2$ be the composition of all the above blowing-ups. Let E_i be a unique (-1) curve contained in $\sigma^{-1}(P_i)$. Write the total transform of l_i as follows:

$$\sigma^*(l_i) \sim E_0+u_iE_i+(\text{other boundary components}) \quad \text{for} \quad 1\leq i\leq 3$$

$$\sigma^*(l_0) \sim \sum_{i=1}^{3} v_iE_i+(\text{other boundary components})\,.$$

Set $D=\sigma^*(l_0+l_1+l_2+l_3)_{\mathrm{red}}-(E_1+E_2+E_3)$ and $X=V-D$. Then the order of $H_1(X;\boldsymbol{Z})$ is $|d|$, where $d=u_1u_2u_3-u_1u_2v_3-u_1u_3v_2-u_2u_3v_1$ (cf. [2]). The equation $d=\pm 1$ has the following solutions for any prime number $p\neq 2$:

$$\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\} \quad \text{is a permutation of}$$
$$\{(2, 1), (4r+1, 1), (2r+1, r)\} \quad \text{if} \quad p=4r+1\,,$$
$$\{(u_1, v_1), (u_2, v_2), (u_3, v_3)\} \quad \text{is a permutation of}$$
$$\{(2, 1), (4r+3, 1), (2r+1, r)\} \quad \text{if} \quad p=4r+3\,.$$

Since the role of l_2 and l_3 is symmetric, there are three homology planes according to which one of three pairs is assigned to l_1. Among them, we consider only those homology planes with $v_1=1$ and therefore we may assume that $(u_3, v_3)=(2r+1, r)$. For a different prime number p, we denote by $X_m^{(p)}$ a homology plane with $u_1=m$, m being 2 or $4r+1$ if $p=4r+1$ (resp. 2 or $4r+3$ if $p=4r+3$). We shall show that $X_m^{(p)}$ admits an automorphism of order m.

Let $\eta=\exp(2\pi\sqrt{-1}/m)$. Define an action of $\boldsymbol{Z}/m$ on $\boldsymbol{P}^2$ by $(X_0: X_1: X_2)\to(\eta^{-1}X_0: \eta^{-1}X_1: X_2)$. Then $P_0=(0:0:1)$ is a fixed point under this action and l_0 is pointwise fixed. We state explicitly the process of blowing-ups $\sigma: V\to P^2$;

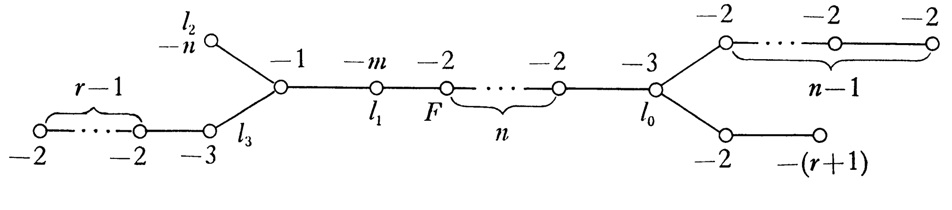
(i) blow up P_0,

(ii) blow up P_2 and its infinitely near points over l_2 altogether n times,

(iii) blow up P_3 and its infinitely near points over l_3 altogether s times, in such a way that $\sigma^*(l_3)\sim u_3E_3+\cdots$ and $\sigma^*(l_0)\sim v_3E_3+\cdots$,

(iv) blow up P_1 and its infinitely near points over l_1 altogether m times and blow up an arbitrary point on the last exceptional curve which is not an intersection point,

where $(m, n, s)=(u_1, u_2, u_3)$. Define D on V as stated before. Then the dual graph of D looks like:



We note that the last blowing-up on the line l_1 is performed on F. Since the first $1+n+s+m$ blowing-ups are performed at the intersection points of two components of fibers, it is easy to see that the action of $\boldsymbol{Z}/m$ is liftable to the blown-up surface. Then at the last step, F becomes pointwise fixed under the induced action. Thus the action of $\boldsymbol{Z}/m$ is liftable onto V and induces an action on $X_m^{(p)}$. It is clear that this action on X has a unique isolated fixed point which lies on the unique (-1) curve contained in $\sigma^{-1}(P_1)$, i.e., on E_1.

Summarizing the results in this section we obtain the following theorem:

Theorem 2. (1) *There exist infinitely many homology planes with* $\kappa=2$, *each of which admits an involution* ι. *This involution* ι *has a unique isolated fixed point.*
(2) *For every prime number* $p\neq 2$, *there exist homology planes* $X_m^{(p)}$ *of* $\kappa=1$ *on which a cyclic group* $\boldsymbol{Z}/m$ *acts, where* $m=2$, *or* $4r+1$ *if* $p=4r+1$ *and* $m=2$ *or* $4r+3$ *if* $p=4r+3$. *This action has a unique isolated fixed point.*

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