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ON ALMOST RELATIVE PROJECTIVES OVER PERFECT RINGS

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We have defined a new concept of almost relative projectivity [7]. If a module M_0 is M_i-projective for a finite set of modules M_i, then M_0 is $\Sigma_i \oplus M_i$-projective [2]. However this fact is not true for almost relative projectives [7]. We have filled this gap in [6], when a ring R is a semiperfect ring with radical nil and M_0 is a local R-module and the M_i are LE R-modules. As we investigate further several properties of almost relative projectives, it seems for us that the gap is one of essential structures of almost relative projectives. Thus we shall fill completely that gap in this paper, when R is a perfect ring (Main theorem). M_0 was cyclic in [6] and hence the proof was rather simple. Modifying its proof, we shall give a generalization of [6], Theorem 2.

We shall give several applications of the main theorem in forthcoming paper [8], and give the properties dual to this paper in [9].

1. Preliminaries

In this paper we always assume that R is a ring with identity and that every module is a unitary right R-module and e, e' are primitive idempotents unless otherwise stated. We recall here the definition of almost relative projectivity [7]. Let M and N be R-modules. For any diagram with K a submodule of M:

$$\begin{array}{ccc} M_1 & \overset{\tilde{h}}{\dashrightarrow} & N \\ \cap_{\oplus} & \swarrow^{\tilde{h}} & \downarrow h \\ M & \xrightarrow{\nu} & M/K \rightarrow 0 \end{array}$$

if either there exists $\tilde{h}$: $N \rightarrow M$ with $\nu\tilde{h}=h$ or there exist a nonzero direct summand M_1 of M and $\tilde{h}$: $M_1 \rightarrow N$ with $h\tilde{h}=\nu|M_1$, then N is called *almost M-projective* [7] (if we obtain only the first case, we say that N is *M-projective* [2]).

We note the following fact.

(#) *When N is almost M-projective and M is indecomposable, if the h in the diagram* (1) *is not an epimorphism, then there exists always an $\tilde{h}$: $N \rightarrow M$ with $\nu\tilde{h}=h$.*

We frequently use this fact without any references.

Lemma 1. *Let R be a right perfect ring with Jacobson radical J and let M_0 and M_1 be R-modules and $M_0 \cong P/Q$ for R-modules $P \supset Q$ with $Q \subset PJ$. Let g be an element in* $\mathrm{Hom}_R(P, M_1)$. *We assume one of the following*:

a) *M_0 is M_1-projective, and*

b) *M_0 is almost M_1-projective, M_1 is indecomopsable and g is not an epimorphism.*

Then $g(Q)=0$ (cf. [3], Lemma 6).

Proof. Consider the derived diagram from g

$$\begin{array}{ccc} & & M_0=P/Q\,. \\ & & \downarrow \bar{g} \\ M_1 & \xrightarrow{\nu} & M_1/g(Q) \to 0 \end{array}$$

From assumption and (#) there exists $\tilde{h}$: $P/Q \to M_1$ with $\nu\tilde{h}=\bar{g}$. Let ρ be the natural epimorphism: $P \to P/Q$ and put $h=\tilde{h}\rho$: $P \to M_1$. Since $\nu\tilde{h}=\bar{g}$, for any $p \in P$

$$g(p)+g(Q)=\bar{g}(p+Q) = \nu\tilde{h}(p+Q) = \nu\tilde{h}\rho(p) = h(p)+g(Q)\,.$$

Hence

$$(2) \qquad g(p)-h(p) = g(q(p));\ q(p) \quad \text{is an element in } Q\,.$$

Let $\{p_i\}$ be a set of generators of P, i.e., $P=\Sigma p_i R$ and put

$$(3) \qquad g(p_i)-h(p_i) = g(q_i) \quad \text{for each } i \text{ from (2)}\,,$$

where q_i is some element in Q.
Now $Q \subset PJ = \Sigma p_i J$ by assumption, and $q=\Sigma p_i x_i$; $x_i \in J$ for any q in Q. Then

$$\begin{aligned} 0 = \tilde{h}(q+Q) &= h(q) = \Sigma h(p_i)\, x_i \\ &= \Sigma(g(p_i)\, x_i - g(q_i)\, x_i) \quad \text{from (3)} \\ &= g(\Sigma p_i x_i) - \Sigma g(q_i)\, x_i = g(q) - \Sigma g(q_i)\, x_i\,. \end{aligned}$$

Accordingly $g(Q) \subset g(Q)J = g(QJ) \subset g(Q)$. Therefore $g(Q)J=g(Q)$ implies $g(Q)=0$.

In Lemma 1 we take a projective cover P of M_0, i.e., there exists an epimorphism ν: $P \to M_0$ where P is projective and $\ker \nu = K$ is small in P. Then the following is clear from Lemma 1.

Corollary 1 ([1], p. 22, Exercise 4). *Let P and M_0 be as above and M_1 an R-module. Then M_0 is M_1-projective if and only if $h(K)=0$ for any h in* $\mathrm{Hom}_R(P, M_1)$.

If $\mathrm{End}_R(M)$ is a local ring for an R-module M, then we call M an LE module. It is clear that an LE module is indecomposable. By $\mathrm{J}(M)$ we denote *the Jacobson radical* of M. Let eR/A and eR/B be local modules, i.e., e is primitive. We say that $eR/A \oplus eR/B$ has *the lifting property of simple modules modulo radical* (briefly LPSM) if and only if for any isomorphism f of eR/eJ onto itself, there exists a g in $\mathrm{Hom}_R(eR/A, eR/B)$ (or in $\mathrm{Hom}_R(eR/B, eR/A)$) such that g induces f (or f^{-1}). If eR/A and eR/B are LE, then the concept of LPSM coincides with one in [5], §9. See [10] for the definition of the lifting module.

Proposition 1. *Let R be a perfect ring and let M_1, M_2 be indecomposable R-modules and M_0 an R-module. Assume that M_0 is almost M_1-projective, but not M_1-projective. Then* 1): *if M_0 is M_2-projective, M_1 is M_2-projective.* 2): *If M_0 is almost M_2-projective, but not M_2-projective, then M_1 is $\mathrm{J}(M_2)$-projective and further we obtain the following two cases; i) if $M_1/\mathrm{J}(M_1) \not\approx M_2/\mathrm{J}(M_2)$, M_1 is M_2-projective and M_2 is M_1-projective, ii) if $M_1/\mathrm{J}(M_1) \approx M_2/(\mathrm{J}(M_2)$, we have the following equivalent conditions*:

a) *M_1 is almost M_2-projective.*

a′) *M_2 is almost M_1-projective.*

b) *$M_1 \oplus M_2$ has* LPSM.

Proof. 1) Assume that M_0 is M_2-projective. Since M_0 is not M_1-projective, $M_1 \approx eR/A$ by [6], Corollary 1, where e is a primitive idempotent and $A \subset eR$. Further from [6], Corollary 2 there exists a homomorphism f: $M_1 = eR/A \to M_0$ such that $f(\tilde{e}) = m_0 = m_0 e \notin J(M_0)$, where $\tilde{e} = e + A$ in eR/A. Since $m_0 \notin J(M_0)$, there exists a projective cover $P = eR \oplus e_2R \oplus \cdots$ of M_0 and the natural epimorphism ν: $P \to M_0$ such that $\nu(e) = m_0$. Put $K = \ker \nu$ and $B = K \cap eR$ ($eR \subset P$). Since $f(eR/A) = m_0 R \approx eR/B$, there exists a unit x in eRe with $xA \subset B$. Since $eR/A \approx eR/xA$, we may assume $A = xA \subset B$. Let h be any element in $\mathrm{Hom}_R(eR, M_2)$. Then we can naturally extend h to an element h' in $\mathrm{Hom}_R(P, M_2)$, since eR is a direct summand of P. M_0 being M_2-projective and P being a projective cover of M_0, $h'(K) = 0$ by Corollary 1. Hence

$$h(A) \subset h(B) \subset h'(K) = 0\,,$$

and so eR/A is M_2-projective again by Corollary 1.

2) Assume that M_0 is not M_2-projective. Then $M_2 \approx e'R/C$ for some primitive idempotent e' by [6], Corollary 1. First assume i): $e \not\approx e'$. Then the above h is not an epimorphism. Hence we can find a non-epic homomorphism h' in $\mathrm{Hom}_R(P, M_2)$, which is an extension of h. Then since $h'(K) = 0$ by Lemma 1, M_1 is M_2-projective (and so $J(M_2)$-projective) as the last sentence of the proof of 1). Similarly M_2 is M_1-projective by symmetric assumption. Finally assume ii): $e \approx e'$. We may assume $e = e'$. Take a diagram with row exact:

$$\begin{array}{ccc} & & eR/A \\ & & \downarrow h \\ eJ/C & \xrightarrow{\nu} & eJ/D \rightarrow 0 \end{array}$$

Since eR is projective, there exists h': $eR \rightarrow eJ/C \subset eR/C = M_2$ with $\nu h' = h\rho$, where ρ: $eR \rightarrow eR/A$ is the natural epimorphism. Then since h' is not an epimorphism onto M_2, $h'(A)=0$ by Lemma 1 as before, and so h' induces $\tilde{h}$: $eR/A \rightarrow eJ/C$ with $\nu\tilde{h}=h$. Hence eR/A is eJ/C-projective (similarly eR/C is eJ/A-projective). Now suppose that $M_1 \oplus M_2$ has LPSM. Let u be any unit in eRe. Then $(u+j)A \subset C$ or $(u+j)\,C \subset A$ for some j in eJe by definition. j_l, the multipliaction of j from the left side, gives an element in $\mathrm{Hom}_R(eR, eR/C)$ and j_l is not an epimorphism. Further j_l induces an element in $\mathrm{Hom}_R(P, M_2)$ as in the proof of 1). Since M_0 is almost M_2-projective, $jA \subset C$ by Lemma 1 and the last fact of the proof of 1). Similarly we obtain $jC \subset A$. Therefore $uA \subset C$ or $uC \subset A$. Hence M_1 and M_2 are mutually almost relative projective by [3], Proposition 2. a) implies b) by definition.

2. Main theorem

Let M_0 be an R-module and $\{M_i\}_{i=1}^s$ a set of indecomposable R-modules. If M_0 is almost $\sum_{i=1}^s \oplus M_i$-projective, clearly M_0 is almost M_i-projective for all i. We assume conversely that M_0 is almost M_i-projective for all i. In [6] we have given a condition under which M_0 is almost $\sum_{i=1}^s \oplus M_i$-projective, when R is semiperfect and $M_0 = eR/A$ for a primitive idempotent e and a submodule A in eR. In this section we shall generalize this condition, when R is a perfect ring and M_0 is an R-module.

Now we assume that R is a semiperfect ring with radical J. Let M_0 be an R-module such that $M_0 \neq M_0J$. Then M_0/M_0J is semisimple. Put $M_0/M_0J = \Sigma \oplus S_i$, where the S_i are simple modules isomorphic to e_iR/e_iJ for some primitive idempotent e_i. We take m_j in M_0 such that $(m_jR + M_0J)/M_0J = S_j$; $m_je_j = m_j$, and fix one simple component S_1 among S_j.

Lemma 2. *Let R, M_0, $\{m_i\}$ and e_1 be as above and M an R-module. Let x be an element in M with $xe_1 = x$. If*

i) *M_0 is M-projective, or*

ii) *M_0 is almost M-projective, M is indecomposable and $xR \subsetneq M$,*

then there exists a homomorphism $\tilde{h}$: $M_0 \rightarrow M$ such that

1) *$\tilde{h}(m_1) = x + xj$; $j \in eJe$ and*

2) *$\tilde{h}(m_i) \in xJ$ for $i \neq 1$, and hence $\tilde{h}(M_0) = xR$.*

Proof. Since $xe_1 = x$, $xR/xJ \approx e_1R/e_1J$. Further $M_0/M_0J = \Sigma \oplus \overline{m_iR}$; $\overline{m_iR} = (m_iR + M_0J)/M_0J$. Hence we can take a submodule B in M_0 such that $B \supset M_0J$, $M_0/B \approx m_1R$ and $m_j \in B$ for $j \neq 1$. Take a diagram:

$$\begin{array}{c} M_o \\ \downarrow \nu_o \\ M_o/B \\ \wr\!\wr\, g \\ xR/xJ \\ \cap \\ M \xrightarrow{\nu'} M/xJ \to 0 \end{array}$$

where $g(\bar{m}_1)=x+xJ$. Then from the assumption i) or ii) together with (#) there exists $\tilde{h}$: $M_o\to M$ such that $\nu'\tilde{h}=g\nu_o$. Hence $\tilde{h}(m_1)=x+xj$; $j\in J$ and $\tilde{h}(m_i)\in xJ$ for $i\neq 1$. Clearly $\tilde{h}(M_o)=xR$.

Corollary 2. *We assume in Lemma 2 that J is left T-nilpotent. Then we can find $\tilde{h}$: $M_o\to M$ with $\tilde{h}(m_1)=x$ and $\tilde{h}(m_i)\in xJ$ for $i\neq 1$.*

Proof. We obtain $\tilde{h}_1$: $M_o\to xR\subset M$ such that $\tilde{h}_1(m_1)=x-xj_1$; $j_1\in J$. Being $xj_1e_1=xj_1$ and $xj_1R\subset xR\neq M$ (in case ii)), we have $\tilde{h}_2$: $M_o\to xj_1R\subset xR\subset M$ such that $\tilde{h}_2(m_1)=xj_1-xj_1j_2$; $j_2\in J$ and $\tilde{h}_2(m_i)\in xJ$ for $i\neq 1$. Hence $(\tilde{h}_1+\tilde{h}_2)(m_1)=x-xj_1j_2$ and $(\tilde{h}_1+\tilde{h}_2)(m_i)\in xJ$ for $i\neq 1$. Since J is left T-nilpotent, we can find $\{\tilde{h}_i\}$ such that $(\tilde{h}_1+\tilde{h}_2+\cdots+\tilde{h}_n)(m_2)=x$ for some n and $(\tilde{h}_1+h_2\cdots+\tilde{h}_n)(m_i)\in xJ$ for $i\neq 1$.

Similarly to Lemma 2 we obtain

Lemma 2′. *Let R be a semiperfect rnig with J left T-nilpotent. Let $M_1=eR/A_1$, $M_2=eR/A_2$ be mutually almost relative projective. Then for any element x_i in $M_i-J(M_i)$ with $x_i=x_i e$ $(i=1, 2)$ there exists either h_1: $M_1\to M_2$ (or h_2: $M_2\to M_1$) with $h_1(x_1)=x_2$ (or $h_2(x_2)=x_1$), where e is a primitive idempotent.*

Proof. Take a diagram

$$\begin{array}{c} M_2 \\ \downarrow \nu_2 \\ M_2/J(M_2) \\ \wr\!\wr\, f \\ M_1 \xrightarrow{\nu_1} M_1/J(M_1) \to 0 \end{array}$$

where $f(x_2+J(M_2))=\nu_1(x_1)$. Then there exists $\tilde{h}_2$: $M_2\to M_1$ (or $\tilde{h}_1$: $M_1\to M_2$) with $\tilde{h}_2(x_2)=x_1-x_1j$; $j\in eJe$ (or $\tilde{h}_1(x_1)=x_2-x_2j$). Further from Corollary 2 there exist $\tilde{h}_2'$: $M_2\to M_1$ and $\tilde{h}_1'$: $M_1\to M_2$ with $\tilde{h}_2'(x_2)=x_1j$ and $\tilde{h}_1'(x_1)=x_2j$, respectively. Therefore $(\tilde{h}_2+\tilde{h}_2')(x_2)=x_1$ or $(\tilde{h}_1+\tilde{h}_1')(x_1)=x_2$.

The following simple lemma is useful in this paper.

Lemma 3. *Let R be a perfect ring and let M_o be an R-module and $M_1=eR/A$ for a primitive idempotent e. Let $x=xe$ be an element in $M_1-J(M_1)$ and*

$h: M_1 \to M_0$ any homomorphism such that $h(x)\ (=m_0=m_0 e) \notin J(M_0)$. Under those assumptions if M_0 is almost M_1-projective, then for each element j in eJe, there exists an endomorphism f of M_0 such that $f(m_0)=m_0+m_0 j$.

Proof. Since $xj \in J(M_1)e$, there exists $g: M_0 \to M_1$ such that $g(m_0)=xj$ by Corollary 2. Hence $f=1_{M_0}+hg$ is the desired endomorphism.

Before stating Main Theorem, we give here a simple remark, which is helpful for us to understand the argument in [3], §1.

Let $D=D_1 \oplus D_2 \oplus D_3$ be a direct sum of modules D_i, and $\pi_i: D \to D_i$ the projection. Take any submodule K of D and put $K^i=\pi_i(K)$. Then we have the following commutative diagram:

$$(4) \qquad \begin{array}{ccccc} & \nearrow & D_1/K^1 \oplus (D_2 \oplus D_3)/(\pi_2 \oplus \pi_3)(K) & \searrow & \\ D/K & & & & D_1/K^1 \oplus D_2/K^2 \oplus D_3/K^3 \\ & \searrow & D_2/K^2 \oplus (D_1 \oplus D_3)/(\pi_1 \oplus \pi_3)(K) & \nearrow & \end{array}$$

Now we assume that R is a perfect ring. Let M_0 be an R-module and $\{M_i, N_k\}_{i=1,k=1}^{t\ \ n}$ a set of LE R-modules. Further assume that M_0 is almost $\sum_{i=1}^t \oplus M_i \oplus \sum_{k=1}^n \oplus N_k$-projective. Therefore we may suppose that

(*) *M_0 is N_k-projective for all k and*
M_0 is almost M_i-projective, but not M_i-projective for all i.

Then from [6], Corollary 1, $\{M_i\}$ is divided into the following subsets

$$(5) \qquad \{M_i\}_{i=1}^t = \{M_{ij} = e_1 R/A_{ij}\}_{j=1}^{a(1)} \cup \{M_{2j} = e_2 R/A_{2j}\}_{j=1}^{a(2)} \cup \cdots$$

where the e_i are primitive idempotents.

We give some remarks related with [6], Proposition 5. We assumed there that M_0 was finitely generated. However we assume here that R is perfect and so we can find a maximal submodule B given in its proof. Hence [6], Proposition 5 is true for any module M_0, provided R is perfect. Therefore $M_i \oplus M_j$ has LPSM for any $i \neq j$. Moreover since M_0 is almost M_i-projective, M_{ks} is almost $M_{ks'}$-projective for all k and $s \neq s'$ by Proposition 1–2).

We are ready to obtain a generalization of [6], Theorem 2, when R is a perfect ring.

Theorem. *Let R be a perfect ring and M_0 an R-module and let $\{M_{ij}, N_k\}_{i=1,j=1,k=1}^{m\ a(i)\ n}$ be the above set of* LE *modules with* (*) *and* (5). *Then the following conditions are equivalent:*

1) *M_0 is almost $(\Sigma_{ij} \oplus M_{ij} \oplus \Sigma_k \oplus N_k)$-projective.*

2) *M_{ij} is almost $M_{i'j'}$-projective for all $(i'j') \neq (ij)$ and hence $\Sigma_{ij} \oplus M_{ij}$ is a lifting module.*

3) *For each i and any pair j, j' $(j \neq j')$ either M_{ij} is almost $M_{ij'}$-projective or $M_{ij'}$ is almost M_{ij}-projective.*

4) $M_{ij} \oplus M_{i'j'}$ *has* LPSM *for each* $(ij) \neq (i'j')$, *and hence* $\Sigma_{ij} \oplus M_{ij}$ *has* LPSM.

Proof. 1)→2), 2)↔3)↔4). These are clear from Proposition 1, [6], Corollary 1, Proposition 5 together with above remark and [3], Theorem 1.
2)→1). Take any diagram with row exact:

$$\begin{array}{ccc} & & M_0 \\ & & \downarrow h \\ 0 \to K \to M = \Sigma_{ij} \oplus M_{ij} \oplus \Sigma_k \oplus N_k & \overset{\nu}{\to} & M/K \to 0 \end{array} \tag{6}$$

We shall show that

(7) there exists $\tilde{h}$: $M_0 \to M$ with $\nu \tilde{h} = h$ or there exist a non-zero direct summand M^* of M and $\tilde{h}$: $M^* \to M_0$ with $h\tilde{h} = \nu | M^*$.

Now we shall prove (7) by induction on the number $\Sigma\, a(\mathrm{i})$ of direct summands M_{ij}. Since the argument is very long, we shall divide it into several steps.
Step 1 $\Sigma\, a(i) = 0$. We are done from Azumaya's theorem [2].

Hence we assume $\Sigma\, a(i) \neq 0$. Let π_{ij}: $M \to M_{ij}$ be the projection and put $\pi_{ij}(K) = K^{ij}$.
Step 2 $K^{ij} = M_{ij}$ *for some* (ij). We can reduce, by the proof of [3], Lemma 1, a new diagram from (6), which is essentially same as (6) and in which M_{ij} disappears, i.e.

$$\begin{array}{ccc} & M_0 & \\ & \downarrow h & \\ M & \to\ M/K & \to 0 \\ \cup & \wr\wr & \\ M' & \to\ M'/K' & \to 0 \end{array}$$

where $M' = \Sigma_{(i'j') \neq (ij)} \oplus M_{i'j'} \oplus \Sigma_k \oplus N_k$ and $K' = K \cap M'$. Hence we obtain (7) by induction hypothesis (cf. the proof of [3], Lemma 1). Thus we may assume always

$$K^{ij} = \pi_{ij}(M) \neq M_{ij} \quad \text{for all } i \text{ and } j. \tag{8}$$

Following the argument in [3], §1, we can derive the new diagram from (6):

$$\begin{array}{ccc} & & M_0 \\ & & \downarrow \nu'_{ij}\, h \\ M_{ij} & \xrightarrow{\nu'_{ij}\, \nu | M_{ij}} & M_{ij}/K^{ij} \to 0 \end{array} \tag{9}$$

where ν'_{ij}: $M/K \to M_{ij}/K^{ij} \oplus (1_M - \pi_{ij})(M)/(1_M - \pi_{ij})(K) \overset{\overline{\pi}_{ij}}{\to} M_{ij}/K^{ij}$ (cf. (4)).

Step 3 *Existence* $\tilde{h}_{ij}$: $M_o \to M_{ij}$ *for all* i *and* j. We shall show under the assumption (8)

(10) if there exists $\tilde{h}_{ij}$: $M_o \to M_{ij}$ with $\nu'_{ij}\,\nu\tilde{h}_{ij} = \nu'_{ij}\,h$ in (9) for all i and j, then we can find $\tilde{h}$: $M_o \to M$ such that $\nu\tilde{h} = h$, i.e. (7).

We shall prove (10) again by induction on the mumber $\Sigma a(i)$ of direct summands M_{ij}. If $\Sigma a(i)=0$, we obtain (10) from Azumaya's theorem [2]. Put $\Sigma_{(ij)\neq(11)}\oplus M_{ij}\oplus\Sigma_k\oplus N_k = M - M_{11}$. Then since $M = M_{11}\oplus(M-M_{11})$, we obtain from (3) and (3′) in [3] (see (9))

(11)
$$\begin{array}{ccc} & & M_o \\ & & \downarrow \nu'_{11}\,h \\ M_{11} & \xrightarrow{\nu'_{11}\,\nu\,|\,M_{11}} & M_{11}/K^{11} \to 0 \end{array}$$

and

(12)
$$\begin{array}{ccc} & & M_o \\ & & \downarrow \nu^*_{11}\,h \\ (M-M_{11}) & \xrightarrow{\nu^*_{11}\,\nu\,|\,(M-M_{11})} & (M-M_{11})/(1_M-\pi_{11})(K) \to 0 \end{array}$$

where ν^*_{11}: $M/K \to (M-M_{11})/(1_M-\pi_{11})(K)$.
We want to apply the induction hypothesis on (12). Now for each $(ij)\neq(11)$ we derive a diagram (9′) similar to (9) from (12)

(9′)
$$\begin{array}{ccc} & & M_o \\ & & \downarrow \nu'_{ij}\,h \\ M_{ij} & \xrightarrow{\nu'_{ij}\,\nu\,|\,M_{ij}} & M_{ij}/K^{ij} \to 0 \quad \text{(cf. [3])}. \end{array}$$

We remark that the diagram (9′) satisfies the assumption in (10). It is clear that the assumption (8) holds true in the diagram (12). Recalling the diagram (4), we know that ν'_{ij}: $M/K \to M_{ij}/K^{ij}$ in the diagram (9) is essentially determined by π_{ij}. Hence the assumption of existence of $\tilde{h}_{ij}$ in (10) guarantees an existence of $\tilde{h}_{ij}$ in the diagrams (9′). Accordingly we can apply the induction hypothesis on (12), and hence there exists $\tilde{h}'$: $M_o \to (M-M_{11})$ such that $\nu^*_{11}\,\nu\tilde{h}' = \nu^*_{11}\,h$. Further from the assumption (10) we obtain also $\tilde{h}''$: $M_o \to M_{11}$ which makes (11) commutative. Therefore from (♯), (8) and the argument in [3], §1, we obtain $\tilde{h}$: $M_o \to M$ such that $\nu\tilde{h} = h$. Thus we have shown (10). As a consequence

Step 4 *Existence* $\tilde{h}_{ij}$: $M_{ij} \to M_o$ *for some* (ij). We can assume that for some (ij) there exists $\tilde{h}_{ij}$: $M_{ij} \to M_o$ which makes the following diagram commutative:

$$\begin{array}{ccc} & \overset{\tilde{h}_{ij}}{\nearrow} & M_o \\ & & \downarrow \nu'_{ij}\,h \\ M_{ij} & \xrightarrow{\nu'_{ij}\,\nu\,|\,M_{ij}} & M_{ij}/K^{ij} \to 0 \end{array}$$

We note that the above diagram is actually given from the following one:

$$\begin{array}{c} M_o \\ \xrightarrow{\qquad \nu | M_{ij} \qquad} \downarrow h \xrightarrow{\qquad\qquad \nu'_{ij} \qquad\qquad} \end{array}$$

$$M_{ij} \to M_{ij} \oplus (M - M_{ij}) \xrightarrow{\nu} M/K \xrightarrow{\nu'} M_{ij}/K^{ij} \oplus (M - M_{ij})/(1_M - \pi_{ij})(K)$$
$$\rightrightarrows M_{ij}/K^{ij} \to 0$$

Hence $\ker \nu'_{ij} = \nu(K^{ij} \oplus (M - M_{ij}))$. Put $M_{ij} = e_i R/A_{ij}$, $\tilde{e}_i = e_i + A_{ij}$ ((5)) and

$$\tilde{h}_{ij}(\tilde{e}_i) = m_o \quad (= m_o e_i),$$

It is clear from (8) and the above diagram that $m_o \notin J(M_o)$. Since $h(m_o) - \nu(\tilde{e}_i) \in \ker \nu'_{i'j'}$, $h(m_o) - \nu(\tilde{e}_i) = \nu(k_{ij} + \Sigma_{(i'j') \neq (ij)} x_{i'j'} + \Sigma_k y_k)$, where $k_{ij} = k_{ij} e_i \in K^{ij}$, $x_{i'j'} = x_{i'j'} e_i \in M_{i'j'}$ and $y_k = y_k e_i \in N_k$. Further $K^{ij} \subset J(M_{ij}) = \tilde{e}_i J$ by (8) and hence $k_{ij} = \tilde{e}_i b$; $b \in e_i J e_i$. Therefore

$$h(m_o) = \nu(x_{ij} + \Sigma_{(i'j') \neq (ij)} x_{i'j'} + \Sigma_k y_k) \quad (= \nu(x)),$$

where

$$x_{ij} = \tilde{e}_i(e_i + b) \quad \text{is a generator of } M_{ij}$$

and

$$x = x_{ij} + \Sigma_{(i'j') \neq (ij)} x_{i'j'} + \Sigma y_k .$$

Here we consider $\{x_{i1}, x_{i2}, \cdots, x_{ij}, \cdots, x_{ia(i)}\}$. Among those elements we put $X = \{x_{it} \notin J(M_{it})\} \ni x_{ij}$. Since M_{it} is almost $M_{it'}$-projective for $t \neq t'$, we can find an x_{is} in X and

$$g_{i'j'}\colon M_{is} \to M_{i'j'} \quad \text{with} \quad g_{i'j'}(x_{is}) = x_{i'j'} \quad \text{for any} \quad (i'j') \neq (is)$$

by Lemma 2' (use induction) and Corollary 2, and we obtain

$$g_k\colon M_{is} \to N_k \quad \text{with} \quad g_k(x_{is}) = y_k \quad \text{for all } k.$$

by Proposition 1 and Corollary 2.

Step 5–1 $s \neq j$. Putting $g = \Sigma_{(i'j') \neq (is)} g_{i'j'} + \Sigma_k g_k\colon M_{is} \to M - M_{is}$,

$$x = x_{ij} + \Sigma_{(i'j') \neq \{(ij), (is)\}} x_{i'j'} + \Sigma_k y_k + x_{is} = (1 + g)(x_{is})$$

is a generator of $M_{is}(g) = \{z + g(z) \mid z \in M_{is}\}$. Hence we obtain $M = M_{is}(g) \oplus (M - M_{is})$ and $x \in M_{is}(g)$. On the other hand

$$\tilde{h}_{ij}(x_{ij}) = \tilde{h}_{ij}(\tilde{e}_i(e_i + b)) = m_o + m_o b (= m'_o = m'_o e_i) .$$

Since $e_i + b$ is a unit in $e_i R e_i$, we can put $(e_i + b)^{-1} = e_i + b'$; $b' \in e_i J e_i$. Then $m_o = m'_o(e_i + b') = m'_o + m'_0 b'$. By Lemma 3 there exists an endomorphism f of M_o

such that

$$f(m_o') = m_o \quad (\text{note } m_o' \notin J(M_o)) .$$

Further we have an isomorphism $p: M_{is}(g) \to M_{is}$ with $p(x) = x_{is}$. Put

$$\tilde{h} = f\tilde{h}_{ij}\, g_{ij}\, p \colon M_{is}(g) \to M_o ,$$

and $\tilde{h}(x) = f\tilde{h}_{ij}\, g_{ij}(x_{is}) = f\tilde{h}_{ij}(x_{ij}) = f(m_o') = m_o$. Hence $h\tilde{h}(x) = h(m_o) = v(x)$, i.e.

$$h\tilde{h} = v\,|\,M_{is}(g)\ ((7)) .$$

Step 5–2 $s=j$. Then again by the assumption 2) and Corollary 2, there exist

$$g'_{i'j'}\colon M_{ij} \to M_{i'j'} \quad \text{with} \quad g'_{i'j'}(x_{ij}) = x_{i'j'} \quad \text{for all} \quad (i'j') \neq (ij)$$

and

$$g'_k\colon M_{ij} \to N_k \quad \text{with} \quad g'_k(x_{ij}) = y_k \quad \text{for all } k .$$

Putting $g' = \Sigma_{(i'j') \neq (ij)}\, g'_{i'j'} + \Sigma_k\, g'_k$ as above,

$$x = x_{ij} + \Sigma_{(i'j') \neq (ij)}\, x_{i'j'} + \Sigma_k\, y_k = (1+g')\ (x_{ij})$$

Hence we obtain $M = M_{ij}(g') \oplus (M - M_{ij})$ and $x \in M_{ij}(g')$. Now there exists an isomorphism $p'\colon M_{ij}(g') \to M_{ij}$ with $p'(x) = x_{ij}$. Put

$$\tilde{h} = f\tilde{h}_{ij}\, p'\colon M_{ij}(g') \to M_o$$

and $\tilde{h}(x) = m_o$. Therefore

$$h\tilde{h} = v\,|\,M_{ij}(g') .$$

Thus we have proved (7), i.e. M_o is almost M-projective.

Corollary 3. *Let R be perfect. Let M_o be an R-module and let M_1 and M_2 be finite direct sums of* LE *R-modules. Assume that M_o is M_1-projective and almost M_2-projective. Then M_o is almost $M_1 \oplus M_2$-projective.*

Proof. We take a direct decomposition $M_2 = \Sigma_j \oplus T_j \oplus \Sigma_k \oplus N_k$ into LE modules T_j, N_k such that M_o is N_k-projective and M_o is almost T_j-projective, but not T_j-projective. Then $\Sigma_j \oplus T_j$ is a lifting module by Theorem. Hence M_o is almost $M_1 \oplus M_2$-projective by Theorem.

REMARK. We know from the proof of Theorem that 2) implies 1) without assumption "LE modules".

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