Maeda, T.
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ON AMALGAMATED DECOMPOSITIONS OF FINITELY PRESENTED GROUPS ALONG SPLITTING INFINITE CYCLIC GROUPS

Dedicated to Prof. K. Murasugi on his 60th birthdy

Toru MAEDA

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The structures of finitely generated groups or finitely presented groups have been studied through their decomposition-trees consisted of HNN extensions of treed amalgamated products.

In this paper we study the existence and the uniqueness problems for amalgamated free product decompositions of some finitely presented groups with certain infinite cyclic amalgamated subgruops.

Theorem. *Let G be a finitely presented group so that the Betti number of the first homology group $H_1(G)$ is one. Let x be an element of G mapped to a non-trivial element in an infinite cyclic group Z under some homomorphism. Then G can be decomposed into a finite number of irreducible factors uniquely in the sense of amalgamated product with the amalgamated subgroup generated by x.*

1. Definitions and Problems

NOTATION 1.1 Let G be a finitely presented group and Z be the infinite cyclic group generated by t. Let g be a monomorphism from Z into G. Then we denote this pair by (G, g).

NOTE 1.2 If $g(t)$ is mapped to t or t^{-1} in Z under some homomorphism h from G onto Z, G is always an HNN extension of some finitley generated (not necessarily finitely presented) base group by the infinite cyclic group generated by $g(t)$.

Proof. Let G have a presentation

$$\langle x_1, \cdots, x_m : r_1, \cdots, r_n \rangle$$

for some non-negative integers m and n. Let W be a word on $x_1, \cdots, x_m$ representing $g(t)$. Add a generating symbol T and a relator TW^{-1} to the presentation. Now we replace each generator x_k except T with a new symbol a_i by

$a_i=x_i T^{-p_i}$, where $h(x_i)=t^{p_i}$. Let R_j, $1\leqq j\leqq n$, and R_{n+1} be the defining relators rewritten from r_j and TW^{-1} respectively. Since h is a homomorphism onto Z, we can construct a new set of defining relators R_j^*, $1\leqq j\leqq n+1$, on the set of the generating symbols T and a_i's so that the exponent sum $\sigma_T\ (R_j^*)=0$ with respect to T (as same as Theorem 3.5 in [7]). Then each relator R_j^* is a product of a finite number of $a_{ik}=T^{-k}\,a_i\,T^k$. Since the number of defining relators R_j^* is finite, there are integers u and v satisfying that all R_j^* are products of some of a_{ik}, $1\leqq i\leqq m$ and $u\leqq k\leqq v$. Therefore we can take the subgroup generated by a_{ik}, $1\leqq i\leqq m$ and $u\leqq k\leqq v+1$, as an expecting base group of G with the free generator $g(t)$. One of the associated subgroups is generated by $a_{ik'}$, $u\leqq k'\leqq v$ and the other is generated by $a_{ik''}$, $u+1\leqq k''\leqq v+1$. The action of $g(t)$ on $a_{ik'}$ is $g(t)^{-1}\,a_{ik'}\,g(t)=a_{ik'+1}$.

DEFINITION 1.3 We call (G_1, g_1) and (G_2, g_2) are *isomorphic*, $(G_1, g_1)\cong(G_2, g_2)$, if there exists an isomorphism f from G_1 onto G_2 satisfying $f\,g_1=g_2$.

DEFINITION 1.4 The *product* of (G_1, g_2) and (G_2, g_2) is a pair (G, g) given by

$$G=G_1 \underset{g_1(t)=g_2(t)}{*} G_2, \text{ amalgamated product}$$

and $g(t)=g_1(t)=g_2(t)$. We denote this by $(G_1, g_1)*(G_2, g_2)$.

REMARK 1.5 If we only assume G, G_1 and G_2 are finitely generated, then the following are equivalent;

(1) G is finitely presented
(2) G_1 and G_2 are finitely presented.

DEFINITION 1.6 A pair $(G, g)\not\cong(Z, id_z)$ is *irreducible* if $(G, g)=(G_1, g_1)*(G_2, g_2)$ implies that one of (G_1, g_1) and (G_2, g_2) is isomorphic to (Z, id_z).

DEFINITION 1.7 If (G, g) is the product

$$(G_1, g_1)*(G_2, g_2)*\cdots*(G_n, g_n) \tag{1.8}$$

of irreducible factors $(G_i, g_i)'s$, then we call (1.8) *a prime decomposition of* (G, g).

Existence Problem: Does there exist a prime decomposition for any (G, g)?

Uniqueness Problem: Is prime decomposition of (G, g) unique?

DEFINITION 1.9 If (G, g) have no sequence of the following decompositions into two non-trivial factors;

$$(B_n, b_n)=(G_{n+1}, g_{n+1})*(B_{n+1}, b_{n+1})$$

so that $$(G, g) = (A_n, a_n)*(B_n, b_n)$$

and $$(A_{n+1}, a_{n+1}) = (A_n, a_n)*(G_{n+1}, g_{n+1}),$$

then we call that (G, g) *has only prime decompositions.*

Strong Existence Problem: Does (G, g) have only prime decompositions?

Our theorem gines an affirmative case of these problems.

REMARK 1.10 In general (star) decomposition ([5]), decompositions are not unique, even the numbers of irreducible factors are not unique [6].

2. Proof of Existence

Let h be a homomorphism from G into Z in the theorem. We can assume h is onto.

Assume that the theorem is false. Let (G, g) be a counter example which has a base group S, as an HNN extension in Note 1.2, of the minimal rank among all HNN structures of all counter examples. Then for any integer n, (G, g) has the following decompositions;

$$(G, g) = (A_n, a_n)*(B_n, b_n)$$
$$(B_n, b_n) = (G_{n+1}, g_{n+1})*(B_{n+1}, b_{n+1})$$
$$(A_{n+1}, a_{n+1}) = (A_n, a_n)*(G_{n+1}, g_{n+1}).$$

1) First we will prove the case $g(t)$ mapped to a generator of Z under the homomorphism h.

Sublemma 2.1 S is a free group.

Proof. Let n be any integer. Since S is a base group, S is contained in Ker $h(=(A_n \cap \text{Ker } h)*(B_n \cap \text{Ker } h)$, the free product). By Kurosh subgroup theorem [4], S is a free product of the following three types of its subgroups; (I) $S \cap a^{-1} A_n a$, (II) $S \cap b^{-1} B_n b$, (III) free group, for some a, b in Ker h.

We will show that type (I) does not exist for any n. If type (I) eixsts, say H, for some n, consider $(G/H^G, pg)$, where H^G is the normal closure of H in G and p is the natural epimorphism from G onto G/H^G. $(G/H^G, pg) \cong (p(A_n), p|_{A_n} g)*(B_n, b_n) \cong (A_n/H^{A_n}, k\, a_n)*(B_n, b_n)$ where k is the natural epimorphism from A_n onto A_n/H^{A_n}. Since S is finitely generated, H is also finitely generated by Grushko-Neumann-Wagner's theorem ([2], [8], [9]). Therefore this is again a counter example. Moreover, this has an HNN strucutre for the subgroup $p(S)$ as its base group with free part generated by $pg(t)$. The rank of $p(S)$ is less than one of S because of the Grushko-Neumann-Wagner's theorem. This contradicts the minimality of rank of S.

Next claiming that there are no non-trivial elements belonging to a type (II)-subgroup at each n, we can complete the proof. Let x be a non-trivial element in type-(II) subgroup at each n. Similar to the argument for type (I), we consider $(G/x^G, qg)$ where q is the natural epimorphism from G onto G/x^G. For each n, $(G/x^G, qg)\cong(A_n, a_n)*(q(B_n), qb_n)\cong(A_n, a_n)*(B_n/(bxb^{-1})^{B_n}, sg)$ where s is the natural epimorphism from B_n onto $B_n/(bxb^{-1})^{B_n}$. Therefore this is a counter example and has less rank of base group with free part generated by $qh(t)$. This means x can not contained in type (II)-subgroups for any n greater than some large integer. Again since H is finitely generated, type (II) never appears for large integers.

Let us denote the subgroup $g(t)^{-k}\, Sg(t)^k$ by S_k

Sublemma 2.2 The subgroup generated by $S_0\cup S_1$ is a free group.

Proof. Let $\langle S_0\cup S_1\rangle$ be the subgroup generated by $S_0\cup S_1$. Consider $\langle S_0\cup S_1\rangle$ as a subgroup of Ker h. This is a free product of the following three types of its subgroups; (I) $\langle S_0\cup S_1\rangle\cap a^{-1}A_n a$, (II) $\langle S_0\cup S_1\rangle\cap b^{-1}B_n b$, (III) free group, for some a, b in Ker h.

On the other hand, $\langle S_0\cup S_1\rangle=S_0 \underset{C_{01}}{*} S_1$, an amalgamated product with amalgamated subgroup C_{01}.

In the type (I), $\langle S_0\cap S_1\rangle\cap a^{-1}A_n a$ is a subgroup of $S_0 \underset{C_{01}}{*} S_1$. Applying the Karrass-Solitar's subgroup theorem [3], we have an HNN extension of tree product group with a finitely generated free part. But each vertex group of this tree product is trivial as we have seen in Sublemma 2.1. Hence this case subgroups are all free for any n. Type (II) case is the same for large n's. Therefore $\langle S_0\cup S_1\rangle$ is a free product of free groups.

We now use the condition $H_1(G)$ has the Betti number 1.

Sublemma 2.3 S_0 and $\langle S_0\cup S_1\rangle$ have same rank.

Proof. Let m be the rank of S. Under the abelainization, the Ker h goes to the torsion part of $H_1(G)$. The other hand, Ker h is generated by all S_k's. The image of Ker h under the abelianization can be taken through the following steps; (1st) S_k's are identified onto S, (2nd) S goes to the free abelian group A_m of rank m, (3rd) A_m is mapped onto a finite abelain group by using the images of the relaters in A_m induced from C_{01}. Hence the amalgamated subgroup C_{01} has at least m generators. This implies the rank of $\langle S_0\cup S_1\rangle$ is less than or equal to $m=2m-m$. But $\langle S_0\cup S_1\rangle$ is also a base group of G with the same free part, i.e. the infinite cyclic group generated by $g(t)$. So by the minimality of m, we have this sublemma.

Now we know that the free subgroup $\langle S_0 \cup S_1 \rangle$ can also take the place of S as a minimam rank base group of (G, g). So the free subgroup $\langle S_i \cup S_{i+1} \cup \cdots \cup S_j \rangle$ also can for any $i \leqq j$. Let complete the proof. Take a non-trivial element, say c, in $A_1 \cap \text{Ker } h \subset \text{Ker } h$. So c is in $S^* = \langle S_u \cup S_{u+1} \cup \cdots \cup S_v \rangle$ for some $u \leqq v$. Then $(G/\langle c \rangle^G, yg) \cong (A_1/\langle c \rangle^{A_n}, ya_1)*(B_1, b_1)$ is also a counter example. Moreover, $S^*/(S^* \cap \langle c \rangle^G)$ is a base group, the rank of this base group is less than m, the rank of S, because this is a proper factor group of the free group S^* which is non-Hopfian. Contradiction.

2) General case ($h(g(t))=t^r$ for non zero integer r):
Let F be the infinite clclic group generated by z. Let f be the monomorphism from Z into F defined by $f(t)=z^r$. Take $(K, k)=(G, g)*(F, f)$. Then the Betti number of $H_1(K)$ is one and z is mapped to t under a homomorphism from K onto Z. So (K, k') has only prime decompositions, where k' is the monomorphism from Z into K defined by $k'(t)=z$. Since $(G, g)=(G_1, g_1)*(G_2, g_2)$ imples

$$K = \underset{g_1(t)=f(t)}{(G_1 * F)} \underset{z=z}{*} \underset{f(t)=g_2(t)}{(F * G_2)},$$

(G, g) has also only prime decompositions.

3. Proof of Uniqueness

It is suficient to prove this for the case $h(g(t))=t$.
Take two prime decompositions of (G, g)

$$(K_1, k_1)*(K_2, k_2)*\cdots*(K_m, k_m)$$
$$(H_1, h_1)*(H_2, h_2)*\cdots*(H_n, h_n).$$

Let $(A_p, a_p)=(H_1, h_1)*\cdots*(H_p, h_p)$, $(B_p, b_p)=(H_{p+1}, h_{p+1})*\cdots*(H_n, h_n)$. Now consider K_1 as a subgroup of amalgamated product $(A_1, a_1)*(B_1, b_1)$. K_1 is an HNN extension of tree product group. But $k_1(t)$ is in the amalgamated subgroup of $(A_1, a_1)*(B_1, b_1)$. Since $H_1(K_1)=Z=H_1(G)$, K_1, must be just a tree product.

Here we reduce this tree product such that each vertex group is properly embedded into its vertex groups. First we consider the case the resulting tree product has edge groups. Take any edge group, say $E=\langle z^{-1} g(t)^r z \rangle$, z in G. Then tree product induces on K_1 an amalgamated product structure $A*B$ along this edge group E. We can assume one of these factors, say A, has $g(t)$, because, before reducing the tree product, the vertex groups $K_1 \cap A_1$ and $K_1 \cap B_1$ contained $g(t)$.

Sublemma 3.1 A is the infinite cyclic group generated by $g(t)$.

Proof. First we show that z is replaced by d an element d in K_1. Write down the z into the canonical form associated with right coset representative systems $\mathcal{A}_1=\langle g(t)\rangle\backslash K_1$ and $\mathcal{B}_1=\langle g(t)\rangle\backslash K_2*\cdots*K_n$. $z=\theta\,\alpha_1\,\beta_1\,\alpha_2\,\beta_2\cdots\alpha_s\,\beta_s$ where $\theta\in\langle g(t)\rangle$, α_p's$\in\mathcal{A}_1$ and β_p's$\in\mathcal{B}_1$. Since $z^{-1}g(t)\,z$ is in K_1, its canonical form is $\theta\,\alpha$ for some $\theta\in\langle g(t)\rangle$ and $\alpha\in\mathcal{A}_1$. Thereofre, if $\beta_s\neq 1$, all α_p and β_p must commute with $g(t)$. So we can take $d=g(t)^q$ for some integer q. If $\beta_s=1$ then all α_p's and β_p's execept α_s must commute with $g(t)$. So can take $d=g(t)^q$ α_s for some integer q.

Similarly d is also replaced by an element a in A.

Now, if A is not the infinite cyclic group generated by $g(t)$, K_1 is decomposed along $C=\langle a^{-1}g(t)\,a\rangle$. That is $K_1=A\underset{C}{*}(C\underset{F}{*}B)$. This contradicts the irreducibility of (K_1, k_1). So $A=\langle g(t)\rangle$.

This sublemma implies that

$$K_1=\langle g(t)\rangle\underset{\langle g(t)^r\rangle}{*}(K_1\cap x^{-1}A_1\,x)\quad\text{or}\quad\langle g(t)\rangle\underset{\langle g(t)^r\rangle}{*}(K_1\cap y^{-1}B_1\,y)\,.$$

Applying the same argument to other facotrs K_i's and H_j's, we have one to one correspondence between i's and j's so that, for some x_i in G,

(1) $K_i=x_i^{-1}H_j\,x_i$ and $H_j=x_i\,K_i\,x_i^{-1}$,

or (2) $K_i=\langle g(t)\rangle\underset{\langle g(t)^{r_i}\rangle}{*}(K_i\cap x_i^{-1}H_j\,x_i)$ and $H_j=\langle g(t)\rangle\underset{\langle g(t)^{r_i}\rangle}{*}(H_j\cap x_i\,K_i\,x_i^{-1})$, and $r_i\geqq 2$.

In both cases, we have $(K_i, k_i)\cong(H_j, h_j)$. Moreover we can take x_i commutes with $g(t)$ in (1), and commutes with $g(t)^{r_i}$ but not with $g(t)$ in (2). This completes the proof.

Corollary (Dunwoody-Fenn [1]). Let G be a finitely presented group and x be an element of G such that $H_1(G)\cong Z$ and the normal closure of x in G is G. Then G is decomposed into a finite number of irreducible factors along the infinite cyclic group generated by x.

Our proof has been independent from theirs. Usefully their proof has given an upper bound of numbers of irreducible factors by certain number from a geven presentation of G.

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Faculty of Engineering
Kansai University
3–3–35 Yamatechou Suita
Osaka 564
Japan