

STABILITY OF SURFACES WITH CONSTANT MEAN CURVATURE IN 3-DIMENSIONAL RIEMANNIAN MANIFOLDS

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0. Introduction

In [1] Barbosa and do Carmo adopted a new approach to the stability of minimal surfaces. In particular they discussed the stability of simply connected compact domains with boundary on minimal surfaces in space forms (cf. [2], [8] and [10]). Their method was applied also to the stability of surfaces with constant mean curvature in 3-dimensional space forms (see [5], [9] and [14]). It is natural to ask if these arguments can be generalized for general ambient spaces. In the case of minimal surfaces, a positive answer to this question was given in our previous paper [11]. In this paper we give a positive answer in the case of surfaces with constant mean curvature. Namely we prove:

Theorem. *Let $f: M \rightarrow N$ be an immersion of a 2-dimensional orientable manifold M into a 3-dimensional orientable Riemannian manifold N . Assume that the mean curvature of the immersion f is constant. Let D be a simply connected compact domain on M with piecewise smooth boundary. We denote by A and dM the second fundamental form and the area element of M induced by f , respectively. Suppose that the sectional curvature of N and the norm of the covariant derivative of the curvature tensor of N are bounded. Then there is a positive constant c_1 depending only on N such that if $\int_D (1 + |A|^2/2) dM < c_1$, then D is stable.*

REMARK. (i) The method in [11] is not available to the stability of surfaces with constant mean curvature.

(ii) If we omit the hypothesis that D is simply connected, it is not known whether the theorem is true or not (cf. [11, Theorem 0.3]).

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1. Preliminaries

Let $f: M \rightarrow N$ be an immersion of an m -dimensional orientable manifold

M into an $(m+1)$ -dimensional orientable Riemannian manifold N . Assume that the mean curvature of the immersion f is constant. Let D be a compact domain on M with piecewise smooth boundary ∂D . We choose a unit normal vector field ν to $f(M)$. We denote by $F(D)$ the space of smooth functions ψ on D such that $\psi=0$ on ∂D and $\int_D \psi dM=0$, where dM is the volume element of M with respect to the metric ds^2 induced by f .

For $\psi \in F(D)$, we consider a smooth map $F: [0, 1] \times D \rightarrow N$ such that $F_t: D \rightarrow N$ for $t \in [0, 1]$ defined by $F_t(p) = F(t, p)$ for $p \in D$ is an immersion, $F_0 = f$, $F_t|_{\partial D} = f|_{\partial D}$, $(d/dt)F_t|_{t=0} = \psi\nu$ and $(d/dt) \int_{[0, t] \times D} F^*dN = 0$, where dN is the volume element of N . The second variation $I(\psi)$ of the volume functional of D for the variational vector field $\psi\nu$ is defined by $I(\psi) = (d^2/dt^2) \text{vol}(D, t)|_{t=0}$, where $\text{vol}(D, t)$ is the volume of D with respect to the metric induced by F_t . Let ∇ , A and Ric denote the Riemannian connection of (M, ds^2) , the second fundamental form of f and the Ricci tensor of N , respectively. Then by the second variation formula of the volume for hypersurfaces with constant mean curvature (see [3] and [4]),

$$(1) \quad I(\psi) = \int_D \{ |\nabla\psi|^2 - (\text{Ric}(\nu, \nu) + |A|^2)\psi^2 \} dM.$$

The domain D is stable if $I(\psi) > 0$ for any $\psi \in F(D)$ which is not identically zero, and D is unstable if $I(\psi) < 0$ for some $\psi \in F(D)$.

2. A curvature estimate

The purpose of this section is to prove the following:

Lemma. *Let $f: M \rightarrow N$ be an immersion of a 2-dimensional orientable manifold M into a 3-dimensional orientable Riemannian manifold N . Assume that the mean curvature of the immersion f is constant. We denote by A and ds^2 the second fundamental form and the metric of M induced by f , respectively. Suppose that the sectional curvature of N and the norm of the covariant derivative of the curvature tensor of N are bounded. Then the Gaussian curvature \tilde{K} of M with respect to the metric $d\tilde{s}^2 = (1 + |A|^2/2)ds^2$ satisfies $\tilde{K} \leq c_2$ for a positive constant c_2 depending only on N .*

Proof. By the hypothesis we may assume that the sectional curvature of N is bounded from above by a and below by b , and the norm of the covariant derivative of the curvature tensor of N is not greater than ξ .

Let K , ∇ and Δ denote the Gaussian curvature, the Riemannian connection and the Laplacian of (M, ds^2) , respectively. Then we have

$$\begin{aligned}
 (2) \quad \tilde{K} &= \frac{K}{1+|A|^2/2} - \frac{1}{2(1+|A|^2/2)} \Delta \log\left(1 + \frac{1}{2}|A|^2\right) \\
 &= \frac{K}{1+|A|^2/2} - \frac{\langle A, \Delta A \rangle}{2(1+|A|^2/2)^2} \\
 &\quad + \frac{1}{2(1+|A|^2/2)^3} \left\{ -|\nabla A|^2 - \frac{1}{2}|A|^2|\nabla A|^2 + \left| \frac{1}{2} \nabla(|A|^2) \right|^2 \right\}.
 \end{aligned}$$

We shall make a pointwise argument at a point p on M . Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for the tangent space of N at $f(p)$ such that e_1, e_2 are tangent to $f(M)$. In what follows, we use the following convention on the ranges of indices: $1 \leq i, j, k, \dots \leq 2, 1 \leq B, C, D, \dots \leq 3$. We denote by $h_{ij}, h_{ijk}, R_{CDE}^B$ and R_{CDEF}^B the components of $A, \nabla A$, the curvature tensor of N and the covariant derivative of the curvature tensor of N with respect to the basis, respectively. We may choose $\{e_1, e_2\}$ so that $h_{11} = \lambda, h_{12} = h_{21} = 0$ and $h_{22} = \mu$ for some λ and μ . By the Gauss equation we have

$$(3) \quad K = R_{212}^1 + \lambda\mu \leq a + \frac{1}{2}(\lambda^2 + \mu^2) = a + \frac{1}{2}|A|^2.$$

Using the equations (2.7), (2.9), (2.17) and (2.21) in [7], we find that

$$\begin{aligned}
 (4) \quad -\langle A, \Delta A \rangle &= (R_{2121}^3 + R_{1122}^3)\lambda + (R_{1212}^3 + R_{2211}^3)\mu + (R_{323}^2 - 2R_{212}^1)\lambda^2 \\
 &\quad + (R_{313}^1 - 2R_{212}^1)\mu^2 + 2\left(2R_{212}^1 - \frac{1}{2}R_{313}^1 - \frac{1}{2}R_{323}^2\right)\lambda\mu - \lambda\mu(\lambda - \mu)^2 \\
 &\leq \sqrt{2} \{(R_{2121}^3)^2 + (R_{1122}^3)^2 + (R_{1212}^3)^2 + (R_{2211}^3)^2\}^{1/2}(\lambda^2 + \mu^2)^{1/2} \\
 &\quad + (a - 2b)(\lambda^2 + \mu^2) + \max\{|2a - b|, |a - 2b|\}(\lambda^2 + \mu^2) + (\lambda^2 + \mu^2)^2 \\
 &\leq \sqrt{2}\xi|A| + c_3|A|^2 + |A|^4,
 \end{aligned}$$

where

$$c_3 = a - 2b + \max\{|2a - b|, |a - 2b|\}.$$

Noting that the components h_{ijk} satisfy $h_{11i} + h_{22i} = 0, h_{12i} = h_{21i}$ and $h_{ijk} - h_{ikj} = R_{ijk}^3$ by the Codazzi equation, we have

$$\begin{aligned}
 (5) \quad &-\frac{1}{2}|A|^2|\nabla A|^2 + \left| \frac{1}{2} \nabla(|A|^2) \right|^2 \\
 &= -\frac{1}{2} \sum_{i,j} (h_{ij})^2 \sum_{i,j,k} (h_{ijk})^2 + \sum_k \left(\sum_{i,j} h_{ij} h_{ijk} \right)^2 \\
 &= -(\lambda^2 + \mu^2) \{(h_{111})^2 + (h_{121})^2 + (h_{112})^2 + (h_{122})^2\} \\
 &\quad + (\lambda h_{111} + \mu h_{221})^2 + (\lambda h_{112} + \mu h_{222})^2 \\
 &= -(\lambda^2 + \mu^2) \{(h_{111})^2 + (h_{112} + R_{112}^3)^2 + (h_{112})^2 + (-h_{111} + R_{221}^3)^2\} \\
 &\quad + (\lambda - \mu)^2 \{(h_{111})^2 + (h_{112})^2\}
 \end{aligned}$$

$$\begin{aligned}
&= -(\lambda + \mu)^2 \{(h_{111})^2 + (h_{112})^2\} - (\lambda^2 + \mu^2) \{(R_{112}^3)^2 + (R_{221}^3)^2\} \\
&\quad + 2(\lambda^2 + \mu^2)(h_{111}R_{221}^3 - h_{112}R_{112}^3) \\
&\leq 2|A|^2(h_{111}R_{221}^3 - h_{112}R_{112}^3) \\
&\leq 2\{(h_{111})^2 + (h_{112})^2\} + \frac{1}{2}\{(R_{221}^3)^2 + (R_{112}^3)^2\}|A|^4 \\
&\leq |\nabla A|^2 + \frac{1}{4}(a-b)^2|A|^4,
\end{aligned}$$

where for the last inequality we use the following (see [6]):

$$|R_{221}^3| \leq \frac{1}{2}(a-b), \quad |R_{112}^3| \leq \frac{1}{2}(a-b).$$

By (2), (3), (4) and (5) we get

$$\begin{aligned}
\mathcal{R} &\leq \frac{a + |A|^2/2}{1 + |A|^2/2} + \frac{\sqrt{2}\xi|A| + c_3|A|^2 + |A|^4}{2(1 + |A|^2/2)^2} + \frac{(a-b)^2|A|^4}{8(1 + |A|^2/2)^3} \\
&\leq \sup_{t \geq 0} \left\{ \frac{a + t^2/2}{1 + t^2/2} + \frac{\sqrt{2}\xi t + c_3 t^2 + t^4}{2(1 + t^2/2)^2} + \frac{(a-b)^2 t^4}{8(1 + t^2/2)^3} \right\} = c_2.
\end{aligned}$$

It is easy to see that $0 < c_2 < \infty$ and c_2 depends only on N . Thus the proof is complete.

REMARK. For example, let us consider the case where N is the 3-dimensional unit sphere. Then $a=b=1$, $c_3=0$ and $\xi=0$. Hence we have

$$c_2 = \sup_{t \geq 0} \left\{ 1 + \frac{t^4}{2(1 + t^2/2)^2} \right\} = 3,$$

which is worse than Proposition 1 of [5].

3. Proof of Theorem

Proof of Theorem. Let $F(D)$, $I(\cdot)$, ∇ and ds^2 be as in Section 1. By the hypothesis we may assume that the sectional curvature of N is bounded from above by a . Then by (1) we have

$$(6) \quad I(\psi) \geq \int_D \{ |\nabla \psi|^2 - (2a + |A|^2)\psi^2 \} dM$$

for $\psi \in F(D)$. Set $d\tilde{s}^2 = (1 + |A|^2/2)ds^2$. Let $\tilde{\nabla}$ and $d\tilde{M}$ denote the Riemannian connection and the area element of $(M, d\tilde{s}^2)$, respectively. Then we see that

$$(7) \quad |\tilde{\nabla} \psi|^2 = \frac{|\nabla \psi|^2}{1 + |A|^2/2}$$

and

$$(8) \quad d\tilde{M} = \left(1 + \frac{1}{2}|A|^2\right)dM.$$

By (6), (7) and (8) we have

$$(9) \quad \begin{aligned} I(\psi) &\geq \int_D \left\{ |\nabla\psi|^2 - \frac{2a + |A|^2}{1 + |A|^2/2} \psi^2 \right\} d\tilde{M} \\ &\geq \int_D (|\nabla\psi|^2 - 2\eta\psi^2) d\tilde{M}, \end{aligned}$$

where $\eta = \max\{a, 1\}$. We denote by $\lambda_1(D)$ the first eigenvalue on D of the Laplacian of $(M, d\tilde{s}^2)$ with Dirichlet boundary condition. The inequality (9) says that D is stable if $\lambda_1(D) > 2\eta$.

Let c_2 be as in Section 2. Set

$$c_1 = \frac{4\pi}{c_2 + \eta},$$

which is positive and depends only on N . We note that $c_2 \geq \eta$ by the definitions of c_2 and η . Using Proposition 3.3 and 3.10 of [1] with Lemma, we can see that $\lambda_1(D) > 2\eta$ if $\tilde{a}(D) < c_1$, where $\tilde{a}(D)$ denotes the area of D with respect to the metric $d\tilde{s}^2$. By (8) we have $\tilde{a}(D) = \int_D (1 + |A|^2/2)dM$. Thus the proof is complete.

REMARK. In fact we prove that D is strongly stable in the sense of [14] if $\int_D (1 + |A|^2/2)dM < c_1$.

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