Kawanago, T.
Osaka J. Math.
27 (1990), 769–796

THE BEHAVIOR OF SOLUTIONS OF QUASI-LINEAR HEAT EQUATIONS

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(Received November 10, 1989)

0. Introduction

We shall consider the behavior of solutions of the following initial-boundary value problems:

$$(D)\begin{cases} u_t=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(a^{ij}(x,u)\frac{\partial u}{\partial x_j}) & \text{in} \quad \Omega\times\boldsymbol{R}^+,\\ u(x,t)=0 \quad \text{on} \quad \partial\Omega\times\boldsymbol{R}^+, \\ u(x,0)=u_0 \quad \text{in} \quad \Omega, \end{cases}$$

$$(N)\begin{cases} u_t=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}(a^{ij}(x,u)\frac{\partial u}{\partial x_j}) & \text{in} \quad \Omega\times\boldsymbol{R}^+,\\ \sum_{i,j=1}^{N}a^{ij}(x,u)\,\nu_i(x)\frac{\partial u}{\partial x_j}=0 \quad \text{on} \quad \partial\Omega\times\boldsymbol{R}^+, \\ u(x,0)=u_0 \quad \text{in} \quad \Omega. \end{cases}$$

Here, $\Omega\subset\boldsymbol{R}^N(N\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and $\nu=(\nu_1,\cdots,\nu_N)$ denotes the outward normal on $\partial\Omega$. We assume that $a^{ij}=a^{ji}$ and set $\boldsymbol{R}^+=(0,\infty)$. These equations arise in heat flow through solids. In this case, $u(x,t)$ represents the temperature of a position x at a time t in a solid Ω. If Ω is isotropic, we can set $a^{ij}(x,u)=k(x,u)\,\delta^{ij}$ (δ^{ij} is the Kronecker's dela delta) and $k(x,u)>0$ represents the thermal conductivity of the substance, which generally depends on a position $x\in\Omega$ and the temperature u (see [6].) When the thermal conductivity is a function of the temperature only, by setting $\phi(u)=\int_0^u k(s)\,ds$, we can reweite (D) and (N) as the following equation $(\tilde{D})$ and $(\tilde{N})$ respectively:

$$(\tilde{D})\begin{cases} u_t=\Delta\phi(u) \quad \text{in} \quad \Omega\times\boldsymbol{R}^+,\\ u(x,t)=0 \quad \text{on} \quad \partial\Omega\times\boldsymbol{R}^+, \\ u(x,0)=u_0(x) \quad \text{in} \quad \Omega, \end{cases}$$

$$(\tilde{N})\begin{cases} u_t=\Delta\phi(u) \quad \text{in} \quad \Omega\times\boldsymbol{R}^+,\\ \frac{\partial}{\partial\nu}u(x,t)=0 \quad \text{on} \quad \partial\Omega\times\boldsymbol{R}^+, \\ u(x,0)=u_0(x) \quad \text{in} \quad \Omega. \end{cases}$$

We remark that (D) and (N) also model diffusion of moleculars in mediums (see [7]).

On the other hand, the equations $(\tilde{D})$ and $(\tilde{N})$ model, for example, an gas flowing in homogeneous porous mediums when $(\tilde{D})$ and $(\tilde{N})$ are degenerate at $u=0$, i.e.

$$(0.1) \qquad \phi'(0) = 0 \quad \text{and} \quad \phi'(r)>0 \quad \text{if} \quad r \neq 0 .$$

Many aothors ([1], [2], [4], [5], [11], [13], [16] and the references in them) studied the behavoir of solutions of $(\tilde{D})$ and $(\tilde{N})$ under the condition (0.1). Alikakos and Rostamian [1] slso investigated the nondegenerate case in deriving results for the degenerate case. It seems, however, that the nondegenerate case has not been fully studied yet. In this paper we intend to study the problems (D), (N), $(\tilde{D})$ and $(\tilde{N})$ when they are nondegenerate, with applications to the degenerate case.

In section 1 we mention basic known results about (D), (N), $(\tilde{D})$ and $(\tilde{N})$ including the existence of weak solutions of these problems.

We shall give the statement of our main results for the nondegenerate case in section 2 and their proofs in section 3. First if we assume that

$$(0.2) \qquad \phi \in C^1(\boldsymbol{R}) \text{ and } k(r) = \phi'(r) \geq k_0 \quad \text{for some constant } k_0>0$$

and

$$(0.3) \qquad k(r) \geq k(0) - \theta/(-\log |r|)^{1+\rho} \quad \text{for} \quad r \in (-1, 1)$$

for some $\theta, \rho>0$, then the weak soluiton $u(x, t)$ of $(\tilde{D})$ satisfies the following estimate:

$$(0.4) \qquad ||u(t)||_{L^\infty} \leq Ce^{-\lambda k(0)t} \quad \text{for} \quad t \geq 0 ,$$

where $\lambda>0$ is the smallest positive eigenvalue of $-\Delta$ with Dirichlet condition, and $C>0$ is a constant depending only on $||u_0||_\infty$, k_0, θ, ρ, N and Ω. Similar results hold for problems (D) and (N) (see Theorem 2.1). And it seems that (0.3) is also almost a necessary condition for (0.4) (see Remark 2.1). Theoerm 2.1 is an extension of Theorem 3.3 in Alkakos and Rostamian [1]. In [1] they obtained an exponential-decay estimate for solutions of $(\tilde{N})$ with $\phi(r)=|r|^{m-1}r$ $(m>1)$ and $\operatorname{ess.inf}_{x\in\Omega} u_0>0$, but did not determine the precise exponent of exponential in their decay estimate. Next, Evans [8] studied the differentiability of weak solutions of $(\tilde{D})$ under the conditions:

$$(0.5) \qquad \phi\colon \boldsymbol{R} \to \boldsymbol{R} \text{ is a strictly increasing, continuous function with } \phi(0) = 0$$

and

$$(0.6) \qquad \phi^{-1}\colon \boldsymbol{R} \to \boldsymbol{R} \text{ is uniformly Lipschitz continuous.}$$

Under the same conditions (0.5) and (0.6) we shall establish L^2-L^∞ estimates for weak solutions of $(\tilde{D})$ and $(\tilde{N})$ (see Theorem 2.2). We remark that Evans [9] has alrrady obtained this type of estimate for solutions of the linear equation with certain nonlinear boundary conditions. Finally, under the conditions: (0.1) and

(0.7) $0<\alpha<\phi(r)\,\phi''(r)/[\phi'(r)]^2\leq 1$ in a neighborhood of $r=0$ for some $\alpha\in(0, 1)$,

Bertsch and Peletier [4] determined $y(t)$ and $f(x)$ such that

$$\|u(\cdot, t)/y(t)-f(x)\|_{L^\infty}\to 0 \quad \text{as} \quad t\to\infty\,, \tag{0.8}$$

where u denotes the positive solution of $(\tilde{D})$. Clearly $y(t)$ is the precise decay order of u. And $f(x)$ is usually called the *asymtotic profile* of u. When $(\tilde{D})$ is nondegenerate and $u_0\geq 0$, we shall establish a L^2-version of (0.8) (see Theorem 2.3). In our case we can take $y(t)=(u(t), e_1)$, $f(x)=e_1$ and show that $(u(t), e_1)\sim \exp\{-\phi'(0)\,\lambda t\}$, where $\lambda>0$ denotes the positive smallest eigenvalue of $-\Delta$ with zero-Dirichlet condition and $e_1>0$ is the unit eigenfunction corresponding to λ. The proof of our Theorem 2.3 depends on the energy method; while the proof of (0.8) in [4] on the comparison principle. It seems difficult to apply the comparison principle to our case. Indeed, in establishing (0.8) by the comparison principle, [4] essentially uses the property

$$\lim_{t\to\infty} f(t+c)/f(t) = 1 \quad \text{for any} \quad c\in \boldsymbol{R}\,. \tag{0.9}$$

But in our case $f(t)\sim\exp\{-\phi'(0)\,\lambda t\}$ does not have the property (0.9).

In section 4 we will study the case when $(\tilde{D})$ is degenerate at $u=0$ only, and in particular when

$$k(r) = \phi'(r)\sim 1/(-\log|r|)^\eta \text{ in a neighborhood of } r = 0 \tag{0.10}$$

for some $\eta>0$. Bertsch and Peletier [4] fully investigated the behavior of nonnegative solutions of $(\tilde{D})$ under the condition (0.7). We remark that (0.10) does not satisfy (0.7). [4] proved that the solutions of $(\tilde{D})$ with (0.7) decay polynomially; while we will prove that the solutions of $(\tilde{D})$ with (0.10) decay exponentially (see Corollary 4.2).

In section 5 we will consider the case when (N) is degenerate. Alikakos and Rostamian [1] proved that

(0.11) the solution $u(t)$ of $(\tilde{N})$ converges its average in $L^\infty(\Omega)$ at $t\to\infty$

under the conditions that the dimension $N=1$ and ϕ is an odd smooth function with $\phi(0)=\phi'(0)=0$ and $\phi'(r)>0$ if $r\neq 0$ (Theorem 3.4 in [1]). (For $N\geq 2$ [1] has also obtained L^p-version of (0.8) $(1\leq p<\infty)$.) (0.11) implies that

(0.12) the solution $u(x, t)$ with $\int_\Omega u_0\,dx>0$ eventaully becomes strictly positive even if $u_0(x)$ has compact support in Ω.

We are interested in extending these results for $N\geq 2$. We will show that it is possible if we assume that

(0.13) the initial value $u_0(x)$ is nonnegative and does not identically vanish in Ω (see Theorem 5.1). We can prove Theorem 5.1 mainly with the aid of the comparison principle. When $(\tilde{N})$ is degenerate at $u=0$ only, (0.12) means that $u(x, t)$ behaves as a solution of a nondegenerate equation after a finite time. Hence, in this case we can apply Theorem 2.1 and obtain (0.12) with an estimate like (0.4) (see Corollary 5.1). A related positivity property for $(\tilde{D})$ was established by Bertsch and Peletier [5].

While typing this manuscript we knew the related works Berryman and Holland [19] and Nagasawa [18] which has genealized and extended [19]. They studied the asymptotic behavior of classical solutions of the one-dimensional nondegenerate equations related to $(\tilde{D})$. In particular they obtained H_0^1-versions of (0.8). The main difference between their works and our results for the nondegenerate case is that in our paper we study the behavior of *weak* solutions of the *multi-dimensional* equations.

Acknowledgment. I would like to express my deepest gratitude to Professor Hiroki Tanabe for his proper guidance and his constant encouragement, to Professor Takashi Senba for suggesting some problems, and to Professor Mitsuru Ikawa and Professor Kenji Maruo for their useful advices.

Natation.

1. $||\cdot||_p$ denotes the norm of $L^p(\Omega)$.
2. $|A|$ is the measure of A for Lebesque's measurable set $A\subset \boldsymbol{R}^N$.
3. We set $\bar{f}=1/|\Omega|\int_\Omega f\,dx$ for $f\in L^1(\Omega)$.
4. We sometimes denote $\{x; f(x)\geq 0\}$ by $[f(x)\geq 0]$.
5. $(\cdot, \cdot)_2$ denotes the inner product in $L^2(\Omega)$.
6. $\boldsymbol{R}^+=(0, \infty)$.
7. $B(P; r)=\{Q\in \boldsymbol{R}^N; \overline{PQ}<r\}$ is the open ball at center P of radius r in $\boldsymbol{R}^N$.

1. Preliminary

In this section we collect some basic known results which are needed later. At first we shall define the weak solutions of (D) and (N) following essentially Oleinik and Kruzhkov [12].

Definition 1.1. (i) A function $u(x, t)$ will be called a weak solution of (D) if the following coditions a)-d) are sasisfied:

a) $u(x, t)$ is a (locally) Hölder continuous in $\Omega\times \boldsymbol{R}^+$,

b) $u\in L^\infty(\Omega\times(0, T))$ and $\partial u/\partial x_j\in L^2(\Omega\times(0, T))$, $1\leq j\leq N$, for any $T>0$,

c) $$\int_\Omega u_0(x)\,\eta(x,0)\,dx-\int_\Omega u(x,T)\,\eta(x,T)\,dx+\int_0^T\int_\Omega u\eta_t\,dx\,dt$$

(1.1) $$-\int_0^T\int_\Omega a^{ij}(x,u)\frac{\partial u}{\partial x_i}\frac{\partial \eta}{\partial x_j}\,dx\,dt=0$$

for any $T>0$ and for any $\eta\in C^1(\bar{\Omega}\times[0,T])$ such that $\eta(x,t)=0$ on $\partial\Omega\times[0,T]$,

d) $u(x,t)=0$ on $\partial\Omega\times(0,T)$.

(ii) A function $u(x,t)$ will be called a weak solution of (N) if the condition a), b) and the following c') are satisfied:

c') The equality (1.1) holds for any $T>0$ and for any $\eta\in C^1(\bar{\Omega}\times[0,T])$.

Proposition 1.1. *Assume that*

(1.2) $$a^{ij}(x,r)\in C(\bar{\Omega}\times\boldsymbol{R}),$$

(1.3) *there exists a positive non-increasing function* k_0: $[0,\infty)\to\boldsymbol{R}$ *such that* $\sum_{i,j=1}^N a^{ij}(x,r)\,\xi_i\,\xi_j\ge k_0(|r|)|\xi|^2$ *for any* $(x,r)\in\bar{\Omega}\times\boldsymbol{R}$ *and any* $\xi=(\xi_1,\cdots,\xi_N)\in\boldsymbol{R}^N$,

(1.4) $$u_0\in L^\infty(\Omega).$$

Then (D) (resp. (N)) possesses at least one weak solution. Furthermore, if we aslo ssume that

(1.5) $a^{ij}(x,r)$ *is locally Lipschitz continuous with respect to* r, *i.e.*

$$\forall L>0,\exists C>0;\ \forall r_1,r_2\in[-L,L],\ \forall x\in\bar{\Omega},\ |a^{ij}(x,r_1)-a^{ij}(x,r_2)|\le C|r_1-r_2|,$$

then (D) (resp. (N)) has a unique solution.

Proof. The proof is the same as that of Theorem 15 and 16 in [12]. However, we shall construct weak solutions by a different way for later use. We shall only show the existence of weak solutions for (N) because we can similarly do for (D). We choose $\{a_n^{ij}\}_{n=1}^\infty$ such that

(1.6) $a_n^{ij}\in C^\infty(\boldsymbol{R}^N\times\boldsymbol{R})\underset{n\to\infty}{\longrightarrow}a^{ij}$ uniformly on $\bar{\Omega}\times[-T,T]$ for any $T>0$,

(1.7) $a_n^{ij}(x,r)$ is uniformly continuous on every compact subset of $\bar{\Omega}\times\boldsymbol{R}$ with respect to r without depending on n, i.e.

$$\forall\varepsilon>0,\forall L>0,\exists\delta>0;\ \forall n\in\boldsymbol{N},\ \forall r_1,r_2\in[-L,L]\ \text{ with }\ |r_1-r_2|\le\delta,\ \forall x\in\bar{\Omega};$$
$$|a_n^{ij}(x,r_1)-a_n^{ij}(x,r_2)|\le\varepsilon.$$

For example, we can construct $a^{ij}(x,r)$ in the following way: Let $\hat{a}^{ij}\in C(\boldsymbol{R}^N\times\boldsymbol{R})$ be such that $\hat{a}^{ij}=a^{ij}$ on $\bar{\Omega}\times\boldsymbol{R}$. It is sufficient to set $a_n^{ij}=\rho_{1/n}*\hat{a}^{ij}$, where $\rho_\varepsilon*$ ($\varepsilon>0$) is the standard mollifier. We also choose u_0^n such that

(1.8) $$u_0^n\in C_0^\infty(\Omega)\underset{n\to\infty}{\longrightarrow}u_0\quad\text{in}\quad L^2(\Omega),$$

(1.9) $$||u_0^n||_\infty \leq ||u_0||_\infty .$$

We denote by $u_n(x, t)$ the unique classical solution of the following (N_n):

$$(N_n)\begin{cases} u_{nt} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_n^{ij}(x, u_n)\frac{\partial u_n}{\partial x_j}\right) & \text{in } \Omega\times \boldsymbol{R}^+ , \\ \sum_{i,j=1}^{N} a_n^{ij}(x, u_n)\, \nu_i(x)\frac{\partial u_n}{\partial x_j} = 0 & \text{on } \partial\Omega\times \boldsymbol{R}^+ , \\ u_n(x, 0) = u_0^n(x) & \text{in } \Omega . \end{cases}$$

By the same argument as in the proof of Theorem 16 in [12], we can obtain a weak solution $u(x, t)$ as the pointwise limit function of an appropriate subseqeunce of $\{u_n(x, t)\}_n$. ■

Next we shall briefly show how the weak solutions of $(\tilde{D})$ and $(\tilde{N})$ are defined from the nonlinear semigroup theory. We define operators A, B: $L^1(\Omega)\to L^1(\Omega)$ by

$$Au = -\Delta\phi(u) \quad \text{for} \quad u\in D(A)$$

with $D(A)=\{u\in L^1(\Omega);\ \phi(u)\in W_0^{1,1}(\Omega),\ \Delta\phi(u)\in L^1(\Omega)\}$, and

$$Bu = -\Delta\phi(u) \quad \text{for} \quad u\in D(B)$$

with $D(B)=\{u\in L^1(\Omega);\ \phi(u)\in W^{1,1}(\Omega),\ \Delta\phi(u)\in L^1(\Omega)$ and $\int_\Omega h\Delta\phi(u)\,dx+\int_\Omega \nabla h\cdot\nabla\phi(u)\,dx=0$ for any $h\in C^1(\bar{\Omega})\}$.
Under the condition

(1.10) ϕ: $\boldsymbol{R}\to\boldsymbol{R}$ is a strictly increasing, continuous function with $\phi(0)=0$,

both A and B are m-accretive in $L^1(\Omega)$. Therefore A and B generate the contraction semigroups $S_A(t)$ and $S_B(t)$ respectively. Hence we can define the weak solution of $(\tilde{D})$ (resp. $(\tilde{N})$) by $S_A(t)\,u_0$ (resp. $S_B(t)\,u_0$) for any $u_0\in\overline{D(A)}=\overline{D(B)}=L^1(\Omega)$. For the details, see [8], [10] and [3]. Throughout this paper, we shall always assume the condition (1.10).

We shall mention a few properties of the weak solutions of $(\tilde{D})$ and $(\tilde{N})$.

Proposition 1.2. *We assume that ϕ satisfies* (1.10).

(i) *If $u(x, t)$ is the* (*weak*) *solution of $(\tilde{D})$, then the following hold*:
(1) (*The maximum principle*) *For any $u_0\in L^p(\Omega)$ $(p\in[1,\infty])$, $u(t)\in L^p(\Omega)$ for $t\geq 0$, and $||u(t)||_p$ is non-increasing.*
(2) (*The order-preserving property*) *If $u_0, v_0\in L^1(\Omega)$ and $u_0\geq v_0$, then $S(t)\,u_0\geq S(t)\,v_0$ a.e. in Ω for any $t\in\boldsymbol{R}^+$. Here $S(t)\,u_0$ and $S(t)\,v_0$ denote the solution corresponding to u_0 and v_0 respectively.*

(ii) *If $u(x, t)$ is the* (*weak*) *solution of $(\tilde{N})$, then the following hold*:
(3) (*The maximum principle*) *For any $u_0\in L^p(\Omega)$ with $p\in[1,\infty]$, $u(t)\in L^p(\Omega)$*

for $t \geq 0$, *and* $\|u(t)-\overline{u}_0\|_p$ *is non-increasing.*
(4) (*The property preserving the quanity of heat*) $\overline{u(t)}=\overline{u}_0$ *for any* $u_0 \in L^1(\Omega)$ *and any* $t \geq 0$.
(5) (*The order-preserving property*) $u(x, t)$ *has the same property as stated in* (2).

Proof. Proposition 1.2 was proved by [10] and [1]. Or we can prove by a different way: Using the following Corollary 1.1 and 1.2 (the smoothing technique), it suffices to prove (1)−(5) under the additional conditions that $u_0 \in C_0^\infty(\Omega)$ and $\phi \in C^\infty(\boldsymbol{R})$. Since $u(x, t)$ is smooth, the classical maximum principle implies (2) and (5). We will prove (1) and (3) in the proof of Lemma 3.1, and (4) is obtained in a similar way. ■

REMARK 1.1. In view of the proof of Proposition 1.1, we can similarly prove that the statement of Proposition 1.2 is valid for the weak solutions of (D) and (N) under the condition $u_0 \in L^\infty(\Omega)$.

We shall describe the smoothing technique.

Proposition 1.3. *Let* ϕ *and* ϕ_n *satisfy* (1.10). *We assume that* $\bigcap_{n=1}^\infty R(\phi_n) \supset R(\phi)$ *and that* ϕ_n^{-1} *converges to* ϕ^{-1} *unifomly on every compact subset of* $R(\phi)$. *We aslo assume that* $u_0^n \underset{n\to\infty}{\to} u_0$ *in* $L^1(\Omega)$.

(i) *Let* $u(x, t)$ *be the solution of* $(\tilde{D})$ *and* $u_n(x, t)$ *to the following* $(\tilde{D}_{\varepsilon_n})$:

$$(\tilde{D}_{\varepsilon_n}) \begin{cases} u_{nt} = \Delta\phi_n(u_n) & in \quad \Omega \times \boldsymbol{R}^+ , \\ u_n(x, t) = \varepsilon_n & on \quad \partial\Omega \times \boldsymbol{R}^+ , \\ u_n(x, 0) = u_0^n(x) & in \quad \Omega . \end{cases}$$

We assume that $\varepsilon_n \underset{m\to\infty}{\to} 0$. *Then it follows that*

$$u_n \underset{n\to\infty}{\to} u \quad in \quad C([0, T]; L^1(\Omega)) , \tag{1.11}$$

where $T>0$ *is an arbitrary time.*

(ii) *Let* $u(x, t)$ *be the solution of* $(\tilde{N})$ *and* $u_n(x, t)$ *of* $(\tilde{N})$ *with* ϕ *and* u_0 *replaced by* ϕ_n *and* u_0^n *respectively. Then* (1.11) *holds.*

We can prove Proposition 1.3, as in Evans [8, section 4], with the aid of Proposition II. 2.17 of Benilan [3] and the convergence theorem on the nonlinear semigroup (see e.g. Evans [10, p. 168]). We have two corollaries from Proposition 1.3. Corollay 1.1 shows that the weak solution of a nondegeneate equation can be approximated by a sequence of classical solutions; Corollary 1.2 shows that the solution of a degenerate equation can be approximated by a sequence of solutions of nondegenerate equations.

Corollary 1.1. *Assume that* ϕ *satisfies* (1.10) *and that* ϕ^{-1}: $\boldsymbol{R} \to \boldsymbol{R}$ *is uni-*

formly Lipschitz continuous with a Lipschitz constant $1/k_0$. *We define* ϕ_n *by* $\phi_n^{-1}(r)=(\rho_{1/n}*\phi^{-1})(r)+(1/n)r+c_n$, *where* $\rho_\varepsilon*$ *is the standard mollifier, and* c_n *is the constant such that* $\phi_n^{-1}(0)=0$.

Then the following (1), (2) *and* (3) *hold*:

(1) ϕ_n: $\boldsymbol{R}\to\boldsymbol{R}$ *is a* C^∞*-function satisfirg* (1.10) *and* $\phi_n'\geq k_0/(1+k_0/n)$.

(2) *If* ϕ *is uniformly Lipschitz continuous with a Lipschitz constnat* $k_1>0$, *then* $\phi_n'\leq k_1/(1+k_1/n)$.

(3) *If we assume that* $u_0^n\in C_0^\infty(\Omega)\to u_0$ *in* $L^1(\Omega)$ $(n\to\infty)$, *then* (i) *and* (*ii*) *of Proposition* 1.3 *hold.* (*We remark that* $u_n(x,t)$ *is a classical solution.*)

We obtain Corollary 1.1, following Evans [8, section 4].

RRMARK 1.2. Assume that ϕ satisfies all the conditions of Corollary 1.1 and that $u_0\in L^\infty(\Omega)$. Then the weak solutions of $(\tilde{D})$ and $(\tilde{N})$ are locally Hölder continuous in $\Omega\times\boldsymbol{R}^+$. We can derive this fact from Corollary 1.1 and Theorem 2 in [12]. The proof is the same as that of Thoerem 16 in [12].

Corollary 1.2. *Assume that* ϕ *satisfies* (1.10). *Ww set* $\phi_n(r)=\phi(r)+r/n$ $(n\in\boldsymbol{N})$. *We also assume that* $u_0^n\to u_0$ *in* $L^1(\Omega)$ $(n\to\infty)$. *Then* (i) *and* (ii) *of Proposition* 1.3 *hold.*

Corollary 1.2 was used in the proof of Theorem 3.4 of Alikakos and Rostamian [1]. We shall sketch the proof because it is omitted in [1].

Proof. From the inequality:

$$|\phi_n^{-1}(r)-\phi^{-1}(r)|\leq|\phi^{-1}(r-1/n\phi^{-1}(r))-\phi^{-1}(r)|,$$

we can see that ϕ_n^{-1} converges to ϕ^{-1} uniformly on every compact subsets of $R(\phi)$, which implies Corollary 1.2 in view of Proposition 1.3. ■

Following Aronson, Crandall and Peletier [15], we define supersolutions and subsolutions of $(\tilde{D})$ and $(\tilde{N})$.

DEFINITION 1.2. (i) A subsolution $u(x,t)$ of $(\tilde{D})$ on $[0,T]$ is a function with the following properties (1) and (2).

$$u\in C([0,T]:L^1(\Omega))\cap L^\infty(\Omega\times[0,T]),\tag{1}$$

$$\int_\Omega(u(t)\,\varphi(t)-u(0)\,\varphi(0))\,dx-\int_0^t\int_\Omega(u\varphi_t+\phi(u)\,\Delta\varphi)\,dx\,dt\leq 0\tag{2}$$

for all $t\in[0,T]$ and $\varphi\in C^2(\bar{\Omega}\times[0,T])$ such that φ is nonnegative and $\varphi=0$ on $\partial\Omega\times[0,T]$. A supersolution of $(\tilde{D})$ is defined by (1) and (2) with $\leq$ replaced by $\geq$.

(ii) A subsolution $u(x,t)$ of $(\tilde{N})$ on $[0,T]$ is a function with the following properties (3) and (4).

(3) $u\in C([0, T]: L^1(\Omega))\cap L^\infty(\Omega\times[0, T])$ and $\phi(u)\in L^1([0, T]: H^1(\Omega))$,

(4) $$\int_\Omega (u(t)\,\varphi(t)-u(0)\,\varphi(0))\,dx-\int_0^t\int_\Omega (u\varphi_t-\nabla\phi(u)\cdot\nabla\varphi)\,dx\,dt\leq 0$$

for all $t\in[0, T]$ and $\varphi\in C^2(\bar{\Omega}\times[0, T])$ such that φ is nonnegative.

By Corollary 1.1, 1.2 and Lemma 3.3, we can verify that all weak solutions of $(\tilde{D})$ (resp. $(\tilde{N})$) are also sub- and supersolutions of $(\tilde{D})$ (resp. $(\tilde{N})$) under the conditions that $u_0\in L^\infty(\Omega)$ and ϕ is locally Lipschitz continuous.

Proposition 1.4. (*The comparison principle*) *Assume that ϕ is a locally Lipschitz continuous function. Let $\hat{u}$ be a supersolution of $(\tilde{D})$ (resp. $(\tilde{N})$) on $[0, T]$ $(T>0)$ and $\tilde{u}$ be a subsolution of $(\tilde{D})$ (resp. $(\tilde{N})$) on $[0, T]$. Then, if $\hat{u}(x, 0)\geq\tilde{u}(x, 0)$ in Ω, we have*

$$\hat{u}(x, t)\geq\tilde{u}(x, t)\quad a.e.\ in\quad \Omega\times[0, T].$$

Proof. The proof for $(\tilde{D})$ is just the same as that of Proposition 9 given in [15]. The proof for $(\tilde{N})$ is similar to that for $(\tilde{D})$. So we leave it to the reader. ■

2. The nondegenerate case

In this section we give the statement of our main theorems. We begin with a rasult on the behavior of weak solutions of (D) and (N).

Theorem 2.1. *We assume that all the conditions of Proposition 1.1 are valid.*

(i) *Assume that*

(2.1) *there exists $\theta, \rho>0$ such that*

$$\sum_{i,j=1}^N (a^{ij}(x, r)-a^{ij}(x, 0))\,\xi_i\,\xi_j\geq -\theta\,|\xi|^2/(-\log|r|)^{1+\rho}$$
$$\text{for any}\quad (x, r)\in\bar{\Omega}\times(-1, 1)\quad\text{and any}\quad \xi\in\boldsymbol{R}^N.$$

Let $u(x, t)$ be the weak solution of (D). Then,

(2.2) $$\|u(t)\|_\infty\leq C_1\,e^{-\lambda_1 t}\quad for\quad t\geq 0,$$

where $\lambda_1>0$ is the smallest positive eigenvalue of $-\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a^{ij}(x, 0)\frac{\partial}{\partial x_j}\cdot)$ with Dirichlet condition, and $C_1>0$ depends only on N, Ω, $\|u_0\|_\infty$, θ, ρ and $k_0(\|u_0\|_\infty)$.

(ii) *Assume that*

(2.3) *there exist $\theta, \rho>0$ such that*

$$\sum_{i,j=1}^N (a^{ij}(x, r)-a^{ij}(x, \bar{u}_0))\,\xi_i\,\xi_j\geq -\theta\,|\xi|^2/(-\log|r-\bar{u}_0|)^{1+\rho}$$
$$\text{for any}\quad (x, r)\in\bar{\Omega}\times(\bar{u}_0-1, \bar{u}_0+1)\quad\text{and any}\quad \xi\in\boldsymbol{R}^N.$$

Let $u(x, t)$ be the weak solution of (N). Then,

$$\|u(t)-\bar{u}_0\|_\infty \le C_2 e^{-\mu_1 t} \quad \text{for} \quad t \ge 0, \tag{2.4}$$

where $\mu_1>0$ is the smallest positive eigenvalue of $-\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a^{ij}(x, \bar{u}_0)\frac{\partial}{\partial x_j}\cdot)$ with Neumann condition, and $C_2>0$ depends only on N, Ω, $\|u_0\|_\infty$, θ, ρ and $k_0(\|u_0\|_\infty)$.

REMARK 2.1. We consider the case when $a^{ij}(x, r)=k(r)\,\delta^{ij}$ and $k(r)>0$. Let $u(x, t)$ be the solution of $(\tilde{D})$ with $u_0\ge 0$ and $\phi(r)=\int_0^r k(s)\,ds$. Taking account of the results in Bertsch and Peletier [4] and the proof of Remark 4.1 in section 4, it is expected that the decay rate of $u(x, t)$ corresponds with that of the solution $x(t)$ of the ordinary differential equation $dx/dt=-\lambda\phi(x)$, where $\lambda>0$ is the positive smallest eigenvalue of $-\Delta$. Indeed this is true under some conditions for ϕ (see Theorem 2.3). When $\phi(r)=r-r/(-\log r)$, $x(t)\sim t\times\exp(-\lambda t)$. By this it seems that (2.1) is almost a necessary condition for (2.2). See also Remark 2.2.

Below we consider the behavior of weak solutions of $(\tilde{D})$ and $(\tilde{N})$. First we are interested in the case when

(2.5) ϕ^{-1}: $\boldsymbol{R}\to\boldsymbol{R}$ is a uniformly Lipschitz continuous function with a Lipschitz constant $1/k_0(k_0>0)$.

Theorem 2.2. *We assume that ϕ satisfies* (1.10) *and* (2.5).

(i) *Let $u(x, t)$ be the (weak) solution of $(\tilde{D})$. Then for any $u_0\in L^2(\Omega)$, $u(t)\in L^\infty(\Omega)$ for $t>0$ with the estimate:*

$$\|u(t)\|_\infty \le \frac{C_1}{(k_0 t)^{N/4}}\|u_0\|_2 \quad \text{for} \quad t>0, \tag{2.6}$$

$$\|u(t)\|_\infty \le C(N, k_0, t_0)\, e^{-\lambda k_0 t}\|u_0\|_2 \quad \text{for} \quad t\ge t_0, \tag{2.7}$$
$$\text{where} \quad C(N, k_0, t_0) = C_1\, e^{\lambda k_0 t_0}/(k_0\, t_0)^{N/4},$$

where $t_0>0$ is an arbitrary time, $C_1>0$ depends only on N, and $\lambda>0$ is the smallest positive eigenvalue of $-\Delta$ with Dirichlet condition.

(ii) *Let $u(x, t)$ be the solution of $(\tilde{N})$. For any $u_0\in L^2(\Omega)$, $u(t)\in L^\infty(\Omega)$ for $t>0$ with the estimate:*

$$\|u(t)-\bar{u}_0\|_\infty \le \frac{C_2}{(k_0 t)^{N/4}}\|u_0-\bar{u}_0\|_2 \quad \text{for} \quad t>0, \tag{2.8}$$

$$\|u(t)-\bar{u}_0\|_\infty \le C(N, \Omega, k_0, t_0)\, e^{-\mu k_0 t}\|u_0-\bar{u}_0\|_2 \quad \text{for} \quad t>t_0, \tag{2.9}$$
$$\text{where} \quad C(N, \Omega, k_0, t_0) = C_2\, e^{\mu k_0 t_0}/(k_0\, t_0)^{N/4},$$

where $t_0>0$ is an arbitrary time, $C_2>0$ depends only on N and Ω, and $\mu>0$ is the smallest positive eigenvalue of $-\Delta$ with Neumann condition.

Finally we are interested in the behavior of nonnegative solutions of $(\tilde{D})$.

Theorem 2.3. *Assume that ϕ satisfies* (1.10) *and that $k=\phi'$: $\boldsymbol{R}\to\boldsymbol{R}$ is continuous* (*i.e. ϕ belongs to C^1-class*). *We also assume that*
(2.10) *there exist a positive non-increasing function k_0: $\boldsymbol{R}^+\to\boldsymbol{R}$ and a non-decreasing function k_1: $\boldsymbol{R}^+\to\boldsymbol{R}$ such that $k_0(|r|)\le k(r)\le k_1(|r|)$ for any $r\in\boldsymbol{R}$,*
(2.11) *there exist θ, $\rho>0$ such that $|k(r)-k(0)|\le\theta/(-\log|r|)^{1+\rho}$ for any $r\in(-1, 1)$,*
(2.12) *$u_0\ge 0$, $u_0(x)$ does not identically vanish in Ω and $u_0\in L^\infty(\Omega)$. Let $u(x, t)$ be the weak solution of $(\tilde{D})$. Then, the following estimates hold:*

$$C_1\, e^{-\lambda_1 t}\le\|u(t)\|_\infty\le C_2\, e^{-\lambda_1 t} \quad \text{for} \quad t\ge 0\,, \tag{2.13}$$

$$\Big\|\frac{u(t)}{(u(t), e_1)_2}-e_1\Big\|_2\le\frac{C_3}{(1+t)^{(1+\rho)/2}} \quad \text{for} \quad t\ge 0\,, \tag{2.14}$$

Moreover, when $N=1$, $u(t)/(u(t), e_1)_2\to e_1$ in $L^\infty(\Omega)$ as $t\to\infty$ with the estimate:

$$\Big\|\frac{u(t)}{(u(t), e_1)_2}-e_1\Big\|_\infty\le\frac{C_4}{(1+t)^{(1+\rho)/4}} \quad \text{for} \quad t\ge 0\,, \tag{2.15}$$

where $\lambda_1>0$ is the smallest positive eigenvalue of $-k(0)\Delta$ with Dirichlet condition, and C_1, C_2, C_3, $C_4>0$ depend only on N, Ω, $\|u_0\|_\infty$, $(u_0, e_1)_2$, ρ, θ, $k_0(\|u_0\|_\infty)$ and $k_1(\|u_0\|_\infty)$. We denote by $e_1\ge 0$ the unit eigenvector corresponding to λ_1.

REMARK 2.2. The left-hand side of (2.13) does not always hold without the condition (2.11). Indeed if $k(r)=1+1/(-\log|r|)^\rho$ for some $\rho\in(0, 1)$ and $\|u_0\|_\infty<1$, then the corresponding solution $u(x, t)$ satisfies the following estimate:

$$\|u(t)\|_\infty\le C\exp(-\lambda_1 t-(\lambda_1 t)^{1-\rho}) \quad \text{for} \quad t\ge 0\,. \tag{2.16}$$

To obtain (2.16), we have olny to substitute $\varepsilon=C\exp\{-\lambda_1 t-(\lambda_1 t)^{1-\rho}\}$ into (4.4) of Proposition 4.1 in section 4.

REMARK 2.3. It seems difficult to have the result about $(\tilde{N})$ which corresponds to Theorem 2.3, because it is difficult to find the condition corresponding to (2.12). Let $u(x, t)$ be the solution of $(\tilde{D})$. (2.12) is a simple sufficient condition to imply that

$$(u(t), e_1)\neq 0 \quad \text{for} \quad t\ge 0\,. \tag{2.17}$$

We give an example to show that (2.13) does not always hold without the condition (2.12). Assume that ϕ: $\boldsymbol{R}\to\boldsymbol{R}$ is a smooth odd function with $\phi'>0$. We assume that $N=1$, $\Omega=(0, \pi)$ and $u_0(x)=\sin mx$ ($m\in\boldsymbol{N}$). Let $u(x, t)$ be the solution of $(\tilde{D})$. Then the following estmimate holds:

$$C_1\, e^{-m^2k(0)t}\le\|u(t)\|_\infty\le C_2\, e^{-m^2k(0)t} \quad \text{for} \quad t\ge 0\,. \tag{2.18}$$

We shall derive (2.18). We define by $v(x, t)$ the solution corresponding to $u_0(x)=\sin x$. Then, we obtain that

(2.19) $u(x, t) = (-1)^j v(m(x-j\pi/m), m^2 t)$ if $x \in [j\pi/m, (j+1)\pi/m]$.
$(j = 0, 1, 2, \cdots, m-1)$

We immediately obtain (2.18) from (2.19) and Theorem 2.3.

3. Proofs of results in section 2

We need some lemmas to prove Theorem 2.1.

Lemma 3.1. *We assume that* $a^{ij}(x, r) \in C^\infty(\bar{\Omega})$ *and that there exsists a constant* $k_0 > 0$ *such that*

$$\sum_{i,j=1}^N a^{ij}(x, r)\, \xi_i\, \xi_j \geq k_0 |\xi|^2$$

for any $(x, r) \in \bar{\Omega} \times \boldsymbol{R}$ *and any* $\xi = (\xi_1, \cdots, \xi_N) \in \boldsymbol{R}^N$.

(i) *Let* $u(x, t)$ *be the classicla solution of* (D). *Then estimates* (2.6) *and* (2.7) *hold.*

(ii) *Let* $u(x, t)$ *be the classical solution of* (N). *Then estimates* (2.8) *and* (2.9) *hold.*

Proof. At first we shall prove (i).

$$\frac{d}{dt}\int_\Omega |u|^p\, dx = p\int |u|^{p-1} \operatorname{sign} u \cdot u_t\, dx$$

$$= -p(p-1)\int |u|^{p-2} \sum_{i,j=1}^N a^{ij}(x, u) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx$$

$$\leq -p(p-1)\, k_0 \int |u|^{p-2} |\nabla u|^2\, dx \tag{3.1}$$

$$\leq -2k_0 \int_\Omega |\nabla |u|^{p/2}|^2\, dx \quad \text{for} \quad p \in [2, \infty). \tag{3.2}$$

We can prove (2.6) with the aid of (3.2) and a basic Sobolev's inequality:

$$\|f\|_{2N/(N-1)} \leq C\|\nabla f\|_2^{1/2} \|f\|_2^{1/2} \quad \text{for any} \quad f \in H_0^1(\Omega). \tag{3.3}$$

(Here we set $2N/(N-1) = \infty$ for $N=1$). Indeed when $N=1$, we set $p=2$ in (3.2) (or (3.1)) and integrate in t to obtain

$$\|u(t)\|_2^2 - \|u_0\|_2^2 \leq -2k_0 \int_0^t \|\nabla u(s)\|_2^2\, ds. \tag{3.4}$$

It follows from (3.3), (3.4), Proposition 1.2 (1) and Remark 1.1 that

$$\|u_0\|_2^2 \geq 2k_0 \int_0^t \left(\frac{\|u(s)\|_\infty^2}{C\|u(s)\|_2}\right)^2 ds \geq C k_0 t \times \frac{\|u(t)\|_\infty^4}{\|u_0\|_2^2}, \tag{3.5}$$

which implies (2.6). When $N \geq 2$, the proof is essentially the same, but we need Moser's iteration technique, which is used in Evans [9, section 4]. We omit the details because the argument is the same as in [9].

Next we shall derive (2.7). Following the proof of Theorem 3.3 in [1], we shall get a L^2-decay estimate. If we substitute $p=2$ into (3.1) and use Poincaré inequality, then we have

$$\frac{d}{dt}\int u^2\,dx \leq -2k_0\,\lambda \int u^2\,dx\,.$$

Therefore, we obtain that

$$\|u(t)\|_2 \leq e^{-k_0\lambda t}\,\|u_0\|_2 \quad \text{for} \quad t \geq 0\,. \tag{3.6}$$

Hence, with the aid of (2.6) and (3.6),

$$\|u(t)\|_\infty \leq \frac{C_1}{(k_0\,t_0)^{N/4}}\,e^{-k_0\lambda(t-t_0)}\,\|u_0\|_2\,,$$

which implies (2.7).

We can prove (ii) in the same manner as above with the aid of the following inequality corresponding to (3.3):

$$\|\,|f-\bar{f}|^p\|_{2N/(N-1)} \leq C\|\nabla|f-\bar{f}|^p\|_2^{1/2}\|\,|f-\bar{f}|^p\|_2^{1/2} \tag{3.7}$$

for any $p\in[1,\infty)$ and any measurable function f such that $|f-\bar{f}|^{p-1}(f-\bar{f})\in H^1(\Omega)$, where $C>0$ depends only on Ω and N. (3.7) is not trivial, but is implied by the following Lemma 3.2. ■

Lemma 3.2. (*A version of Poincaré inequality*) *Assume that* $p\in[1,\infty)$ *and* f *is any measurable function such that* $|f|^{p-1}f\in H^1(\Omega)$ *and* $\int_\Omega f\,dx=0$. *Then,*

$$K\int_\Omega (|f|^p)^2\,dx \leq \int_\Omega |\nabla|f|^p|^2\,dx\,,$$

where $K>0$ *depends only on* N *and* Ω.

REMARK 3.1. 1) This refines Lemma 3.2 in Alikakos and Rostamian [1] in that K *does not* depends on p.

2) We cannot know from the proof below how large $K>0$ is. But we can take $K=|\Omega|^{-2}$ when $N=1$. We can prove this fact in the same way as in the well-known case: $N=1$ and $p=1$.

Proof. We shall proceed by contradiction. We set $\varphi_p(x)=|x|^p$ sign x. We assume that there exist a sequence of measurable functions $\{\xi_n\}_{n=1}^\infty$ and $\{p_n\}_{n=1}^\infty\subset[1,\infty)$ such that

$$\int \xi_n\,dx = 0\,, \tag{3.8}$$

$$\int |\varphi_{p_n}(\xi_n(x))|^2\,dx = 1\,, \tag{3.9}$$

$$\int |\nabla\varphi_{p_n}(\xi_n(x))|^2\,dx \underset{n\to\infty}{\to} 0\,. \tag{3.10}$$

We set $\nu_n=\varphi_{p_n}\circ\xi_n$. Then we obtain from (3.9) and (3.10) that $\{\nu_n\}_{n=1}^{\infty}$ is bounded in $H^1(\Omega)$. Therefore, there exist $\nu\in H^1(\Omega)$ and an appropriate subsequence of $\{\nu_n\}_{n=1}^{\infty}$ such that

$$\nu_n\to\nu \quad \text{weakly in} \quad H^1(\Omega) \quad (n\to\infty)\,, \tag{3.11}$$

$$\nu_n\to\nu \quad \text{in} \quad L^2(\Omega) \quad (n\to\infty)\,. \tag{3.12}$$

We obtain from (3.10) and (3.11) that

$$\int |\nabla\nu|^2\,dx \leq \liminf_{n\to\infty}\int |\nabla\nu_n|^2\,dx = 0\,.$$

It follows that $\nabla\nu=0$. Therefore, $\nu=c=$constant. We obtain $c\neq 0$ from (3.9) and (3.12). We shall consider two cases.

(I) the case $\liminf_{n\to\infty} p_n<\infty$.

In this case, the argument is essentially the same as the proof of Lemma 3.2 of [1]. Choosing a subsequence, if necessary, we may assume that there exists $p_0\in[1, \infty)$ such that $p_n\to p_0(n\to\infty)$. Then,

$$\xi_n=\varphi_{p_n}^{-1}\circ\nu_n\to\varphi_{p_0}^{-1}\circ\nu=\xi \quad \text{in} \quad L^2(\Omega). \quad (n\to\infty)$$

It follows that

$$0=\int \xi_n\,dx\to\int \xi\,dx=\varphi_{p_0}^{-1}(c)|\Omega| \text{ as } n\to\infty. \quad \text{This contradicts } c\neq 0\,.$$

(II) the case $\liminf_{n\to\infty} p_n=\infty$.

Choosing a subsequence, if necessary, we may assume that

$$p_1<p_2<p_3<\cdots<p_n\to\infty \quad (n\to\infty)\,.$$

We fix $\varepsilon>0$ sufficiently small. We assume without loss of generality that $c>0$. We set $A_n=[|\nu_n-c|>\varepsilon]$. By (3.12),

$$\begin{cases} |A_n|\to 0 \quad (n\to\infty)\,, \\ \underset{x\in A_n^c}{\text{ess. sup}}\ |\xi_n(x)-1|\to 0 \quad (n\to\infty)\,, \end{cases}$$

where $A_n^c=\Omega-A_n$. Therefore,

$$\int_{A_n^c}\xi_n(x)\,dx\to|\Omega| \quad (n\to\infty)\,. \tag{3.13}$$

By (3.8) and (3.13),

$$\int_{A_n} \xi_n \, dx \to -|\Omega| \quad (n \to \infty).$$

On the other hand, we have

$$\int_{A_n} \xi_n dx = \Big(\int_{[\xi_n \le -2]} + \int_{[-2 \le \xi_n < 0]} + \int_{A_n \cap [\xi_n \ge 0]}\Big) \xi_n dx \ge \int_{[\xi_n \le -2]} \xi_n dx - 2|A_n| .$$

Hence,

$$\int_{[\xi_z \le -2]} \xi_n dx \le \int_{A_n} \xi_n \, dx + 2|A_n| \to -|\Omega| \quad (n \to \infty).$$

It follows that

$$\liminf_{n\to\infty} \int_{[\xi_n \le -2]} |\xi_n| dx \ge |\Omega| .$$

Therefore,

$$\int_{[\xi_n \le -2]} v_n^2 \, dx = \int_{[\xi_n \le -2]} |\xi_n|^{2p_n - 1} \cdot |\xi_n| dx \ge 2^{2p_n - 1} \int_{[\xi_n \le -2]} |\xi_n| dx \to \infty .$$

This contradicts (3.9). ■

Proof of Theorem 2.1. We shall prove (i) only, because the proof of (ii) is similar to that of (i). We shall proceed in two steps.

Step 1. Assume the additional hypothesises that $a^{ij}(x, r) \in C^\infty(\bar{\Omega} \times \boldsymbol{R})$ and $u_0 \in C_0^\infty(\Omega)$. Then $u(x, t)$ is a smooth solution. By Lemma 3.1,

$$\|u(t)\|_\infty \le \theta_1 e^{-\theta_2 t} \quad \text{for} \quad t \ge 0 , \tag{3.14}$$

$$\|u(t)\|_\infty \le \theta_3 \|u(t-t_0)\|_2 \quad \text{for} \quad t \ge t_0 , \tag{3.15}$$

where $t_0 > 0$ is an arbitrary but fixed time, and $\theta_1, \theta_2, \theta_3 > 0$ are some constants. θ_1 depends only on N, Ω, $\|u_0\|_\infty$ and $k_0(\|u_0\|_\infty)$, θ_2 only on N, Ω and $k_0(\|u_0\|_\infty)$, and θ_3 only on N, $k_0(\|u_0\|_\infty)$ and t_0.

In view of (3.14), we may assume without loss of generality that $\|u_0\|_\infty > 0$ and $\theta_1 > 0$ are sufficiently small. By (3.15), the proof is complete if we show that

$$\|u(t)\|_2 \le \theta_4 e^{-\lambda_1 t} \quad \text{for} \quad t \ge 0 , \tag{3.16}$$

where $\theta_4 = \theta_4(N, \Omega, \|u_0\|_\infty, \theta, \rho, k_0(\|u_0\|_\infty)) > 0$. With the aid aid of (2.1),

$$\begin{aligned} \frac{d}{dt} \int_\Omega u^2 \, dx &= -2 \int \sum_{i,j=1}^N a^{ij}(x, u) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx \\ &\le -2 \int \Big[\sum_{i,j=1}^N a^{ij}(x, 0) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \frac{\theta}{(-\log|u|)^{1+\rho}} |\nabla u|^2\Big] dx . \end{aligned} \tag{3.17}$$

We set $K_0 = k_0(\|u_0\|_\infty)$. Then by (1.3),

$$(3.18) \qquad |\nabla u|^2 \leq \frac{1}{K_0} \sum_{i,j=1}^{N} a^{ij}(x, 0) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} .$$

Since $(-\log r)^{-(1+\rho)}(0 \leq r < 1)$ is an increasing function, we obtain that

$$(3.19) \qquad \frac{1}{(-\log |u|)^{1+\rho}} \leq \frac{1}{(-\log ||u||_\infty)^{1+\rho}} .$$

It follows from (3.17), (3.18), (3.19) and the eigenfunction expansion that

$$\frac{d}{dt} \int_\Omega u^2\, dx \leq -2 \left[1 - \frac{\theta}{K_0(-\log ||u||_\infty)^{1+\rho}}\right] \left(-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (a^{ij}(x, 0) \frac{\partial u}{\partial x_j}),\ u_2\right)$$

$$(3.20) \qquad = -2 \left[1 - \frac{\theta}{K_0(-\log ||u||_\infty)^{1+\rho}}\right] \sum_{j=1}^{\infty} \lambda_j (u, e_j)_2^2$$

$$(3.21) \qquad \leq -2\lambda_1 \left[1 - \frac{\theta}{K_0(-\log ||u||_\infty)^{1+\rho}}\right] \int_\Omega u^2\, dx ,$$

where λ_j is the j-th largest eigenvalue of $-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} (a^{ij}(x, 0) \frac{\partial}{\partial x_j} \cdot)$ and e_j is the eigenvector corresponding to λ_j. We can assume that $\{e_j\}_{j=1}^{\infty}$ are C.O.N.S. in $L^2(\Omega)$ and $e_1 \geq 0$. It follows from (3.21) that

$$(3.22) \qquad ||u(t)||_2 \leq ||u_0||_2 \exp \{-\lambda_1 t + \int_0^t \frac{\lambda_1 \theta}{K_0(-\log ||u(s)||_\infty)^{1+\rho}}\, ds\} .$$

Here, we obtain from (3.14) that

$$(3.23) \qquad \int_0^t \frac{1}{(-\log ||u(s)||_\infty)^{1+\rho}}\, ds \leq \int_0^t \frac{ds}{(-\log \theta_1 + \theta_2 s)^{1+\rho}} .$$

The right-hand side of (3.23) is less than some constant depending on θ_1 and θ_2 because we may assume that $\theta_1 \in (0, 1)$. Therefore (3.16) holds.

Step 2. Using Step 1, we shall complete the proof of Theorem 2.1. We approximate $u(x, t)$ by a sequence of classical solutions. We can choose $\{a_n^{ij}(x, r)\}_{n=1}^{\infty} \subset C^\infty(\boldsymbol{R}^N \times \boldsymbol{R})$ such that a_n^{ij} satisfies (1.6) and (1.7) in the proof of Proposition 1.1 and also satisfies

$$(3.24) \qquad \sum_{i,j=1}^{N} a_n^{ij}(x, r)\, \xi_i\, \xi_j \geq \sum_{i,j=1}^{N} a^{ij}(x, r)\, \xi_i\, \xi_j$$

for any n, any $\xi \in \boldsymbol{R}^N$ and any $(x, r) \in \bar{\Omega} \times [-||u_0||_\infty, ||u_0||_\infty]$. And we choose $\{u_0^n\}_{n=1}^{\infty} \subset C_0^\infty(\Omega)$ such that u_0^n satisfies (1.8) and (1.9). If we denote by $u_n(x, t)$ the classical solution of (N_n), then $u(x, t)$ is the pointwise limit function of an appropriate subsequence of $\{u_n(x, t)\}_n$. We can apply Step 1 to $u_n(x, t)$ and obtain estimates for $u_n(x, t)$ corresponding to (3.14)–(3.23). We remark that the estimate corresponding to (3.17) is, with the aid of (3.24), the following:

$$\frac{d}{dt} \int_\Omega u_n^2\, dx \leq -2 \int [\sum_{i,j=1}^{N} a^{ij}(x, 0) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} - \frac{\theta}{(-\log |u_n|)^{1+\rho}} |\nabla u_n|^2]\, dx .$$

Let $n\to\infty$ and we obtain (3.15) and (3.16) for the weak solution $u(x, t)$. The estimates (3.15) and (3.16) imply (2.2). ■

Proof of Theorem 2.2. Lemma 3.1 implies Theorem 2.2 when $\phi\in C^\infty(\boldsymbol{R})$ and $u_0\in C_0^\infty(\Omega)$, because in this case the solution $u(x, t)$ is smooth. For general ϕ and u_0, we apply Corollary 1.1 (the smoothing technique). The argument is the same as Step 2 of the proof of Theorem 2.1 and we omit the details. ■

We need a lemma for the proof of Theorem 2.3.

Lemma 3.3. *Assume that* ϕ *satisfies* (1.10). *We also assume that* $\Phi(u_0)\in L^1(\Omega)$, *where we set* $\Phi(r)=\int_0^r \phi(s)\,ds$.

(i) *Let* $u(x, t)$ *be the weak solution of* $(\tilde{D})$. *Then the following estimate holds*:

$$||\nabla\phi(u(t))||_2\leq t^{-1/2}\,\{\int_\Omega \Phi(u_0)\,dx\}^{1/2} \quad for \quad t>0\,, \tag{3.25}$$

(ii) *Let* $u(x, t)$ *be the weak solution of* $(\tilde{N})$. *Then the estimate* (3.25) *holds.*

Lemma 3.3 is a generalization of (2.8) of Theorem 3 in Nakao [16]. Since we can prove Lemma 3.3 in the same way as in [16], we omit its proof.

Proof of Theorem 2.3. We shall prove (2.13)–(2.15) only under the additional assumptions that $u_0\in C_0^\infty(\Omega)$ and $\phi\in C^\infty(\boldsymbol{R})$. For general ϕ and u_0, we omit the details because we have only to apply the smoothing technique in the same way as in the proof of Theorems 2.1 and 2.2. In what follows, we use the notains in the proof of Theorem 2.1, First we shall prove (2.13). The right-hand side of (2.13) is (2.2) of Theorem 2.1. We shall show the left-hand side of (2.13). It suffices to derive

$$(u(t), e_1)_2\geq C_1\,e^{-\lambda_1 t} \quad \text{for} \quad t\geq 0\,. \tag{3.26}$$

With the aid of integration by parts,

$$\frac{d}{dt}(u(t), e_1)_2 = (\phi(u), \Delta e_1)\geq -\frac{\lambda_1}{k(0)}\,k_1(||u_0||_\infty)\,(u(t), e_1)\,,$$

which implies that

$$(u(t), e_1)\geq (u_0, e_1)\exp\,[-\lambda_1\,k_1(||u_0||_\infty)/k(0)] \quad \text{for} \quad t\geq 0\,. \tag{3.27}$$

We may assume from (2.7) of Theorem 2.2 and (3.27) that $||u_0||_\infty>0$ is small enough. By (3.27) and (2.11), we obtain that

$$\frac{d}{dt}(u(t), e_1)_2 = -\frac{\lambda_1}{k(0)}\,(\int_0^u k(s)\,ds, e_1)_2\geq (-\lambda_1\,u-\frac{\lambda_1\theta}{k(0)}\int_0^u \frac{ds}{(-\log s)^{1+\rho}}, e_1)_2\,.$$

Here,

$$\int_0^u \frac{ds}{(-\log s)^{1+\rho}} \leq \frac{1}{(-\log ||u(t)||_\infty)^{1+\rho}} \int_0^u ds \leq \frac{u}{(-\log ||u||_\infty)^{1+\rho}}.$$

Therefore,

$$\frac{d}{dt}(u(t), e_1)_2 \geq \left(-\lambda_1 - \frac{\lambda_1 \theta}{k(0)(-\log ||u(t)||_\infty)^{1+\rho}}\right)(u(t), e_1)_2, \tag{3.28}$$

which implies that for $t>0$,

$$(u(t), e_1)_2 \geq (u_0, e_1)_2 \exp\left\{-\lambda_1 t - \frac{\lambda_1 \theta}{k(0)} \int_0^t \frac{ds}{(-\log ||u(s)||_\infty)^{1+\rho}}\right\}. \tag{3.29}$$

We obtain (3.26) from (3.29) and (3.14). The detailed argument is the same as the proof of Theorem 2.1. Next we shall prove (2.14). With the aid of (3.20) and (3.28),

$$\frac{d}{dt} ||u(t) - (u(t), e_1)_2 e_1||_2^2$$

$$= \frac{d}{dt}(u, u)_2 - 2(u, e_1)_2 \frac{d}{dt}(u, e_1)_2 \tag{3.30}$$

$$\leq \left(\frac{2\theta\lambda_1}{K_0} + \frac{2\lambda_1\theta}{k(0)}\right) \frac{(u, e_1)_2^2}{(-\log ||u(t)||_\infty)^{1+\rho}}$$

$$-2\left\{1 - \frac{\theta}{K_0(-\log ||u||_\infty)^{1+\rho}}\right\} \sum_{j=2}^\infty \lambda_j (u, e_j)_2^2$$

$$\leq \frac{4\lambda_1\theta}{K_0} \frac{(u, e_1)_2^2}{(-\log ||u(t)||_\infty)^{1+\rho}} \tag{3.31}$$

$$-2\lambda_2\left\{1 - \frac{\theta}{K_0(-\log ||u||_\infty)^{1+\rho}}\right\} ||u(t) - (u(t), e_1)_2 e_1||_2^2.$$

If we set $y(t) = ||u(t) - (u(t), e_1)_2 e_1||_2^2 / (u(t), e_1)^2$, then by (3.28) and (3.31),

$$y'(t) \leq \left\{-2(\lambda_2 - \lambda_1) + \frac{2\theta(\lambda_1 + \lambda_2)}{K_0(-\log ||u(t)||_\infty)^{1+\rho}}\right\} y(t) \tag{3.32}$$

$$+ \frac{4\theta\lambda_1}{K_0(-\log ||u||_\infty)^{1+\rho}}.$$

Since $||u(t)||_\infty$ is nonincreasing, we may assume that $||u_0||_\infty > 0$ is so small that for $t \geq 0$,

$$\frac{\theta(\lambda_1 + \lambda_2)}{K_0(-\log ||u(t)||_\infty)^{1+\rho}} \leq \theta_5 = (\lambda_2 - \lambda_1)/2. \tag{3.33}$$

It follows from (3.32) and (3.33) that

$$y'(t) \leq -2\theta_5 y(t) + \frac{4\theta\lambda_1}{K_0(-\log ||u||_\infty)^{1+\rho}}.$$

Hence, for $t \geq 0$,

$$y(t) \leq y(0)\, e^{-2\theta_5 t} + \frac{4\lambda_1 \theta}{K_0} \int_0^t \frac{e^{-2\theta_5(t-s)}\, ds}{(-\log \|u(s)\|_\infty)^{1+\rho}}. \tag{3.34}$$

Here, with the aid of (3.14) and easy computation

$$\int_0^t \frac{e^{-2\theta_5(t-s)}\, ds}{(-\log \|u(s)\|_\infty)^{1+\rho}} \leq \int_0^t \frac{e^{-2\theta_5(t-s)}\, ds}{(\theta_2 s - \log \theta_1)^{1+\rho}} \tag{3.35}$$

$$\leq \frac{C}{(\theta_2 t - \log \theta_1)^{1+\rho}}, \tag{3.36}$$

where $C = C(\theta_1, \theta_2, \theta_5)$. We immediately derive (2.14) from (3.34) and (3.36). Finally we shall prove (2.15). By a basic Sobolev's inequality and (2.14),

$$\begin{aligned} &\left\| \frac{u(t)}{(u(t), e_1)_2} - e_1 \right\|_\infty \\ &\leq \sqrt{2} \left\| \frac{u(t)_x}{(u(t), e_1)_2} - e_{1x} \right\|_2^{1/2} \times \left\| \frac{u(t)}{(u(t), e_1)_2} - e_1 \right\|_2^{1/2} \\ &\leq \left\| \frac{u(t)_x}{(u(t), e_1)_2} - e_{1x} \right\|_2^{1/2} \times \frac{C}{(1+t)^{(1+\rho)/4}}. \end{aligned} \tag{3.37}$$

On the other hand, (3.25) of Lemma 3.3 and (3.16) imply that

$$K_0 \|u(t)_x\|_2 \leq \sqrt{\frac{K_1}{2t_0}}\, \|u(t-t_0)\|_2 \leq \theta_4 \sqrt{\frac{K_1}{2t_0}}\, e^{-\lambda_1(t-t_0)}. \tag{3.38}$$

Here $t_0 > 0$ is any time and we set $K_1 = k_1(\|u_0\|_\infty)$. By (3.26) and (3.38),

$$\frac{\|u(t)_x\|_2}{(u(t), e_1)_2} \leq \frac{C e^{\lambda_1 t_0}}{\sqrt{t_0}}. \tag{3.39}$$

The estimates (3.37) and (3.39) imply (2.15). ■

4. The case when $(\tilde{D})$ is degenerate at $u=0$

Throughout this section, we assume that

(4.1) $\phi: \boldsymbol{R} \to \boldsymbol{R}$ is in $C^1(\boldsymbol{R})$ and is a strictly increasing function with $\phi(0)=0$,

(4.2) There exists a strictly increasing function $K: [0, \infty) \to \boldsymbol{R}$ such that $K(0) \geq 0$ and $k(r) = \phi'(r) \geq K(|r|)$ for any $r \in \boldsymbol{R}$.

We begin with a result about the smoothing effect:

Proposition 4.1. *Assume that* ϕ *satisfies* (4.1) *and* (4.2) *and that* $u_0 \in L^2(\Omega)$. *Let* $u(x, t)$ *be the weak solution of* $(\tilde{D})$. *Then* $u(t) \in L^\infty(\Omega)$ *for* $t>0$ *and* $u(t) \underset{t\to\infty}{\longrightarrow} 0$ *in* $L^\infty(\Omega)$ *with the estimates*:

$$\|u(t)\|_\infty \leq \varepsilon + \frac{C_1}{(K(\varepsilon)\, t)^{N/4}}\, \|u_0\|_2 \quad \textit{for any} \quad \varepsilon > 0 \ \textit{and} \ t > 0. \tag{4.3}$$

$$\|u(t)\|_\infty \leq \varepsilon + \frac{C_2 e^{-\lambda K(\varepsilon)(t-t_0)}}{(K(\varepsilon)\, t_0)^{N/4}}\, \|u_0\|_2 \quad \textit{for any} \quad \varepsilon > 0 \ \textit{and} \ t > t_0. \tag{4.4}$$

Here, $t_0>0$ is an arbitrary time, C_1, $C_2>0$ are some constants dependent only on N and in particular not independent of ε, and $\lambda>0$ is the smallest positive eigenvalue of $-\Delta$ with Dirichlet condition.

Proof. Following Bertsch and Peletier [5], we compare $u(x,t)$ with the solution $v(x,t)$ of the following (I_ε):

$$(I_\varepsilon)\quad \begin{cases} v_t=\Delta\phi(v) & \text{in} \quad \Omega\times\boldsymbol{R}^+\,, \\ v(x,t)=\varepsilon>0 & \text{on} \quad \partial\Omega\times\boldsymbol{R}^+\,, \\ v(x,0)=\sup\,(u_0(x),\varepsilon) & \text{in} \quad \Omega. \end{cases}$$

With the aid of the comparison principle (Proposition 1.4),

$$u(x,t)\leq v(x,t) \quad \text{in} \quad \Omega\times\boldsymbol{R}^+\,. \tag{4.5}$$

On the other hand, by (2.6) of Theorem 2.2,

$$||v(t)-\varepsilon||_\infty\leq\frac{C}{(K(\varepsilon)\,t)^{N/4}}||v(0)-\varepsilon||_2\leq\frac{C}{(K(\varepsilon)\,t)^{N/4}}||u_0||_2 \quad \text{for} \quad t>0\,. \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$u(x,t)\leq\varepsilon+\frac{C}{(K(\varepsilon)\,t)^{N/4}}||u_0||_2 \quad \text{in} \quad \Omega\times\boldsymbol{R}^+\,. \tag{4.7}$$

If we replace ε by $-\varepsilon$ and 'sup' by 'inf' in (I_ε), then we obtain from the same argument as above that

$$u(x,t)\geq-\varepsilon-\frac{C}{(K(\varepsilon)\,t)^{N/4}}||u_0||_2 \quad \text{in} \quad \Omega\times\boldsymbol{R}^+\,.$$

Hence we obtain (4.3). We similarly obtain (4.4) from (2.7) of Theorem 2.2. ■

If $(\tilde{D})$ is degenerate at $u=0$, then, as is expected, the solution $u(x,t)$ *nevr* satisfies such a estimate as (2.2).

Corollary 4.1. *Assume that ϕ satisfies* (4.1), (4.2) *and* $k(0)=\phi'(0)=0$. *We aslo assume that $u_0\in L^2(\Omega)$, $u_0\geq0$ and $u_0(x)$ does not identically vanish in Ω. Let $u(x,t)$ be the weak solution of $(\tilde{D})$. Then, for all $\eta>0$ there exists a time $T>0$ such that*

$$||u(t)||_\infty\geq e^{-\eta t} \quad \textit{for} \quad t\geq T\,. \tag{4.8}$$

Proof. It follows from Proposition 4.1 that there exsists a time $T>0$ such that

$$||u(t)||_\infty\leq R \quad \text{for} \quad t\geq T\,, \tag{4.9}$$

where $R>0$ is a constant such that $\max_{0\leq|r|\leq R} k(r)\leq\eta/\lambda_1$ and λ_1 denotes the smallest positive eigenvalue of $-\Delta$ with Dirichlet condition. We denote by e_1 the unit positive eigenvalue of $-\Delta$ corresponding to λ_1. By integration by parts and (4.9),

$$\frac{d}{dt}(u(t), e_1)_2 = (\phi(u), \Delta e_1)_2 = -\lambda_1(\frac{\eta}{\lambda_1} u, e_1)_2 \geq -\eta(u(t), e_1)_2 \quad \text{for} \quad t \geq T,$$

which implies (4.8). ■

However, the solutions of some degenerate equations decay fairly fast.

Corollary 4.2. *Assume that ϕ satisfies* (4.1) *and that there exist $r_0 \in (0, 1)$ and η, k_0, $\theta > 0$ such that*

$$k(r) \geq \frac{\theta}{(-\log|r|)^\eta} \quad \textit{for} \quad r \in [-r_0, r_0], \tag{4.10}$$

$$k(r) \geq k_0 \quad \textit{for} \quad r \in \boldsymbol{R} \setminus [-r_0, r_0]. \tag{4.11}$$

Let $u(x,t)$ be the weak solution of $(\tilde{D})$ with $u_0 \in L^2(\Omega)$. Then the following estimate holds:

$$||u(t)||_\infty \leq C(t+1)^{N\eta/4(\eta+1)} \exp\{(-(\theta\lambda t)^{1/(\eta+1)}\} \quad \textit{for} \quad t \geq 0, \tag{4.12}$$

where $\lambda > 0$ is the smallest positive eigenvalue of $-\Delta$ with zero-Dirichlet condition, and $C > 0$ depends only on $||u_0||_2$, r_0, k_0, η, θ, N and Ω.

Proof. We assume, by Proposition 4.1, without loss of generality that $u_0 \in L^\infty(\Omega)$ and $||u_0||_\infty \leq r_0$. Substituting $\varepsilon = C \exp\{-(\theta\lambda t)^{1/(\eta+1)}\}$ to (4.4), we immediately obtain (4.12). ■

REMARK 4.1. The estimate (4.12) seems to be fairly sharp. Assume that there exist $r_0 \in (0, 1)$ and η, $\theta > 0$ such that

$$\phi(r) = \frac{\theta r}{(-\log|r|)^\eta} \quad \text{for} \quad r \in [-r_0, r_0].$$

We assume for simplicity that $u_0 \in L^\infty(\Omega)$, $||u_0||_\infty \leq r_0$ and $\inf_{x \in \Omega} u_0(x) \geq \delta$ for some $\delta \in (0,1)$. Then the following lower estimate holds:

$$||u(t)||_1 \geq C \exp\{-((\eta+1)\,\theta\lambda t)^{1/(\eta+1)}\} \quad \text{for} \quad t \geq 0. \tag{4.13}$$

Here $\lambda > 0$ is the same constant as defined in Corollary 4.2, and $C > 0$ depends only on r_0, η, θ, δ, Ω and N.

Now we prove (4.13).

Proof of (4.13). The main tools for the proof are the smoothing technique and the comparison principle. Let $u_\varepsilon(x, t)$ be the solution of the following problem:

$$\begin{cases} u_{\varepsilon t} = \Delta\phi(u_\varepsilon) & \text{in} \quad \Omega\times\boldsymbol{R}^+ , \\ u_\varepsilon(x, t) = \varepsilon>0 & \text{on} \quad \partial\Omega\times\boldsymbol{R}^+ , \\ u_\varepsilon(x, 0) = u_0+\varepsilon & \text{in} \quad \Omega . \end{cases}$$

It follows from Proposition 1.3 that

$$u_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} u(t) \quad \text{in} \quad L^1(\Omega) . \tag{4.14}$$

On the other hand, following closely Bertsch and Peletier [4], we shall construct a separable subsolution of $u_\varepsilon(x, t)$ for all $\varepsilon>0$. Let $w(x)$ be a solution of

$$\begin{cases} -\Delta w = \lambda w & \text{in} \quad \Omega , \\ w = 0 & \text{on} \quad \partial\Omega , \\ 0\leq w<1/e,\ w\neq 0 & \text{in} \quad \Omega . \end{cases}$$

And let $y(t)$ be the solution of

$$\begin{cases} y'(t) = -\lambda\phi(y) \\ y(0) = \delta . \end{cases}$$

Here, δ is the same constnat as stated in Remark 4.1. We set $w_\varepsilon=w+\varepsilon$ and $z_\varepsilon(x, t)=w_\varepsilon(x)\, y(t)$ ($\varepsilon>0$). It follows that

$$\begin{aligned} \mathcal{L}(z_\varepsilon) &= (z_\varepsilon)_t-\Delta\phi(z_\varepsilon) \\ &\leq -\lambda\phi(y)\, w_\varepsilon+k(yw_\varepsilon)\, y\cdot\lambda w \\ &\leq \frac{\lambda\theta yw}{(-\log y)^\eta}\{-1+\frac{(-\log y)^\eta\,(-\log y-\log w_\varepsilon+\eta)}{(-\log y-\log w_\varepsilon)^{\eta+1}}\} . \end{aligned}$$

Here, if we set $a=-\log y>0$ and $b=-\log w_\varepsilon$, then

$$\{\quad\}\leq -1+\frac{a^{\eta+1}+(\eta+1)\, a^\eta b}{(a+b)^{\eta+1}}\leq 0 .$$

Hence we obtain that

$$\mathcal{L}(z_\varepsilon)\leq 0 \quad \text{in} \quad \Omega\times\boldsymbol{R}^+ .$$

Futhermore, we have $z_\varepsilon(x, t)\leq u_\varepsilon(x, t)$ on the parabolic boundary of $\Omega\times\boldsymbol{R}^+$. Therefore, we apply Proposition 1.4 (the comprrison principle) to obtain that

$$z_\varepsilon(x, t)\leq u_\varepsilon(x, t) \quad \text{for} \quad \varepsilon>0 \quad \text{and} \quad (x, t)\in\Omega\times\boldsymbol{R}^+ .$$

It follows that

$$||z_\varepsilon(t)||_1\leq ||u_\varepsilon(t)||_1 \quad \text{for} \quad \varepsilon>0 \quad \text{and} \quad t\geq 0 . \tag{4.15}$$

Let $\varepsilon\to 0$ in (4.15), then by (4.14), (4.16) $$\|w\|_1\, y(t) \leq \|u(t)\|_1 \quad \text{for} \quad t \geq 0\,,$$

which implies (4.13). ■

5. The case when $(\tilde{N})$ is degenerate

Throughout this section we always assume that

(5.1) $\phi: \boldsymbol{R} \to \boldsymbol{R}$ is a strictly increasing, locally Lipschitz continuous function with $\phi(0)=0$,

(5.2) $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ a.e. in Ω and the set $S_\nu = \{x \in \Omega;\ u_0(x) \geq \nu\}$ contains a nonempty open subset of Ω for some $\nu > 0$.

We remark that the weak solutions of $(\tilde{N})$ become nonnegative under the condition (5.2). We give a result on the behavior of the support for weak solutions of $(\tilde{N})$ for finite values of time:

Theorem 5.1. *We assume* (5.1) *and* (5.2). *Let* $u(x, t)$ *be the solution of* $(\tilde{N})$. *Then there exist* $\delta > 0$ *and* $T > 0$ *such that*

$$u(x, t) \geq \delta \quad \textit{for} \quad (x, t) \in \bar{\Omega} \times [T, \infty)\,.$$

If we assume the following stronger condition (5.3) instead of (5.1):

(5.3) $\phi: \boldsymbol{R} \to \boldsymbol{R}$ is in $C^1(\boldsymbol{R})$ and is a strictly increasing function with $\phi(0)=0$ and $\phi'(r) > 0$ if $r \neq 0$,

then Theorem 5.1 implies that the solution $u(x, t)$ of $(\tilde{N})$ behaves as a solution of a nondegenerate equation after a finite time even if the initial value $u_0(x)$ has compact support in Ω. If we apply Theorem 2.1, then we immediately obtain the following result:

Corollary 5.1. *We assume* (5.2), (5.3) *and* (2.3) (*we set* $a^{ij}(x, r) = k(r)\, \delta^{ij}$). *Let* $u(x, t)$ *be the solution of* $(\tilde{N})$. *Then the following estimate holds*:

$$\|u(t) - \bar{u}_0\|_\infty \leq C \exp\left[-\mu \phi'(\bar{u}_0)\, t\right] \quad \textit{for} \quad t \geq 0\,,$$

where $\mu > 0$ *is the smallest positive eigenvalue of* $-\Delta$ *with Neumann condition, and* $C > 0$ *is a constant depending on* u_0.

We need several lemmas to prove Theorem 5.1.

Lemma 5.1. *We assume that all the assumptions of Theorem* 5.1 *are satisfied. Then,*

$\forall$ *compact subset* $S \subset \Omega$, $\exists T_1 > 0, \forall t \geq T_1$, $\exists \eta = \eta(t) > 0$;

$$u(x, t) \geq \eta \quad \textit{on} \quad S\,.$$

Proof. Since the proof is the same as that of Proposition 4 given in Aronson and Peletier [2], we omit it (see also the proof of Theorem 5.1). ■

We set $\Gamma=\partial\Omega$. We denote by $\boldsymbol{v}_Q$ the unit outward vector at $Q\in\Gamma$. We set $\Gamma_d=\{P\in\boldsymbol{R}^N;$ there exists $Q\in\Gamma$ such that $\overrightarrow{PQ}=d\boldsymbol{v}_Q\}$,

$$B(P;d)=\{Q\in\boldsymbol{R}^N;\ \overline{PQ}<d\}\quad\text{and}\quad \Omega_d=\Omega\setminus\bigcup_{\delta\in(0,d]}\Gamma_\delta\,.$$

The following result is well-known (See e.g. Theorem IV. 1.1 in [17]).

Lemma 5.2. *There exists a constant $d_0>0$ such that the following* (1)-(3) *hold*:

(1) $PQ>d_0$ *for any* $P\in\Gamma_{d_0}$ *and* $Q\in\Gamma$ *such that* $PQ\neq d_0\boldsymbol{v}_Q$.

(2) Γ_{d_0} *is a smooth* $(N-1)$*-dimensional manifold with* $\Gamma\ni Q\overset{\sim}{\to}\overrightarrow{OQ}-d_0\boldsymbol{v}_Q\in\Gamma_{d_0}$: *diffeomrophism.*

(3) $\bigcup_{P\in\Gamma_{d_0}} B(P;\,d_0/3)\subset\overline{\Omega_{d_0/3}}\subset\Omega.$

Lemma 5.3. *Let d_0 be the same constant as stated in Lemma* 5.2. *Then there exists a constant $d_1>d_0$ such that $(PQ,\boldsymbol{v}_Q)\geq 0$ for all $P\in\Gamma_{d_0}$ and $Q\in\Gamma\cap\overline{B(P;d_1)}$.*

Proof. We define a continuous map

$$F\colon \Gamma_{d_0}\times\Gamma\ni(P,Q)\mapsto(\overrightarrow{PQ},\boldsymbol{v}_Q)\in\boldsymbol{R}\,.$$

We also define a map $G\colon\Gamma_{d_0}\to\boldsymbol{R}$ by

$$\Gamma_{d_0}\ni P\mapsto\sup\,\{d_0<d\leq 2d_0;\ (\overrightarrow{PQ},\boldsymbol{v}_Q)>0\quad\text{for all}\quad Q\in\Gamma\cap\overline{B(P;d)}\}\,.$$

G is well-defined in view of Lemma 5.2. Furthermore G is lower semicontinuous by the continuity of F. Hence

$$\liminf_{P_j\to P} G(P_j)\geq G(P)\,. \tag{5.4}$$

It follows from (5.4) and the compactness of Γ_{d_0} that G takes the minimum $d_2(>d_0)$. We have only to choose d_1 such that $d_0<d_1<d_2$. ■

Proof of Theorem 5.1. We often use the notations in Lemmas 5.2 and 5.3. By Lemma 5.1, we may assume without loss of generality that $u(x,t)$ is strictly positive in $\overline{\Omega_{d_0/3}}$ for all $t\in[0,\infty)$, i.e.

$$u(x,t)\geq\eta(t)>0\ \text{ in }\ \overline{\Omega_{d_0/3}}\quad\text{for}\quad t\in[0,\infty)\,. \tag{5.5}$$

Hence the proof is complete if we show the existence of a time $T_1>0$ and a constant $\delta_1>0$ such that

$$u(x,T_1)\geq\delta_1\quad\text{on}\quad\overline{B(P;d_0)}\quad\text{for all}\quad P\in\Gamma_{d_0}\,. \tag{5.6}$$

Indeed, by (5.5) and (5.6) there exists a constant $\delta>0$ such that $u(x,T_1)\geq\delta$ in

Ω, which, combined with (5) of Proposition 1.2 (set $v_0=\delta$), implies that $u(x,t)\geq\delta$ for all $(x,t)\in\bar\Omega\times[T_1,\infty)$. Therefore we shall prove (5.6). By (5.5) and (3) of Lemma 5.2 we may assume that there exists a constant $\delta_2>0$ sucht hat

$$(5.7)\qquad u_0\geq\delta_2 \quad\text{on}\quad B(P;d_0/3) \quad\text{for all}\quad P\in\Gamma_{d_0}.$$

We fix an arbitrary point $P\in\Gamma_{d_0}$ and assume for simplicity that $P=O$. We choose $v_0\in C_0^\infty(B(0;d_0/3))$ such that $v_0=v_0(r)$ is a nonincreasing function of $r=|x|$, $0\leq v_0\leq\delta_2$ and v_0 does not identically vanish in $B(0;d_0/3)$. Let $v(x,t)$ be the solution of $(\tilde D)$ with Ω and $u_0(x)$ replaced by $B(0;d_1)$ and $v_0(x)$ respectively. Then v is a nonincreasing function of $r=|x|$ for every $t\in[0,\infty)$. For the proof of this fact, see the proof of Lemma 2.2 in Aronson and Caffarelli [14]. There exists a time $T_1>0$ such that

$$(5.8)\qquad \text{support}\,(v(x,T_1))=\overline{B(0;d_1)}.$$

Indeed, if otherwise, $v(x,t)$ has compact support in $B(0;d_1)$ for all $t\geq 0$. Then we can easily verify that $v(x,t)$ is also a solution of $(\tilde N)$ with Ω and $u_0(x)$ replaced by $B(0;d_1)$ and $v_0(x)$ respectively. Hence, by (4) of Proposition 1.2, $\int_{B(0;d_1)}v(x,t)\,dx=\int_{B(0;d_1)}v_0\,dx$ for all $t\in[0,\infty)$. This contradicts Lemma 3.3. We set

$$w(x,t)=\begin{cases}v(x,t) & \text{if}\quad x\in\Omega\cap B(0;d_1),\\ 0 & \text{if}\quad x\in\Omega\backslash B(0;d_1).\end{cases}$$

We caim that

(5.9) $w(x,t)$ is a subsolution of $(\tilde N)$ on $t\in[0,\infty)$.

Since $u_0\geq v_0$, (5.9) and Proposition 1.4 show that $u(x,t)\geq w(x,t)$ for $(x,t)\in\bar\Omega\times[0,\infty)$. This leads us to (5.6). Hence we shall now prove (5.9). We proceed in two steps.

Step 1. We shall show (5.9) under the following additional condition:

(5.10) ϕ is smooth.

We set $\phi_n(r)=\phi(r)+r/n$. Let $v_n(x,t)$ be the (smooth) solution of $(\tilde D)$ with ϕ, Ω and u_0 replaced by ϕ_n, $B(0;d_1)$ and v_0 respectively. By the choice of v_0, v_n is a nonincreasing function of $r=|x|$ for every $t\in[0,\infty)$. It follows from this observation and Lemma 5.3 that

$$(5.11)\qquad \frac{\partial}{\partial\nu}\phi_n(v_n(x,t))\leq 0 \quad\text{a.e. on}\quad (x,t)\in\partial(\Omega\cap B(P;d_1))\times[0,\infty).$$

We obtain from (5.11) that $v_n(x,t)$ is a subsolution of $(\tilde N)$ with Ω replaced by $\Omega'=\Omega\cap B(P;d_1)$, i.e.

$$(5.12)\quad \int_{\Omega'}(v_n(T)\varphi(T)-v_0(0)\varphi(0))\,dx-\int_0^T\int_{\Omega'}(v_n\varphi_t-\nabla\phi_n(v_n)\cdot\nabla\varphi)\,dx\,dt\leq 0$$

for all $T\in(0, \infty)$ and $\varphi\in C^2(\bar{\Omega}\times[0, \infty))$ such that φ is nonnegative. Let $n\to\infty$ in (5.12) and we obtain from Corollary 1.2 and Lemma 3.3 that

$$(5.13)\quad \int_\Omega (w(T)\varphi(T)-v_0(0)\varphi(0))\,dx-\int_0^T\int_\Omega (w\varphi_t-\nabla\phi(w)\cdot\nabla\varphi)\,dx\,dt\leq 0$$

for all $T\in(0, \infty)$ and $\varphi\in C^2(\bar{\Omega}\times[0, \infty))$ such that φ is nonnegative. This leads us to the claim (5.9).

Step 2. We shall obtain (5.9) without assuming (5.10). Let $v_n(x, t)$ be defined as in Step 1. Then we can obtain (5.12), applying the smoothing technique in the same way as in Step 1. (We use Corollary 1.1 instead of Corollary 1.2.) Then the argument used to derive (5.9) is just the same as in Step 1. ■

Appendix

We shall describe a rseult on the behavior of solutions of the following equation with absorption:

$$(N_p)\begin{cases} u_t = \Delta(|u|^{m-1}u)-\lambda u^p & \text{in } \Omega\times\boldsymbol{R}^+, \\ \dfrac{\partial}{\partial\nu}(|u|^{m-1}u) = 0 & \text{on } \partial\Omega\times\boldsymbol{R}^+, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $m>1$, $p\geq 1$ and $\lambda>0$ are constants. We immediately obtain the following Theorem A.1. from Theorem 5.1, the agrumentation used by Alikakos and Rostamian [11, section 2] and the proof of Lemma 7 in Bertsch, Nanbu and Peletier [13].

Theorem A.1. *Assume that $p\geq m>1$ and u_0 satisfies the condition* (5.2). *Let $u(x, t)$ be the* (*nonnegative*) *weak solution of* (N_p). *Then $u(x, t)$ eventually becomes strictly positive even if $u_0(x)$ has compact support in Ω. And $u(t)\to 0$ in $L^\infty(\Omega)$ as $t\to\infty$ with the estimate*:

$$\|[(p-1)\,t]^{1/(p-1)}u(t)-1\|_\infty\leq Ct^{-1/(p-1)} \quad \textit{for} \quad t\geq 0\,.$$

Here $C>0$ is a constant depending on u_0.

REMARK A.1. 1) Alikakos and Rostamian [11] have obtained the L^q-estimate $(1\leq q<\infty)$ without the sign condition of $u_0(x)$.

2) Bertsch, Nanbu and Peletier [13] fully discussed the nonnegative solution of the Dirichlet probelm corresponding to (N_p). In particular, as a result, they proved that when $1<p<m$, for certain initial functions with compact support in Ω, the support of solutions $u(x, t)$ remain compact for all time. This result also holds for (N_p) because the solution of the Dirichlet problem with compact support is also a solution of (N_p).

Note added in proof. The author has noticed that we can derive the better estimate than (2.14):

$$\left\|\frac{u(t)}{(u(t), e_1)_2}-e_1\right\|_{H_0^1}\leq\frac{C_3}{(1+t)^{1+\rho}} \quad \text{for} \quad t\geq 0$$

by combining the proof of Theorem 2.5 in Nagasawa [18] with that of our Theorem 2.3.

References

[1] N.D. Alikakos and R. Rostamian: *Large time behavior of solutions of Neumann boundary value problem for the porous medium equation,* Indiana University Math. Journal **30** (1981), 749–785.

[2] D.G. Aronson and L.A. Peletier: *Large time behaviour of solutions of the porous mediem equation in bounded domains,* J. Diff. Equations **39** (1981), 378–412.

[3] P. Benilan: *Équations d'évolution dans un espace de Banach quelconque et applications,* Thesis, Univ. Paris XI, Orsay, 1972.

[4] M. Bertsch and L.A. Peletier: *The Asymptotic Profile of Solutions of Degenerate Diffusion Equations,* Arch. Rat. Mech. Anal. **91** (1986), 207–229.

[5] M. Bertsch and L.A. Peletier: *A positivity property of solutions of nonlinear diffusion equations,* J. Diff. Eqns. **53** (1984), 30–47.

[6] H.S. Carslaw and J.C. Jaeger: Conduction of Heat in Solids, 2nd ed., Oxford at the Clarendon Press, Oxford, 1959.

[7] J. Crank: The Mathematics of Diffusion, 2nd ed., Oxford University Press, Oxford, 1975.

[8] L.C. Evans: *Differentiability of a nonlinear semigroup in* L^1, J. Math. Analysis and Application **60** (1977), 703–715.

[9] L.C. Evans: *Regularity properties for the heat equation subject to nonlinear boundary constraints,* Nonlinear Analysis **1** (1977), 593–602.

[10] L.C. Evans: *Application of nonlinear semigroup theory to certain partial differential equations,* in Nonlinear Evolution Equations (M.G. Crandall ed.) Academic Press, New York, 1978.

[11] N.D. Alikakos and R. Rostamian: *Stabilization of solutions of the equation* $\partial u/\partial t=\Delta\phi(u)-\beta(u)$, Nonlinear Analysis **6** (1982), 637–647.

[12] O.A. Oleinik and S.N. Kruzhkov: *Quasi-linear second-order parabolic equations with many independent variables,* Russian Math. Surveys **16** (1961), 105–146.

[13] M. Bertsch, T. Nanbu and L.A. Peletier: *Decay of solutions of a degenerate nonlinear diffusion equation,* Nonlinear Analysis **6** (1982), 539–554.

[14] D.G. Aronson and L.A. Caffarelli: *The initial trace of a solution of the porous medium equation,* Trans. Amer. Math. Soc. **280** (1983), 351–366.

[15] D.G. Aronson, M.G. Crandall and L.A. Peletier: *Stabilization of solutions of a degenerate nonlinear diffusion problem,* Nonlinear Analysis **6** (1982), 1001–1022.

[16] M. Nakao: *Global solutions for some nonlinear parabolic equations with nonmonotonic perturbations,* Nonlinear Analysis **10** (1986), 299–314.

[17] H. Nakagawa: Global Riemannian Geometry (in Japanese), Kaigai-shhuppan-

boeki, Inc., Tokyo, 1977.
[18] T. Nagasawa: *Boundary value problems to a certain class of nonlinear diffusion equations,* (manuscript).
[19] J.G. Berryman and C.J. Holland: *Asymptotic behavior of the nonlinear diffusion equation* $n_t=(n^{-1}n_x)_x$, J. Math. Phys. **23** (1982), 983–987.

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