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MAX-FLOW MIN-CUT THEOREM IN AN ANISOTROPIC NETWORK

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1. Introduction

Max-flow problems and min-cut problems have been investigated in domains in Euclidean spaces as well as on graphs. In this paper, we shall formulate general optimization problems which contain the problems such as in [9], [15], [16] and establish max-flow min-cut theorems related to these problems.

To clarify our idea more precisely, let us begin with recalling a standard max flow min-cut theorem on networks due to Ford and Fulkerson [6]. Let α and β be two distinguished nodes of a finite connected graph G, and let c be a capacity function, that is, a nonnegative function on the set $Y=E(G)$ of arcs (= edges) in G. Let $X=V(G)$ be the set of all nodes (=vertices) in G. For each node x, denote by $Y_+(x)$ (resp. $Y_-(x)$) the set of all arcs which come from (resp. go to) x. A flow σ from α to β is a real-valued function on Y such that its net flow out of x, which is defined by

$$\sum_{y\in Y_+(x)} \sigma(y)-\sum_{y\in Y(x)} \sigma(y)\,,$$

is required to vanish for each x in X except α and β. The value of a flow σ from α to β is defined by its net flow out of α. A max-flow problem is to find the maximal flow value of σ subject to the constraint that σ is a flow from α to β and $|\sigma|\leq c$ on Y. On the other hand, a subset Q of Y is called a cut separating α and β if there exists a partition (X', X'') of X such that $\alpha\in X'$, $\beta\in X''$ and Q is the set of all arcs joining X' and X''. For a cut Q, we call the quantity $\sum_{y\in Q} c(y)$ the cut capacity. The min-cut problem related to the above max-flow problem is to find the minimal cut capacity of all cuts separating α and β. The celebrated max-flow min-cut theorem in [6] assures that the values of those problems are equal.

Now we state continuous versions of the above problems. In stead of G and $\{\alpha, \beta\}$, we take a bounded domain Ω in the n-dimensional Euclidean space R^n and mutually disjoint nonempty two subsets $\{A, B\}$ of the boundary $\partial\Omega$ of Ω. A flow σ from A to B is a vector field which satisfies the following conditions:

$$\operatorname{div} \sigma = 0 \quad \text{on } \Omega, \quad \sigma \cdot \nu = 0 \quad \text{on} \quad \partial\Omega - (A \cup B),$$

where ν is the unit outer normal to Ω. Note that this definition has no ambiguity in case $\partial\Omega$, A, B and σ are sufficiently smooth. The net flow into a node x in $\Omega \cup \partial\Omega$ is equal to div $\sigma(x)$ if $x \in \Omega$ and to $-\sigma \cdot \nu(x)$ if $x \in \partial\Omega$.

Given a nonnegative function c on Ω, we formulate a max-flow problem (MFI) as maximizing the flow value $\int_A \sigma \cdot \nu \, ds$ subject to the constraint that σ is a flow from A to B and $|\sigma| \leq c$ on Ω, where ds denotes the surface element.

A cut S separating A and B is a subset of Ω such that $A \subset \partial S$ and $B \cap \partial S = \phi$. Furthermore we require that ∂S is sufficiently smooth so that we can define $\int_{\partial S \cap \Omega} c ds$. Then a min-cut problem (MCI) corresponding to (MFI) is to minimize the cut capacity $\int_{\partial S \cap \Omega} c ds$ subject to the constraint that S is a cut separating A and B.

A continuous version of the max-flow min-cut theorem is to assure the equality of the values of two dual problems such as problems (MFI) and (MCI). Notice that (MFI), (MCI) are special cases of problems studied in Iri [9], Taguchi and Iri [16]. We mean by "I" in (MFI) and (MCI) that these problems are of Iri's type. It should be noted that no mathematical conditions which assure the max-flow min-cut theorem are given in [9] and [16].

In connection with problems raised in capillarity and plasticity, Strang [15] studied continuous versions of the max-flow problem and the min-cut problem of another type. We denote them by (MFS), (MCS) and call them Strang's version. He used some mathematical tools such as functions of bounded variation and coarea formula. Furthermore he noticed that (MFI) and (MCI) could be treated by his method.

Our aim of this paper is to give rigorous formulations of max-flow problems and min-cut problems on a bounded domain Ω in R^n with Lipschitz boundary which contain problems in [9], [15], [16] and to prove max-flow min-cut theorems for them. For example, we have to give a rigorous notion of a flow σ, the outer unit normal ν, the inner product $\sigma \cdot \nu$ and the cut capacity of a cut. To do so, we basically follow Strang's idea in [15]. The space of essentially bounded vector fields with divergence in $L^n(\Omega)$ and the space of functions of bounded variation will play important roles in our study. In fact, a static flow is represented by an element in the former space and a static cut is represented by a characteristic function which belongs to the latter space. One of our mathematical tools is a generalized Greens' formula, due to Kohn and Temam [12], for functions of bounded variation and essentially bounded vector fields with divergence in $L^n(\Omega)$. Furthermore as in [9] we treat the general case when our network is anisotropic. Namely, the capacity constraint is described as follows:

$$\sigma(x)\in\Gamma(x) \qquad \text{for all} \quad x\in\Omega\,,$$

where $\Gamma(x)$ is a compact convex set in $\boldsymbol{R}^n$ containing 0 for each $x\in\Omega$. To deal with anisotropic domain, we need an extended version of the coarea formula. The mathematical tools will be explained in §2.

We shall introduce in §3 optimization problems (MF) and (MF_0) which may be regarded as generalized max-flow problems on linear spaces. In relation to these problems, we consider formal dual problems (MF^*), (MF_0^*) and generalized min-cut problems (MC), (MC_0) on linear spaces. Duality relations will be studied for these problems by means of the minimax theorem in [5].

Our principal results will be given in §4 and §5. In §4 we consider a max-flow problem ($\mathrm{M}\Phi_1$) and several problems related to it. The problem ($\mathrm{M}\Phi_1$) is of Strang's type. However, a constraint on $\sigma\cdot\nu$ in ($\mathrm{M}\Phi_1$) is weaker than that in (MFS). In (MFS), $\sigma\cdot\nu$ must be proportional to a prescribed distribution on $\partial\Omega$. In our max-flow problem, $\sigma\cdot\nu$ can be free on a part of $\partial\Omega$ while the difference between $\sigma\cdot\nu$ and the product of a constant and the prescribed distribution is constrained by two given functions on another part of $\partial\Omega$ for each feasible flow σ.

In §5 we deal with another max-flow problem ($\mathrm{M}\Psi_2$) which is an extension of (MFI). Capacity constraints and feasible flows depend on discrete time in ($\mathrm{M}\Psi_2$), so we call it a dynamic version of the max-flow problem. A max-flow min-cut theorem on a network in a dynamic version can be found in Anderson, Nash and Philpott [2]. The duality theorem for (MF_0) is utilized to prove max-flow min-cut theorems in §4 and §5.

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2. Functions of bounded variation and a space of vector fields

In the present section, we introduce functions of bounded variation and a class of vector fields on Ω, which are used to define flows and cuts in the next sections. Throughout this paper we assume that Ω is a bounded domain with Lipschitz boundary $\partial\Omega$ in R^n. For such domains, we refer to Maz'ja [11; Definition 1.1.9/1]. We set

$$H_{n-1}(E)=\sup_{\delta\downarrow 0}\inf\{v_{n-1}\textstyle\sum_j r_j^{n-1};\ \cup_j B(x_j,r_j)\supset E,\ r_j<\delta\}$$

for any subset E of R^n, where v_{n-1} is the volume of the unit ball in R^{n-1} and $B(x_j,r_j)$ is the open ball in R^n with center x_j and radius r_j. It is called the Hausdorff measure of $(n-1)$-dimension. By m_n we denote the Lebesgue measure on R^n. Note that the outer unit normal ν to Ω is defined H_{n-1}-a.e. on $\partial\Omega$

and ν is H_{n-1}-measurable.

In this paper, all derivatives of locally integrable functions on Ω are understood in the sense of distributions unless otherwise stated. Then the space of functions of bounded variation is defined by

$$BV(\Omega) = \{u \in L^1(\Omega);\ \partial u/\partial x_j \text{ is a Radon measure of bounded variation for each } j = 1, \cdots, n\} .$$

Let $C^\infty(\Omega)$ be the space of infinitely differentiable functions in Ω. Denoting the closure of Ω by $\bar{\Omega}$, we set

$$C^\infty(\bar{\Omega}) = \{f|_{\bar{\Omega}};\ f \in C^\infty(R^n)\} ,$$

where $f|_{\bar{\Omega}}$ is the restriction of f to $\bar{\Omega}$, and set

$$C_0^\infty(\Omega) = \{f \in C^\infty(\Omega);\ \text{supp}\, f \text{ is a compact subset of } \Omega\} ,$$

where supp f is the closure of $\{x \in \Omega;\ f(x) \neq 0\}$. By $C_0^\infty(\Omega; R^n)$ we denote $\{\sigma = (\sigma_1, \cdots, \sigma_n);\ \sigma_i \in C_0^\infty(\Omega)$ for all $i = 1, \cdots, n\}$. We define $C^\infty(\bar{\Omega}; R^n)$ and $C^\infty(\Omega; R^n)$ in a similar way. Note that, for $u \in L^1(\Omega)$, $u \in BV(\Omega)$ if and only if

$$||\nabla u||_\Omega = \sup \{\int_\Omega u \operatorname{div} \sigma\, dx;\ \sigma \in C_0^\infty(\Omega; R^n),\ |\sigma| \leq 1 \text{ on } \Omega\}$$

is finite, where $\operatorname{div} \sigma = \sum_{i=1}^n \partial\sigma_i/\partial x_i$ and $|\sigma| = (\sum_{i=1} \sigma_i^2)^{1/2}$. We set

$$||\nabla u||_U = \sup \{\int_U u \operatorname{div} \sigma\, dx;\ \sigma \in C_0^\infty(U; R^n),\ |\sigma| \leq 1 \text{ on } U\}$$

for $u \in BV(\Omega)$ and any open subset U of Ω, and thus define a Radon measure on Ω. We shall denote it by $|\nabla u|$ and call it the measure of total variation of ∇u. Evidently $W^{1,1}(\Omega) = \{u \in L^1(\Omega);\ \partial u/\partial x_i \in L^1(\Omega)$ for each $i = 1, \cdots, n\}$ is a subspace of $BV(\Omega)$. We say that $\{u^j\}$ *converges to* u *in* $BV(\Omega)$ if $u^j \to u$ in $L^1(\Omega)$ and $||\nabla u^j||_\Omega \to ||\nabla u||_\Omega$.

Let $L^1(\partial\Omega)$ be the space of H_{n-1}-integrable functions on $\partial\Omega$. By Giusti [8; Theorems 1.17, 2.10, 2.11 and Remark 2.12] we have

Theorem 2.1. (1) *There exists a linear mapping γ from $BV(\Omega)$ to $L^1(\partial\Omega)$ such that*

$$\lim_{\rho \downarrow 0} \frac{1}{m_n(\Omega \cap B(x, \rho))} \int_{\Omega \cap B(x,\rho)} |u(y) - \gamma u(x)|\, dy = 0$$

for H_{n-1}-a.e. $x \in \partial\Omega$, and that $\gamma u^j \to \gamma u$ in $L^1(\partial\Omega)$ if $\{u^j\}$ converges to u in $BV(\Omega)$.

(2) *For each $u \in BV(\Omega)$, there exists $\{u^j\} \subset BV(\Omega) \cap C^\infty(\Omega)$ such that $\gamma u^j = \gamma u$ H_{n-1}-a.e. on $\partial\Omega$ for all j and $\{u^j\}$ converges to u in $BV(\Omega)$.*

The function γu is called the trace of u on $\partial\Omega$. We note that

$$\gamma u(x) = \lim_{\rho \downarrow 0} \frac{1}{m_n(\Omega \cap B(x, \rho))} \int_{\Omega \cap B(x,\rho)} u(y)\, dy$$

for H_{n-1}-a.e. $x \in \partial\Omega$.

Since by Adams [1; Theorem 3.18] the family $\{f|_\Omega; f \in C^\infty(\bar{\Omega})\}$ is dense in $W^{1,1}(\Omega)$ with respect to the norm $||u||_{W^{1,1}(\Omega)} = ||u||_{L^1(\Omega)} + ||\nabla u||_\Omega$, for each $u \in BV(\Omega)$ there exists $\{u^j\} \subset \{f|_\Omega; f \in C^\infty(\bar{\Omega})\}$ such that $\{u^j\}$ converges to u in $BV(\Omega)$. It is known that $BV(\Omega) \subset L^{n/(n-1)}(\Omega)$, more precisely, there is a positive constant k such that the inequality $||u||_{L^{n/(n-1)}(\Omega)} \leq k(||u||_{L^1(\Omega)} + ||\nabla u||_\Omega)$ holds for all $u \in BV(\Omega)$. This is a direct conclusion from (2) of Theorem 2.1 and the Sobolev imbedding theorem for $W^{1,1}(\Omega)$. For the proof of Sobolev's theorem, we refer to [1; Theorem 5.4].

Furthermore γ is a surjection from $W^{1,1}(\Omega)$ to $L^1(\partial\Omega)$. (See Gagliardo [7; Theorem 1.2].) By the equality in (1) of Theorem 2.1, $\gamma u = u|_{\partial\Omega}$ for any $u \in BV(\Omega) \cap C(\bar{\Omega})$. In order to give another characterization of γu, which is due to [11; §6.5], we state the definition of the reduced boundary. Let S be a subset of R^n with $\chi_S \in BV(R^n)$. Then the reduced boundary $\partial^* S$ of S is the set of all $x \in \partial S$ such that there exists Federer's normal $\nu = \nu(x)$ to S. This normal ν is characterized by the relation

$$\lim_{\rho \downarrow 0} \rho^{-n}\, m_n(B(x, \rho) \cap A_+ \cap S) = \lim_{\rho \downarrow 0} \rho^{-n}\, m_n(B(x, \rho) \cap A_- - S) = 0\,,$$

where $A_+ = \{y \in R^n; \nu \cdot (y-x) \geq 0\}$ and $A_- = \{y \in R^n; \nu \cdot (y-x) \leq 0\}$. It is proved that $\partial^* S$ is a measurable set and Federer's normal is a measurable mapping from $\partial^* S$ to R^n with respect to both $|\nabla \chi_S|$ and H_{n-1} and that $|\nabla \chi_S|(R^n - \partial^* S) = 0$. Furthermore it is known that E is H_{n-1}-measurable,

$$|\nabla \chi_S|(E) = H_{n-1}(E) \quad \text{and} \quad \nabla \chi_S(E) = -\int_E \nu(x)\, d|\nabla \chi_S|(x)$$

for each $|\nabla \chi_S|$-measurable set $E \subset \partial^* S$. The proof of these facts is given in [11; §6.2].

Now let $u \in BV(\Omega)$ and set $N_r = \{y \in \Omega; u(y) \geq r\}$ for $r \in R$. Coarea formula [8; Theorem 1.23] yields that $\chi_{N_r} \in BV(\Omega)$ for a.e. $r \in R$. For $x \in \partial\Omega$, we set

$$u^*(x) = \sup \{r \in R; \chi_{N_r} \in BV(\Omega) \quad \text{and} \quad x \in \partial^* N_r\}$$

if there is $r \in R$ such that $\chi_{N_r} \in BV(\Omega)$ and $x \in \partial^* N_r$, and set $u^*(x) = -\infty$ otherwise. Since $\partial^* N_r$ is H_{n-1}-measurable, u^* is also H_{n-1}-measurable. Furthermore according to [11; Theorem 6.5.4], $u^* \in L^1(\partial\Omega)$. (As noted after Corollary 6.5.5/3 in [11], conditions in [11; Theorem 6.5.4] are satisfied and the theorem can be applied to our case.) Hence from [11; Theorem 6.6.2] and the equality stated in (1) of Theorem 2.1, it follows that $u^* = \gamma u$ H_{n-1}-a.e. on $\partial\Omega$. If $S \subset \Omega$ and $\chi_S \in BV(\Omega)$, then $\chi_S \in BV(R^n)$ by [11; Lemma 6.5.1/1] and $\chi_S^* = \chi_{\partial^* S \cap \partial\Omega}$. Therefore $\gamma \chi_S = \chi_{\partial^* S \cap \partial\Omega}$ H_{n-1}-a.e. on $\partial\Omega$.

Next we define a space of vector fields. We set

$$H(\Omega)=\{\sigma=(\sigma_1,\cdots,\sigma_n);\ \sigma_i\in L^\infty(\Omega)\quad\text{for all}\quad i=1,\cdots,n\quad\text{and}\quad \operatorname{div}\sigma\in L^n(\Omega)\}\ .$$

Every $\sigma\in H(\Omega)$ can be approximated by functions in $C^\infty(\bar{\Omega};R^n)$ in the following sense. Let $\sigma\in H(\Omega)$, and $\{\sigma^j\}$ be a sequence in $H(\Omega)$. If $\sigma^j\to\sigma$ a.e. on Ω, $\operatorname{div}\sigma^j\to\operatorname{div}\sigma$ in $L^n(\Omega)$ and $\{\sigma^j\}$ is uniformly bounded, then $\{\sigma^j\}$ is said to *approximate* σ *in* $H(\Omega)$ or *tend to* σ *as* $j\to\infty$ *in the sense of* $H(\Omega)$. The following proposition is proved in a way similar to Kohn and Temam [10; Lemma 2.3].

Proposition 2.2. *Given $\sigma\in H(\Omega)$ there exists $\{\sigma^j\}$ in $C^\infty(\bar{\Omega};R^n)$ which approximates σ in $H(\Omega)$.*

Proof. We give only a sketch. There exist a positive number r_0, a finite number of open sets $U_1,\cdots,U_N$ with $\bigcup_{k=1}^N U_k\supset\partial\Omega$ and open cones $C_1,\cdots,C_N$ with vertices at the origin of R^n such that $U_k\cap\partial\Omega\neq\phi$ and $(x+C_k\cap B(0,r_0))\cap\bar{\Omega}=\phi$ for each k and $x\in\partial\Omega\cap U_k$. Furthermore we may assume that $U_k\cap(\Omega-y)\subset\Omega$ for all $y\in C_k\cap B(0,r_0)$ and all k. Let U_0 be an open subset of Ω such that $U_0\supset\Omega-\bigcup_{k=1}^N U_k$ and the closure of U_0 is contained in Ω. Then $\bigcup_{k=0}^N U_k\supset\bar{\Omega}$. Let $\{\psi_k;k=0,\cdots,N\}$ be a partition of unity subordinate to $\{U_k;k=0,\cdots,N\}$, that is, $\sum_{k=0}^N\psi_k=1$ on $\bar{\Omega}$, $\psi_k\geq 0$, $\operatorname{supp}\psi_k\subset U_k$ and $\psi_k\in C_0^\infty(R^n)$ for all k. We take mollifiers $\eta_0\in C_0^\infty(B(0,1))$ and $\eta_k\in C_0^\infty(C_k\cap B(0,1))$ for $k=1,\cdots,N$ such that $\eta_k\geq 0$ and the integral of η_k over R^n is equal to 1 for each $k=0,\cdots,N$. For $\eta_{k,r}(x)=\eta_k(x/r)\,r^{-n}$, we set

$$(\psi_k\sigma)*\eta_{k,r}(x)=\int_{\Omega\cap U_k}\psi_k(y)\,\sigma(y)\,\eta_{k,r}(x-y)\,dy$$

and $\sigma^j=\sum_{k=0}^N(\psi_k\sigma)*\eta_{k,r}$ for $r=1/j$. One can prove along the same lines as in the proof of [10; Lemma 2.3] that $\operatorname{div}((\psi_k\sigma)*\eta_{k,r})=\operatorname{div}(\psi_k\sigma)*\eta_{k,r}$ a.e. on Ω for all sufficiently large j. Hence $\{\sigma^j\}$ satisfies the required conditions.

The following theorem is due to Kohn and Temam (see [10; Proposition 1.1]).

Theorem 2.3. *Let $\sigma\in H(\Omega)$ and $u\in BV(\Omega)$.*

(1) *The distribution $(\sigma\nabla u)$ defined by*

$$\langle(\sigma\nabla u),\psi\rangle=-\int_\Omega u\psi\operatorname{div}\sigma\,dx-\int_\Omega u\sigma\cdot\nabla\psi\,dx$$

for $\psi\in C_0^\infty(\Omega)$ is a Radon measure of bounded variation and thus $(\sigma\nabla u)(\Omega)$ is defined and finite.

(2) *There exists a function $g\in L^\infty(\partial\Omega)$ such that*

$$\int_{\partial\Omega}g\,\gamma v dH_{n-1}=\int_\Omega\sigma\cdot\nabla v dx+\int_\Omega v\operatorname{div}\sigma\,dx$$

for all $v\in W^{1,1}(\Omega)$. *We shall write* $\sigma\cdot\nu$ *for* g.

(3) (*Green's formula*) *For* $(\sigma\nabla u)$ *and* $\sigma\cdot\nu$ *as in* (1) *and* (2), *the following equality holds*:

$$\int_{\partial\Omega}\sigma\cdot\nu\gamma u H_{n-1}=(\sigma\nabla u)(\Omega)+\int_{\Omega}u\operatorname{div}\sigma\,dx\,.$$

By Theorem 2.3, $\sigma^j\cdot\nu\to\sigma\cdot\nu$ in $L^\infty(\partial\Omega)$ with respect to the weak* topology and $(\sigma^j\nabla u)(\Omega)\to(\sigma\nabla u)(\Omega)$ for all $u\in BV(\Omega)$ if $\{\sigma^j\}$ approximates σ in $H(\Omega)$.

Next we are concerned with equalities of the coarea formula type. Let Γ be a set-valued mapping from Ω to R^n, that is, $\Gamma(x)$ is a subset of R^n. Throughout this paper we assume

(2.1) $\Gamma(x)$ is a compact convex set containing 0 for each $x\in\Omega$.

Furthermore in this section we assume that the following two conditions are fulfilled unless otherwise stated:

(2.2) Let Ω_0 be a compact subset of Ω and $\varepsilon>0$. Then there is $\delta>0$ such that $\Gamma(x)\subset\Gamma(y)+B(0,\varepsilon)$ whenever $x,y\in\Omega_0$ and $|x-y|<\delta$.

(2.3) $\bigcup_{x\in\Omega}\Gamma(x)$ is bounded.

We define a function β_Γ on $R^n\times\Omega$ by

$$\beta_\Gamma(v,x)=\sup_{w\in\Gamma(x)}v\cdot w$$

for $v\in R^n$ and $x\in\Omega$ and define a functional ψ_Γ on $BV(\Omega)$ by

$$\psi_\Gamma(u)=\int_\Omega\beta_\Gamma(\nabla u/|\nabla u|,\cdot)\,d|\nabla u|$$

for $u\in BV(\Omega)$, where $\nabla u/|\nabla u|$ is the Radon-Nikodym derivative of ∇u with respect to $|\nabla u|$. If $u=\chi_S\in BV(\Omega)$ with $S\subset\Omega$, then $\nabla u/|\nabla u|=-\nu$ H_{n-1}-a.e. on $\Omega\cap\partial^*S$ by [11; Theorem 6.2.1] so that

$$\psi_\Gamma(u)=\int_{\Omega\cap\partial^*S}\beta_\Gamma(-\nu(x),x)\,dH_{n-1}(x)\,.$$

where ν is Federer's normal to S.

Note that β_Γ is a continuous function on $R^n\times\Omega$ by (2.1) and (2.2). Thus $\beta_\Gamma(\nabla u/|\nabla u|,\cdot)$ is $|\nabla u|$-integrable. Furthermore we set

$$K_\Gamma=\{\sigma\in L^\infty(\Omega;R^n);\ \sigma(x)\in\Gamma(x)\quad\text{for a.e. }x\in\Omega\}\,.$$

We prove a variant of coarea formula.

Proposition 2.4. *Let* $u\in BV(\Omega)$ *and set* $N_r=\{x\in\Omega;\ u(x)\geq r\}$. *Then* $\psi_\Gamma(u)=\int_{-\infty}^{\infty}\psi_\Gamma(\chi_{N_r})\,dr$.

Before proving Proposition 2.4, we prepare several lemmas. We set $K_\Gamma^0=\{\sigma=(\sigma_1,\cdots,\sigma_n);\ \sigma_i$ is a Borel measurable function for each i and $\sigma(x)\in\Gamma(x)$ for all $x\in\Omega\}$. Evidently $K_\Gamma^0\subset K_\Gamma$.

Lemma 2.5. (1) *For each positive Radon measure μ on Ω and each μ-measurable mapping v from Ω to R^n, there is $\sigma\in K_\Gamma^0$ such that $\beta_\Gamma(v(x), x)=v(x)\cdot\sigma(x)$ for μ-a.e. $x\in\Omega$.*

(2) *Assume that*

(2.4) *there is $\rho_0>0$ such that $\Gamma(x)\supset B(0, \rho_0)$ for each $x\in\Omega$,*
let σ be a vector field in K_Γ^0 and let μ be a positive Radon measure on Ω. Then there is $\{\sigma^j\}\subset K_\Gamma^0\cap C_0^\infty(\Omega; R^n)$ such that $\sigma^j\to\sigma$ μ-a.e.

(3) *Let σ be a vector field in $K_\Gamma\cap H(\Omega)$ and Ω_0 be a compact subset of Ω. Then there is $\{\sigma^j\}\subset C^\infty(\bar{\Omega}; R^n)$ such that $\sigma^j\to\sigma$ in the sense of $H(\Omega)$, $\|\sigma^j\|_{L^\infty(\Omega; R^n)}\leq\|\sigma\|_{L^\infty(\Omega; R^n)}+1$ and $\sigma^j(x)\in\Gamma(x)+B(0, 1/j)$ for all $x\in\Omega_0$.*

Proof. (1) Let μ be a Radon measure on Ω, $B(R^n)$, $B(\Omega)$ be the classes of Borel subsets of R^n, Ω respectively, $\hat{B}$ be the completion of $B(\Omega)$ with respect to μ and $\hat{B}\times B(R^n)$ be the σ-algebra generated by $\{E\times E'; E\in\hat{B}, E'\in B(R^n)\}$. Since β_Γ is continuous on $R^n\times\Omega$, for any Radon measure μ and any μ-measurable mapping v from Ω to R^n we see that $\{(x, w)\in\Omega\times R^n; \beta_\Gamma(v(x), x)=v(x)\cdot w, w\in\Gamma(x)\}\in\hat{B}\times B(R^n)$. Thus from a measurable selection theorem, it follows that there is a μ-measurable vector field σ^0 such that $\sigma^0(x)\in\Gamma(x)$ and $\beta_\Gamma(v(x), x)=v(x)\cdot\sigma^0(x)$ for all $x\in\Omega$. By considering $\sigma\in K_\Gamma^0$ such that $\sigma=\sigma^0$ μ-a.e. on Ω, we complete the proof of (1). (As for the measurable selection theorem, we refer to Castaing and Valadier [3; Theorem 3.22] or Rockafellar [13; Theorem 2 and Corollary 1.1].)

(2) Let $\sigma\in K_\Gamma^0$. We may assume that $\mu(\Omega)$ is finite. It suffices to show that there is $\{\sigma^j\}\subset K_\Gamma^0\cap C_0^\infty(\Omega; R^n)$ such that

$$\int_\Omega |\sigma^j-\sigma|\,d\mu\to 0$$

as $j\to\infty$. In view of (2.4), considering $t\sigma$ with $0<t<1$, we may assume that $\sigma(x)+B(0, \varepsilon)\subset\Gamma(x)$ for all $x\in\Omega$, where ε is a positive number. Then σ can be approximated by simple vector fields in K_Γ^0. Thus we may assume that σ is simple, that is, $\sigma=\sum_{k=1}^N w_k\chi_{U_k}$, where $w_k\in R^n$ and $\{U_k\}$ is a class of disjoint Borel sets such that $\Omega=\cup_{k=1}^N U_k$. Furthermore by approximating U_k by compact sets, we may assume that U_k is compact for each k. Finally by approximating χ_{U_k} by continuous functions and considering a regularization, we obtain the desired $\{\sigma^j\}$.

(3) Let Ω_0 be a compact subset of Ω. We set

$$\sigma^r=\sum_{k=0}^N(\psi_k\sigma)*\eta_{k,r}$$

with ψ_k and η_k considered in the proof of Proposition 2.2. Let ε be a positive number. In virtue of (2.2), there exists $\delta>0$ such that $d(\Omega_0, \partial\Omega)>2\delta$, and that $\psi_k(x-y)\leq\psi_k(x)+\varepsilon$ for all k and $\Gamma(x-y)\subset\Gamma(x)+B(0, \varepsilon)$ for all $x\in\Omega_0$ and $y\in B(0, \delta)$. Then $\sigma(x-y)\in\Gamma(x-y)\subset\Gamma(x)+B(0, \varepsilon)$ for all $x\in\Omega_0$ and a.e.

$y \in B(0, \delta)$. Thus

$$(\psi_k \sigma) * \eta_{k,r}(x) = \int_{B(0,r)} \psi_k(x-y)\, \sigma(x-y)\, \eta_{k,r}(y)\, dy$$
$$\in (\psi_k(x)+\varepsilon)\,(\Gamma(x)+B(0,\varepsilon)),$$

so that $\sigma^r(x) \in (1+(N+1)\varepsilon)(\Gamma(x)+B(0,\varepsilon))$ for all $x \in \Omega_0$ and $0<r<\delta$. Since $\sigma^r \to \sigma$ as $r \to \infty$ in the sense of $H(\Omega)$, there exists a sequence $\{r(j)\}$ of positive numbers such that $\{\sigma^{r(j)}\}$ approximates σ in $H(\Omega)$ and $\sigma^{r(j)}(x) \in \Gamma(x)+B(0, 1/j)$ for all $x \in \Omega_0$. Furthermore we may assume that $\|\sigma^{r(j)}\|_{L^\infty(\Omega\,;\,R^n)} \leq \|\sigma\|_{L^\infty(\Omega\,;\,R^n)}+1$. This completes the proof.

Lemma 2.6. *Let $u \in BV(\Omega)$. Then for each $\sigma \in K_\Gamma \cap H(\Omega)$*

$$(\sigma \nabla u)(\Omega) \leq \psi_\Gamma(u).$$

Furthermore if Γ satisfies (2.4), *then*

$$\psi_\Gamma(u) = \sup_{\sigma \in K_\Gamma \cap H(\Omega)} (\sigma \nabla u)(\Omega) = \sup_{\sigma \in K_\Gamma^0 \cap C_0^\infty(\Omega\,;\,R^n)} \int_\Omega \sigma d\nabla u.$$

Proof. First assume that Γ satisfies (2.4). Let σ be an element in $K_\Gamma \cap H(\Omega)$. For $\varepsilon>0$, there exists a compact subset Ω_0 of Ω such that $|\nabla u|(\Omega-\Omega_0) <\varepsilon$. Let $\{\sigma^j\}$ be a sequence as stated in (3) of Lemma 2.5. Then

$$\begin{aligned}(\sigma \nabla u)(\Omega) &= \lim_{j\to\infty} (\sigma^j \nabla u)(\Omega)\\ &\leqq (\|\sigma\|_{L^\infty(\Omega\,;\,R^n)}+1)\,\varepsilon + \liminf_{j\to\infty} \int_{\Omega_0} \sigma^j d\nabla u\\ &\leqq (\|\sigma\|_{L^\infty(\Omega\,;\,R^n)}+1)\,\varepsilon + \int_{\Omega_0} \beta_\Gamma(\nabla u/|\nabla u|, \cdot)\, d|\nabla u|.\end{aligned}$$

Hence

$$\begin{aligned}\sup_{\sigma \in K_\Gamma^0 \cap C_0^\infty(\Omega\,;\,R^n)} \int_\Omega \sigma d\nabla u &\leq \sup_{\sigma \in K_\Gamma \cap H(\Omega)} (\sigma \nabla u)(\Omega)\\ &\leq \int_\Omega \beta_\Gamma(\nabla u/|\nabla u|, \cdot)\, d|\nabla u|.\end{aligned}$$

Since $\cup_{x\in\Omega} \Gamma(x)$ is bounded, the converse inequality follows from (1) and (2) of Lemma 2.5 and Lebesgue's convergence theorem.

Next we consider the general case and prove $(\sigma \nabla u)(\Omega) \leq \psi_\Gamma(u)$ for each $\sigma \in K_\Gamma \cap H(\Omega)$. Let $\sigma \in K_\Gamma \cap H(\Omega)$. We set $\Gamma_j(x) = \Gamma(x)+\overline{B(0, 1/j)}$ for each $x \in \Omega$ and each positive integer j. Then Γ_j is a set-valued mapping satisfying (2.1)–(2.4) and $\sigma \in K_\Gamma \subset K_{\Gamma_j}$. Thus from the first part of this proof it follows that $(\sigma \nabla u)(\Omega) \leq \psi_{\Gamma_j}(u)$ for each j. Since $\beta_{\Gamma_j}(v, x) \to \beta_\Gamma(v, x)$ as $j\to\infty$ for all $v \in R^n$ and $x \in \Omega$, letting $j \to \infty$ we obtain $(\sigma \nabla u)(\Omega) \leq \psi_\Gamma(u)$. This completes the proof.

Using Lemma 2.6, we obtain

Lemma 2.7. *Let u be a function in $BV(\Omega)$ and $\{u_j\}$ be a sequence in $BV(\Omega)$ such that $u_j \to u$ as $j \to \infty$ in $L^1(\Omega)$. Assume that Γ satisfies* (2.4) *or $\{\|\nabla u_j\|_\Omega\}$ is bounded. Then $\psi_\Gamma(u) \leq \liminf_{j\to\infty} \psi_\Gamma(u^j)$.*

Proof. Let u, u_j be functions in $BV(\Omega)$ such that $u_j \to u$ in $L^1(\Omega)$. First assume that Γ satisfies (2.4). Then in virtue of Lemma 2.6

$$\int_\Omega \sigma d\nabla u = -\int_\Omega u \operatorname{div} \sigma \, dx = \lim_{j\to\infty} -\int_\Omega u_j \operatorname{div} \sigma \, dx$$
$$= \lim_{j\to\infty} \int_\Omega \sigma d\nabla u_j \leq \liminf_{j\to\infty} \psi_\Gamma(u_j)$$

for each $\sigma \in K_\Gamma^0 \cap C_0^\infty(\Omega; R^n)$. Hence using the last part of Lemma 2.6 we see that $\psi_\Gamma(u) \leq \liminf_{j\to\infty} \psi_\Gamma(u_j)$.

Next we consider the case where Γ does not satisfy (2.4) and assume that $\{\|\nabla u_j\|_\Omega\}$ is bounded. Let Γ_j be as defined in the proof of Lemma 2.6. Then

$$\psi_{\Gamma_k}(u) \leq \liminf_{j\to\infty} \psi_{\Gamma_k}(u_j)$$

for each k. Since $\psi_{\Gamma_k}(u_j) \leq \psi_\Gamma(u_j) + \sup_i \|\nabla u_i\|_\Omega \cdot k^{-1}$ for each j, k and $\psi_{\Gamma_k}(u) \to \psi_\Gamma(u)$ as $k \to \infty$, we conclude that $\psi_\Gamma(u) \leq \liminf_{j\to\infty} \psi_\Gamma(u_j)$. This completes the proof.

The following lemma is proved in a way similar to the proof of Ohtsuka [12; Lemma 10].

Lemma 2.8. *Assume* (2.4) *and let $u \in BV(\Omega)$. Let H and I be open subsets of Ω such that $H + B(0, \varepsilon_1) \subset I$ for some $\varepsilon_1 > 0$, and η be a nonnegative function in $C_0^\infty(B(0, 1))$ satisfying $\int_{B(0,1)} \eta(x)\, dx = 1$. Then for each $t > 1$ there exists $r_0 > 0$ such that*

$$\sup_{\sigma \in K_\Gamma^0 \cap C_0^\infty(H\,;\,R^n)} \int_H \sigma \cdot \nabla (u * \eta_r)\, dx \leq t \cdot \sup_{\sigma \in K_\Gamma^0 \cap C_0^\infty(I\,;\,R^n)} \int_I \sigma d\nabla u$$

for $0 < r < r_0$, where $\eta_r(x) = \eta(x/r)\, r^{-n}$.

Proof. For $\sigma \in K_\Gamma^0 \cap C_0^\infty(H; R^n)$ and $0 < r < \varepsilon_1$,

$$\int_H \sigma \cdot \nabla (u * \eta_r)\, dx = -\int_H (u * \eta_r) \operatorname{div} \sigma dx$$
$$= -\int_H \{\int_{B(0,r)} u(x-y)\, \eta_r(y)\, dy\} \operatorname{div} \sigma(x)\, dx$$
$$= -\int_{B(0,r)} \eta_r(y) \{\int_H u(x-y) \operatorname{div} \sigma(x)\, dx\}\, dy$$
$$= -\int_{B(0,r)} \eta_r(y) \{\int_{H-y} u(z) \operatorname{div} \sigma(y+z)\, dz\}\, dy$$
$$= \int_{B(0,r)} \eta_r(y)\, dy \int_I \sigma(y+z)\, d\nabla u(z)$$

$$= \int_I \hat{\sigma} * \eta_r \, d\nabla u ,$$

where $\hat{\sigma} * \eta_r(z) = \int_{B(0,r)} \eta_r(y)\, \sigma(y+z)\, dy$. Let $t>1$. Then by (2.4), there exists $0<r_0<\varepsilon_1/2$ such that

$$\Gamma(y+z) \subset \Gamma(z) + B(0, (t-1)\,\rho_0) \subset \Gamma(z) + (t-1)\,\Gamma(z) = t\Gamma(z)$$

if $z+y \in H$ and $y \in B(0, r_0)$. It follows that $\hat{\sigma} * \eta_r(z) \in t\Gamma(z)$ for $0<r<r_0$. Thus

$$\int_I (\hat{\sigma} * \eta_r)\, d\nabla u \le \sup_{\sigma \in K^0_{t\Gamma} \cap C_0^\infty(I\,;\,R^n)} \int_I \sigma d\nabla u = t \cdot \sup_{\sigma \in K^0_{\Gamma} \cap C_0^\infty(I\,;\,R^n)} \int_I \sigma d\nabla u \, .$$

This completes the proof.

Lemma 2.9. *Assume* (2.4). *For each bounded continuous nonnegative function* g *on* Ω, *we set* $(g\Gamma)(x) = \{g(x)\, v \in R^n; v \in \Gamma(x)\}$ *and* $\mu(g) = \sup_{\sigma \in K^0_{g\Gamma} \cap C_0^\infty(\Omega\,;\,R^n)} \int_\Omega \sigma d\nabla u$. *Then* μ *is additive.*

Proof. Let $\Omega_j = \{x \in \Omega; g(x) > 1/j\}$. Then $g\Gamma$ satisfies (2.1)–(2.4) on Ω_j. Thus applying Lemma 2.5 (2) with Ω, Γ replaced by Ω_j, $g\Gamma$ respectively, we can prove that

$$\sup_{\sigma \in K^0_{g\Gamma} \cap C_0^\infty(\Omega_j\,;\,R^n)} \int_\Omega \sigma d\nabla u = \sup \{ \int_{\Omega_j} \sigma d\nabla u ; \sigma \in K^0_{g\Gamma} \}$$

for each j. It follows that

$$\begin{aligned} \mu(g) &= \lim_{j\to\infty} \sup_{\sigma \in K^0_{g\Gamma} \cap C_0^\infty(\Omega_j\,;\,R^n)} \int_\Omega \sigma d\nabla u \\ &= \lim_{j\to\infty} \sup \{ \int_{\Omega_j} \sigma d\nabla u ; \sigma \in K^0_{g\Gamma} \} = \sup \{ \int_\Omega \sigma d\nabla u ; \sigma \in K^0_{g\Gamma} \} \, . \end{aligned}$$

Let g_1 and g_2 be bounded continuous nonnegative functions on Ω. It is easy to prove $\mu(g_1) + \mu(g_2) \le \mu(g_1+g_2)$. To prove the converse inequality, we let $\sigma \in K^0_{(g_1+g_2)\Gamma}$. If $\inf_\Omega g_2 > 0$, then

$$\begin{aligned} \int_\Omega \sigma d\nabla u &= \int_\Omega g_1 \sigma/(g_1+g_2)\, d\nabla u + \int_\Omega g_2 \sigma/(g_1+g_2)\, d\nabla u \\ &\le \mu(g_1) + \mu(g_2) \, . \end{aligned}$$

Thus in case $\inf_\Omega g_2 > 0$, we obtain $\mu(g_1+g_2) = \mu(g_1) + \mu(g_2)$. In the general case,

$$\begin{aligned} \mu(g_1+g_2) &= \lim_{\varepsilon \downarrow 0} \mu(g_1+g_2+\varepsilon) \\ &= \lim_{\varepsilon \downarrow 0} \{\mu(g_1) + \mu(g_2+\varepsilon)\} \\ &= \mu(g_1) + \mu(g_2) \, . \end{aligned}$$

Hence we conclude that μ is additive.

In virtue of Lemma 2.9, μ in the lemma is extended to a linear form on

$C_0(\Omega)$. Therefore μ is identified with a Radon measure on Ω.

Lemma 2.10. *Let $u\in BV(\Omega)$. Then there is $\{u^j\}\subset BV(\Omega)\cap C^\infty(\Omega)$ such that $\{u^j\}$ converges to u in $BV(\Omega)$ and $\psi_\Gamma(u^j)\to\psi_\Gamma(u)$ as $j\to\infty$.*

Proof. First we assume (2.4) and let μ be the additive functional as defined in Lemma 2.9. We regard μ as a Radon measure on Ω. Let $\{G_p\}$ be a sequence of open subsets of Ω such that $\bigcup_{p=1}^\infty G_p=\Omega$, $\bar{G}_p\subset G_{p+1}$ and $\mu(\partial G_p)=0$ for all p. We assume that $G_0=\phi$. Now for $k>0$ we take sequences $\{H_p\}$ and $\{I_p\}$ of open subsets of Ω such that $H_p\supset\bar{G}_{p+1}-G_p$ and $I_p\supset\bar{H}_p$ for all p and that

$$\sum_{p=1}^\infty \mu(I_p-(\bar{G}_p-G_{p-1}))<1/k\,. \tag{2.5}$$

Furthermore let $\{\alpha_p\}\subset C_0^\infty(\Omega)$ be a partition of unity subordinate to the covering $\{H_p\}$ of Ω. By using Lemma 2.8, we can easily see that there exists a sequence $\{\lambda_p\}$ of positive numbers such that

$$\max\left(\|\nabla\alpha_p\|_{L^\infty(\Omega\,;\,R^n)}\cdot\sup_{\sigma\in K_\Gamma}\|\sigma\|_{L^\infty(\Omega\,;\,R^n)},\,1\right)\cdot\int_{H_p}|u*\eta_{\lambda_p}-u|\,dx<1/(k2^p)\,, \tag{2.6}$$

$$\|\nabla(u*\eta_{\lambda_p})\|_{H_p}\leq(1+1/k)\|\nabla u\|_{I_p} \tag{2.7}$$

and

$$\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(H_p\,;\,R^n)}\int_{H_p}\sigma\cdot\nabla\,(u*\eta_{\lambda_p})\,dx\leq(1+1/k)\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(I_p\,;\,R^n)}\int_{I_p}\sigma d\nabla u\,. \tag{2.8}$$

We set $u^k=\sum_{p=1}^\infty \alpha_p(u*\eta_{\lambda_p})$. Then $\|u^k-u\|_{L^1(\Omega)}<1/k$ by (2.6), and

$$\begin{aligned}\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(\Omega\,;\,R^n)}\int_\Omega\sigma\cdot\nabla u^k\,dx&\leq\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(\Omega\,;\,R^n)}\sum_{p=1}^\infty\int_\Omega\sigma\cdot(\nabla\alpha_p)\,(u*\eta_{\lambda_p})\,dx\\&\quad+\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(\Omega\,;\,R^n)}\sum_{p=1}^\infty\int_\Omega\sigma\cdot\alpha_p\,\nabla(u*\eta_{\lambda_p})\,dx\,.\end{aligned}$$

Since $\sum_{p=0}^\infty\nabla\alpha_p=\nabla(\sum_{p=0}^\infty\alpha_p)=0$, by (2.6) we see that the first term of the right hand side is less than $1/k$. For the second term, by (2.8) we obtain

$$\begin{aligned}&\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(\Omega\,;\,R^n)}\sum_{p=1}^\infty\int_\Omega\sigma\cdot\alpha_p\,\nabla(u*\eta_{\lambda_p})\,dx\\&\leq\sum_{p=1}^\infty\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(\Omega\,;\,R^n)}\int_\Omega\sigma\cdot\alpha_p\,\nabla(u*\eta_{\lambda_p})\,dx\\&\leq\sum_{p=1}^\infty\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(H_p\,;\,R^n)}\int_{H_p}\sigma\cdot\nabla(u*\eta_{\lambda_p})\,dx\\&\leq(1+1/k)\sum_{p=1}^\infty\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(I_p\,;\,R^n)}\int_{I_p}\sigma d\nabla u\\&=(1+1/k)\sum_{p=1}^\infty\mu(I_p)\,.\end{aligned}$$

Thus by (2.5) we see that the second term is less than $(1+1/k)\,(\mu(\Omega)+1/k)$. In virtue of Lemma 2.6, $\mu(\Omega)=\sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(\Omega;R^n)}\int_\Omega \sigma d\nabla u=\psi_\Gamma(u)$. It follows that $\psi_\Gamma(u^k)\leq(1+1/k)(\psi_\Gamma(u)+1/k)+1/k$. Furthermore since $\sup_{\sigma\in K_\Gamma}||\sigma||_{L^\infty(\Omega;R^n)}\geq \rho_0$ and $\mu(I_p-(\bar{G}_p-G_{p-1}))\geq\rho_0|\nabla u|(I_p-(\bar{G}_p-G_{p-1}))$ by (2.4), using (2.6) and (2.7) we obtain

$$
\begin{aligned}
||\nabla u^k||_\Omega\leq&\sup\{\textstyle\sum_{p=1}^\infty\int_\Omega \sigma\cdot(\nabla\alpha_p)(u*\eta_{\lambda_p})\,dx;\ \sigma\in C_0^\infty(\Omega;R^n),\ |\sigma|\leq 1 \text{ on } \Omega\}\\
&+\sup\{\textstyle\sum_{p=1}^\infty\int_\Omega \sigma\cdot\alpha_p\nabla(u*\eta_{\lambda_p})\,dx;\ \sigma\in C_0^\infty(\Omega;R^n),\ |\sigma|\leq 1 \text{ on } \Omega\}\\
\leq&\textstyle\sum_{p=1}^\infty||\nabla\alpha_p||_{L^\infty(\Omega;R^n)}\cdot\int_{H_p}|u*\eta_{\lambda_p}-u|\,dx+\sum_{p=1}^\infty||\nabla(u*\eta_{\lambda_p})||_{H_p}\\
\leq&(\rho_0 k)^{-1}+(1+1/k)\textstyle\sum_{p=1}^\infty||\nabla u||_{I_p}\\
=&(\rho_0 k)^{-1}+(1+1/k)\textstyle\sum_{p=1}^\infty\{|\nabla u|(\bar{G}_p-G_{p-1})+|\nabla u|(I_p-(\bar{G}_p-G_{p-1}))\}\\
\leq&(\rho_0 k)^{-1}+(1+1/k)(||\nabla u||_\Omega+(\rho_0 k)^{-1}).
\end{aligned}
$$

Hence $\limsup_{k\to\infty}||\nabla u^k||_\Omega\leq||\nabla u||_\Omega$. Since $u^k\to u$ in $L^1(\Omega)$, $\psi_\Gamma(u)\leq\liminf_{k\to\infty}\psi_\Gamma(u^k)$ by Lemma 2.7 and $||\nabla u||_\Omega\leq\liminf_{k\to\infty}||\nabla u^k||_\Omega$ by [11; Lemma 6.1.2/2]. It follows that $\psi_\Gamma(u)=\lim_{k\to\infty}\psi_\Gamma(u^k)$ and $||\nabla u||_\Omega=\lim_{k\to\infty}||\nabla u^k||_\Omega$.

To prove the general case, let $\Gamma_1(x)=\Gamma(x)+\overline{B(0,1)}$ and apply the first part of this proof to Γ_1. Since

$$
\begin{aligned}
\psi_{\Gamma_1}(u)&=\int_\Omega\beta_{\Gamma_1}(\nabla u/|\nabla u|,\cdot)\,d|\nabla u|=\int_\Omega\{\beta_\Gamma(\nabla u/|\nabla u|,\cdot)+1\}\,d|\nabla u|\\
&=\psi_\Gamma(u)+||\nabla u||_\Omega,
\end{aligned}
$$

we obtain the desired $\{u^j\}$. This completes the proof.

Proof of Proposition 2.4. First assume (2.4) and let u be a function in $BV(\Omega)$. Let $\{u^j\}$ be a sequence stated in Lemma 2.10 and set $N_{j,r}=\{x\in\Omega;\ u^j(x)\geq r\}$. By [11; Lemma 6.1.6], we may assume that $\chi_{N_{j,r}}\to\chi_{N_r}$ in $L^1(\Omega)$ for almost all r. Then for each j, by [11; Theorem 1.2.4],

$$
\begin{aligned}
\psi_\Gamma(u^j)&=\int_\Omega\beta_\Gamma(\nabla u^j,\cdot)\,dx=\int_{\Omega'}\beta_\Gamma(\nabla u^j/|\nabla u^j|,\cdot)|\nabla u^j|\,dx\\
&=\int_{-\infty}^\infty dr\int_{\partial N_{j,r}\cap\Omega'}\beta_\Gamma(\nabla u^j/|\nabla u^j|,\cdot)\,dH_{n-1},
\end{aligned}
$$

where $\Omega'=\{x\in\Omega;\ \nabla u^j\neq 0\}$. In view of Sard's theorem, $\nabla u_j\neq 0$ on $\partial N_{j,r}\cap\Omega$ and $\partial N_{j,r}\cap\Omega$ is of C^∞-class for a.e. $r\in R$. Furthermore $\chi_{N_{j,r}}\in BV(\Omega)$ and $\nabla u_j/|\nabla u_j|=\nabla\chi_{N_{j,r}}/|\nabla\chi_{N_{j,r}}|\,H_{n-1}$-a.e. on $\partial N_{j,r}\cap\Omega$ for a.e. r. It follows that

$$
\psi_\Gamma(u^j)=\int_{-\infty}^\infty dr\int_\Omega\beta_\Gamma(\nabla\chi_{N_{j,r}}/|\nabla\chi_{N_{j,r}}|,\cdot)\,d|\nabla\chi_{N_{j,r}}|=\int_{-\infty}^\infty\psi_\Gamma(\chi_{N_{j,r}})\,dr.
$$

By $\overline{\int}$ and $\underline{\int}$ we denote the upper and lower integrals respectively. Since $\chi_{N_{j,r}}\to\chi_{N_r}$ in $L^1(\Omega)$ as $j\to\infty$ for a.e. r, in virtue of Lemma 2.7 $\psi_\Gamma(\chi_{N_r})\leq\liminf_{j\to\infty}\psi_\Gamma(\chi_{N_{j,r}})$. Hence

$$\begin{aligned}\psi_\Gamma(u) &= \lim_{j\to\infty}\psi_\Gamma(u^j)=\lim_{j\to\infty}\int_{-\infty}^{\infty}\psi_\Gamma(\chi_{N_{j,r}})\,dr\\ &\geq\underline{\int}_{-\infty}^{\infty}\liminf_{j\to\infty}\psi_\Gamma(\chi_{N_{j,r}})\,dr\geq\overline{\int}_{-\infty}^{\infty}\psi_\Gamma(\chi_{N_r})\,dr\,.\end{aligned}$$

On the other hand, by [11; Theorem 1.2.3],

$$\begin{aligned}\int_\Omega\sigma d\nabla u &= -\int_\Omega u\operatorname{div}\sigma\,dx=-\int_\Omega u^+\operatorname{div}\sigma\,dx+\int_\Omega u^-\operatorname{div}\sigma\,dx\\ &= -\int_0^\infty dr\int_\Omega\chi_{N_r}\operatorname{div}\sigma\,dx+\int_0^\infty dr\int_\Omega(1-\chi_{N_{-r}})\operatorname{div}\sigma\,dx\\ &= -\int_{-\infty}^{\infty}dr\int_\Omega\chi_{N_r}\operatorname{div}\sigma\,dx=\int_{-\infty}^{\infty}dr\int_\Omega\sigma d\nabla\chi_{N_r}\end{aligned}$$

for all $\sigma\in C_0^\infty(\Omega;R^n)$, where $u^+=\max(u,0)$ and $u^-=\max(-u,0)$. Thus

$$\begin{aligned}\psi_\Gamma(u) &= \sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(\Omega;R^n)}\int\sigma d\nabla u\\ &= \sup_{\sigma\in K_\Gamma^0\cap C_0^\infty(\Omega;R^n)}\int_{-\infty}^{\infty}dr\int\sigma d\nabla\chi_{N_r}\leq\underline{\int}_{-\infty}^{\infty}\psi_\Gamma(\chi_{N_r})\,dr\,.\end{aligned}$$

It follows that $\psi_\Gamma(\chi_{N_r})$ is a measurable function of r and $\psi_\Gamma(u)=\int_{-\infty}^{\infty}\psi_\Gamma(\chi_{N_r})\,dr$.

To prove the general case, we consider $\Gamma_j(x)=\Gamma(x)+\overline{B(0,1/j)}$. Since $\beta_{\Gamma_j}(v,x)\downarrow\beta_\Gamma(v,x)$ for all $v\in R^n$ and $x\in\Omega$, by letting $j\to\infty$ in

$$\psi_{\Gamma_j}(u)=\int_{-\infty}^{\infty}\psi_{\Gamma_j}(\chi_{N_r})\,dr\,,$$

we see that

$$\psi_\Gamma(u)=\int_{-\infty}^{\infty}\psi_\Gamma(\chi_{N_r})\,dr\,.$$

This completes the proof.

REMARK 2.11. Suppose that Γ satisfies (2.1), (2.3), (2.4) but (2.2) is replaced by the following condition:

(2.2′) Let $\varepsilon>0$. Then there is $\delta>0$ such that $\Gamma(x)\subset\Gamma(y)+B(0,\varepsilon)$ whenever $x,y\in\Omega$ and $|x-y|<\delta$.

Then we can choose $\{\sigma^j\}$ in (3) of Lemma 2.5 such that $\sigma^j(x)\in\Gamma(x)+B(0,1/j)$ for all $x\in\Omega$. Furthermore by (2.4), we may assume that $\sigma^j(x)\in\Gamma(x)$ for all $x\in\Omega$ and all j.

Let σ be a vector field in $H(\Omega)$. We take Γ_0 such that $\Gamma_0(x)=\{w\in R^n; |w|\leq||\sigma||_{L^\infty(\Omega\,;\,R^n)}\}$ for all $x\in\Omega$ and apply Remark 2.11 to these σ and Γ_0. Then we obtain a sequence $\{\sigma^j\}$ in $C^\infty(\bar{\Omega}; R^n)$ such that $||\sigma^j||_{L^\infty(\Omega\,;\,R^n)}\leq||\sigma||_{L^\infty(\Omega\,;\,R^n)}$ for all j and $\sigma^j\to\sigma$ as $j\to\infty$ in the sense of $H(\Omega)$. Since $\{\sigma^j\cdot\nu\}$ converges to $\sigma\cdot\nu$ weakly* in $L^\infty(\partial\Omega)$ as stated after Theorem 2.3, we obtain

$$\begin{aligned}||\sigma\cdot\nu||_{L^\infty(\partial\Omega)} &\leq \liminf_{j\to\infty}||\sigma^j\cdot\nu||_{L^\infty(\partial\Omega)}\\ &\leq \liminf_{j\to\infty}||\sigma^j||_{L^\infty(\Omega\,;\,R^n)}\leq||\sigma||_{L^\infty(\Omega\,;\,R^n)}\,.\end{aligned}$$

REMARK 2.12. Instead of (2.2) and (2.3), we assume that $\{(x, w); x\in\Omega, w\in\Gamma(x)\}$ is a bounded Borel subset of $\Omega\times R^n$. Then the statement of Lemma 2.5 (1) is true. Let $u\in W^{1,1}(\Omega)$. Then using Lemma 2.5 (1) we can prove that $\psi_\Gamma(u)=\sup_{\sigma\in K_\Gamma}\int_\Omega\sigma\cdot\nabla u dx$. Furthermore let $\{u^j\}\subset W^{1,1}(\Omega)$ such that $||\nabla u^j-\nabla u||_{L^1(\Omega)}\to 0$ as $j\to\infty$. Then

$$\begin{aligned}|\psi_\Gamma(u^j)-\psi_\Gamma(u)| &\leq \sup_{\sigma\in K_\Gamma}\left|\int_\Omega\sigma\cdot(\nabla u^j-\nabla u)\,dx\right|\\ &\leq \sup_{\sigma\in K_\Gamma}||\sigma||_{L^\infty(\Omega\,;\,R^n)}\cdot||\nabla u^j-\nabla u||_{L^1(\Omega)}\,.\end{aligned}$$

It follows that $\psi_\Gamma(u^j)\to\psi_\Gamma(u)$ as $j\to\infty$.

3. Duality theorem

In this section, we define optimization problems (MF) and (MF*) and state a duality relation for the problems. Furthermore as an application, we prove a duality theorem for a max-flow problem in a general form.

Let X, Y and Z be real linear spaces. We consider two more real linear spaces Y_1 and Z_1 with $Y_1\supset Y$ and $Z_1\subset Z$. Let K be a convex set in Y_1 containing the origin and P be a convex cone in X with vertex at the origin. Furthermore we consider functionals L_X, L_Y and h. Let L_X, L_Y be bilinear functionals defined on $X\times Z$, $Y\times Z$ respectively and h be a linear functional defined on X. We assume that L_Y is defined and bilinear also on $Y_1\times Z_1$.

We define (MF) and (MF*) as follows:

(MF) Maximize $h(p)$ subject to the constraint that $p\in P$ and $(p, y)\in V$ for some $y\in K\cap Y$,

where $V=\{(p, y)\in X\times Y; L_X(p, z)=L_Y(y, z)$ for all $z\in Z\}$.

(MF*) Minimize $\Psi_K(z)$ subject to $z\in W$,

where $\Psi_K(z)=\sup_{y\in K\cap Y}L_Y(y, z)$ and $W=\{z\in Z; L_X(p, z)\geq h(p)$ for all $p\in P\}$. We denote the values of (MF) and (MF*) by MF and MF^* respectively. Throughout this paper we use the convention that the supremum on the empty set is $-\infty$ and the infimum on the empty set is ∞. If $p\in P$ and $(p, y)\in V$ for some $y\in K\cap Y$, then we call p a feasible element of (MF). Similarly we call $z\in W$ a feasible element of (MF*). Since K contains the origin, $\Psi_K(z)\geq 0$ for all $z \in Z$. We can easily prove

Lemma 3.1. *$MF \leq MF^*$.*

We give a sufficient condition for $MF^* = MF$. In what follows, we assume that Y_1 and Z_1 are locally convex Hausdorff spaces and $L_Y(y, \cdot)$ and $L_X(p, \cdot)$ are continuous on Z_1 for each $y \in Y_1$ and $p \in P$ respectively. We define an auxiliary value MF' by

$$MF' = \sup\{h(p); p \in P \text{ such that } (p, y) \in V_1 \text{ for some } y \in K\},$$

where $V_1 = \{(p, y) \in X \times Y_1; L_X(p, z) = L_Y(y, z) \text{ for all } z \in Z_1\}$. Then we have

Lemma 3.2. *The equality $MF' = \sup_{y \in K} \inf_{z \in W \cap Z_1} L_Y(y, z)$ holds if the following two conditions are satisfied:*

(3.1) *$L_X(P) = \{L_X(p, \cdot); p \in P\}$ is a closed set of the topological dual space Z_1^* of Z_1 with weak* topology.*

(3.2) *For each $p \in P$, $\inf_{z \in W \cap Z_1} L_X(p, z) = h(p)$.*

Proof. For simplicity we set $f(y) = \inf_{z \in W \cap Z_1} L_Y(y, z)$ for $y \in K$ and $A = \sup_{y \in K} f(y)$ in this proof. Let y be an element in K. We define a linear functional μ_y on Z_1 by $\mu_y(z) = L_Y(y, z)$. Then $\mu_y \in Z_1^*$. First we assume that $\mu_y \notin L_X(P)$. By (3.1) and the separation theorem (cf. Schaefer [14; Theorem 9.2 in Chap. 2]), there exists $z_0 \in Z_1$ such that $\mu_y(z_0) < 0$ and $L_X(p, z_0) \geq 0$ for all $p \in P$. Then for $z_1 \in W \cap Z_1$ and $r > 0$, $rz_0 + z_1 \in W \cap Z_1$ and therefore

$$f(y) \leq \inf_{r>0} L_Y(y, rz_0 + z_1) = \lim_{r \to \infty} r\mu_y(z_0) + L_Y(y, z_1) = -\infty .$$

Next we assume that $\mu_y \in L_X(P)$. Then there exists $p_0 \in P$ such that $\mu_y(z) = L_Y(y, z) = L_X(p_0, z)$ for all $z \in Z_1$. By (3.2), $h(p_0) = \inf_{z \in W \cap Z_1} L_X(p_0, z) = f(y)$. Thus $f(y) \leq MF'$ for any $y \in K$ so that $A \leq MF'$.

If $MF' = -\infty$, then naturally $A = MF'$. Suppose there exists $p \in P$ such that $(p, y) \in V_1$ for some $y \in K$. Then $L_Y(y, z) = L_X(p, z)$ for all $z \in Z_1$ and $h(p) = f(y)$ by (3.2). It follows that $h(p) \leq A$ which shows $MF' \leq A$. Thus $MF' = A$. Our lemma is now proved.

It is easy to see that $MF = MF'$ if the following conditions (3.3) and (3.4) are satisfied:

(3.3) Let $y \in Y$ and $p \in P$. If $L_Y(y, z) = L_X(p, z)$ for all $z \in Z_1$, then $L_Y(y, z) = L_X(p, z)$ for all $z \in Z$.

(3.4) Let $y \in Y_1$ and $p \in P$. If $L_Y(y, z) = L_X(p, z)$ for all $z \in Z_1$, then $y \in Y$.

On the other hand,

$$MF' = \sup_{y \in K} \inf_{z \in W \cap Z_1} L_Y(y, z) = \inf_{z \in W \cap Z_1} \sup_{y \in K} L_Y(y, z)$$

by Lemma 3.2 and a minimax theorem (cf. Fan [5; Theorem 2]), if conditions

(3.1), (3.2),
(3.5) K is compact,
(3.6) $L_Y(\cdot, z)$ is continuous on K for each $z \in Z_1$
are fulfilled. Using these facts we prove

Theorem 3.3. *Assume that conditions* (3.1)–(3.6) *are satisfied. Then the duality relation $MF = MF^*$ holds. Furthermore if MF is finite, then there is an optimal solution of* (MF).

Proof. By the definition of MF^*, $\inf_{z \in W \cap Z_1} \sup_{y \in K \cap Y} L_Y(y, z) \geq MF^*$. From the above observation, it follows that $MF \geq MF^*$. The converse inequality follows from Lemma 3.1. Next we assume that MF is finite. Since $MF = \sup_{y \in K} \inf_{z \in W \cap Z_1} L_Y(y, z)$ and $f(y) = \inf_{z \in W \cap Z_1} L_Y(y, z)$ is upper semicontinuous on K by (3.6), according to (3.5) there is $y_0 \in K$ such that $MF = f(y_0)$. Since MF is finite, as shown in the proof of Lemma 3.2, there is $p_0 \in P$ such that $L_Y(y_0, z) = L_X(p_0, z)$ for all $z \in Z_1$ and $f(y_0) = h(p_0)$. By (3.3) and (3.4), p_0 is a feasible element of (MF). It follows that p_0 is an optimal solution of (MF) and the proof is completed.

Next we are concerned with min-cut problems. By a cut of a domain, we mean a partition of the domain into two parts. We identify any cut with the characteristic function of one of the parts. Let Z_0 be a subset of Z containing 0. Later Z_0 will be taken to be a class of characteristic functions. Here we define (MC) as follows:
(MC) Minimize $\Psi_K(z)/\Pi(z)$ subject to $z \in Z_0$ and $\Pi(z) > 0$,

where $\Pi(z) = \inf \{L_X(p, z); p \in P, h(p) \geq 1\}$.

We denote the value of (MC) by MC. Let $\tilde{Z}$ be the set of all $z \in Z$ satisfying the following condition:
(3.7) There exist Lebesgue measurable subsets J and J' of R and a subset $\{z_r\}_{r \in J}$ of Z_0 such that $J' \subset J \subset R$, and that $\Psi_K(z_r)$ and $\Pi(z_r)$ are integrable functions of r on J and J' and satisfy

$$\Psi_K(z) = \int_J \Psi_K(z_r)\, dr \quad \text{and} \quad \Pi(z) = \int_{J'} \Pi(z_r)\, dr$$

respectively.
Then we have

Theorem 3.4. *Assume that $h \geq 0$ on P and that there is a sequence $\{z^j\} \subset \tilde{Z}$ satisfying $\Pi(z^j) > 0$ for each j and*

$$\limsup_{j \to \infty} \Psi_K(z^j)/\Pi(z^j) \leq \Psi_K(z)/\Pi(z)$$

if $z \in Z$ and $\Pi(z) > 0$. Then $MF^* = MC$.

Proof. If $h(p)=0$ for all $p\in P$, then, by considering $z=0$, we can see $MF^*=MC=0$. Thus we consider the case where there exists $p_0\in P$ such that $h(p_0)>0$. First we define an auxiliary problem $(\tilde{M}^*)$:
$(\tilde{M}^*)$ Minimize $\Psi_K(z)/\Pi(z)$ subject to $z\in Z$ and $\Pi(z)>0$.
We denote the value by $\tilde{M}^*$. We show that $MF^*=\tilde{M}^*$. If there is no feasible element of (MF*), then $MF^*=\infty\geq\tilde{M}^*$. If $z\in Z$ is a feasible element of (MF*), that is, $L_X(p, z)\geq h(p)$ for all $p\in P$, then $\Pi(z)\geq 1$ and therefore $\Psi_K(z)/\Pi(z)\leq \Psi_K(z)$. It follows that $\tilde{M}^*\leq MF^*$. We may now assume that there is a feasible element of $(\tilde{M}^*)$. Let z be such an element. If $p\in P$ and $h(p)>0$, then $h(p)^{-1}p \in P$ and $h(h(p)^{-1}p)=1$. Thus $L_X(p, z/\Pi(z))\geq h(p)$ if $p\in P$ and $h(p)>0$. Furthermore, if $p\in P$ and $h(p)=0$, then by considering $p+tp_0$ and letting $t\to 0$ we obtain $L_X(p, z/\Pi(z))\geq h(p)$. Thus $\Pi(z)^{-1}z$ is a feasible element of (MF*) and $\text{MF}^*\leq\Psi_K(\Pi(z)^{-1}z)=\Psi_K(z)/\Pi(z)$. Since z may be an arbitrary feasible element of $(\tilde{M}^*)$, $MF^*\leq\tilde{M}^*$ so that $MF^*=\tilde{M}^*$.

Next we show that $\tilde{M}^*=MC$. Evidently $\tilde{M}^*\leq MC$. So we assume $\tilde{M}^*<\infty$. First we suppose there exists $z\in\tilde{Z}$ which is a feasible element of $(\tilde{M}^*)$. Then there exist J, J' and $\{z_r\}$ which satisfy the conditions stated in (3.7). In particular, all $z_r\in Z_0$ and

$$\Psi_K(z)=\int_J\Psi_K(z_r)\,dr \quad\text{and}\quad \Pi(z)=\int_{J'}\Pi(z_r)\,dr\,.$$

Since $\Pi(z)>0$, the Lebesgue measure of $\{r\in J'; \Pi(z_r)>0\}$ is positive. If $\Psi_K(z_r)/\Pi(z_r)>\Psi_K(z)/\Pi(z)$ for all $r\in J'$ such that $\Pi(z_r)>0$, then

$$\Psi_K(z)\geq\int_{J'}\Psi_K(z_r)\,dr>(\Psi_K(z)/\Pi(z))\int_{J'}\Pi(z_r)\,dr=\Psi_K(z)\,.$$

This is a contradiction. Thus there exists $r_0\in J'$ such that $\Pi(z_{r_0})>0$ and $\Psi_K(z_{r_0})/\Pi(z_{r_0})\leq\Psi_K(z)/\Pi(z)$. Hence $MC\leq\Psi_K(z)/\Pi(z)$. Next let z be an arbitrary feasible element of $(\tilde{M}^*)$. By our assumption, there is a sequence $\{z^j\}$ in $\tilde{Z}$ such that $\Pi(z^j)>0$ for each j and

$$\limsup_{j\to\infty}\Psi_K(z^j)/\Pi(z^j)\leq\Psi_K(z)/\Pi(z)\,.$$

Since $z^j\in\tilde{Z}$ and z^j is a feasible element of $(\tilde{M}^*)$, $MC\leq\Psi_K(z^j)/\Pi(z^j)$. It follows that

$$MC\leq\limsup_{j\to\infty}\Psi_K(z^j)/\Pi(z^j)\leq\Psi_K(z)/\Pi(z)\,.$$

Thus $MC\leq\tilde{M}^*$. This completes the proof.

Let us consider special X, Y, Z, Y_1, Z_1, L_X, L_Y and h. Let T be the set of all nonnegative integers. We set

$$X=\{(\xi_{1,t},\xi_{2,t})_{t\in T}\in(L^n(\Omega)\times L^\infty(\partial\Omega))^T;\ \textstyle\sum_{t\in T}(\|\xi_{1,t}\|_{L^n(\Omega)}+\|\xi_{2,t}\|_{L^\infty(\partial\Omega)})<\infty\}\,,$$

$$Y_1 = \{(\sigma_t, \kappa_t)_{t\in T} \in (L^\infty(\Omega; R^n)\times L^\infty(\partial\Omega))^T; \sum_{t\in T}\|\sigma_t\|_{L^\infty(\Omega; R^n)}<\infty \text{ and } \sum_{s=0}^\infty \kappa_s \text{ exists in } L^\infty(E)\},$$

where E is a Borel subset of $\partial\Omega$. Let K be a convex set in Y_1 containing the origin and P be a convex cone in X with vertex at the origin as required in the beginning of the present section. We assume that $\kappa_t=0$ H_{n-1}—a.e. on $\partial\Omega-E$ for all $(\sigma_t, \kappa_t)_{t\in T}\in K$. Furthermore we set

$$\begin{aligned} Y &= \{(\sigma_t, \kappa_t)_{t\in T}\in Y_1;\ \sigma_t\in H(\Omega) \quad \text{for all} \quad t\in T\},\\ Z &= \{(z_t)_{t\in T}\in BV(\Omega)^T;\ \sup_{t\in T}(\|z_t\|_{L^1(\Omega)}+\|\nabla z_t\|_\Omega)<\infty,\\ &\qquad \sum_{t\in T}\|\gamma z_t-\gamma z_{t+1}\|_{L^1(E)}<\infty\},\\ Z_1 &= \{(z_t)_{t\in T}\in Z;\ z_t\in W^{1,1}(\Omega) \quad \text{for all} \quad t\in T\}; \end{aligned}$$

in the definition of Z, γ is the mapping defined in Theorem 2.1 (1).

Let Ω' and A' be Borel subsets of Ω and $\partial\Omega$ respectively. We define L_X, L_Y, h by

$$\begin{aligned} L_X(\xi, z) &= \sum_{t\in T}\{\int_\Omega \xi_{1,t}\, z_t\, dx+\int_{\partial\Omega} \xi_{2,t}\, \gamma z_t\, dH_{n-1}\},\\ L_Y(y, z) &= \sum_{t\in T}\{(\sigma_t\, \nabla z_t)(\Omega)-\int_E \kappa_t\, \gamma z_t\, dH_{n-1}\},\\ h(\xi) &= \sum_{t\in T}\{\int_{\Omega'} \xi_{1,t}\, dx+\int_{A'} \xi_{2,t}\, dH_{n-1}\} \end{aligned}$$

for $\xi=(\xi_{1,t}, \xi_{2,t})_{t\in T}\in X$, $z=(z_t)_{t\in T}\in Z$ and $y=(\sigma_t, \kappa_t)_{t\in T}\in Y$. Since γ is a continuous mapping from $W^{1,1}(\Omega)$ to $L^1(\partial\Omega)$, using Theorem 2.1 (2) we observe for $(z_t)_{t\in T}\in Z$ that

$$\sup_{t\in T}\|\gamma z_t\|_{L^1(\partial\Omega)}\le \sup_{t\in T} k(\|z_t\|_{L^1(\Omega)}+\|\nabla z_t\|_\Omega)<\infty,$$

where k is a constant depending on Ω. We infer easily that $L_X(\xi, z)$ and $h(\xi)$ are finite. Setting $\Theta_t=\sum_{s=0}^t \kappa_s$ and $\Theta_{-1}=0$, we have

$$\begin{aligned} \sum_{s=0}^t \int_E \kappa_s\, \gamma z_s\, dH_{n-1} &= \sum_{s=0}^t \int_E (\Theta_s-\Theta_{s-1})\, \gamma z_s\, dH_{n-1}\\ &= \sum_{s=0}^{t-1} \int_E \Theta_s(\gamma z_s-\gamma z_{s+1})\, dH_{n-1}+\int_E \Theta_t\, \gamma z_t\, dH_{n-1}. \end{aligned}$$

Letting $t\to\infty$, we see that $L_Y(y, z)$ is also finite and

$$L_Y(y, z) = \sum_{t\in T}\{(\sigma_t\, \nabla z_t)(\Omega)-\int_E \Theta_t(\gamma z_t-\gamma z_{t+1})\, dH_{n-1} + \int_E \Theta(\gamma z_t-\gamma z_{t-1})\, dH_{n-1}\}, \tag{3.8}$$

where we set $\Theta=\sum_{s=0}^\infty \kappa_s$ and $z_{-1}=0$. For $y=(\sigma_t, \kappa_t)_{t\in T}\in Y_1$ and $z=(z_t)_{t\in T}\in Z_1$, $L_Y(y, z)$ is defined by

$$L_Y(y, z) = \sum_{t\in T}\{\int_\Omega \sigma_t\cdot\nabla z_t\,dx - \int_E \kappa_t\,\gamma z_t\,dH_{n-1}\}\ .$$

With the above data we consider (MF) and (MF*) as in the beginning of this section and denote them by (MF_0) and (MF_0^*) respectively. As usual we denote the values of (MF_0), (MF_0^*) by MF_0, MF_0^* respectively. From Lemma 3.1 it follows that $MF_0 \leq MF_0^*$.

We define a max-flow problem ($M\Phi_0$) as follows:

$$(M\Phi_0)\quad \text{Maximize } \sum_{t\in T}(\int_{\Omega'} -\text{div}\,\sigma_t\,dx + \int_{A'}(\sigma_t\cdot\nu - \kappa_t)\,dH_{n-1})$$
$$\text{subject to}\quad (\sigma_t, \kappa_t)_{t\in T}\in K \quad\text{and}\quad (-\text{div}\,\sigma_t, \sigma_t\cdot\nu-\kappa_t)_{t\in T}\in P\ ,$$

where Theorem 2.3 (2) is used to see that $\sigma\cdot\nu$ may be regarded as a function of $L^\infty(\partial\Omega)$ when $\sigma\in H(\Omega)$. We denote the value of ($M\Phi_0$) by $M\Phi_0$. For any feasible element $(\sigma_t, \kappa_t)_{t\in T}$ of ($M\Phi_0$), one may call $(\sigma_t)_{t\in T}$ a feasible flow of ($M\Phi_0$). We prove

Proposition 3.5. $M\Phi_0 = MF_0$.

Proof. Let $p=(p_{1,t}, p_{2,t})_{t\in T}\in P$ be a feasible element of (MF_0). Then there is $y=(\sigma_t, \kappa_t)_{t\in T}\in K\cap Y$ such that $L_X(p, z)=L_Y(y, z)$ for all $z\in Z$. In particular, $\int_\Omega p_{1,t}\,v dx + \int_{\partial\Omega} p_{2,t}\,\gamma v dH_{n-1} = \int_\Omega \sigma_t\cdot\nabla v dx - \int_E \kappa_t\gamma v dH_{n-1}$ for all $t\in T$ and $v\in W^{1,1}(\Omega)$. Hence $\int_\Omega p_{1,t}\,v dx = \int_\Omega \sigma_t\cdot\nabla v dx$ for all $v\in C_0^\infty(\Omega)$ and thus $p_{1,t} = -\text{div}\,\sigma_t$ a.e. on Ω for all $t\in T$. Furthermore since $\kappa_t=0$ H_{n-1}—a.e. on $\partial\Omega - E$, by Green's formula we obtain $\int_{\partial\Omega} p_{2,t}\,\gamma v dH_{n-1} = \int_{\partial\Omega}(\sigma_t\cdot\nu-\kappa_t)\,\gamma v dH_{n-1}$ for all $t\in T$ and $v\in W^{1,1}(\Omega)$. From $\{\gamma v;\ v\in W^{1,1}(\Omega)\} = L^1(\partial\Omega)$ it follows that $p_{2,t} = \sigma_t\cdot\nu - \kappa_t$ H_{n-1}—a.e. on $\partial\Omega$. Thus $(\sigma_t, \kappa_t)_{t\in T}$ is a feasible element of ($M\Phi_0$) and $M\Phi_0 \geq MF_0$.

Conversely let $y=(\sigma_t, \kappa_t)_{t\in T}$ be a feasible element of ($M\Phi_0$) and set

$$p = (p_{1,t}, p_{2,t})_{t\in T} = (-\text{div}\,\sigma_t, \sigma_t\cdot\nu - \kappa_t)_{t\in T}\ .$$

Then $p\in P$ and $\sigma_t\in H(\Omega)$ for all $t\in T$. Since

$$L_Y(y, z) = \sum_{t\in T}\{-\int_\Omega z_t\,\text{div}\,\sigma_t\,dx + \int_{\partial\Omega}(\sigma_t\cdot\nu-\kappa_t)\,\gamma z_t\,dH_{n-1}\} = L_X(p, z)$$

for each $z=(z_t)_{t\in T}\in Z$, it follows that p is a feasible element of (MF_0) and $MF_0 \geq M\Phi_0$. This completes the proof.

To state a duality theorem for (MF_0) and (MF_0^*), we define topologies on Z_1 and Y_1. Let $W^{1,1}(\Omega)^*$ be the topological dual space of $W^{1,1}(\Omega)$ and set

$$||v||_{W^{1,1}(\Omega)^*} = \sup\{v(u);\ ||u||_{W^{1,1}(\Omega)} \leq 1\}$$

for $v \in W^{1,1}(\Omega)^*$. Furthermore we set

$$Z_1^* = \{(v_t, \eta_t)_{t\in T} \in (W^{1,1}(\Omega)^* \times L^\infty(E))^T;\ \sum_{t\in T} ||v_t||_{W^{1,1}(\Omega)^*} \text{ is finite and } \sum_{s=0}^\infty \eta_s \text{ exists in } L^\infty(E)\}.$$

We define a bilinear form $\langle \cdot, \cdot \rangle$ on $Z_1 \times Z_1^*$ as follows:

$$\langle z, (v, \eta) \rangle = \sum_{t\in T} \{v_t(z_t) + \int_E \eta_t \gamma z_t \, dH_{n-1}\}$$

for $z = (z_t)_{t\in T} \in Z_1$ and $(v, \eta) = (v_t, \eta_t)_{t\in T} \in Z_1^*$. In the same way as the finiteness of $L_Y(y, z)$ one can show the finiteness of $\langle z, (v, \eta) \rangle$ and hence that $(Z_1, Z_1^*, \langle \cdot, \cdot \rangle)$ is a paired space. On Z_1 and Z_1^* we consider the weak topologies defined by the pairing, and on Y_1 the topology induced by the product topology of the weak* topologies on $L^\infty(\Omega; R^n) \times L^\infty(\partial\Omega)$. Then Z_1^* is regarded as the topological dual space of Z_1.

Now we prove

Theorem 3.6. *Assume that K is compact and that conditions* (3.1), (3.2) *and the following condition are satisfied*:

(3.9) *There are sequences $\{a_t\}$ and $\{b_t\}$ of positive numbers such that $\sum_{t\in T} a_t < \infty$, $\lim_{t\to\infty} b_t = 0$, and that $||\sigma_t||_{L^\infty(\Omega; R^n)} \leq a_t$, $||\sum_{s=t}^\infty \kappa_s||_{L^\infty(\partial\Omega)} \leq b_t$ for all $(\sigma, \kappa) \in K$.*

Then $MF_0 = MF_0^*$. Furthermore if MF_0 is finite, then there is an optimal solution of (MF_0).

Proof. We note that $L_Y((\sigma, \kappa), \cdot)$ and $L_X(p, \cdot)$ are continuous on Z_1 for each $p \in X$ and $(\sigma, \kappa) \in Y_1$. In fact, define $v_t \in W^{1,1}(\Omega)^*$ by

$$v_t(w) = \int_\Omega \sigma_t \cdot \nabla w dx$$

for $w \in W^{1,1}(\Omega)$, and set $\eta_t = -\kappa_t|_E$ for each $t \in T$. Then $(v_t, \eta_t)_{t\in T} \in Z_1^*$ and $L_Y((\sigma, \kappa), z) = \langle z, (v, \eta) \rangle$ for all $z \in Z_1$. It follows that $L_Y((\sigma, \kappa), \cdot)$ is continuous on Z_1. On the other hand, by letting

$$v_t'(w) = \int_\Omega p_{1,t} w dx + \int_{\partial\Omega} p_{2,t} \gamma w dH_{n-1}$$

for $w \in W^{1,1}(\Omega)$ and considering $(v_t', 0)_{t\in T} \in Z_1^*$, we obtain the continuity of $L_X(p, \cdot)$. In order to prove this theorem it suffices, according to Theorem 3.3, to check that (3.1)—(3.6) are satisfied. By assumptions, (3.1), (3.2) and (3.5) are satisfied. Using Theorem 2.1 (2), we can easily prove that (3.3) is fulfilled. Since P is a subset of $(L^n(\Omega) \times L^\infty(\partial\Omega))^T$, (3.4) is also fulfilled. Finally (3.6) follows from (3.9) and the equality in (3.8). This completes the proof.

We give a lemma that will be used to check (3.1).

Lemma 3.7. *Assume that there exist two convex cones P^1 and P^2 in $(L^n(\Omega)\times L^\infty(\partial\Omega))^T$ satisfying the following conditions*:

(3.10) *P^1 and P^2 are closed with respect to the canonical weak* topology on $(L^n(\Omega)\times L^\infty(\partial\Omega))^T$.*

(3.11) *P^1 is a finitely generated convex cone contained in X.*

(3.12) *$p_{1,t}=0$ a.e. on Ω for all $t\in T$ and $p_{2,t}=0$ H_{n-1}-a.e. on E for all $t\geq N$ if $(p_{1,t}, p_{2,t})_{t\in T}\in P^2$, where N is a positive integer independent of $(p_{1,t}, p_{2,t})_{t\in T}$.*

(3.13) *$P=P^1+P^2\cap X$ and $P^1\cap(-P^2)=\{0\}$.*

Then $L_X(P)$ is a closed set in Z_1^*.

Proof. Since $L_X(P^1)$ is a finitely generated convex cone which is closed in Z_1^*, $L_X(P^1)$ is locally compact.

We show that $L_X(P^2\cap X)$ is closed in Z_1^*. Let $\{p^i\}=\{(0, p^i_{2,t})_{t\in T}\}$ be a net in $P^2\cap X$ and $(v_t, \eta_t)_{t\in T}$ be an element in Z_1^* such that

$$L_X(p^i, z)\to \sum_{t\in T}\{v_t(z_t)+\int_E \eta_t\,\gamma z_t\,dH_{n-1}\}$$

for each $z=(z_t)_{t\in T}\in Z_1$. In particular,

$$\int_{\partial\Omega} p^i_{2,t}\,\gamma w dH_{n-1}\to v_t(w)+\int_E \eta_t\,\gamma w dH_{n-1}$$

for all $w\in W^{1,1}(\Omega)$ and $t\in T$. We set $\tau_t(w)=v_t(w)+\int_E \eta_t\,\gamma w dH_{n-1}$. Then $\tau_t\in W^{1,1}(\Omega)^*$. Since $\tau_t(w)=0$ if $\gamma w=0$ H_{n-1}-a.e. on $\partial\Omega$, we can regard τ_t as a continuous linear functional on $W^{1,1}(\Omega)/W_0^{1,1}(\Omega)$ which is topologically isomorphic to $L^1(\partial\Omega)$. Hence there is $p_{2,t}\in L^\infty(\partial\Omega)$ such that $\tau_t(w)=\int_{\partial\Omega} p_{2,t}\,\gamma w dH_{n-1}$ for all $w\in W^{1,1}(\Omega)$. Using the fact that $\{\gamma w;\, w\in W^{1,1}(\Omega)\}=L^1(\partial\Omega)$, we see that $p^i_{2,t}\to p_{2,t}$ with respect to the weak* topology on $L^\infty(\partial\Omega)$ for each $t\in T$. Hence (3.10) implies $(0, p_{2,t})_{t\in T}\in P^2$. To prove $(v_t, \eta_t)_{t\in T}\in L_X(P^2\cap X)$, it suffices to show that $(0, p_{2,t})_{t\in T}\in X$. Since $p^i_{2,t}=0$ H_{n-1}-a.e. on E for all $t\geq N$ and all i by (3.12), $p_{2,t}=0$ H_{n-1}-a.e. on E for all $t\geq N$. We claim that $\sum_{t\in T}\|p_{2,t}\|_{L^\infty(\partial\Omega)}$ is finite. Suppose that $\sum_{t\in T}\|p_{2,t}\|_{L^\infty(\partial\Omega)}=\infty$. Then there is $\{q_t\}_{t\in T}\subset L^1(\partial\Omega)$ such that $\|q_t\|_{L^1(\partial\Omega)}\leq 1$, $q_t=0$ H_{n-1} a.e. on E, $p_{2,t}\,q_t\geq 0$ H_{n-1}-a.e. on $\partial\Omega$ for each $t\in T$ and

$$\sum_{t\in T}\int_{\partial\Omega-E} p_{2,t}\,q_t\,dH_{n-1}=\infty\,.$$

Furthermore in virtue of [8; Theorem 2.16] there is $\{w_t\}_{t\in T}\subset W^{1,1}(\Omega)$ such that $\gamma w_t=q_t$ H_{n-1}-a.e. on $\partial\Omega$ and $\sup_{t\in T}\|w_t\|_{W^{1,1}(\Omega)}<\infty$. Then $(w_t)_{t\in T}\in Z_1$ and thus $\sum_{t\in T}\tau_t(w_t)=\sum_{t\in T}\int_{\partial\Omega-E} p_{2,t}\,\gamma w_t\,dH_{n-1}<\infty$. This is a contradiction. It follows that $\sum_{t\in T}\|p_{2,t}\|_{L^\infty(\partial\Omega)}$ is finite and $(0, p_{2,t})_{t\in T}\in X$. This shows that $L_X(P^2\cap X)$

is closed in Z_1^*.

It follows from the relation $P^1 \cap (-p^2) = \{0\}$ in (3.13) that $L_X(P^1) \cap (-L_X(P^2 \cap X)) = \{0\}$. Hence in virtue of Dieudonné [4; Proposition 1], $L_X(P^1) + L_X(P^2 \cap X)$ is closed. Since $L_X(P) = L_X(P^1 + P^2 \cap X) = L_X(P^1) + L_X(P^2 \cap X)$ by (3.13), we conclude that $L_X(P)$ is closed. This completes the proof.

We set

$$Q(0) = \{(S_t)_{t\in T};\ S_t \subset \Omega,\ \chi_{S_t} \in BV(\Omega) \quad \text{for all } t \in T\}\ ,$$
$$Z_0 = \{(z_t)_{t\in T} \in Z;\ (z_t)_{t\in T} = (\chi_{S_t})_{t\in T} \quad \text{or} \quad (z_t)_{t\in T} = (-\chi_{S_t})_{t\in T} \quad \text{for some} \quad (S_t)_{t\in T} \in Q(0)\}$$

and denote the problem corresponding to (MC) by (MC_0). Theorem 3.4 gives some conditions under which the value MC_0 of (MC_0) equals MF_0^*.

The formulation of (MF_0) is owing to an advice given by H. Aikawa at Gakushuin University.

4. The first max-flow min-cut theorem

In this section, we consider a special case of (MF_0). Let $E = \partial\Omega$ and let $T, X, Y, Y_1, Z, Z_1, L_X, L_Y, \Omega', A', h$ be as defined for (MF_0) and (MF_0^*) in §3.

Let α_0, α_0' be nonnegative functions in $L^\infty(\partial\Omega)$, let Γ_0 be a set-valued mapping from Ω to R^n which satisfies (2.1), (2.2), (2.3) and define K_0, K_0' by

$$K_0 = K_{\Gamma_0},\ K_0' = \{\kappa \in L^\infty(\partial\Omega);\ -\alpha_0 \le \kappa \le \alpha_0'\ H_{n-1}\text{-a.e. on } \partial\Omega\}$$

respectively. Let Λ be a Borel set such that $A' \subset \Lambda \subset \partial\Omega$, $F \in L^n(\Omega)$, $f \in L^\infty(\Lambda)$ and set

$$K = \{(\sigma_t, \kappa_t)_{t\in T} \in Y_1;\ \sigma_0 \in K_0,\ \kappa_0 \in K_0' \quad \text{and} \quad \sigma_t = 0 \text{ a.e. on } \Omega\ ,\ \kappa_t = 0\ H_{n-1}\text{-a.e. on } \partial\Omega \quad \text{for all } t \ge 1\}\ ,$$
$$P = \{(p_{1,t}, p_{2,t})_{t\in T} \in X;\ p_{1,0} = \lambda F \text{ a.e. on } \Omega,\ p_{2,0} = \lambda f\ H_{n-1}\text{-a.e. on } \Lambda \quad \text{for some } \lambda \ge 0 \text{ and } p_{1,t} = 0 \text{ a.e. on } \Omega\ ,\ p_{2,t} = 0\ H_{n-1}\text{-a.e. on } \partial\Omega \quad \text{for all } t \ge 1\}\ .$$

Since Γ_0 satisfies (2.1)–(2.3), K is a compact set in Y_1. We denote the problems (MF_0), (MF_0^*), the quantities MF_0, MF_0^* for the above data by (MF_1), (MF_1^*), MF_1, MF_1^* respectively. Furthermore let Z_0 be a subset of Z as defined at the end of §3 and denote (MC_0), MC_0 for the above data by (MC_1), MC_1 respectively.

We define a max-flow problem of Strang's type and its dual problem as follows:

($\mathrm{M\Phi}_1$) Maximize $\lambda \int_{\Omega'} F dx + \lambda \int_{A'} f dH_{n-1}$ subject to $\lambda \ge 0$,

div $\sigma=-\lambda F$ a.e. on Ω and $\sigma\cdot\nu-\kappa=\lambda f\ H_{n-1}$-a.e. on Λ
for some $(\sigma,\kappa)\in K_0\times K_0'$.

($M\Phi_1^*$) Minimize $\psi_0(u)+\zeta_0(\gamma u)$ subject to $u\in BV(\Omega)$,

$$L(u)\geq\int_{\Omega'}Fdx+\int_{A'}fdH_{n-1}\text{ and }\gamma u=0\ H_{n-1}\text{-a.e. on }\partial\Omega-\Lambda,$$

where $L(u)=\int_\Omega Fudx+\int_\Lambda f\gamma duH_{n-1}$, $\psi_0(u)=\psi_{\Gamma_0}(u)$ and

$$\zeta_0(\gamma u)=\sup_{\kappa\in K_0'}\int_{\partial\Omega}-\kappa\gamma udH_{n-1}=\int_{\partial\Omega}(\alpha_0\,\gamma u^++\alpha_0'\,\gamma u^-)\,dH_{n-1}\,.$$

We denote the values of ($M\Phi_1$), ($M\Phi_1^*$) by $M\Phi_1$, $M\Phi_1^*$ respectively. Let $\sigma\in K_0$. If there are $\lambda\geq 0$ and $\kappa\in K_0'$ such that div $\sigma=-\lambda F$ a.e. on Ω and $\sigma\cdot\nu-\kappa=\lambda f$ H_{n-1}-a.e. on Λ, then we call σ a feasible flow of ($M\Phi_1$). Furthermore if $\lambda(\int_{\Omega'}Fdx+\int_{A'}fdH_{n-1})=M\Phi_1$, then σ is called an optimal flow of ($M\Phi_1$). We observe that $M\Phi_0$ corresponding to the above data is equal to $M\Phi_1$.

Using Proposition 3.5, we obtain

Lemma 4.1. $M\Phi_1=MF_1$.

For the dual problem, we have

Lemma 4.2. *The inequalities* $MF_1^*\leq M\Phi_1^*<\infty$ *and*

$$\Psi_K((z_t)_{t\in T})\leq\psi_0(z_0)+\zeta_0(\gamma z_0)$$

hold for all $(z_t)_{t\in T}\in Z$. *If* Γ_0 *satisfies* (2.4), *then* $MF_1^*=M\Phi_1^*$ *and*

$$\Psi_K((z_t)_{t\in T})=\psi_0(z_0)+\zeta_0(\gamma z_0)$$

for all $(z_t)_{t\in T}\in Z$.

Proof. Let us show that $M\Phi_1^*$ is finite. Set $a=\int_{\Omega'}Fdx+\int_{A'}fdH_{n-1}$. If $a=0$, then $u=0$ is a feasible element of ($M\Phi_1^*$). If $a\neq 0$, then $\int_\Omega|F|dx+\int_\Lambda|f|dH_{n-1}>0$. In case $\int_\Omega|F|dx>0$, there exists $u\in C_0^\infty(\Omega)$ such that $L(u)=\int_\Omega Fudx>a$ so that u is feasible. In case $\int_\Omega|F|dx=0$ and $\int_\Lambda|f|dH_{n-1}>0$, there exists $u\in W^{1,1}(\Omega)$ such that $L(u)=\int_\Lambda f\gamma udH_{n-1}>a$ and $\gamma u=0$ H_{n-1}-a.e. on $\partial\Omega-\Lambda$. This shows that u is a feasible element of ($M\Phi_1^*$). Since $M\Phi_1^*\geq 0$, $M\Phi_1^*$ is finite.

Let $(z_t)_{t\in T}\in Z$. Then in virtue of Lemma 2.6,

$$\Psi_K((z_t)_{t\in T})\leq\psi_0(z_0)+\zeta_0(\gamma z_0)\,.$$

Furthermore from the second part of Lemma 2.6 it follows that

$$\Psi_K((z_t)_{t\in T})=\psi_0(z_0)+\zeta_0(\gamma z_0)$$

if Γ_0 satisfies (2.4).

Let z_0 be a feasible element of $(M\Phi_1^*)$ and set $z_t=0$ for all $t\geq 1$. Then $(z_t)_{t\in T}$ is a feasible element of (MF_1^*). Thus from the fact stated above it follows that $MF_1^*\leq M\Phi_1^*$.

Conversely let $(z_t)_{t\in T}\in Z$ be a feasible element of (MF_1^*). Then it is easy to see that z_0 is a feasible element of ($\mathrm{M}\Phi_1^*$). If Γ_0 satisfies (2.4), then $M\Phi_1^*\leq MF_1^*$ and hence $MF_1^*=M\Phi_1^*$.

Using Theorem 3.6, we shall prove

Lemma 4.3. $MF_1=MF_1^*$.

Proof. To prove this lemma, it suffices to check the conditions in Theorem 3.6. We note that the topologies on Y_1, Z_1 and Z_1^* are defined before Theorem 3.6. To check (3.1), we set two cones P^1, P^2 in $(L^n(\Omega)\times L^\infty(\partial\Omega))^T$ as follows:

$$\begin{aligned}P^1 = \{&(p_{1,t}, p_{2,t})_{t\in T}\in(L^n(\Omega)\times L^\infty(\partial\Omega))^T; p_{1,0} = \lambda F \text{ a.e. on } \Omega,\\ &P_{2,0} = \lambda f\, H_{n-1}\text{-a.e. on } \Lambda,\ p_{2,0} = 0\ H_{n-1}\text{-a.e. on } \partial\Omega-\Lambda,\\ &p_{1,t} = 0 \text{ a.e. on } \Omega \text{ and } p_{2,t} = 0\ H_{n-1}\text{-a.e. on } \partial\Omega \quad \text{for all} \quad t\geq 1\},\\ P^2 = \{&(p_{1,t}, p_{2,t})_{t\in T}\in(L^n(\Omega)\times L^\infty(\partial\Omega))^T; p_{1,t} = 0 \text{ a.e. on } \Omega\\ &\text{for all} \quad t\in T,\ p_{2,0} = 0\ H_{n-1}\text{-a.e. on } \Lambda \text{ and } p_{2,t} = 0\\ &H_{n-1}\text{-a.e. on } \partial\Omega \quad \text{for all} \quad t\geq 1\}.\end{aligned}$$

Then P^1, P^2 satisfy conditions (3.10)-(3.13) in Lemma 3.7. Hence in virtue of Lemma 3.7, $L_X(P)$ is closed in Z_1^*. It follows that condition (3.1) is satisfied.

To check (3.2) we let $p\in P$ and let λ be a nonnegative number such that $p_{1,0}=\lambda F$ a.e. on Ω and $p_{2,0}=\lambda f\ H_{n-1}$-a.e. on Λ. Since the set W of all feasible elements of (MF_1^*) is given by

$$\begin{aligned}W= &\{z\in Z;\ L_X(p, z)\geq h(p) \quad \text{for all } p\in P\}\\ = &\{(z_t)_{t\in T}\in Z;\ \gamma z_0 = 0\ H_{n-1}\text{-a.e on } \partial\Omega-\Lambda \quad \text{and}\\ &L(z_0)\geq\int_{\Omega'} Fdx+\int_{A'} fdH_{n-1}\},\end{aligned}$$

it follows that

$$h(p) = \inf_{(z_t)_{t\in T}\in W\cap Z_1} \lambda L(z_0) = \inf_{(z_t)_{t\in T}\in W\cap Z_1} L_X(p, (z_t)_{t\in T})$$

and (3.2) is fulfilled. Finally we note that (3.9) is evidently satisfied. Thus applying Theorem 3.6 to (MF_1) and (MF_1^*) we conclude the proof.

Now we obtain a duality theorem.

Theorem 4.4. $MF_1=MF_1^*=M\Phi_1=M\Phi_1^*$. *Furthermore they are all finite and each of* (MF_1) *and* ($M\Phi_1$) *has an optimal solution.*

Proof. In virtue of Lemmas 4.1, 4.2 and 4.3, $M\Phi_1=MF_1=MF_1^*\leq M\Phi_1^*$

$<\infty$. Since $M\Phi_1\geq 0$, all the quantities in the theorem are finite. We have seen in the proof of Lemma 4.3 that we can apply Theorem 3.6. Therefore (MF_1) has an optimal solution. It is easy to see that also $(\mathrm{M}\Phi_1)$ has an optimal solution.

Next we prove $M\Phi_1=M\Phi_1^*$. If Γ_0 satisfies (2.4), then noting Lemma 4.2 we obtain $M\Phi_1=M\Phi_1^*$. In the general case, since $M\Phi_1\leq M\Phi_1^*$, it suffices to prove $M\Phi_1\geq M\Phi_1^*$. If $\int_{\Omega'}Fdx+\int_{A'}fdH_{n-1}=0$, then $M\Phi_1=M\Phi_1^*=0$. Hence we assume that $\int_{\Omega'}Fdx+\int_{A'}fdH_{n-1}\neq 0$. We set $\Gamma_{0j}(x)=\Gamma_0(x)+\overline{B(0, 1/j)}$ for all $x\in\Omega$ and j. Using Γ_{0j} instead of Γ_0, we can define two problems similar to $(\mathrm{M}\Phi_1)$ and $(\mathrm{M}\Phi_1^*)$. We denote the two values by M_j and M_j^* respectively. Since Γ_{0j} satisfies (2.4), $M_j=M_j^*$ for all j. Choose $\lambda_j\geq 0$ and $(\sigma^j, \kappa^j)\in K_{\Gamma_{0j}}\times K_0'$ for each j such that $-\mathrm{div}\,\sigma^j=\lambda_j F$ a.e. on Ω, $\sigma^j\cdot\nu-\kappa^j=\lambda_j f$ H_{n-1}-a.e. on Λ and $\lambda_j(\int_{\Omega'}Fdx+\int_{A'}fdH_{n-1})>M_j-1/j$. Since K_0 and K_0' are weak*-compact sets in $L^\infty(\Omega; R^n)$ and $L^\infty(\partial\Omega)$ respectively, there is a subsequence $\{(\sigma^{j'}, \kappa^{j'})\}$ of $\{(\sigma^j, \kappa^j)\}$ such that $\sigma^{j'}\to\sigma\in K_0$ and $\kappa^{j'}\to\kappa\in K_0'$. Then $-\mathrm{div}\,\sigma=\lambda F$ a.e. on Ω and $\sigma\cdot\nu-\kappa=\lambda f$ H_{n-1}-a.e. on Λ, where $\lambda=\lim_{j\to\infty}\lambda_j=\lim_{j\to\infty}M_j/(\int_{\Omega'}Fdx+\int_{A'}fdH_{n-1})$. Thus $M\Phi_1\geq\lim_{j\to\infty}M_j=\lim_{j\to\infty}M_j^*\geq M\Phi_1^*$. It follows that $M\Phi_1=M\Phi_1^*$. This completes the proof.

Before defining a min-cut problem corresponding to $(\mathrm{M}\Phi_1)$, we prove two lemmas.

Lemma 4.5. *Let $u\in BV(\Omega)$, and set $N_r=\{x\in\Omega; u(x)\geq r\}$ and $N_r'=\{x\in\partial\Omega; \gamma u\geq r\}$ for $r\in R$. Then the relation*

$$\chi_{N_r'}=\chi_{\partial^*N_r\cap\partial\Omega}=\gamma\chi_{N_r}$$

holds H_{n-1}-a.e. on $\partial\Omega$ for a.e. $r\in R$.

Proof. We use u^* which is defined in §2. First we note that $\chi_{N_r}\in BV(\Omega)$ for a.e. $r\in R$ and $H_{n-1}(\partial\Omega-\partial^*\Omega)=0$. Let r be such a real number. Suppose there exists $x\in\partial^*\Omega$ with $u^*(x)>r$, and take r' such that $u^*(x)>r'>r$ and $\chi_{N_{r'}}\in BV(\Omega)$. Then $x\in\partial^*N_{r'}\cap\partial^*\Omega\subset\partial^*N_r\cap\partial^*\Omega$ so that

$$\{y\in\partial^*\Omega; u^*(y)>r\}\subset\partial^*N_r\cap\partial^*\Omega .$$

By the definition of u^* we have

$$\partial^*N_r\cap\partial^*\Omega\subset\{y\in\partial^*\Omega; u^*(y)\geq r\} .$$

Since $u^*\in L^1(\partial\Omega)$, $H_{n-1}(\{y\in\partial^*\Omega; u^*(y)=q\})=0$ for a.e. $q\in R$. Thus for a.e. $q\in R$, $\chi_{\{u^*\geq q\}}=\chi_{\partial^*N_q\cap\partial\Omega}$ H_{n-1}-a.e. on $\partial^*\Omega$. It is easy to see $\chi_{\partial^*N_q\cap\partial\Omega}=\chi_{N_q}^*$ for a.e. $q\in R$. Noting that $u^*=\gamma u$ holds H_{n-1}-a.e. on $\partial\Omega$ as was stated in §2, we conclude the proof.

Lemma 4.6. *Let $u\in BV(\Omega)$. Then*

$$L(u)=\int_0^\infty L(\chi_{N_r})\,dr+\int_{-\infty}^0 L(-\chi_{M_r})\,dr\,,$$

where $N_r=\{x\in\Omega;\, u(x)\geq r\}$ and $M_r=\{x\in\Omega;\, u(x)\leq r\}$. Furthermore if $\int_\Omega Fdx+\int_\Lambda fdH_{n-1}=0$, then $L(u)=\int_{-\infty}^\infty L(\chi_{N_r})\,dr$.

Proof. We have

$$\int_\Omega Fu^+\,dx=\int_0^\infty dr\int_{\{u\geq r\}}Fdx,\ \int_\Omega Fu^-\,dx=\int_0^\infty dr\int_{\{u\leq -r\}}Fdx$$

by [11; Theorem 1.2.3]. Using Lemma 4.5, we can show that

$$\int_\Lambda f(\gamma u)^+dH_{n-1}=\int_0^\infty dr\int_{\{\gamma u\geq r\}\cap\Lambda}fdH_{n-1}=\int_0^\infty dr\int_\Lambda f\gamma\chi_{N_r}\,dH_{n-1}\,,$$
$$\int_\Lambda f(\gamma u)^-dH_{n-1}=\int_0^\infty dr\int_{\{\gamma u\leq -r\}\cap\Lambda}fdH_{n-1}=\int_0^\infty dr\int_\Lambda f\gamma\chi_{M_{-r}}\,dH_{n-1}\,.$$

Thus

$$\begin{aligned}L(u)=&\int_0^\infty\Big(\int_\Omega F\chi_{N_r}\,dx+\int_\Lambda f\gamma\chi_{N_r}\,dH_{n-1}\Big)\,dr\\&-\int_0^\infty\Big(\int_\Omega F\chi_{M_{-r}}\,dx+\int_\Lambda f\gamma\chi_{M_{-r}}\,dH_{n-1}\Big)\,dr\,.\end{aligned}$$

It follows that $L(u)=\int_0^\infty L(\chi_{N_r})\,dr+\int_{-\infty}^0 L(-\chi_{M_r})\,dr$.

Next assume $\int_\Omega Fdx+\int_\Lambda fdH_{n-1}=0$. Since $\gamma\chi_{N_{-r}}+\gamma\chi_{M_{-r}}=\gamma(\chi_{N_{-r}}+\chi_{M_{-r}})=1$ H_{n-1}-a.e. on $\partial\Omega$ for a.e. r,

$$\begin{aligned}\int_\Omega F\chi_{M_{-r}}\,dx+\int_\Lambda f\gamma\chi_{M_{-r}}\,dH_{n-1}&=-\int_\Omega F\chi_{N_{-r}}\,dx-\int_\Lambda f\gamma\chi_{N_{-r}}\,dH_{n-1}\\&=-L(\chi_{N_{-r}})\,.\end{aligned}$$

Hence

$$L(u)=\int_{-\infty}^\infty dr\Big(\int_\Omega F\chi_{N_r}\,dx+\int_\Lambda f\gamma\chi_{N_r}\,dH_{n-1}\Big)=\int_{-\infty}^\infty L(\chi_{N_r})\,dr\,.$$

Now we define a min-cut problem $(\mathrm{M}\Gamma_1)$ associated with $(\mathrm{M}\Phi_1)$. Let

$$Q(1)=\{S\subset\Omega;\ \chi_S\in BV(\Omega),\ \gamma\chi_S=0\quad H_{n-1}\text{-a.e. on }\partial\Omega-\Lambda\}\,.$$

Then $(\mathrm{M}\Gamma_1)$ is defined as follows:

($\mathrm{M}\Gamma_1$) Minimize $(\int_{\Omega'} Fdx+\int_{A'} fdH_{n-1})\,(\psi_0(u)+\zeta_0(\gamma u))/L(u)$
subject to the constraint that $u=\chi_S$ or $u=-\chi_S$ for some $S\in Q(1)$ and $L(u)>0$.

We denote the value of ($\mathrm{M}\Gamma_1$) by $M\Gamma_1$. If $S\in Q(1)$ and if χ_S or $-\chi_S$ is a feasible element of ($\mathrm{M}\Gamma_1$), then S is called a feasible cut of ($\mathrm{M}\Gamma_1$).

We shall assume

$$\int_{\Omega'} Fdx+\int_{A'} fdH_{n-1}>0 \tag{4.1}$$

in Lemmas 4.7, 4.8 and Theorem 4.9. The case when (4.1) does not hold will be examined in Remark 4.10.

Lemma 4.7. *Assume* (4.1). *Let* $(z_t)_{t\in T}\in Z$ *and* Π *be the functional on* Z *as defined before Theorem* 3.4. *Then*

$$\Pi((z_t)_{t\in T})=(\int_{\Omega'} Fdx+\int_{A'} fdH_{n-1})^{-1}L(z_0)$$

if $\gamma z_0=0$ H_{n-1}*-a.e. on* $\partial\Omega-\Lambda$ *and* $L(z_0)\geq 0$, *and* $\Pi((z_t)_{t\in T})=-\infty$ *otherwise.*

Furthermore $MC_1\leq M\Gamma_1$, *and* $MC_1=M\Gamma_1$ *if* Γ_0 *satisfies* (2.4).

Proof. Since (4.1) is satisfied, the first assertion follows from the definition of Π, L_X and h. Assuming $M\Gamma_1<\infty$, let z_0 be a feasible element of ($\mathrm{M}\Gamma_1$) and set $z_t=0$ for all $t\geq 1$. Then $\Pi((z_t)_{t\in T})>0$ and thus $(z_t)_{t\in T}$ is a feasible element of (MC_1). Hence the inequality $MC_1\leq M\Gamma_1$ follows from Lemma 4.2.

Conversely assume that Γ_0 satisfies (2.4) and suppose that there exists a feasible element $(z_t)_{t\in T}$ of (MC_1). By the aid of the first part of the present lemma we see that z_0 is a feasible element of ($\mathrm{M}\Gamma_1$). Thus Lemma 4.2 yields $MC_1\geq M\Gamma_1$. This completes the proof.

Lemma 4.8. *Assume that* Γ_0 *satisfies* (2.4) *and that* (4.1) *is fulfilled. Then the equality* $MF_1^*=MC_1$ *holds.*

Proof. In order to prove $MF_1^*=MC_1$ it suffices to check the conditions in Theorem 3.4. By (4.1), $h(p)\geq 0$ for all $p\in P$. Suppose there exists $z=(z_t)_{t\in T}\in Z$ with $\Pi(z)>0$, and set $z_0^j=z_0$ and $z_t^j=0$ for all $t\geq 1$ and j. We set $z_{t,r}^j=\chi_{N_{t,r}^j}$ for $r\geq 0$ and $z_{t,r}^j=-\chi_{M_{t,r}^j}$ for $r<0$, where $N_{t,r}^j=\{x\in\Omega;\, z_t^j(x)\geq r\}$ and $M_{t,r}^j=\{x\in\Omega;\, z_t^j\leq r\}$. Then $(z_{t,r}^j)_{t\in T}\in Z_0$ for a.e. $r\in R$. As in the proof of Lemma 4.6 we have

$$\begin{aligned}\zeta_0(\gamma z_0^j)&=\int_{\partial\Omega}\alpha_0(\gamma z_0^j)^+dH_{n-1}+\int_{\partial\Omega}\alpha_0'(\gamma z_0^j)^-dH_{n-1}\\&=\int_0^\infty dr\int_{\partial\Omega}\alpha_0\,\gamma\chi_{N_{0,r}^j}\,dH_{n-1}+\int_{-\infty}^0 dr\int_{\partial\Omega}\alpha_0'\,\gamma\chi_{M_{0,r}^j}\,dH_{n-1}\end{aligned}$$

$$= \int_{-\infty}^{\infty} \zeta_0(\gamma z^j_{0,r})\, dr\,.$$

It follows from this relation and Proposition 2.4 that

$$\begin{aligned}\Psi_K((z^j_t)_{t\in T}) &= \psi_0(z^j_0)+\zeta_0(\gamma z^j_0)\\ &= \int_{-\infty}^{\infty} (\psi_0(z^j_{0,r})+\zeta_0(\gamma z^j_{0,r}))\, dr = \int_{-\infty}^{\infty} \Psi_K((z^j_{t,r})_{t\in T})\, dr\end{aligned}$$

and from Lemmas 4.6 and 4.7 that

$$\Pi((z^j_t)_{t\in T}) = \int_{-\infty}^{\infty} \Pi((z^j_{t,r})_{t\in T})\, dr\,.$$

Thus $\{(z^j_t)_{t\in T}\}$ satisfies the conditions required in Theorem 3.4 and this theorem yields $MF_1^*=MC_1$. This completes the proof.

Now we obtain the first max-flow min-cut theorem.

Theorem 4.9. *Assume* (4.1). *Then* $M\Phi_1=M\Gamma_1=MC_1$.

Proof. If Γ_0 satisfies (2.4), then in virtue of Lemmas 4.2, 4.7 and 4.8 $M\Phi_1^*=M\Gamma_1$ and hence by Theorem 4.4 $M\Phi_1=M\Gamma_1$. In the general case, one can prove the equality along the same lines as in the proof of Theorem 4.4. From Theorem 4.4 and Lemma 4.7 it follows that $M\Phi_1=MF_1^*\le MC_1\le M\Gamma_1$. Hence $M\Phi_1=M\Gamma_1=MC_1$ and the proof is completed.

REMARK 4.10. Suppose (4.1) does not hold, namely, $\int_{\Omega'} F dx+\int_{A'} f dH_{n-1}\le 0$. Then $M\Phi_1=0$. We note that $\Pi(z)=\infty$ for all $z\in Z_0$. Thus each element in Z_0 is a feasible element of (MC_1) and $MC_1=0$. Let us examine ($\mathrm{M}\Gamma_1$). If $\int_\Omega |F|\, dx+\int_\Lambda |f|\, dH_{n-1}=0$, then there is no feasible element of ($\mathrm{M}\Gamma_1$) and hence $M\Gamma_1=\infty$. If $\int_\Omega |F|\, dx+\int_\Lambda |f|\, dH_{n-1}>0$ and $\int_{\Omega'} F dx+\int_{A'} f dH_{n-1}=0$, then $M\Gamma_1=0$. If $\int_{\Omega'} F dx+\int_{A'} f dH_{n-1}<0$, then $M\Gamma_1\le 0$ and both the case when $M\Gamma_1=0$ and the case when $M\Gamma_1<0$ may happen.

REMARK 4.11. Assume that one of the following two conditions is satisfied:

(H1-a) $\Lambda = \partial\Omega,\ \int_\Omega F dx+\int_\Lambda f dH_{n-1} = 0$ and $\alpha_0 = \alpha_0' = 0$ H_{n-1}-a.e. on $\partial\Omega$.

(H1-b) $F\ge 0$ a.e. on Ω and $f\ge 0$ H_{n-1}-a.e. on Λ.

Then

$$M\Gamma_1 = \inf\{(\int_{\Omega'} F dx+\int_{A'} f dH_{n-1})\, (\psi_0(\chi_S)+\zeta_0(\gamma\chi_S))/L(\chi_S);\ S\in Q(1) \text{ such that } L(\chi_S)>0\}\,. \tag{4.2}$$

In fact if (H1-a) is satisfied and $u=-\chi_S$ is a feasible element of $(M\Gamma_1)$, then $1+u=\chi_{\Omega-S}$ is also a feasible element of $(M\Gamma_1)$, $\psi_0(u)=\psi_0(1+u)$ and $L(u)=L(1+u)$. If (H1-b) is satisfied and u is a feasible element of $(M\Gamma_1)$, then $L(u)>0$ and thus $u\geq 0$. Hence (4.2) holds.

5. The second max-flow min-cut theorem

In this section, we consider another special case of (MF_0). Let A, B be disjoint Borel subsets of $\partial\Omega$, take $\partial\Omega-(A\cup B)$ as E, take ϕ as Ω' and take A as A' in §3. Furthermore let T, X, Y, Y_1, Z, Z_1, L_X, L_Y, h be as defined for (MF_0) and (MF_0^*) in §3.

Let α_t, α'_t be nonnegative functions in $L^\infty(\partial\Omega)$ such that $\alpha_t=\alpha'_t=0$ H_{n-1}-a.e. on $A\cup B$ and Γ_t be a set-valued mapping from Ω to R^n which satisfies (2.1), (2.2) and (2.3) for each $t\in T$. We set

$$K_t = K_{\Gamma_t} \quad \text{and} \quad K'_t = \{\kappa\in L^\infty(\partial\Omega);\ -\alpha_t\leq\kappa\leq\alpha'_t\ H_{n-1}\text{-a.e. on } \partial\Omega\}\ .$$

In the present case, we take

$$K = \{(\sigma_t, \kappa_t)_{t\in T}\in Y_1;\ \sigma_t\in K_t,\ \textstyle\sum_{s=0}^t \kappa_s\in K'_t \quad \text{for all} \quad t\in T\}\ ,$$
$$P = \{(p_{1,t}, p_{2,t})_{t\in T}\in X;\ p_{1,t} = 0 \text{ a.e. on } \Omega,\ p_{2,t}\geq 0\ H_{n-1}\text{-a.e.}$$
$$\text{on } A \text{ and } p_{2,t}=0\ H_{n-1}\text{-a.e. on } E \quad \text{for all} \quad t\in T\}\ .$$

With these data we consider the problems corresponding to (MF_0), (MF_0^*), $(M\Phi_0)$ and denote them by (MF_2), (MF_2^*), $(M\Phi_2)$ respectively, In addition we denote by (MC_2) the problem which corresponds to (MC_0) given at the end of §3. We denote the values of (MF_2), (MF_2^*), $(M\Phi_2)$, (MC_2) by MF_2, MF_2^*, $M\Phi_2$, MC_2 respectively. Throughout this section we assume that

$$(5.1) \qquad \textstyle\sum_{t\in T} \sup_{\sigma\in K_t} \|\sigma\|_{L^\infty(\Omega;R^n)}<\infty \quad \text{and}$$
$$\lim_{t\to\infty}(\|\alpha_t\|_{L^\infty(\partial\Omega)}+\|\alpha'_t\|_{L^\infty(\partial\Omega)}) = 0\ .$$

Then K is a compact convex subset of Y_1.

Now we define a max-flow problem $(M\Psi_2)$ of Iri's type and its dual problem $(M\Psi_2^*)$ as follows:

$(M\Psi_2)$ Maximize $\sum_{t\in T}\int_A \sigma_t\cdot\nu dH_{n-1}$ subject to $(\sigma_t)_{t\in T}\in\Pi_{t\in T} K_t$, $\operatorname{div}\sigma_t = 0$ a.e. on Ω, $\sigma_t\cdot\nu\geq 0$ H_{n-1}-a.e. on A and $(\sum_{s=0}^t \sigma_s\cdot\nu)\,\chi_E\in K'_t$ for all $t\in T$.

$(M\Psi_2^*)$ Minimize $\sum_{t\in T}(\psi_t(z_t)+\zeta_t(\gamma z_t-\gamma z_{t+1}))$ subject to $(z_t)_{t\in T}\in BV(\Omega)^T$, $\gamma z_t\geq 1$ H_{n-1}-a.e. on A and $\gamma z_t = 0$ H_{n-1}-a.e. on B for each $t\in T$,

where $\psi_t(v)=\psi_{\Gamma_t}(v)$ for $v\in BV(\Omega)$ and

$$\zeta_t(\varphi) = \sup_{\kappa\in K_t'} \int_{\partial\Omega} -\kappa\varphi dH_{n-1} = \int_{\partial\Omega} \alpha_t\, \varphi^+ dH_{n-1} + \int_{\partial\Omega} \alpha_t'\, \varphi^- dH_{n-1}$$

for $\varphi\in L^1(\partial\Omega)$.

We denote the values of $(\mathrm{M}\Psi_2)$, $(\mathrm{M}\Psi_2^*)$ by $M\Psi_2$, $M\Psi_2^*$ respectively. We call each feasible element $(\sigma_t)_{t\in T}$ of $(\mathrm{M}\Psi_2)$ a feasible flow of $(\mathrm{M}\Psi_2)$. We note that $(\mathrm{M}\Psi_2)$ is slightly different from $(\mathrm{M}\Phi_2)$.

An application of Theorem 3.6 yields

Lemma 5.1. $MF_2 = MF_2^*$.

Proof. We check the conditions in Theorem 3.6. From (5.1) condition (3.9) directly follows. Hence we prove that (3.1) and (3.2) are satisfied. Let $p=(0, p_{2,t})_{t\in T}\in P$. Since $h(p)=\sum_{t\in T}\int_A p_{2,t}\, dH_{n-1}$ and the set W of all feasible elements of (MF_2^*) is given by

$$W= \{z = (z_t)_{t\in T}\in Z;\ \gamma z_t\geq 1\ H_{n-1}\text{-a.e. on } A \quad\text{and}\quad \gamma z_t = 0\ H_{n-1}\text{-a.e. on } B\}\ ,$$

$\inf_{z\in W\cap Z_1} L_X(p, z)=h(p)$. Thus (3.2) follows.

To prove that (3.1) is satisfied, we set two cones P^1 and P^2 in $(L^n(\Omega)\times L^\infty(\partial\Omega))^T$ as follows:

$$P^1 = \{(p_{1,t}, p_{2,t})_{t\in T}\in(L^n(\Omega)\times L^\infty(\partial\Omega))^T;\ p_{1,t} = 0 \text{ a.e. on } \Omega \text{ and } p_{2,t} = 0\ H_{n-1}\text{-a.e. on } \partial\Omega \quad \text{for all}\quad t\in T\}\ ,$$

$$P^2 = \{(p_{1,t}, p_{2,t})_{t\in T}\in(L^n(\Omega)\times L^\infty(\partial\Omega))^T;\ p_{1,t} = 0 \text{ a.e. on } \Omega\ ,\ p_{2,t}\geq 0\ H_{n-1}\text{-a.e. on } A \quad\text{and}\quad p_{2,t} = 0\ H_{n-1}\text{-a.e. on } E \quad \text{for all}\quad t\in T\}\ .$$

Then P^1, P^2 satisfy conditions (3.10)–(3.13) in Lemma 3.7. Thus Lemma 3.7 yields that $L_X(P)$ is closed in Z_1^* and (3.1) is satisfied.

Hence we can apply Theorem 3.6 to (MF_2) and (MF_2^*) and obtain $MF_2= MF_2^*$. This completes the proof.

Let $(z_t)_{t\in T}\in Z$ and let Π be a functional on Z as defiend before Theorem 3.4. We set

$$\operatorname{essinf}_{x\in A} \gamma z_t(x) = \sup\{r\in R;\ H_{n-1}(A\cap M_{t,r}') = 0\}\ ,$$

where $M_{t,r}'=\{x\in\partial\Omega;\ \gamma z_t(x)\leq r\}$. Then it is easy to prove that $\Pi((z_t)_{t\in T})= \inf_{t\in T} \operatorname{essinf}_{x\in A} \gamma z_t(x)$ if $\gamma z_t=0$ H_{n-1}-a.e. on B and $\gamma z_t\geq 0$ H_{n-1}-a.e. on A for all $t\in T$, and $\Pi((z_t)_{t\in T})=-\infty$ otherwise.

We prove

Lemma 5.2. *Let $(z_t)_{t\in T}\in BV(\Omega)^T$ be a feasible element of $(M\Psi_2^*)$ such that*

$\sum_{t\in T}(\psi_t(z_t)+\zeta_t(\gamma z_t-\gamma z_{t+1}))$ is finite. We set $\Pi_0=\inf_{t\in T}$ essinf$_{x\in A}\,\gamma z_t(x)$. Then there is a sequence $\{(z_t^j)_{t\in T}\}$ in Z such that

$$\limsup_{t\to\infty}\sum_{t\in T}(\psi_t(z_t^j)+\zeta_t(\gamma z_t^j-\gamma z_{t+1}^j))$$
$$\leq\sum_{t\in T}(\psi_t(z_t)+\zeta_t(\gamma z_t-\gamma z_{t+1}))/\Pi_0$$

and $(z_t^j)_{t\in T}$ is a feasible element of $(M\Psi_2^*)$ satisfying the following condition for each j:

(5.2) *There is $t_j\in T$ such that $z_t^j=z_{t_j}^j$ for all $t\geq t_j$.*

Furthermore if z_t is a characteristic function for each $t\in T$, then we can take $\{(z_t^j)_{t\in T}\}$ such that z_t^j is a characteristic function for all j and t.

Proof. Let $(z_t)_{t\in T}\in BV(\Omega)^T$ be a feasible element of $(M\Psi_2^*)$. Considering z_t/Π_0, we may assume that $\Pi_0=1$. We set $\bar{z}_t=\min(\max(z_t,0),1)$. Then using Proposition 2.4 we see that $\psi_t(z_t)\geq\psi_t(\bar{z}_t)$. Furthermore using the fact that $\gamma u=u^*$ H_{n-1}-a.e. on $\partial\Omega$ for any $u\in BV(\Omega)$, we obtain $\gamma\bar{z}_t=\min(\max(\gamma z_t,0),1)$ H_{n-1}-a.e. on $\partial\Omega$. Thus $(\gamma\bar{z}_t-\gamma\bar{z}_{t+1})^+\leq(\gamma z_t-\gamma z_{t+1})^+$ and $(\gamma\bar{z}_t-\gamma\bar{z}_{t+1})^-\leq(\gamma z_t-\gamma z_{t+1})^-$ H_{n-1}-a.e. on $\partial\Omega$. It follows that

$$\sum_{t\in T}(\psi_t(z_t)+\zeta_t(\gamma z_t-\gamma z_{t+1}))\geq\sum_{t\in T}(\psi_t(\bar{z}_t)+\zeta_t(\gamma\bar{z}_t-\gamma\bar{z}_{t+1}))\,.$$

Now we fix an arbitrary characteristic function w in $BV(\Omega)$ such that $\gamma w=1$ H_{n-1}-a.e. on A and $\gamma w=0$ H_{n-1}-a.e. on B. For each positive integer j, we set $z_t^j=\bar{z}_t$ if $0\leq t\leq j$, and $z_t^j=w$ if $t\geq j+1$. Then

$$\sum_{t\in T}(\psi_t(z_t^j)+\zeta_t(\gamma z_t^j-\gamma z_{t+1}^j))$$
$$=\sum_{t=0}^{j}\psi_t(\bar{z}_t)+\sum_{t=j+1}^{\infty}\psi_t(w)+\sum_{t=0}^{j-1}\zeta_t(\gamma\bar{z}_t-\gamma\bar{z}_{t+1})+\zeta_j(\gamma\bar{z}_j-\gamma w)\,.$$

In virtue of (5.1),

$$\sum_{t=j+1}^{\infty}\psi_t(w)\to 0\quad\text{as}\quad j\to\infty$$

and

$$\zeta_j(\gamma\bar{z}_j-\gamma w)\leq\max(||\alpha_j||_{L^\infty(\partial\Omega)},||\alpha_j'||_{L^\infty(\partial\Omega)})(||\gamma\bar{z}_j||_{L^1(\partial\Omega)}+||\gamma w||_{L^1(\partial\Omega)})$$
$$\leq 2H_{n-1}(\partial\Omega)\cdot\max(||\alpha_j||_{L^\infty(\partial\Omega)},||\alpha_j'||_{L^\infty(\partial\Omega)})\to 0$$

as $j\to\infty$. Hence

$$\lim_{j\to\infty}\sum_{t\in T}(\psi_t(z_t^j)+\zeta_t(\gamma z_t^j-\gamma z_{t+1}^j))=\sum_{t\in T}(\psi_t(\bar{z}_t)+\zeta_t(\gamma\bar{z}_t-\gamma\bar{z}_{t+1}))\,.$$

If z_t is a characteristic function for each $t\in T$, then z_t^j is also a characteristic function for each $t\in T$ and j. Hence $\{(z_t^j)_{t\in T}\}$ satisfies the required conditions and the proof is completed.

Now we prove a duality theorem.

Theorem 5.3. *$MF_2=MF_2^*=M\Phi_2=M\Psi_2=M\Psi_2^*$. Furthermore they are all finite and each of (MF_2), $(M\Phi_2)$ and $(M\Psi_2)$ has an optimal solution.*

Proof. Let $z=(z_t)_{t\in T}\in Z$ and $y=(\sigma_t, \kappa_t)_{t\in T}\in K\cap Y$. Then from (5.1) we infer that $\sum_{s=0}^{\infty}\kappa_s=0$ H_{n-1}-a.e. on $\partial\Omega$ and from (3.8) that

$$L_Y(y, z) = \textstyle\sum_{t\in T}\{(\sigma_t\ \nabla z_t)(\Omega)-\int_E \Theta_t(\gamma z_t-\gamma z_{t+1})\,dH_{n-1}\}\ ,$$

where $\Theta_t=\sum_{s=0}^{t}\kappa_s$. Thus using Lemmas 2.6 and 5.2 we see that $MF_2^*\leq M\Psi_2^*$. The equality $MF_2=MF_2^*$ holds according to Lemma 5.1 and $MF_2=M\Phi_2$ follows from Proposition 3.5. Therefore the relation $M\Phi_2=MF_2=MF_2^*\leq M\Psi_2^*$ is derived.

Let us show that $M\Psi_2^*$ is finite. Let w be a function in $BV(\Omega)$ such that $\gamma w=1$ H_{n-1}-a.e. on A and $\gamma w=0$ H_{n-1}-a.e. on B. Taking w as z_t for all $t\in T$, we see that

$$\textstyle\sum_{t\in T}(\psi_t(z_t)+\zeta_t(\gamma z_t-\gamma z_{t+1})) = \sum_{t\in T}\psi_t(w)<\infty$$

by (5.1). Thus $M\Psi_2^*$ is finite.

In case each Γ_t satisfies (2.4), the equality $MF_2^*=M\Psi_2^*$ follows from Lemma 2.6 and the relation $M\Psi_2^*=MF_2^*=MF_2=M\Phi_2$ is obtained. In the general case, we prove the desired equalities along the same lines as in the proof of Theorem 4.4. We set $\Gamma_{tj}(x)=\Gamma_t(x)+\overline{B(0, j^{-1}2^{-t})}$ for each $x\in\Omega$ and positive integer j. By M_j and M_j^* we denote $M\Phi_2$ and $M\Psi_2^*$ corresponding to $\{\Gamma_{tj}\}_{t\in T}$ respectively. As shown above, each M_j^* is finite. Since Γ_{tj} satisfies (2.4), $M_j=M_j^*$. Choose $(\sigma_t^j, \kappa_t^j)_{t\in T}$ in $(L^\infty(\Omega; R^n)\times L^\infty(\partial\Omega))^T$ for each j such that $\sigma_t^j(x)\in\Gamma_{tj}(x)$ for a.e. $x\in\Omega$, $-\alpha_t\leq\sum_{s=0}^{t}\kappa_s^j\leq\alpha_t'$ H_{n-1}-a.e. on $\partial\Omega$ for all $t\in T$, $(-\mathrm{div}\,\sigma_t^j, \sigma_t^j\cdot\nu-\kappa_t^j)_{t\in T}\in P$ and

$$\textstyle\sum_{t\in T}\int_A \sigma_t^j\cdot\nu dH_{n-1}>M_j-1/j\ .$$

Then we may assume that $\{\sigma_t^j\}_j$ converges to an element σ_t in $L^\infty(\Omega; R^n)$ with respect to the weak* topology for each $t\in T$. The relation $(-\mathrm{div}\,\sigma_t^j, \sigma_t^j\cdot\nu-\kappa_t^j)_{t\in T}\in P$ implies that $\mathrm{div}\,\sigma_t^j=0$ a.e. on Ω and $\sigma_t^j\cdot\nu-\kappa_t^j=0$ H_{n-1}-a.e. on E. Hence $\mathrm{div}\,\sigma_t=0$ a.e. on Ω and according to Theorem 2.3 (2) $\sigma_t^j\cdot\nu\to\sigma_t\cdot\nu$ as $j\to\infty$ with respect to the weak* toploogy on $L^\infty(\partial\Omega)$. Furthermore we may assume that $\{\kappa_t^j\}_j$ converges to an element κ_t in $L^\infty(\partial\Omega)$ with respect to the weak* topology for each t. Then $\sigma_t\cdot\nu-\kappa_t\geq 0$ H_{n-1}-a.e. on A, $\sigma_t\cdot\nu-\kappa_t=0$ H_{n-1}-a.e. on E and $(\sigma_t, \kappa_t)_{t\in T}\in K$. We note that $\kappa_t=0$ H_{n-1}-a.e. on $A\cup B$ since $\alpha_t=\alpha_t'=0$ H_{n-1}-a.e. on $A\cup B$ for each $t\in T$. Using the fact stated before Remark 2.12 and the relation $\sigma_t\cdot\nu=\kappa_t$ valid H_{n-1}-a.e. on E, we obtain

$$\sum_{t\in T}\|\sigma_t\cdot\nu-\kappa_t\|_{L^\infty(\partial\Omega)}=\sum_{t\in T}\|\sigma_t\cdot\nu\|_{L^\infty(A\cup B)}\le\sum_{t\in T}\|\sigma_t\|_{L^\infty(\Omega;R^n)}<\infty$$

by (5.1). It follows that $(-\mathrm{div}\,\sigma_t,\sigma_t\cdot\nu-\kappa_t)_{t\in T}\in P$. Hence $(\sigma_t,\kappa_t)_{t\in T}$ is a feasible element of $(M\Phi_2)$. Furthermore using (5.1) we can prove that

$$\sum_{t\in T}\int_A\sigma_t^j\cdot\nu dH_{n-1}\to\sum_{t\in T}\int_A\sigma_t\cdot\nu dH_{n-1}$$

as $j\to\infty$. Hence

$$M\Psi_2^*\le\lim_{j\to\infty}M_j^*=\lim_{j\to\infty}M_j=\lim_{j\to\infty}\sum_{t\in T}\int_A\sigma_t^j\cdot\nu dH_{n-1}$$
$$=\sum_{t\in T}\int_A\sigma_t\cdot\nu dH_{n-1}\le M\Phi_2.$$

It follows that $M\Psi_2^*=MF_2^*=MF_2=M\Phi_2$.

Next we prove that $M\Phi_2\le M\Psi_2\le M\Psi_2^*$. Since $(\sigma_t)_{t\in T}$ is a feasible flow of $(M\Psi_2)$ for each feasible element $(\sigma_t,\kappa_t)_{t\in T}$ of $(M\Phi_2)$, the first inequality directly follows. To prove the second inequality, let $(\sigma_t)_{t\in T}$ be a feasible flow of $(M\Psi_2)$ and $(z_t)_{t\in T}$ be a feasible element of $(M\Psi_2^*)$. We set $\tilde{z}_t=\min(\max(z_t,0),1)$. Then as noted in the proof of Lemma 5.2,

$$\sum_{t\in T}(\psi_t(z_t)+\zeta_t(\gamma z_t-\gamma z_{t+1}))\ge\sum_{t\in T}(\psi_t(\tilde{z}_t)+\zeta_t(\gamma\tilde{z}_t-\gamma\tilde{z}_{t+1})).$$

Thus using Green's formula in Theorem 2.3 (3) we obtain

$$\sum_{t\in T}(\psi_t(z_t)+\zeta_t(\gamma z_t-\gamma z_{t+1}))$$
$$\ge\sum_{t=0}^k\{(\sigma_t\nabla\tilde{z}_t)(\Omega)-\int_E(\sum_{s=0}^t\sigma_s\cdot\nu)(\gamma\tilde{z}_t-\gamma\tilde{z}_{t+1})dH_{n-1}\}$$
$$=\sum_{t=0}^k\{(\sigma_t\nabla\tilde{z}_t)(\Omega)-\int_E\sigma_t\cdot\nu\gamma\tilde{z}_t\,dH_{n-1}\}+\int_E\sum_{s=0}^k\sigma_s\cdot\nu\gamma\tilde{z}_{k+1}\,dH_{n-1}$$
$$=\sum_{t=0}^k\int_A\sigma_t\cdot\nu dH_{n-1}+\int_E\sum_{s=0}^k\sigma_s\cdot\nu\gamma\tilde{z}_{k+1}\,dH_{n-1}$$

for each positive integer k. In virtue of (5.1)

$$\left|\int_E\sum_{s=0}^k\sigma_s\cdot\nu\gamma\tilde{z}_{k+1}\,dH_{n-1}\right|\le\|\sum_{s=0}^k\sigma_s\cdot\nu\|_{L^\infty(E)}\|\gamma\tilde{z}_{k+1}\|_{L^1(E)}$$
$$\le\max(\|\alpha_k\|_{L^\infty(\partial\Omega)},\|\alpha_k'\|_{L^\infty(\partial\Omega)})\cdot H_{n-1}(E)\to 0$$

as $k\to\infty$. Hence

$$\sum_{t\in T}(\psi_t(z_t)+\zeta_t(\gamma z_t-\gamma z_{t+1}))\ge\sum_{t\in T}\int_A\sigma_t\cdot\nu dH_{n-1}.$$

Thus $M\Psi_2\le M\Psi_2^*$. Since $M\Phi_2=M\Psi_2^*$, we conclude that $M\Psi_2=M\Psi_2^*$. Now we have proved that $MF_2=MF_2^*=M\Phi_2=M\Psi_2=M\Psi_2^*$ and the value is finite.

Finally as shown in the proof of Lemma 5.1 one can apply Theorem 3.6 and conclude that (MF_2) has an optimal solution. It is easy to see that also

(MΦ_2) and (MΨ_2) have optimal solutions. This completes the proof.

To define a min-cut problem (MΓ_2) corresponding to (MΨ_2), we set

$$Q(2) = \{S \subset \Omega;\ \chi_S \in BV(\Omega) \ \text{ such that } \ \gamma\chi_S = 1\ H_{n-1}\text{-a.e. on } A \ \text{ and } \ \gamma\chi_S = 0\ H_{n-1}\text{-a.e. on } B\}\ .$$

Then (MΓ_2) is defined as follows:

(MΓ_2) Minimize $\sum_{t\in T}(\psi_t(\chi_{S_t})+\zeta_t(\gamma\chi_{S_t}-\gamma\chi_{S_{t+1}}))$ subject to $(S_t)_{t\in T}\in Q(2)^T$.

We denote the value by $M\Gamma_2$. We call each feasible element $(S_t)_{t\in T}$ of (MΓ_2) a feasible cut of (MΓ_2).

Before stating the second max-flow min-cut theorem, we prepare a lemma.

Lemma 5.4. *Let a and b be nonnegative functions in $L^\infty(\partial\Omega)$ and set*

$$\tilde{\zeta}(\varphi) = \int_{\partial\Omega} a\varphi^+ dH_{n-1} + \int_{\partial\Omega} b\varphi^- dH_{n-1}$$

for $\varphi\in L^1(\partial\Omega)$. Let $\varphi_1, \varphi_2 \in L^1(\partial\Omega)$. We set $N_{i,r}=\{x\in\partial\Omega;\ \varphi_i(x)\geq r\}$, $\varphi_{i,r}=\chi_{N_{i,r}}$ for $r>0$ and $M_{i,r}=\{x\in\partial\Omega;\ \varphi_i(x)\leq r\}$, $\varphi_{i,r}=-\chi_{M_{i,r}}$ for $r<0$, where $i=1, 2$. Then

$$\tilde{\zeta}(\varphi_1-\varphi_2) = \int_{-\infty}^{\infty} \tilde{\zeta}(\varphi_{1,r}-\varphi_{2,r})\, dr\ .$$

Proof. Set $D^+ = \{x\in\partial\Omega;\ \varphi_1(x)\geq\varphi_2(x)\}$ and $I^+=\int_{\partial\Omega} a(\varphi_1-\varphi_2)^+ dH_{n-1}$. Then by using [11; Theorem 1.2.3] we obtain

$$\begin{aligned} I^+ &= \int_{D^+} a(\varphi_1-\varphi_2)\, dH_{n-1} \\ &= \int_0^\infty dr \Big(\int_{D^+\cap\{\varphi_1^+\geq r\}} - \int_{D^+\cap\{\varphi_1^-\geq r\}} - \int_{D^+\cap\{\varphi_2^+\geq r\}} + \int_{D^+\cap\{\varphi_2^-\geq r\}}\Big)\, a dH_{n-1} \\ &= \int_0^\infty dr \int_{D^+\cap(N_{1,r}-N_{2,r})} a dH_{n-1} + \int_{-\infty}^0 dr \int_{D^+\cap(M_{2,r}-M_{1,r})} a dH_{n-1}\ . \end{aligned}$$

Since $(\partial\Omega-D^+)\cap(N_{1,r}-N_{2,r})=\phi$ for $r>0$ and $(\partial\Omega-D^+)\cap(M_{2,r}-M_{1,r})=\phi$ for $r<0$,

$$\begin{aligned} I^+ &= \int_0^\infty dr \int_{N_{1,r}-N_{2,r}} a dH_{n-1} + \int_{-\infty}^0 dr \int_{M_{2,r}-M_{1,r}} a dH_{n-1} \\ &= \int_0^\infty dr \int_{\partial\Omega} a(\chi_{N_{1,r}}-\chi_{N_{2,r}})^+ dH_{n-1} \\ &\quad + \int_{-\infty}^0 dr \int_{\partial\Omega} a(\chi_{M_{2,r}}-\chi_{M_{1,r}})^+ dH_{n-1}\ . \end{aligned}$$

Similarly

$$\int_{\partial\Omega} b(\varphi_1-\varphi_2)^- dH_{n-1} = \int_{\partial\Omega} b(\varphi_2-\varphi_1)^+ dH_{n-1}$$

$$= \int_0^\infty dr \int_{\partial\Omega} b(\chi_{N_{2,r}} - \chi_{N_{1,r}})^+ dH_{n-1}$$
$$+ \int_{-\infty}^0 dr \int_{\partial\Omega} b(\chi_{M_{1,r}} - \chi_{M_{2,r}})^+ dH_{n-1}$$
$$= \int_0^\infty dr \int_{\partial\Omega} b(\chi_{N_{1,r}} - \chi_{N_{2,r}})^- dH_{n-1}$$
$$+ \int_{-\infty}^0 dr \int_{\partial\Omega} b(\chi_{M_{2,r}} - \chi_{M_{1,r}})^- dH_{n-1}.$$

Hence

$$\tilde{\zeta}(\varphi_1 - \varphi_2) = \int_0^\infty dr \int_{\partial\Omega} \{a(\chi_{N_{1,r}} - \chi_{N_{2,r}})^+ + b(\chi_{N_{1,r}} - \chi_{N_{2,r}})^-\} dH_{n-1}$$
$$+ \int_{-\infty}^0 dr \int_{\partial\Omega} \{a(-\chi_{M_{1,r}} + \chi_{M_{2,r}})^+ + b(-\chi_{M_{1,r}} + \chi_{M_{2,r}})^-\} dH_{n-1}$$
$$= \int_{-\infty}^\infty \tilde{\zeta}(\varphi_{1,r} - \varphi_{2,r})\, dr$$

and the proof is completed.

Lemma 5.5. *Assume that* Γ_t *satisfies* (2.4) *for all* $t \in T$. *Then* $MF_2^* = MC_2$.

Proof. We check the conditions in Theorem 3.4. Evidently $h(p) \geq 0$ for all $p \in P$. In virtue of Theorem 5.3, MF_2^* is finite. Let $(z_t)_{t\in T}$ be a feasible element of (MF_2^*). Then $(z_t)_{t\in T}$ is a feasible element of ($M\Psi_2^*$). Let $\{(z_t^j)_{t\in T}\}$ be a sequence as in Lemma 5.2. As noted before Lemma 5.2, $\Pi((z_t^j)_{t\in T}) = \inf_{t\in T} \operatorname{essinf}_{x\in A} \gamma z_t^j(x)$. Hence

$$\limsup_{j\to\infty} \Psi_K((z_t^j)_{t\in T})/\Pi((z_t^j)_{t\in T})$$
$$\leq \textstyle\sum_{t\in T}(\psi_t(z_t) + \zeta_t(\gamma z_t - \gamma z_{t+1}))/\Pi((z_t)_{t\in T}).$$

Now we set $N_{t,r}^j = \{x \in \Omega;\ z_t^j(x) \geq r\}$, $z_{t,r}^j = \chi_{N_{t,r}^j}$ for $r \geq 0$ and $M_{t,r}^j = \{x \in \Omega;\ z_t^j(x) \leq r\}$, $z_{t,r}^j = -\chi_{M_{t,r}^j}$ for $r < 0$. Since $z_t^j = z_{t+1}^j$ for all sufficiently large t if we fix j, $(z_{t,r}^j)_{t\in T} \in Z_0$ for a.e. $r \in R$ and for each j, where Z_0 is the subset of Z defined at the end of §3. In virtue of Lemma 4.5 $\gamma z_{t,r}^j = \chi_{\{\gamma z_{t,r}^j \geq r\}}$ H_{n-1}-a.e. on $\partial\Omega$ for a.e. $r \geq 0$ and $\gamma z_{t,r}^j = -\chi_{\{\gamma z_{t,r}^j \leq r\}}$ H_{n-1}-a.e. on $\partial\Omega$ for a.e. $r < 0$. Hence taking $a = \alpha_t$ and $b = \alpha_t'$ in Lemma 5.4 we obtain

$$\zeta_t(\gamma z_t^j - \gamma z_{t+1}^j) = \int_{-\infty}^\infty \zeta_t(\gamma z_{t,r}^j - \gamma z_{t+1,r}^j)\, dr.$$

Thus by Proposition 2.4

$$\Psi_K((z_t^j)_{t\in T}) = \textstyle\sum_{t\in T}(\psi_t(z_t^j) + \zeta_t(\gamma z_t^j - \gamma z_{t+1}^j))$$
$$= \int_{-\infty}^\infty \textstyle\sum_{t\in T}(\psi_t(z_{t,r}^j) + \zeta_t(\gamma z_{t,r}^j - \gamma z_{t+1,r}^j))\, dr$$
$$= \int_{-\infty}^\infty \Psi_K((z_{t,r}^j)_{t\in T})\, dr.$$

Furthermore we can easily prove that

$$\Pi((z^j_{t,r})_{t\in T}) = 0 \quad \text{for a.e. } r\in(-\infty, 0)\cup(\Pi((z^j_t)_{t\in T}), \infty)$$

and

$$\Pi((z^j_{t,r})_{t\in T}) = 1 \quad \text{for a.e. } r\in(0, \Pi((z^j_t)_{t\in T}))\,.$$

Hence

$$\Pi((z^j_t)_{t\in T}) = \int_{-\infty}^{\infty} \Pi((z^j_{t,r})_{t\in T})\,dr\,.$$

It follows that the conditions in Theorem 3.4 are satisfied. Applying Theorem 3.4 to (MF_2^*) and (MC_2), we complete the proof.

Now we obtain the second max-flow min-cut theorem.

Theorem 5.6. $M\Psi_2 = M\Gamma_2 = MC_2$.

Proof. First assume that Γ_t satisfies (2.4) for each $t\in T$. Then using Lemma 2.6 and the last part of Lemma 5.2 we can prove $MC_2 = M\Gamma_2$. Hence from Theorem 5.3 and Lemma 5.5 it follows that $M\Psi_2 = M\Gamma_2$.

In the general case, one can prove the relation $M\Psi_2 = M\Gamma_2$ along the same lines as in the proof of Theorem 5.3. Furthermore using Lemma 5.2, we can prove that $MC_2 \le M\Gamma_2$. Since $M\Psi_2 = MF_2 = MF_2^* \le MC_2$ by Theorem 5.3, we see that $M\Psi_2 = MC_2 = M\Gamma_2$. This completes the proof.

REMARK 5.7. The following equality holds:

$$M\Psi_2 = \sup\{\textstyle\sum_{t\in T}\int_A \sigma_t\cdot\nu dH_{n-1};\ (\sigma_t)_{t\in T}\in\Pi_{t\in T}K_t \quad \text{such that}$$
$$\operatorname{div}\sigma_t = 0 \text{ a.e. on } \Omega,\ (\textstyle\sum_{s=0}^t \sigma_s\cdot\nu)\,\chi_E \in K'_t \quad \text{for all } t\in T\}\,.$$

To prove this, we denote the right hand side of the equality by M for a moment. Then evidently $M\Psi_2 \le M$. On the other hand, we can prove $M \le M\Psi_2^*$ in the same way as in the last part of the proof of Theorem 5.3. Hence from Theorem 5.3 it follows that $M\Psi_2 = M$.

In the case where $K_t = \{0\}$ for all $t \ge 1$ and $\alpha_t = \alpha'_t = 0$ on $\partial\Omega$ for all $t\in T$, problems similar to ($M\Psi_2$) and ($M\Gamma_2$) are investigated in Iri [9; §4.2]. Furthermore if c is a nonnegative bounded function in $C(\Omega)$ and $\Gamma_0(x) = \{w\in R^n;\ |w| \le c(x)\}$ for all $x\in\Omega$, then in virtue of Remark 5.7 ($M\Psi_2$) corresponds to (MFI) in §1. A problem similar to ($M\Psi_2$) on networks with continuous time is treated in [2]. The equality $M\Psi_2 = M$ in Remark 5.7 was orally noted by H. Aikawa.

References

[1] R.A. Adams: Sobolev spaces, Academic press, New York-London, 1975.
[2] E.J. Anderson, P. Nash and A.B. Philpott: *A class of continuous network flow problems,* Math. Oper. Res. **7** (1982), 501–514.
[3] C. Castaing and M. Valadier: Convex analysis and measurable multifunctions, Lecture Notes in Math. 580, Springer-Verlag, Berlin-New York, 1977.
[4] J. Dieudonné: *Sur la séparation des ensembles convexes,* Math. Ann. **163** (1966), 1–3.
[5] K. Fan: *Minimax theorems,* Proc. Nat. Acad, Sci. U.S.A. **39** (1953), 42–47.
[6] L.R. Ford and D.R. Fulkerson: Flows in networks, Princeton Univ. Press, Princeton, N.J., 1962.
[7] E. Gagliardo: *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili,* Rend, Sem. Mat. Univ. Padova **27** (1957), 284–305.
[8] E. Giusti: Minimal surfaces and functions of bounded vairation, Birkhäuser Verlag, Basel-Boston, Mass., 1984.
[9] M. Iri: *Theory of flows in continua as approximation to flows in networks,* Survey of Math. Programming edited by A. Prekopa, 263–278, North-Holland, Amsterdam, 1979.
[10] R. Kohn and R. Temam: *Dual spaces of stresses and strains, with applications to Hencky plasticity,* Appl. Math. Optim. **10** (1983), 1–35.
[11] W. Maz'ja: Sobolev spaces, Springer-Verlag, Berlin-New York, 1985.
[12] M. Ohtsuka: Perimeter, Lecture notes, Gakushuin Univ., 1983.
[13] R.T. Rockafellar: *Measurable dependence of convex sets and functions on parameters,* J. Math. Anal. Appl. **28** (1969), 4–25.
[14] H.H. Schaefer: Topological vector spaces, Springer-Verlag, Berlin-New York, 1966.
[15] G. Strang: *Maximal flow through a domain,* Math. Programming **26** (1983), 123–143.
[16] A. Taguchi and M. Iri: *Continuum approximation to dense networks and its application to the analysis of urban road networks,* Math. Programming Study **20** (1982), 178–217.

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