Kambara, H.
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ON DIRECTLY FINITE REGULAR RINGS

Dedicated to Professor Manabu Harada on his sixtieth birthday

HIKOJI KAMBARA

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This paper is concerned with the following open problem for directly finite, von Neuman regular rings. The problem was given by Goodearl and Handelman [3]: what conditions on a regular ring R induce that the maximal right quotient ring of R is right and left self-injective. In [4], the auther showed an example of directly finite, right self-injective regular ring which is not left self-injective. So we have an interest in this problem. In Theorem 17 in §3, we give necessary and sufficient conditions for this problem. In §2, we consider the maximal left quotient ring Q of a directly finite, right self-injective regular ring. We show that Q is directly finite (Theorem 7) and the factor ring $Q/\mathcal{M}$ is the maximal left quotient ring of the factor ring R/m for every maximal ideal $\mathcal{M}$ (resp. $\supset m$) of Q (resp. R) (Theorem 9). In §3, we give one generalization of a result in [5]: the maximal left quotient ring of a directly finite, right self-injective regular ring is left and right self-injective. Further we obtain necessary and sufficient conditions for the maximal right quotient ring of a regular ring to be directly finite (Theorem 16).

1. Preliminaries

All rings in this paper are associative with unit and ring homomorphisms are assumed to preserve the unit. A ring R is said to be *directly finite* if $xy=1$ implies $yx=1$ for all $x, y \in R$. A ring is said to be *directly infinite* if R is not directly finite. A regular ring means von Neumann regular ring.

A *rank function* on a regular ring R is a map $N: R \rightarrow [0, 1]$ satisfying the following conditions:

(a) $N(1)=1$,

(b) $N(xy) \leqq N(x)$ and $N(xy) \leqq N(y)$ for all $x, y \in R$,

(c) $N(e+f)=N(e)+N(f)$ for all orthogonal idempotents $e, f \in R$,

(d) $N(x)>0$ for all non-zero $x \in R$.

If R is a regular ring with a rank function N, then $\delta(x, y)=N(x-y)$ defines a metric on R, this metric δ is called *N-metric* or *rank metric* and the (Hausdorff) completion of R with respect to δ is a ring $\bar{R}$ which we call the *N-completion* of R.

An idempotent e of a regular ring R is said to be *abelian* if eRe is strongly regular i.e., all idempotents of eRe are central idempotents of eRe. An idempotent e of R is said to be *directly finite* if eRe is directly finite as a ring. For a ring R, we use $B(R)$ to denote the central idempotents in R. We note that $B(R)$ is a Boolean algebra in which $e\vee f=e+f-ef$ and $e\wedge f=ef$, while $e'=1-e$. If R is regular and right self-injective, then $B(R)$ is complete [2].

Let R be a regular, right self-injective ring. For a given element x in R, put $H=\{g\in B(R)\,|\,xg=0\}$ and $1-h=\bigvee_{g\in H} g$ in $B(R)$. The idempotent h is called the *central cover* of x, denote $c.c(x)$.

Let R be a directly finite, right self-injective regular ring. Then R is said to be *Type* II_f if R contains no abelian idempotents. And R is said to be *Type* I_f if R contains an abelian idempotent f with $c.c(f)=1$. Note that R is uniquely a direct product of rings of Type I_f, II_f ([2] Theorem 10.13).

For a regular ring R and elements $a, b\in R$, we use $aR\lesssim bR$ to mean that aR is isomorphic to a direct summand of bR. A regular ring satisfies *general comparability* provided that for any $x, y\in R$, there exists $g\in B(R)$ such that $gxR\lesssim RgyR$ and $(1-g)xR\gtrsim(1-g)yR$. Note that every regular right self-injective ring satisfies general comparability ([2] Corollary 9.15).

Let R be a subring of a ring Q. For every element x of Q and right ideal I of Q and left ideal J of Q, , we use $(x^{\bullet}.I)$, $(J.^{\bullet}x)$ to denote the right ideal $\{a\in R\,|\,xa\in I\}$, the left ideal $\{a\in R\,|\,ax\in J\}$, respectively.

Lemma A. *For two idempotents e, f of a ring R, the following conditions are equivalent.*

1). $eR\simeq fR$.
2). *There exist elements $x\in eRf$, $y\in fRe$ such that $yx=f$, $xy=e$.*
3). $Re\simeq Rf$.

Proof. It is trivial.

Lemma B. *Let R be a subring of a ring* Q *and $\bar{R}$ be a factor ring of R. For two idempotents e, f with $eR\simeq fR$, the followings hold.*

1). $\bar{e}\bar{R}\simeq\bar{f}\bar{R}$ *and* $\bar{R}\bar{e}\simeq\bar{R}\bar{f}$.
2). $eQ\simeq fQ$ *and* $Qe\simeq Qf$.

Proof. By Lemma A, it is easy.

Lemma C. *Let R be a ring. For two idempotents e, f and an integer n, the followings are equivalent.*

1) $n(eR)\simeq fR$.
2) $n(Re)\simeq Rf$.

where $n(eR)$, $n(Re)$ are direct sums of n-copies of eR, Re, respectively.

Proof. It is easy.

Lemma D. *Let R be a regular ring. For a right ideal $\sum_{i=1}^{\infty} a_iR$, there exist pairwise orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ which satisfy the following:*

(1). $\sum_{i=1}^{m} \oplus e_iR = \sum_{i=1}^{m} a_iR$ *for all m.*

(2). $e_iR \lesssim a_iR$ *for all i.*

If $\sum_{i=1}^{\infty} \oplus a_i R$ *is directsum, then* $\{e_i\}$ *satisfy* (1), (3).

(3) $e_iR \simeq a_iR$ *for all i.*

Proof. We prove Lemma by induction on m. For $m=1$ it is trivial. Assume that w $\{e_i\}_{i=1}^{m}$ satisfy (1), (2) or (3). Now $\sum_{i=1}^{m+1} a_iR = \sum_{i=1}^{m} \oplus e_iR + a_{m+1}R = \sum_{i=1}^{m} \oplus e_iR \oplus (1-\sum_{i=1}^{m} e_i)a_{m+1}R$. Let e'_{m+1} be an idempotent with $e'_{m+1}R = (1-\sum_{i=1}^{m} e_i)a_{m+1}R$. Put $e_{m+1} = e'_{m+1}(1-\sum_{i=1}^{m} e_i)$. Then $e_{m+1}^2 = e_{m+1}$. And $(1-\sum_{i=1}^{m} e_i)a_{m+1}R = e'_{m+1}R \supset e'_{m+1}(1-\sum_{i=1}^{m} e_i)R \supset e'_{m+1}(1-\sum_{i=1}^{m} e_i)e'_{m+1}R = e'_{m+1}R$, i.e., $e_{m+1}R = (1-\sum_{i=1}^{m} e_i)a_{m+1}R$. Thus $\{e_i\}_{i=1}^{m+1}$ are orthogonal and satisfy (1). Since $(1-\sum_{i=1}^{m} e_i)a_{m+1}R$ is projective, we have $(1-\sum_{i=1}^{m} e_i)a_{m+1}R = e_{m+1}R \lesssim a_{m+1}R$. For (3), we have $A = \sum_{i=1}^{m} \oplus e_iR \oplus a_{m+1}R = \sum_{i=1}^{m} \oplus e_iR \oplus e_{m+1}R$. We denote by p the projection from A to $e_{m+1}R$ induced by the decomposition $A = \sum_{i=1}^{m} e_iR \oplus e_{m+1}R$. Then p induce an isomorphism of $a_{m+1}R$ to $e_{m+1}R$.

2. Directly finite maximal quotient ring

We consider the necessary and sufficient condition for a regular ring R to have the directly finite maximal right quotient ring of R. For a prime regular ring with a rank fucntion, the following theorem is known [2], [3].

Theorem 1. ([2] Theorem 21.18 and 19) *Let R be a prime regular ring with a rank function N. Then $Q(R)$ is directly finite if and only if $Q(R) \subseteq \bar{R}$ as a subring if and only if sup $\{N(x) \mid x \in I\} = 1$ for all essential right ideals I of R wheie $Q(R)$ is the maximal right quotient ring of R and $\bar{R}$ is the completion of R in the N-metric.*

In general case, we have the following Proposition and we consider again this property in Theorem 16.

Proposition 2. *For a regular ring R, the following conditions are equivalent.*

(1) *The maximal right quotient ring Q of R is directly finite.*

(2) *Every right ideal isomorphic to some essential right ideal is an essential right ideal of R.*

Proof. (1)$\Rightarrow$(2): Let Q be directly finite. Suppose that R does not satisfy (2). Let I, J be isomorphic right ideals such that J is essential in R but I is not essential in R. There exists an element $q \in Q$ such that $q: J \to I$ $(x \to qx)$ is a given isomorphism. Since the right R-module J is essential in right R-module Q, the homomorphism $q: Q \to Q$ is a monomorphism. Since I is not essential in right R-module Q, qQ is a proper direct summand of right R-module Q. This contradicts that Q is directly finite.

(2)$\Rightarrow$(1): Assume that Q is directly infinite. Then there exists an element q of such that $r_Q(q) = \{x \in Q \mid qx = 0\} = 0$, $Q \neq qQ$. Then $(q^{\cdot}.R) = \{x \in R \mid qx \in R\}$ is an essential right ideal. By $Q \neq qQ$ and $qQ \cap R \supseteq q(q^{\cdot}.R)$, $q(q^{\cdot}.R)$ is not essential in R_R. On the other hand, by $r_Q(q) = 0$, the homomorphism $q: (q.^{\cdot}R) \to q(q^{\cdot}.R)$ is an isomorphism between two right ideals of R. So (2) does not hold.

We consider the maximal left quotient ring of R which is directly finite, right self-injective regular ring. For the end, we start with the following proposition.

Proposition 3. *Let R be a directly finite, right self-injective regular ring with no abelian idempotents. Then there exists a set $\{e_n^*\}_{n=1}^{\infty}$ of orthogonal idempotents such that $\sum^{\infty} \oplus e_n^* R$ is an essential right ideal and $\sum_{n=m+1}^{\infty} e_n^* R \subset_e (1 - \sum_{n=1}^{m} e_n^*) R \simeq e_m^* R$, $2^m (e_m^* R) \simeq R$ for all $m = 1, 2, \cdots$.*

Proof. By [2] Theorem 10.28, there exists an idempotent $e_1^* \in R$ sucht that $2 e_1^* R \simeq R$, $(1 - e_1^*) R \simeq e_1^* R$. For $R_1 = (1 - e_1^*) R (1 - e_1^*)$, there exists an idempotent $e_2^* \in R_1$ such that $2(e_2^* R_1) \simeq R_1$, i.e., $2^2 (e_2^* R) \simeq R$ and $(1 - e_1^* - e_2^*) R \simeq e_2^* R$. We obtain inductively a set $\{e_n^*\}_{n=1}^{\infty}$ of orthogonal idempotents such that $2^n (e_n^* R) \simeq R$, $(1 - \sum_{i=1}^{n} e_i^*) R \simeq e_n^* R$ for all $n = 1, 2, \cdots$.

For a given nonzero idempotent $f \in R$, suppose that $fR \lesssim e_n^* R$ for all n. Then $\aleph_0 fR \lesssim \sum_{n=1}^{\infty} \oplus e_n^* R \subset R$. Since R is directly finite, we have $f = 0$ from [2] Corollary 9.23. This is a contradiction. So, for $f \in R$, we obtain from general comparability on R that $fR \gtrsim g e_m^* R$ for some integer m and some nonzero idempotent $g \in B(R)$.

Suppose that $\sum_{n=1}^{\infty} \oplus e_n^* R$ is not essential in R_R, i.e., there exists a nonzero idempotent e with $(\sum_{n=1}^{\infty} \oplus e_n^* R) \cap eR = 0$. By the above argument, $eR \gtrsim g e_m^* R$ for some integer m and some nonzero idempotent $g \in B(R)$. Then $\sum_{n=1}^{m} \oplus g e_n^* R \oplus$

$geR \neq gR$, i.e., there exist exists a nonzero idempotent f' such that $\sum_{n=1}^{m} \oplus ge_n^*R \oplus geR \oplus f'R = gR$. While $gR \lesssim \sum_{n=1}^{m} \oplus ge_n^*R \oplus geR$ from $(1-\sum_{n=1}^{m} e_n^*)R \simeq e_m^*R$ and $eR \gtrsim ge_m^*R$, that is, gR is isomorphic to a proper direct summand of gR. This contradicts that R is directly finite. So $\sum_{n=1}^{\infty} \oplus e_n^*R$ is essential right ideal of R.

Now we have $(1-\sum_{n=1}^{m} e_n^*)R \supset \sum_{n=m+1}^{\infty} \oplus e_n^*R$. By the same argument, we obtain that $\sum_{n=m+1}^{\infty} \oplus e_n^*R$ is essential in $(1-\sum_{n=1}^{\infty} e_n^*)R$.

Throughout this paper, we use $\{e_n^*\}$ to denote the orthogonal idempotents as above.

From Proof of Proposition 3, we have the following proposition.

Proposition 4. *Let R be a directly finite right self-injective regular ring with no abelian idempotents. For every idempotent e, there exist an integer m and a nonzero central idepmotent $g \in B(R)$ and an idempotent f of eRe such that $ge_m^*R \simeq fR$.*

Corollary of Proposition 3. *Let R be as above. Then, for every n, $\sum_{i=n+1}^{\infty} \oplus Re_i^*$ is essential in $R(1-\sum_{i=1}^{n} e_i^*)$.*

Proof. Since $\{e_i^*\}_{i=1}^{\infty}$ are pairwise orthogonal, $R(1-\sum_{i=1}^{n} e_i^*)$ contains $\sum_{i=n+1}^{\infty} Re_i^*$. Suppose that there exists a nonzero idempotent $e \in R(1-\sum_{i=1}^{n} e_i^*)$ with $Re \cap \sum_{i=n+1}^{\infty} Re_i^* = 0$. By Proposition 4, we may assume that $eR \simeq c \cdot c(e) e_m^*R$ for some m. Put $g = c \cdot c(e) \in B(R)$.

(i) A case of $m < n$. Then $Re_m^*g \simeq Re \subset R(1-\sum_{i=1}^{n} e_i^*)g \simeq Re_n^*g$ implies $ge_m^*R \lesssim ge_n^*R$. So the following holds:

$$2^m(ge_n^*R) \gtrsim 2^m(ge_m^*R) \simeq gR\,. \tag{1}$$

On the other hand we obtain from $m < n$ that $2^m(ge_n^*R)$ is isomorphic to a proper direct summand of $gR \simeq 2^n(ge_n^*R)$. By (1), we obtain that gR is isomorphic to a proper direct summand of gR. This contradicts that R is directly finite.

(ii) A case of $m \geqq n$. Considering $\sum_{i=1}^{n} \oplus Re_i^* \oplus \sum_{i=n+1}^{m+1} \oplus Re_i^*g \oplus Re$ in a regular ring, we may assume that $\{ge_i^*\}_{i=1}^{m+1} \cup \{e\}$ are pairwise orthogonal. Then $\sum_{i=1}^{m} \oplus ge_i^*R \oplus eR$ is a proper direct summand of $\sum_{i=1}^{m+1} \oplus ge_i^*R \oplus eR$, i.e., of gR. On the other hand $Re \simeq Re_m^*g \simeq R(1-\sum_{i=1}^{m} e_i^*)g$ implies $\sum_{i=1}^{m} \oplus ge_i^*R \oplus eR \simeq \sum_{i=1}^{m} \oplus ge_i^*R \oplus g(1-\sum_{i=1}^{m} e_i^*)R = gR$. This contradicts that R is directly finite. Consequently we obtain that $\sum_{i=n+3}^{\infty} \oplus Re_i^*$ is essential in $R(1-\sum_{i=1}^{n} e_i^*)$.

In a right self-injective regular ring R, $B(R)$ is complete Boolean algebra ([2] Proposition 9.9). For any subset $\{g_i\}_I \subset B(R)$ of pairwise orthogonal idempotents, $(\bigvee_I g_i)R$ is the injective hull of a right ideal $\sum_I \oplus g_iR$, and the natural ring homomorphism $(\bigvee_I g_i)R \to \prod_I g_iR$ is an isomorphism ([2] Proposition 9.9 and 9.10), where $\prod_I g_iR$ is the ring of direct product of rings $\{g_iR\}_I$. So we regard as $(\bigvee_I g_i)R = \prod_I g_iR$ and we denote by $\prod a_i (a_i \in g_iR)$ the element $a \in (\bigvee g_i)R$ such that $ag_i = a_i$ for all $i \in I$. A subset I of R is said to be *centrally closed* if, for every subset $A = \{a_\alpha \in I\}$ satisfying the following condition (*), I contains $a = \prod_A a_\alpha \in \prod_A (c\cdot c(a_\alpha)R) = (\bigvee_A c.\cdot c(a_\alpha))R$.

(*) $\{c\cdot c(a) \mid a \in A\}$ are pairwise orghogoanl idempotents in $B(R)$.
Note that every essentially closed left ideal is centrally closed in right self-injeative regular rings.

Proposition 5. *Let R be a directly finite, right self-injective regular ring with no abelian idempotents and let I be a centrally closed non-zero left ideal of R. Then there exist orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ in I and an central idempptent $1-g \in B(R) \cap I$ which satisfy the following conditions:*

1) *$e_iR \simeq c\cdot c(e_i)e_{n(i)}^*R$ for all i, where $\{n(i)\}_{i=1}^{\infty}$ is a strictly increasing sequence of integers.*

2) *$\sum_{i=1}^{\infty} \oplus Re_i$ is essential in Ig and $0 = Ig \cap B(R)$.*

Proof. Let $\{g_\alpha\}$ be a maximal subset of orthogonal idempotents in $I \cap B(R)$. Since I is centrally closed, I containes $1-g = \prod_A g_\alpha = \bigvee_{h \in B(R) \cap I} h$. So $Ig \cap B(R) = 0$. We may assume that $I \cap B(R) = 0$ and I is centrally closed.

By induction on m, we will show that there exist orthogonal idempotents $\{e_i\}_{i=1}^{m}$ of I and integers $n(1) < n(2) < \cdots < n(m)$ satisfying 1) and the following condition:

2′) For any idempotent f in $I(1-\sum_{i=1}^{m} e_i)$, $fR \simeq c\cdot c(f)e_t^*R$ implies $t > n(m)$.

For $m=1$, let $n(1)$ be the smallest integer of $\{n \mid c\cdot c(f)e_n^*R \simeq fR$ for some nonzero idempotent f in $I\}$ which is not empty by Proposition 4. Let $\{g_\alpha\}_A$ be a maximal subset of family of orthogonal idempotents in $\{g \in B(R) \mid eR \simeq ge_{n(1)}^*R$ for some idempotent e in $I\}$. Let $\{e_\alpha\}_A$ be a set of idempotents of I such that $c\cdot c(e_\alpha) = g_\alpha$, $e_\alpha R \simeq g_\alpha e_{n(1)}^*R$ for all $\alpha \in A$. Put $e_1 = \prod_A e_\alpha$, $g_1 = \prod_A g_\alpha$ in $\prod_A g_\alpha R = (\bigvee_A g_\alpha)R$. Since I is centrally closed, it follows that I contains e_1 and

$I=Re_1\oplus I(1-e_1)$. Let t be an integer such that $fR\simeq c\cdot c(f)e_t^*R$ for some non-zero idempotent f in $I(1-e_1)$. Since $n(1)$ is minimal and $\{g_\alpha\}_A$ is maximal and $I\cap B(R)=0$, we have $n(1)<t$.

Assume that $\{e_i\}_{i=1}^m$ satisfy 1), 2′). Now we consider a case of $I(1-\sum_{i=1}^m e_i)=0$, i.e., $I=\sum_{i=1}^m \oplus Re_i$. By Corollary of Proposition 3, we have $R(1-\sum_{i=1}^{n(m)} e_i^*)\supset_e \sum_{i=n(m)+1}^{\infty} \oplus Re_i^*$. Since $Re_m\simeq Re_{n(m)}^*c\cdot c(e_m)\simeq R(1-\sum_{i=1}^{n(m)}\oplus e_i^*)c\cdot c(e_m)$, it follows that Re_m has an essential submodule isomorphic to $\sum_{i=n(m)+1}^{\infty}\oplus Re_i^*c\cdot c(e_m)$. We can see from Lemma D that there exist orghogonal idempotents $\{e_i'\}_{i=m}^{\infty}$ in e_mRe_m and a sequence $\{n(i)=n(m)+1+(i-m)\}_{i=m}^{\infty}$ such that $e_i'R\simeq c\cdot c(e_m)e_{n(i)}^*R$ for all $i=m, m+1, \cdots, \sum_{i=1}^{\infty}\oplus Re_i$ is essential in I and $\{e_i'\}_{i=m}^{\infty}\cup\{e_i\}_{i=1}^{m-1}$ are orthogonal idempotents. Thus we have orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ and a sequence $\{n(i)\}_{i=1}^{\infty}$ satisfying 1), 2).

Next, we consider a case of $I(1-\sum_{i=1}^m e_i)\neq 0$. Using $I(1-\sum_{i=1}^m e_i)$ in place of I, the same argument for $m=1$ implies that there exist an idempotent e_{m+1}' in $I(1-\sum_{i=1}^m e_i)$ and an integer $n(m+1)$ satisfying the same conditions's above. We can see that the idempotent $e_{m+1}=(1-\sum_{i=1}^m e_i)e_{m+1}'$ satisfy 1) and $\{e_i\}_{i=1}^{m+1}$ are orthogonal and satisfy 2′). Thus we obtain orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ satisfying 1).

We will show that $\sum_{i=1}^{\infty}\oplus Re_i$ is essential in I. Suppose that $\sum_{i=1}^{\infty}\oplus Re_i\cap Rf=0$ for some non zero idempotent of f of I. By Proposition 4, we may assume that that $fR\simeq c\cdot c(f)e_t^*R$ for some integer t. Let m be an integer with $n(m)>t$. Since $I=\sum_{i=1}^m Re_i\oplus I(1-\sum_{i=1}^m e_i)\supset \sum_{i=1}^{=} Re_i\oplus Rf$ implies $I(1-\sum_{i=1}^m e_i)\gtrsim Rf$, there exists an idempotent f' of $I(1-\sum_{i=1}^m e_i)$ such that $Rf'\simeq Rf\simeq Re_t^*c\cdot c(f)$. This contradicts 2′), So $\sum_{i=1}^{\infty}\oplus Re_i$ is essential in I.

Corollary. *Let R, I be as above. Then there exist pairwise orthogonal idempotents $\{f_i\}_{i=1}^{\infty}$ satisfying the following conditions*:

(a) *$f_iR\simeq c\cdot c(f_i)e_{n(i)}^*R$ for all i, where $\{n(i)\}_{i=1}^{\infty}$ is a strictly increasing sequence of integers.*

(b) *$\sum_{i=1}^{\infty}\oplus Rf_i$ is essential in I.*

Proof. Let g, $\{e_i\}$, $\{n(i)\}$ be as in Proposition 5. If $1-g=0$, the assertion is trivial. Suppose that $1-g\neq 0$. Put $f_i=(1-g)e_i^*+e_j$ if $i=n(j)$ for some j, put $f_i=(1-g)e_i^*$ if $i\neq n(j)$ for all j. Put $n(i)=i$ for all i. By Proposition 5, the assertion is clear.

Proposition 6. *Let R be a directly finite, right self-injective regular ring, Q be the maximal left quotient ring of R. Then, for a given element $q \in Q$, there exists a central idempotent $g \in B(R)$ and orthogonal idempotents $\{e_n\}_{n=1}^{\infty}$ which satisfy the following*

(1) $R(1-g) \oplus \sum_{n=1}^{\infty} \oplus Re/_n$ *is essential in* $(R.^{\cdot}q)$, $\sum_{n=1}^{\infty} \oplus Re_n$ *is essential in* Rg.

(2) $e_n R \simeq g e_n^* R$ *for all* n,

Proof. By [2] Theorem 10.13, there is a central idempotent $g^* \in B(R)$ such that $(1-g^*)R$ is a ring of type I_f, g^*R is a ring of type II_f. Then by [2] Corollary 10.25, we have $(1-g^*)R=(1-g^*)Q$, so we have $(1-g^*) \in (R.^{\cdot}q)$. It is sufficient ot show that g^*R satisfy the assertion. So we may assume that R is type II_f.

Let $\{a_\alpha\}_A$ be a subset of $(R.^{\cdot}q)$ such that $\{c \cdot c(a_\alpha)\}_A$ are pairwise orthogonal idempotents in $B(R)$. Since $a_\alpha q \in R$ for all $\alpha \in A$, it follows that $(\prod_A a_\alpha)q = \prod_A (a_\alpha q) \in \prod_A (c \cdot c(a_\alpha)R) = (\bigvee_A c \cdot c(a_\alpha))R$, i.e., $\prod_A a_\alpha \in (R.^{\cdot}q))$. So $(R.^{\cdot}q)$ is centrally colsed. By Proposition 5, there exist orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ and a central idempotent $1-g \in B(R) \cap (R.^{\cdot}q)$ satisfying 1), 2) of Proposition 5. Then we may assume for the sake of simpleness that $R=Rg$, $B(R) \cap (R.^{\cdot}q)=0$. Since $(R.^{\cdot}q)$ is an essential left ideal, $\sum_{i=1}^{\infty} \oplus Re_i$ is essential left ideal from 2) of Proposition 5.

We will show that $c \cdot c(e_i)=1$ and $n(i)=i$ for all $i \in N$. Suppose that $c \cdot c(e_t) \neq 1$ or $n(t) \neq t$ for some t. Then we have $K_1 = \sum_{i=1}^{\infty} \oplus (1-c \cdot c(e_i)) e_{n(i)}^* R \oplus \sum_{m \in N \setminus \{n(i)\}_{i=1}^{\infty}} e_m^* R \neq 0$. While we obtain $\sum_{i=1}^{\infty} \oplus e_i R \simeq \sum_{i=1}^{\infty} \oplus c \cdot c(e_i) e_{n(i)}^* R = K_2$ from 1) of Proposition 5 and and Lemma A. So $\sum_{i=1}^{\infty} \oplus e_i R$ is isomorphic to a proper direct summand K_2 of an essential right ideal $\sum_{i=1}^{\infty} \oplus e_i^* R = K_1 \oplus K_2$. By Proposition 2, $\sum_{i=1}^{\infty} \oplus e_i R$ is not essential in R_R. Since R is right self-injective, there exists an idempotent e in R such that $eR \cap \sum_{i=1}^{m} \oplus e_i R = 0$ and $e(\sum_{i=1}^{\infty} \oplus e_i R) = 0$. Put $x = xe = \sum_{i=1}^{\infty} x_i e_i \in Re \cap \sum \oplus Re_i$. Then we obtain that $0 = xee_i = x_i e_i$ for all $i \in N$, so $Re \cap \sum_{i=1}^{\infty} \oplus Re_i = 0$. This contradicts that $\sum_{i=1}^{\infty} \oplus Re_i$ is essential in ${}_R R$. We obtain that $c \cdot c(e_i)=1$, $n(i)=i$ for all $i \in N$.

Theorem 7. *Let R be a directly finite, right self-injective regular ring. Then the maximal left quotient ring Q of R is directly finite.*

Proof. We may assume that R is Type II_f. Suppose that Q is direatly infinite. Then there is an element q of Q such that $1_Q(q)=0$ and $Qq \neq Q$. There exist a central idempotent g and orthogonal idempotents $\{e_n\}_{n=1}^{\infty}$ of R which satisfy the following conditions:

$$(1) \qquad (R^{\cdot}_{\cdot}q) \supset_e R(1-g) \oplus \sum_{i=1}^{\infty} \oplus Re_i$$

$$(2) \qquad Rg \supset_e \sum_{i=1}^{\infty} \oplus Re_i$$

$$(3) \qquad e_i R \simeq g e_i^* R \quad \text{for all } i.$$

We will show that (i) $Q(1-g)q=Q(1-g)$ and (ii) $Qgq=Qg$.

(i): Since $1-g \in (R^{\cdot}_{\cdot}q)$ implies$(1-g)q \in R$, we have $R(1-g)q \oplus A = R(1-g)$. While we have $R(1-g) \simeq R(1-g)q$ from $1_R/(q) \subset 1_Q(q)=0$. Since R is directly finite, we obtain $A=0$, i.e., $Q(1-g)=Q(1-g)q$.

(ii): Here we claim that $\sum_{n=1}^{\infty} \oplus Re_n q$ is essential in Rg. Suppose that there exists a nonzero idempotent e in Rg such that $Re \cap \sum_{u=1}^{\infty} \oplus Re_n q=0$. Further we may assume from Proposition 4 that $eR \simeq g' e_m^* R$ for some integer m and some nonzero central idempotent g' in Rg. Considering the proper direct summand $\sum_{n=1}^{m+1} \oplus Re_n qg' \oplus Re$ of Rg', there exist orthogonal idempotents $\{e'\} \cup \{e'_n\}_{n=1}^{m+1}$ such that $Re'=Re$, $Re'_n=Re_n qg' \simeq Re_n^* g'$ for all $n=1, 2, \cdots, m+1$. Then $(e'+\sum_{x=1}^{m+2} e'_n)R$ is a proper direct summand of $g'R$. On the other hand it easily follows from (3) and Prlposition 3 that $g'R \simeq \sum_{n=1}^{m} \oplus e'_n R \oplus e'R$. This contradicts that R is directly finite. So we obtain that $\sum_{n=1}^{\infty} \oplus Re_n q$ is essential in Rg.

Since Rg is essential in ${}_RQg$, we have $\sum_{n=1}^{\infty} \oplus Re_u q \subset_e Qg$. While we obtain from (2) and non-singularity of ${}_RQ$ that $\sum_{n=1}^{\infty} \oplus Re_n q$ is essential in Qgq. Thus we obtain $Qgq=Qg$.

From (i) and (ii), we obtain $Qq=Q$. This is a contradiction. Thus Q is directly finite.

Proposition 8. *Let R be a directly finite, right self-injective regular ring which contains no nonzero abelian idempotents and $\mathfrak{M}$ a maximal ideal of $B(R)$. Let m be the maximal ideal of R such that $m \supset \mathfrak{M}R$ are essential right ideal of $\bar{R}$. We denote by $\bar{R}$ the facter ring $R/\mathfrak{M}R$.*

(I) *For a given idempotent e of R, the following conditions are equivalent.*

(a) *m contains e but $\mathfrak{M}R$ does not contain e.*

(b) *$\mathfrak{M}$ does not contain the central cover $c \cdot c(e)$ of e and $\mathfrak{M}$ contains all central idempotents $g \in B(R)$ satisfying $g e_n^* R \lesssim eR$ for some integer n.*

(c) *There exist orthogonal central idempotents $\{g_t\}_{t=1}^{\infty}$ and idempotents $e_1^{\cdot}$, $e_2^{\cdot}$ and integers $\{n(t)\}_{t=1}^{\infty}$ which satisfy the following conditions;*

(i) $g_te_i^\bullet=g_te_{n(t)-1}^*$ *and* $g_te_2^\bullet=g_te_{n(t)}^*/$ *for all* t.

(ii) $\{n(t)\}_{t=1}^\infty$ *is a strictly increasing sequence of integers.*

(iii) $e_2^\bullet R\lesssim eR\lesssim e_1^\bullet R$.

(iv) $\bigvee_{t=1}^\infty g_t=c\cdot c(e)\notin\mathfrak{M}$, $g_t\in\mathfrak{M}$ *for all* t.

(d) $\aleph_0(\bar{e}\bar{R})\lesssim\bar{R}$ *and* $\bar{e}\neq\bar{0}$.

(II) *For an idempotent* e *in* R, m *does not contain* e *if and only if there exist a nonzero central idempotent* g *and an integer* n *such that* $ge_n^*R<eR$ *and* $g\notin\mathfrak{M}$.

Proof. (a)$\Rightarrow$(b): Let e be an idempotent in $m\backslash\mathfrak{M}R$. Then $e\notin\mathfrak{M}R$ implies $c\cdot c(e)\notin\mathfrak{M}$. Suppose that there exist a central idempotent g and an integer n satisfying the following condition:

$$ge_n^*R\lesssim eR \text{ and } g\notin\mathfrak{M}. \tag{1}$$

From $2^n(e_n^*R)\simeq R$, there exist orthogonal idempotents $\{e_i'\}_{i=1}^{2^n}$ such that $\sum_{i=1}^{2^n} e_i'=1$, $e_i'R\simeq e_n^*R$ for all $i=1,2,\cdots,2^n$ and $e_1'=e_n^*$. So we obtain that $\sum_{i=1}^{2^n}\bar{e}_i'=\bar{1}$, $\bar{e}_i\bar{R}\simeq\bar{e}_n^*\bar{R}$, $\bar{e}_n^*\neq\bar{0}$ in $\bar{R}$. So we have the following:

$$\bar{R}=\bar{R}\bar{e}_n^*\bar{R}\,. \tag{2}$$

On the other hand, we obtain from (1) and Lemma B that $\bar{g}=\bar{1}$ and $\bar{e}_n^*\bar{R}=\bar{g}\bar{e}_n^*\bar{R}\lesssim\bar{e}\bar{R}$. Then by (2), we obtain $\bar{R}=\bar{R}\bar{e}_n^*\bar{R}=\bar{R}\bar{e}\bar{R}$. This contradicts $e\in m$. We obtain the last part of (b).

(b)$\Rightarrow$(c): Assume that (b) for an idempotent e holds. Let $n(1)$ be a minimal integer of $\{n\geqq 0\mid eR\gtrsim ge_n^*R$ for some $0\neq g\in B(R)\}=N$ where $e_0^*=1$. Let $\{g_i\}_I$ be a maximal subset of orthogonal idempotents in $\{g\in B(R)\mid eR\gtrsim ge_{n(1)}^*R\}=J$. Since R_R is injective and $\sum_{i\in I}\oplus g_ie_{n(1)}^*R\lesssim eR$, it follows that $g_1e_{n(1)}^*R\lesssim eR$ for $g_1=\bigvee_{i\in I}g_i$. Since $\{g_i\}_I$ is maximal and R satisfies general comparability, we obtain $(1-g_1)eR\lesssim e_{n(1)}^*R$. Since $n(1)$ is minimal, we have $g_1eR\lesssim g_1e_{n(1)-1}^*R$ when $n(1)>0$. From $c\cdot c(e)\notin\mathfrak{M}$ and $g_1\in\mathfrak{M}$ we see that $(1-g_1)e\neq 0$ and $(1-g_1)e$ holds (b). By the same argument as above for $(1-g_1)e$, there exist a central idempotent g_2 and an integer $n(2)$ which satisfy the same conditions as above. Since $n(1)$ is the minimal of N, we have $n(1)<n(2)$. By induction, we can obtain orthogonal central idempotents $\{g_t\}_{t=1}^\infty$ and an increasing sequence $\{n(t)\}_{t=1}^\infty$ of integers, which satisfy the following conditions:

$$eR\gtrsim\sum_{t=1}^\infty\oplus g_te_{n(t)}^*R\,,\quad \sum_{t=1}^\infty\oplus g_te_{nt(-1}^*R\gtrsim\sum_{t=1}^\infty\oplus g_teR\,. \tag{3}$$

$$\bigvee_{t=1}^\infty g_t=c\cdot c(e)\,,\quad n(1)<n(2)<\cdots\,. \tag{4}$$

Because it follows from $(1-g_t)\cdots(1-g_1)eR\lesssim(1-g_t)e_{n(t)}^*R$ that $g'eR\lesssim e_{n(t)}^*R$ for all

$t=1, 2, \cdots$, where $g'=c\cdot c(e)-\bigvee_{t=1}^{\infty} g_t$. Then we obtain $\aleph_0(g'eR)\lesssim\sum_{t=1}^{\infty}\oplus e_{n(t)}^* R\subset \sum_{t=1}^{\infty}\oplus e_t^* R\subset R$. By [2] Corollary 9.23, we obtain that $g'e=0$, i.e. $c\cdot c(e)=\bigvee_{t=1}^{\infty} g_t$, because $eR\gtrsim g_t e_{n(t)}^* R$ implies $c\cdot c(e)>g_t$ for all t.

Put $e_1^{\cdot}=\prod_{t=1}^{\infty} g_t e_{n(t)-1}^*$, $e_2^{\cdot}=\prod_{t=1}^{\infty} g_t e_{n(t)}^*$ in $\prod_{t=2}^{\infty} g_t R=\bigvee_{t=1}^{\infty}(g_t)R$. We see from (3), (4) that $\{g_t\}$, $e_1^{\cdot}$, $e_2^{\cdot}$ satisfy (i), (ii) and (iii) and (iv).

(c)$\Rightarrow$(d): Let e be an idempotent satisfying (c). By (iv) $c\cdot c(e)\notin\mathfrak{M}$, we have $e\notin R\mathfrak{M}$, i.e., $\bar{e}\neq\bar{0}$. By (iii) in (c), we have $\bar{e}_1^{\cdot}\bar{R}\gtrsim\bar{e}\bar{R}$. By (iv) $g_t\in\mathfrak{M}$, we obtain that $(\bar{1}-\sum_{i=1}^{t}\bar{g}_i)\bar{e}_1^{\cdot}=\bar{e}_1^{\cdot}$ for all $t=1, 2, \cdots$. By (i) and general comparability on R, we obtain the following:

$$(5)\qquad (1-\sum_{i=1}^{t} g_i)e_1^{\cdot}R\lesssim e_{n(t)}^* R$$

for all t. Then the following hold:

$$\begin{aligned}\aleph_0(\bar{e}\bar{R})&\lesssim\aleph_0(\bar{e}_1^{\cdot}\bar{R}) &&(\text{from } \bar{e}\bar{R}\lesssim\bar{e}_1^{\cdot}\bar{R})\\ &\lesssim\sum_{t=2}^{\infty}\oplus(\bar{1}-\sum_{i=1}^{t}\bar{g}_i)\bar{e}_1^{\cdot}\bar{R} &&(\text{from } \bar{e}_1^{\cdot}=(\bar{1}-\sum_{i=1}^{t}\bar{g}_i)\bar{e}_i^{\cdot})\\ &\lesssim\sum_{t=2}^{\infty}\oplus\bar{e}_{n(t)+1}^*\bar{R}\subsetsim\bar{R}. &&(\text{from } (5)).\end{aligned}$$

Thus we have $\aleph_0(\bar{e}\bar{R})\lesssim\bar{R}$.

(d)$\Rightarrow$(a): By [2] Theorem 9.32, $\tilde{R}=\bar{R}/\bar{m}=R/m$ is a directly finite, right self-injective simple regular ring. For an idempotent e satisfying (d), we see from $n(\bar{e}\bar{R})\lesssim\bar{R}$ that $n(\tilde{e}\tilde{R})\lesssim\tilde{R}$ for all $n=1, 2, \cdots$. By [2] Corollary 9.23, it follows that $\tilde{e}=\tilde{0}$, i.e., $e\in m$.

(II) It is clear from (I).

Theorem 9. *Let R be a directly finite, right self-injective regular ring and Q the maximal left quoitent ring of R. Let $\mathfrak{M}$ be a maximal ideal of $B(R)$. Let $\mathcal{M}$ and m be the maximal ideals of Q and R including the ideal $\mathfrak{M}R$, respectively. Then the factor ring $Q/\mathcal{M}$ is the maximal left quotient ring of R/m.*

Proof. By Theorem 7 and [2] Theorem 10.13, there exists a decomposition $Q=Q_1\times Q_2$ such that Q_1 is type I_f and Q_2 is type II_f. We denote by $R=R_1\times R_2$ the decomposition of R as same as Q. By [2] Proposition 10.4, we have $R_1\subset Q_1$. Since R_1 is left and right self-injective and $Q_1\cap R_2=0$ ([2] Proposition 10.4), we have $R_1=Q_1$. Then every prime ideal contains R_1 or R_2. So, if m contains R_2, the assertion is clear. Since $\mathfrak{M}R$ is prime ideal of R, we may assume that R is type II_f.

First we prove that $R\cap\mathcal{M}=m$. Suppose that the equality does not hold. By

[2] Corollary 8.23, $\mathfrak{m}$ is a unique maximal ideal of R which contains the minimal prime ideal $\mathfrak{M}R$. Hence $\mathfrak{m}\supsetneqq R\cap\mathcal{M}\supset R\mathfrak{M}$. There exists an idempotent e in $\mathfrak{m}\backslash\mathcal{M}\cap R$. From $e\notin\mathcal{M}$ and Proposition 8 (II), there exists a nonzero central idempotent g in $B(Q)$ such that $g\notin\mathfrak{M}$ and

$$(1)\qquad Qe\gtrsim Qe_m^*g$$

for some integer m. From $e\in\mathfrak{m}$ and Proposition 8 (I)-(c), there exist orthogonal central idempotents $\{g_i\}_{i=1}^{\infty}$ and idempotents $e_2^{\cdot}$, $e_1^{\cdot}$ and integers $\{n(i)\}_{i=1}^{\infty}$ satisfying the conditions of Proposition 8 (1) (c). From $e_2^{\cdot}R\lesssim eR\lesssim e_1^{\cdot}R$, we obtain the following:

$$(2)\qquad Qe_2^{\cdot}\leqq Qe\leqq Qe_1^{\cdot}\,.$$

There exists an integer t such that $n(i)>m$ for all $i\geqq t$. From $g_i\in\mathfrak{M}$, we have $(c\cdot c(e)-\sum_{i=1}^{t-1}g_i)g\neq 0$. There exists an integer $s>t$ satisfying

$$(3)\qquad g_sg\neq 0\,.$$

Then the following relations hold:

$$\begin{aligned} Qe_{n(s)-1}^*g_sg &\simeq Qe_1^{\cdot}g_sg && \text{(from Prop. 8 (c) (i))}\\ &\gtrsim Qeg_sg && \text{(from (2))}\\ &\gtrsim Qe_m^*g_sg && \text{(from (1))}\end{aligned}$$

Thus we obtain

$$(4)\qquad Qe_{n)s)-1}^*g_sg\gtrsim Qe_m^*g_sg.$$

On the other hand we obtain from Proposition 3 that $2^{n(s)-1-m}(e_{n(s)-1}^*R)\simeq e_m^*R$. So we have

$$(5)\qquad 2^{n(s)-1-m}(Qe_{n(s)-1}^*)\simeq Qe_m^*\,.$$

Hence, from (3), (4) and (5), nonzero $Qe_m^*g_sg$ is isomorphic to a proper direct summand of itself. This contradicts that Q is directly finite, So we obtain $\mathfrak{m}=\mathcal{M}\subset R$.

We prove that $\bar{R}=R/\mathfrak{m}$ is essential in $\bar{Q}=Q/\mathcal{M}$ as left $\bar{R}$-module. From Proposition 6, for a given element q in Q, we obtain an essential left ideal $R(1-h)\oplus\sum_{i=1}^{\infty}\oplus Re_i$ in $(R.^{\cdot}q)$ such that $h\in B(R)$ and $Re_i\simeq Re_i^*h$ for all integer i. Now $B(\bar{R})=\{\bar{1},\bar{0}\}$ implies $(\bar{1}-\bar{h})=\bar{1}$ or $\bar{0}$. If $(\bar{1}-\bar{h})=\bar{1}$, then $\bar{R}$ contains $\bar{q}$, i.e., $(\bar{R}.^{\cdot}\bar{q})=\bar{R}$. If $(\bar{1}-\bar{h})=\bar{0}$, then $(\bar{R}.^{\cdot}\bar{q})$ contains $\sum\oplus\bar{R}\bar{e}_n$. Since $\bar{R}$ is a simple regular ring with a unique rank function N, $2^n(\bar{e}_n^*\bar{R})\simeq\bar{R}$ implies $N(\bar{e}_n^*)=1/2^n$. Further $\bar{e}_n^*\bar{R}\simeq\bar{e}_n\bar{R}$ implies $N(\bar{e}_n^*)=N(\bar{e}_n)$. Since $\{\bar{e}_i\}_{i=1}^{\infty}$ are pairwise orthogonal, we obtain $1=\sum_{i=1}^{\infty} N(\bar{e}_i)=\sup\{N(x)\,|\,x\in\sum_{i=1}^{\infty}\oplus\bar{e}_i\bar{R}\}$. Hence $\sum_{i=1}^{\infty}\oplus\bar{R}\bar{e}_n$ is an essential left ideal of $\bar{R}$, that is, $(\bar{R}.\dot{}\,\bar{q})$ an is essential left ideal of $\bar{R}$ for every $\bar{q}\in\bar{Q}$. Thus $\bar{R}$ is essential in $\bar{Q}$ as left $\bar{R}$-module.

By [2] Theorem 9.32, $\bar{Q}$ is a left self-injective regular ring. Thus $\bar{Q}$ is the maximal left quotient ring of $\bar{R}$ from ${}_{\bar{R}}\bar{R}\underset{e}{\subset}\bar{Q}$.

3. Left and right self-injective regular ring

A ring R is said to be right (resp. left) $\aleph_0$-injective if every homomorphism from a countablely generated right (resp. left) ideal of R into R extends to an endomorphism of right (resp. left) R-module R.

By Proposition 6, we obtain the following theorem.

Theorem 10. *Let R be a directly finite, right self-injective regular ring. Then R is a left self-injective ring if and only if R is left $\aleph_0$-injective.*

Proof. Let R be left $\aleph_0$-injective and Q the maximal left quotient ring of R. For any element q in Q, there exist a set $\{e_n\}_{n=1}^{\infty}$ of orthogonal idempotents and a central idempotent $g\in B(R)$ such that $\sum_{i=1}^{\infty}\oplus Re_n\oplus R(1-g)$ is essential in $(R.\dot{}\,q)$ and $Re_n\simeq Re_n^* g$ for all n. Since the right multiplication by q is a homomorphism from $\sum\oplus Re_n\oplus R(1-g)$ to R, there exists an element x in R such that $(\sum\oplus Re_n\oplus R(1-g))(q-x)=0$. Since Q is a nonsingular left R-module, we obtain that R contains $q=x$, i.e., that $Q=R$.

The converse is trivial.

A ring is said to satisfy K_l (resp. K_r) if every non-essential left (resp. right) ideal has a non zero right (resp. left) annihilator ideal. We consider one generalizition of Kobayashi's theorem. For the end we use the following Utsumi's theorem:

Theorem. Let R be a regular ring and Q_1 (resp. Q_r) the maximal left (resp. right) quotient ring of R. Then $Q_1=Q_r$ if and only if R satisfies K_1 and K_r. ([6] Theorem 3.3)

In the following Lemmas 11, 12 and 13 and 14, we denote by R a right self-injective regular ring of type II_f and by Q the maximal left quotient ring of R. We use $\{e_i^*\}_{i=1}^{\infty}$ to denote the orthogonal idempotents of R given by Proposition 3.

Lemma 11. *Let $\{e_i\}_{i=1}^{\infty}$, $\{f_i\}_{i=1}^{\infty}$ be pairwise orthogonal idempotents respectively, which satisfy the following conditions:*

(a). (i) *$Re_i\simeq Re_{n(i)}^*\, c\cdot c(e_i)$ for all $i\in N$,*

(ii) *$\{i\,|\,e_i\neq 0\}$ is infinite, and for every nonzero $g\in B(R)$, $ge_i\neq 0$ for infinite many i,* *where* $\{n(i)\}_{i=1}^{\infty}$ *is a strictly increasing sequence.*

(b). *There exists an integer t such that* $Rf_i \simeq Re^*_{t+i}$ *for all* $i \in N$.

(c). $\sum_{i=1}^{\infty} \oplus Re_i \cap \sum_{i=1}^{\infty} \oplus Rf_i = 0$ *and* $\sum_{i=1}^{\infty} \oplus Re_i \oplus \sum_{i=1}^{\infty} \oplus Rf_i$ *is essential in* ${}_R R$.

Then $Re_i \simeq Re^*_i$ *for all* $i=1, 2, \cdots, t-1$ *and* $Re_i \simeq Re^*_{i+1}$ *for all* $i \geqq t$.

Proof. If $n(i) \neq t$ for all $i \in N$, then the following relations hold:

$$\sum_{i=1}^{\infty} \oplus Re_i \simeq \sum_{i=1}^{\infty} \oplus Re^*_{n(i)}\, c \cdot c(e_i) \qquad \text{(from (a))}$$

$$\lesssim \sum_{i=1, \neq t}^{\infty} \oplus Re^*_i \qquad (\text{from } n(i) \neq t \text{ for all } i)$$

$$\sum_{i=1}^{\infty} \oplus Rf_i \simeq \sum_{i=t+1}^{\infty} \oplus Re^*_i \qquad \text{(from (b))}$$

$$\subset_e R(1 - \sum_{i=1}^{t} e^*_i) \simeq Re^*_t\,. \qquad \text{(from Proposition 3 and its Corollary).}$$

By Theorem 7, R satisfy (2) of Proposition 2 for left ideals of R. Since $\sum_{i=1}^{\infty} \oplus Re_i \oplus \sum_{i=1}^{\infty} \oplus Rf_i$ is an essential left ideal, it follows from Proposition 2 that $Re_i \simeq Re^*_i$ for all $i=1, 2, \cdots t-1$, $Re_i \simeq Re^*_{i+1}$ for all $i > t$. So we will show that $n(i) \neq t$ for all $i \in N$.

We begin by showing that $n(i)=i$ for $i=1, 2, \cdots, t-1$. Suppose that there exists an integer s with $n(i) \neq s > t$ for all i. Now the following relations hold:

$$\sum_{i\,:\,n(i)>s} Re_i \simeq \sum_{i\,:\,n(i)>s} Re^*_{n(i)}\, c \cdot c(e_i) \qquad \text{(from (a))}$$

$$\lesssim \sum_{i=s+1}^{\infty} Re^*_i$$

$$\subset R(1 - \sum_{i=1}^{s} e^*_i) \simeq Re^*_s \qquad \text{(from Proposition 3 and its Corollary)}$$

$$\sum_{i\,:\,n(i)<s} Re_i \simeq \sum_{i\,:\,n(i)<s} Re^*_{n(i)}\, c \cdot c(e_i) \qquad \text{(from (a))}$$

$$\lesssim \sum_{i=1}^{s-1} Re^*_i$$

$$\sum_{i=1}^{\infty} \oplus Rf_i \simeq \sum_{i=t+1}^{\infty} \oplus Re^*_i \qquad \text{(from (b))}$$

$$\subset R(1 - \sum_{i=1}^{t} e^*_i) \simeq Re^*_t \qquad \text{(from Proposition 3 and its Corollary)}$$

Consequently essential left ideal $\sum_{i=1}^{\infty} \oplus Re_i \oplus \sum_{i=1}^{\infty} \oplus Rf_i$ is subisomorphic to a proper direct summand $R(e^*_1 + e^*_2 + \cdots + e^*_s + e^*_t)$ of R. This contradicts that R satisfy (2) of Proposition 2. Thus we obtain $n(i)=i$ for $i=1, 2, \cdots, t-1$.

Here we show that $Re_i \simeq Re^*_i$ for all $i < t$. Suppose that there exists an nonzero idempotent $g \in B(R)$ which $e_s g = 0$ for some $s < t$. Using the simillar argument as above for $e_i g$ and $e_i^* g$, essential left ideal $\sum_{i=1}^{\infty} \oplus Re_i g \oplus \sum_{i=1}^{\infty} \oplus Rf_i g$ of Rg is subisomorphic to a proper direct summand $R(e_1^* + e_2^* + \cdots + e_s^* + e_t^*) g$ of Rg. This is a contradiction as above. So we obtain that for every $s<t$, $e_s g \neq 0$ for every nonzero $g \in B(R)$. So we have

$$(1) \qquad Re_i^* \simeq Re_i$$

for all $i<t$.

Suppose that $n(t)=t$. Put $h=c\cdot c(e_t) \neq 0$. Now $\sum_{i=1}^{t} \oplus Re_i h \oplus \sum_{i=1}^{\infty} \oplus Rf_i h \simeq \sum_{i=1}^{t} \oplus Re_i^* h \oplus \sum_{i=t+1}^{\infty} \oplus Re_i^* h$ from (1) and (b). By (a), $\sum_{i=1}^{t} \oplus Re_i h$ is a proper direct summand of $\sum_{i=1}^{\infty} \oplus Re_i h$, that is, $\sum_{i=1}^{t} \oplus Re_i h \oplus \sum_{i=1}^{\infty} \oplus Rf_i h$ is not essential in Rh. This is a contradiction. Hence we obtain $n(i) \neq t$ for all i.

Lemma 12. *Let $e \in Q$ be an idempotent with $eQ \simeq e_n^* Q$ for some integer n. There exist orthogonal idempotents $\{f_i\}_{i=1, \neq n}^{\infty}$ in $l_R(e)$ such that $\sum_{i=1, \neq n}^{\infty} \oplus Rf_i$ is essential in $l_R(e)$, $Rf_i \simeq Re_i^*$ for all $i(\neq n) \in N$.*

Proof. We see that $l_R(e)$ is a centrally closed left ideal of R. By Corollary of Proposition 5, there exists a set $\{f_i\}_{i=1}^{\infty}$ of orthogonal idempotents in $l_R(e)$ which satisfys the following conditions:

(1) $f_i R \simeq c\cdot c(f_i)\, e_{n(i)}^* R$ for all i where $\{n(i)\}_{i=1}^{\infty}$ is a strictly increasing sequence of integers.

(2) $\sum_{i=1}^{\infty} \oplus Rf_i$ is essential in $l_R(e)$.

Then there exists an essentially closed left ideal K of R such that $\sum_{i=1}^{\infty} \oplus Rf_i \oplus K$ is essential in ${}_R R$. Let a be an element in eQe_n^* such that the right multiplication by a induces a given isomorphism $Qe \simeq Qe_n^*$. By Corollary of Proposition 5, there exists a set $\{f_i'\}_{i=1}^{\infty}$ of orthogonal idempotents in $K \cap (R.\dot{}a)$ which satisfies the following conditions:

(3) $f_i' R \simeq c\cdot c(f_i')\, e_{n'(i)}^*\, R$ for all i where $\{n'(i)\}_{i=1}^{\infty}$ is a strictly increasing sequence of integers.

(4) $\sum_{i=1}^{\infty} \oplus Rf_i'$ is essential in $K \cap (R.\dot{}a)$.

Here we claim that $Rf_i' \simeq Re_{n+i}^*$ for all i. From $l_Q(a) = Q(1-e) \supseteq_e l_R(e) \supseteq_e \sum \bigoplus_{i=1}^{\infty} Rf_i$ and $\sum_{i=1}^{\infty} \oplus Rf_i \cap K = 0$, the right multiplication by a is a monomorphism from $\sum_{i=1}^{\infty} \oplus Rf_i'$ to Re_n^*. Since $\sum_{i=1}^{\infty} \oplus Rf_i \oplus K \cap (R.\dot{}a)$ is essential left ideal, we obtain

$$(5) \qquad \sum_{i=1}^{\infty} \oplus Rf_i' \leq_e Re_n^* .$$

If $n'(i)<n$ for some i, then $n'(i)<n$ implies $Re_n^*\gtrsim Rf_i'\simeq Re_{n'(i)}^*c\cdot c(f_i')$ which contradicts that R is directly finite. So we have $n'(i)\geqq n$ for all i. Suppose that $n(1)=n$. Put $g=c\cdot c(f_1')\in B(R)$. If $f_k'g\neq 0$ for some j, then we obtain $Rf_1'\oplus Rf_j'g\lesssim Re_n^*g$ from (5) and $Rf_1'\simeq Re_n^*$ from $n(1)=n$, and they imply that Re_n^*g is isomorphic to a proper direct summand of itself. This is a contradiction. So we have $f_i'g=0$ for all $i\geqq 2$. By Corollary of Proposition 3, the following holds:

$$Re_n^*\simeq R(1-\sum_{i=1}^{n}e_i^*)\supset_e\sum_{i=n+1}^{\infty}\oplus Re_i^* \tag{6}$$

Then $Rf_1'(\simeq Re_n^*g)$ contains a left ideal which is isomorphic to $\sum_{i=n+1}^{\infty}\oplus Re_i^*g$ and is essential in Rf_1'. Changing suitable $\{f_i'\}_{i=1}^{\infty}$ from Lemma D, we may assume that $n'(i)>n$ for all i. Then we obtain the following relation:

$$\begin{aligned}\sum_{i=1}^{\infty}\oplus Rf_i' &\simeq \sum_{i=1}^{\infty}\oplus Re_{n'(i)}^*c\cdot c(f_i') \qquad &&\text{(from (3))}\\ &\lesssim \sum_{i=n+1}^{\infty}\oplus Re_i^* &&\text{(from } n'(i)>n)\\ &\subset_e R(1-\sum_{i=1}^{n}e_i^*)\simeq Re_n^* &&\text{(from (6))}\end{aligned} \tag{7}$$

Using Proposition 2 and Theorem 7 for two left ideal $\sum_{i=n+1}^{\infty}\oplus Re_i^*$, $\sum_{i=1}^{\infty}\oplus Rf_i'\lesssim Re_n^*$ (from (5), (7)), the homomorphism (7) $\sum_{i=1}^{\infty}\oplus Rf_i'\lesssim\sum_{i=n+1}^{\infty}\oplus Re_i^*$ implies that $Rf_i'\simeq Re_{n+i}^*$ for all i.

Suppose that there exists a nonzero central idempotent $g\in B(R)$ satisfying $gf_i=0$ for all but finite many i. For the sake of simplicity, put $\{i\,|\,gf_i\neq 0\}=\{1, 2, \cdots, m\}$. So we have $Rg\supset_e\sum_{i=1}^{m}\oplus Rf_ig\oplus\sum_{i=1}^{\infty}\oplus Rf_i'g$.

Here we claim that (I): $\{n(i)\,|\,i=1, 2, \cdots, m\}\supseteqq\{1, 2, \cdots, n\}$, (II): $Rf_ig\simeq Re_i^*g$ for all $i\leqq n$, (III): $\{n(i)\,|\,1\leqq i\leqq m\}=\{1, 2, \cdots, n\}$.

(I) Suppose that there exists an integer $s(\leqq n)$ with $s\neq n(i)$ for all $1\leqq i\leqq m$. Now we obtain the following from Proposition 3:

$$\sum_{i=1}^{\infty}\oplus Rf_i'g\lesssim_e Re_n^*g\simeq R(1-\sum_{i=1}^{n}e_i^*)g \tag{8}$$

$$\begin{aligned}\sum_{i\,:\,n(i)<s}\oplus Rf_ig &\simeq \sum_{i\,:\,n(i)<s}\oplus Rc\cdot c(gf_i)\,e_{n(i)}^* \qquad \text{(from (1))}\\ &\subset \sum_{i=1}^{n-1}\oplus ge_i^*Q\end{aligned} \tag{9}$$

$$\begin{aligned}\sum_{i\,:\,n(i)>s}\oplus Rf_ig &\simeq \sum_{i\,:\,n(i)>s}\oplus Re_{n(i)}^*c\cdot c(gf_i)\\ &\lesssim \sum_{i=s+1}^{n(m)}\oplus ge_i^*Q\end{aligned} \tag{10}$$

On the other hand, from Proposition 3, $\sum_{i=s+1}^{n(m)} \oplus ge_i^* Q$ is isomorphic to a proper direct summand of $Re_s^* g\,(\simeq R(1-\sum_{i=1}^{s} e_i^*)g \supset_e \sum_{i=s+1}^{\infty} Re_i^* g)$. Thus we obtain from (8), (9), (10) that $\sum_{i=1}^{\infty} \oplus Rf_i g \oplus \sum_{i=1}^{\infty} \oplus Rf_i' g\,(\subset_e Rg)$ is subisomorphic to a proper direct summand of Rg. This contradict that R is directly finite. So we have (I).

(II). Suppose that for a given nonzero idempotent $h=gh\in B(R)$, $hgf_s=0$ for some $s\leqq n$. Using $\{f_i' gh, ghf_i, ghe_i^*\}$ for $\{f_i g, f_i' g, e_i^* g\}$, the same argument as above implies that an essential left ideal of Rgh is subisomorphic to a proper direct summand of Rgh. This is a contradiction. So we obtain that $Rf_i g \simeq Re_i^* g$ for all $1\leqq i\leqq n$.

(III). Suppose that $n(m)>n$. Then $\sum_{i=1}^{n} \oplus Rf_i g \oplus \sum_{i=1}^{\infty} \oplus Rf_i' g$ is a proer proper direct summand of $\sum_{i=1}^{m} \oplus Rf_i g \oplus \sum_{i=1}^{\infty} \oplus Rf_i' g$. On the other hand we obtain from (I), (II) and Proposition 3 that $\sum_{i=1}^{n} \oplus Rf_i g \oplus \sum_{i=1}^{\infty} \oplus Rf_i' g\,(\simeq \sum_{i=1}^{\infty} \oplus Re_i^* g)$ is essential in Rg. This is a contradiction. Thus we have (III).

Put $J=\{g\in B(R)\,|\,gf_i=0$ for all but finite many $i\}$. Put $h=\bigvee_J g$ in $B(R)$. Then $Rf_i h \simeq Re_i^* h$ for all $i=1, 2, \cdots, n$. By Proposition 3, it follows that Rhf_n contains a left ideal which is isomorphic to $\sum_{i=n+1}^{\infty} \oplus Re_i^* h$. Changing suitable pairwise orthogonal idempotents $\{f_i\}_{i=1}^{\infty}$ from Lemma D, we may assume that for every nonzero central idempotent $g\in B(R)$, $gf_i \neq 0$ hold for infinite many $i\in N$. By Lemma 11, we obtain that $Rf_i \simeq Re_{n(i)}^*$ for all $i\neq n$ and $n(i)=i$ for all $i\leqq n-1$ and $n(i)=i+1$ for all $i\geqq n$.

Lemma 13. *Let I be an essentially closed right ideal of Q such that $I\oplus eQ$ is essential in Q_Q, $eQ\simeq e_t^* Q$ for some integer t. There exist pairwise orthogonal idempotents $\{e_n\}_{n=1,\neq t}^{\infty}$ of I such that $\sum_{n=1,\neq t}^{\infty} \oplus e_n Q$ is essential in I, $e_n Q\simeq e_n^* Q$ for all $n(\neq t)\in N$.*

Proof. Since I is essentially closed, I is centrally closed in Q. From Corollary of Proposition 5, there exist pairwise orthogonal idempotents $\{e_i\in I\}_{i=1}^{\infty}$ which satisfy the following conditions:

(1) $e_i Q\simeq c\cdot c(e_i)\, e_{n(i)}^* Q$ for all i where $\{n(i)\}_{i=1}^{\infty}$ is a strictly increasing sequence.

(2) $\sum_{i=1}^{\infty} \oplus e_i Q$ is essential in I.

Suppose that there exists a nonzero central idempotent $g\in B(R)$ satisfying $ge_i=0$ for all but finite many i. By the similar argument in (*I*), (*II*), (*III*) of Proof of Lemma 12, we obtain that $ge_i Q\simeq ge_i^* Q$ for all $i\leqq t$ and $ge_i=0$ for all

$i>t$.

Put $J=\{g\in B(R)\,|\,ge_i=0$ for all but finite many $i\}$. Put $h=\bigvee_J g$ in $B(Q)$. Then $he_iQ\simeq he_i^*Q$ for all $i=1,2,\cdots,t$. By Proposition 3 and its Corollary, it follows that he_tQ contains a right ideal which is isomorphic to $\sum_{i=t+1}^{\infty}\oplus he_i^*Q$ and essential in he_tQ. Changing suitable pairwise orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ from Lemma D, we may assume that for every nonzero central idempotent $g\in B(R)$, $ge_i\neq 0$ for infinite many $i\in N$. Since $eQ\simeq e_t^*Q\simeq(1-\sum_{i=1}^{t}e_i^*)\,Q\;{}_e\!\supset \sum_{i=t+1}^{\infty}\oplus e_i^*Q$, it follows that there exist pairwise orthogonal idempotents $\{f_i\}_{i=1}^{\infty}$ satisfying $eQ\;{}_e\!\supset\sum_{i=1}^{\infty}\oplus f_iQ$ and $f_iQ\simeq e_{i+t}^*Q$ for all i. Applying Lemma 11 to the present argument, we complete the proof.

Lemma 14. *Let I be an essentially closed right ideal of Q such that $I\oplus eQ$ is essential in Q_Q, $eQ\simeq e_t^*Q$ for some integer t. Then $l_Q(I)$ is nonzero.*

Proof. By Lemma 13, there exist pariwise orthogonal idempotents $\{e_n\}_{n=1,\neq t}^{\infty}$ in I which satisfy the following for every $n\neq t$:

$$e_nQ\simeq e_n^*Q\,. \tag{1}$$

$$\sum_{n=1,\neq t}^{\infty}\oplus e_nQ\subseteq_e I \tag{2}$$

By Lemma 12, for every e_n, there exist pairwise orthogonal idempotents $\{f_{ni}\}_{i=1,\neq n}^{\infty}$ in $l_R(e_n)$ which satisfy the following for every $i\neq n$.

$$Rf_{ni}\simeq Re_i^*\,. \tag{3}$$

Put $f_n=\sum_{i=1,\neq n}^{n+t+2}f_{ni}$ in $l_R(e_n)$ for all $n\neq t$. From [2] Theorem 4.14, R satisfy cancellation property. Since R has two decompositions $R=\sum_{i=1,\neq n}^{n+t+2}\oplus Rf_{ni}\oplus R(1-f_n)\simeq \sum_{i=1,\neq n}^{n+t+2}\oplus Re_i^*\oplus Re_n^*\oplus R(1-\sum_{i=1}^{n+t+2}f_{ni})$, we obtain the following from (3):

$$\begin{aligned} R(1-f_n)&\simeq Re_n^*\oplus R(1-\sum_{i=1}^{n+t+2}f_{ni})\\ &\simeq Re_n^*\oplus Re_{n+t+2}^* \qquad \text{(from Proposition 3)}\end{aligned}$$

So we obtain:

$$(1-f_n)R\simeq e_n^*R\oplus e_{n+t+2}^*R\,. \tag{4}$$

On the other hand $l_R(e_n)\supset Rf_n$ implies $r_Rl_R(e_n)\subset(1-f_n)\,R$. So we have

$$\sum_{n=1,\neq t}^{\infty}r_Rl_R(e_n)\subset\sum_{n=1,\neq t}^{\infty}(1-f_n)\,R\,. \tag{5}$$

By Lemma D, there exist pairwise orthogonal idempotents $\{h_n\}_{n=1,\neq t}^{\infty}$ in R, which satisfy the following for every $n\neq t$.

(6) $$(1-f_n)R \gtrsim h_n R.$$

(7) $$\sum_{n=1,\neq t}^{\infty}(1-f_n)R = \sum_{n=1,\neq t}^{\infty}\oplus h_n R.$$

Consequently we obtain:

$$\sum_{n=1,\neq t}^{\infty}\oplus h_n R \lesssim \sum_{n=1,\neq t}^{\infty}\overline{\oplus}(e_n^* R\oplus e_{n+t+2}^* R) \qquad \text{(from (4), (6))}$$

(8) $$\simeq \sum_{n=1,\neq t}^{\infty}\oplus e_n^* R\overline{\oplus}\sum_{n=1,\neq t}^{\infty}\oplus e_{n+t+2}^* R$$

$$\lesssim (1-e_t^*)R\overline{\oplus}e_{t+2}^* R$$

where we denote by $\overline{\oplus}$ outer direct sum. Since R satisfy cancellation property, $R=(1-e_t^*)R\oplus e_t^* R=\sum_{n=1}^{t}\oplus e_n^* R\oplus(1-\sum_{n=1}^{t}e_n^*)R$ and $(1-\sum_{n=1}^{t}e_n^*)R\simeq e_t^* R$ implies $(1-e_t^*)R\simeq\sum_{n=1}^{t}\oplus e_n^* R$. So we obtain from (8) that $\sum_{n=1,\neq t}^{\infty}\oplus h_n R\lesssim\sum_{n=1}^{t}\oplus e_n^* R\oplus e_{t+2}^* R$. Since $R=\sum_{n=1}^{t}\oplus e_n^* R\oplus e_{t+2}^* R\oplus e_{t+1}^* R\oplus(1-\sum_{n=1}^{t+2}e_n^*)R$, i.e., $\sum_{n=1}^{t}\oplus e_{t+2}^* R\oplus e_{t+2}^* R$ is a proper direct summand of R, it follows from (7) and Proposition 2 that $\sum_{n=1,\neq t}^{\infty}(1-f_n)R$ is not essential in R_R.

Let e' be an idempotent in R such that $\sum_{n=1,\neq t}^{\infty}(1-f_n)R$ is essential in $e'R$. We obtain that $0\neq R(1-e')=l_R(e')\subset l_R((1-f_n)R)=Rf_n\subset l_R(e_n)$ for all $n\neq t$. Thus $\bigcap_{n=1,\neq t} l(e_n)\supset R(1-e')\neq 0$. For any element q in I, we obtain from (2) that $(q^{\bullet}\sum_{n=1,\neq t}^{\infty}\oplus e_n Q)=J$ is an essntial right idela of Q. Since $(1-e')qJ=0$ and Q_Q is nonsingular, we see that $(1-e')q=0$. Thus $l_Q(I)\supset Q(1-e')\neq 0$.

Theorem 15. *Let R be a directly finite, right self-injective regular ring. Then the maximal left quotient ring of R is a left and right self-injective regular ring.*

Proof. By [2] Theorem 10.13, R has a decomposition $R=R_1\times R_2$ such thrt R_1 is type I_f and R_2 is type II_f. Then R_1 is right and left self-injective. So we may asmme that R is type II_f.

Suppose that Q satisfies K_r. Since Q satisfies K_1, it follows by Utsumi's Theorem that the maximal right quotient ring of Q is equal to the maximal left quotient ring of Q, i.e., Q. Thus Q is left and right self-injective ring. So it is sufficient to show that Q satisfies K_r.

Let I be a non-essential right ideal of Q such that $eQ\cap I=0$ for some nonzero idempotent e in Q. Put $e'=e'^2$ in eQe such that $Qe'\simeq Qe_t^* g$, $g=c\cdot c(e')$ for some integer t. We prove that $l_Q(I)\neq 0$. So we may assume that $Qe\simeq Qge_2^*$, $c\cdot c(e)=g$ and $eQ\oplus I$ is an essential right ideal of Q. Since $I\subseteq I^c$ implies $l(I^c)\subseteq l(I)$ and $I^c\cap eQ=0$ where I^c is the essential closure of I, we may assume that I is essentially closed. From $l_{gQ}(gI)\subseteq l_Q(I)$, we can assume that $c\cdot c(e)=1$. By Lemma 14, we have $l_Q(I)\neq 0$, i.e., that Q satisfies K_r.

Corollary. ([5]) *Let R be a regular ring with a rank function N. Suppose that N satisfy $1=\sup\{N(x)\,|\,x\in I\}$ for every essential left ideal I of R. Then the maximal right quotient ring of the maximal left quotient ring of R is left and right self-injective and is isomorphic to the N-completion of R by an extention of natural map φ: $R\to\bar{R}$ (See [2]).*

Proof. From [2] Theorem 21.17, the maximal left quotient ring S of R is directly finite. By Theorem 15, the maximal right quotient ring Q of S is left and right self-injective. By the hypothesis and [2] Theorem 21.17, we consider that S is a subring of the N-completion of R and there exists a rank function N of S as a extention of N. For every essential left ideal I of S, we have ${}_RI\supset_e I\cap R$ and $R\supset_e I\cap R$. So $1=\sup\{N(x)\,|\,x\in I\cap R\}\leqq\sup\{N(x)\,|\,x\in I\}\leqq 1$, i.e. sup $\{N(x)\,|\,x\in I\}=1$ for all essential left ideal I of S. For a given essential right ideal J of S, there exist pairwise orthogonal idempotents $\{g_n\}$ such that $J\supset_e \sum_{n=1}^{\infty}\oplus g_n S$. Suppose $\sum_{n=1}^{\infty}\oplus Sg_n\,{}_e\!\subset S(1-g)$ Then $ga=\sum g_n a_n\in gS\cap\sum g_nS$ implies $0=g_nga=g_na_n$ for all n, so $g=1$, i.e., $\sum Sg_n$ is essential in ${}_SS$. Thus $1=\sum N(g_n)\leqq\sup\{N(x)\,|\,x\in J\}\leqq 1$, i.e., $1=\sup\{N(x)\,|\,x\in J\}$ for all essntial right ideal J of S. From [2] Theorem 21.17, we consider that Q is a subring of the N-completion $\bar{R}$ of R and have a same rank function N. In the same way as S, we obtain that $1=\sup\{N(x)\,|\,x\in K\}$ for all essential right ideal of Q. From [2] Proposition 21.3 and 4, N is countably additive on Q. By [2] Theorem 21.7, Q is complete in the N-metric. So $R\subset Q\subset\bar{R}$ implies $Q=\bar{R}$.

We consider again a necessary and sufficient condition for the maximal right quotient ring of a regular ring to be directly finite.

Theorem 16. *For a regular ring R, the following conditions are equivalent.*

1) *The maximal right quotient ring of R is directly finite.*

2) *Every right ideal isomorphic to some essential right ideal is essentail in R_R.*

3) *The maximal left quotient ring of the maximal right quotient ring of R is right and left self-injective.*

4) *There exists a left and right self-injective regular ring S such that R is a subring of S and S is a non-singular right R-molule*:

Proof. 1) $\Rightarrow$ 2): Proposition 2.

1) $\Rightarrow$ 3): Theorem 15 and Proposition 2.

3) $\Rightarrow$ 4): Let Q be the maximal right quotient ring of R and S the maximal left quotient ring of Q. Suppose that x is a singular element in S such that $r(x)$ is an essential right ideal of R. Since $(Q_{\cdot}^{\cdot}x)=\{q\in Q \mid qx\in Q\}$ is an essential left ideal of Q, $((Q_{\cdot}^{\cdot}x)\,x)\,r_R(x)=0$ implies $0=Z(Q_R)\supset(Q_{\cdot}^{\cdot}x)\,x$. Since S is a non-singular left Q-module, it follows that $x=0$, i.e., S is a non-singular right R-module.

4) $\Rightarrow$ 1): Let S_R be non-singular. Since R is regular, S is a flat left R-module. So S is non-singular injective right R-module ([2] Lemma 6.17). Put $Q=\{s\in S \mid (s^{\cdot}{}_{\cdot}R)$ is an essential right ideal of $R\}$. For every $0\neq q\in Q$, we have $qR\cap R\supset q(q^{\cdot}{}_{\cdot}R)\neq 0$ from nonsingularity of S_R. So Q is essential hull of R in injective module S_R, i.e., Q_R is injective.

We will show that $(t^{\cdot}{}_{\cdot}(s^{\cdot}{}_{\cdot}R))$ is an essential right ideal for every s, $t\in Q$. Suppose that $(t^{\cdot}{}_{\cdot}(s^{\cdot}{}_{\cdot}R))\cap xR=0$ for some nonzero t, $s\in Q$ and $x\in R$. Then $0=txR\cap(s^{\cdot}{}_{\cdot}R)$ and $(s^{\cdot}{}_{\cdot}R)\underset{e}{\subset}R$ implies $txR=0$, i.e., $xR\subset(t_{\cdot}^{\cdot}(s^{\cdot}{}_{\cdot}R))$, which is a contradiction. So $(t^{\cdot}{}_{\cdot}(s^{\cdot\cdot}R))$ is essential right ideal for all t, $s\in Q$. Thus we obtain that Q is a subring of S. So Q is the maximal right quotient ring of R.

Since S is directly finite from Utsumi [7] (see [2] Theorem 9.29), Q is directly finite.

Let R be a regular ring and Q be the maximal right quotient ring of the maximal left quotient ring of R. Here we consider necessary and sufficient conditions for Q to be complete in the N-metric for some rank function N of Q (Corollary 1). And, in Corollary 2, we consider a case that R is a prime regular ring

Corollary 1. *Let R be a regular ring and Q be the maximal right quotient ring of the maximal left quotient ring of R. Then the following conditions are equivalent.*

1). *There exists a rank function on R such that $1=\sup\{N(x)\mid x\in I\}$ for all essential left ideal I of R.*

2). *There exists a rank function N of R such that the N-completion of R is the maximal right quotient ring of the maximal left quotient ring of R.*

3). *There exists a rank function $\bar{N}$ on Q such that Q is complete in the $\bar{N}$-metric.*

4). *There exists a rank function N on R such that the N-completion $\bar{R}$ of R is a nonsigular R-module.*

Proof. Let S be the maximal left quotient ring of R.

1) $\Rightarrow$ 2): Corollary of Theorem 15.

2) $\Rightarrow$ 3): [2] Theorem 19.6.

3) $\Rightarrow$ 4): The restriction of $\bar{N}$ to R is a rank function on R. From 3) and $Q\supset R$, the $\bar{N}$-completion $\bar{Q}=Q$ of Q contains the N-completion $\bar{R}$ of R as a sub-

ring. Suppose that $0 \neq q \in Z({}_RQ)$ is a singular element of Q. Then there exists $b \in (q^{\bullet}.S) = \{x \in S \mid qx \in S\}$ such that $qb \neq 0$. Then $l_R(q)\, qb = 0$, which contradicts that ${}_RS$ is nonsingular.

4) $\Rightarrow$ 1): For a given essential left ideal I of R, there exist pairwise orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ sucht hat $I \supset_e \sum Re_i$. Set $f = \lim_n \sum_{i=1}^{n} e_i$ in $\bar{R}$. Then $\sum Re_i(1-f) = 0$ implies $1-f=0$. So, from [2] Theorem 19.6, $1 = N(f) = \lim \sum_{i=1}^{n} N(e_i) = \sum N(e_i) \leqq \sup \{N(x) \mid x \in I\}$, i.e, $\sup \{N(x) \mid x \in I\} = 1$.

Corollary 2. *Let R be a prime regular ring. Then the following conditions are equivalent.*

1). *There exists a rank function N on R such that $1 = \sup \{N(x) \mid x \in I\}$ for all essential left ideal I of R.*

2). *The maximal left quotient ring of R is directly finite.*

3). *The maximal right quotient ring of the maximal left quotient ring of R is right and left self-injective.*

4). *There exists a rank function N on R such that the N-completion of R is a nonsingular left R-module.*

Proof. 1) $\Rightarrow$ 2): [2] Corollary 21.19.

2) $\Rightarrow$ 3): Theorem 15.

3) $\Rightarrow$ 4): [2] Corollary 21.14 and 3) $\Rightarrow$ 4) in Proof of Corollary 1 of Theorem 16.

4) $\Rightarrow$ 1): See 4) $\Rightarrow$ 1) in Proof of Corollary 1 of Theorem 16.

A regular ring R is said to satisfy K_1^* if $r_R(I) \oplus r_R(J)$ is an essential rihgt ideal of R for every essential left ideal $I \oplus J$. Note that essentiality of $I \oplus J$ implies $r_R(I) \cap r({}_RJ) = 0$.

Let R be a subring of a ring S such that R_R is nonsingular. Then S is said to be *left quotient ring* of R if R is essential in ${}_RS$.

In the following theorem, we consider necessary and sufficient conditions that the maximal right quotient ring of a regular ring is left and right self-injective.

Theorem 17. *For a regular ring R, the following conditions are equivalent.*

1) *The maximal right quotient ring of R is right and left self-injective.*

2) *The maximal left quotient ring of R is directly finite and a right quotient ring of R.*

3) i) *Every left ideal isomorphic to some essential left ideal is an essential left ideal of R.*

ii) *R satisfies K_1^*.*

Proof. Let Q be the maximal right quotient ring of R.

1) $\Rightarrow$ 2): Let Q be a right and left self-injective ring. By [2] Lemma 6.17,

Q is injective left R-modlue. Suppose that x is a singular element of left R-module Q. Since R is a non-singular left R-module, it follows that $l_R(x)\{x(x^{\cdot}.R)\}=0$ implies $x(x^{\cdot}.R)=0$, Since Q is a non-singular right R-module, it follows that $x=0$, i.e., Q is non-singular left R-module.

Put $S=\{x\in Q\,|\,(R.^{\cdot}x)\subset_{e}{}_{R}R\}$. By similar symmetric argument as 3)$\Rightarrow$1) in Proof of Theorem 16, we obtain that S is the maximal left quotient ring of R and S is a right quotient ring of R.

2) $\Rightarrow$ 1): Let S be the maximal left quotient ring of R such that S is a directly finite, right quotient ring of R, i.e., $R_R\subset_{e} S_R$. So we can consider that S is a submodule of Q. For any element s, t of S, we denote by $s\circ t$ the multiplication of s and t in Q. Put $I=(t^{\cdot}.(s^{\cdot}.R))$ and $J=(st^{\cdot}.R)$. Then $t(I\cap J)\subset(s^{\cdot}.R)$ and $(s\circ t)(I\cap J)=s(t(I\cap J))=st(I\cap J)$. Since Q is a non-singular right R-module, we have $s\circ t=st$. Thus S is a subring of Q. Since Q is rihgt self-injective and a flat left S-module, it follows from [2] Lemma 6.17 that Q is right S-injective module. While $Q_R\,{}_e\!\subset R_R$ implies $Q_S\supset_e S_S$. Thus Q is the maximal right quotient ring of S. By Theorem 15, it follows that Q is left and right self-injective.

3) $\Leftrightarrow$ 2): By Proposition 2, 3)-i) is equvalent that the maximal left quotient ring of R is directly finite. Let S be the maximal left quotient ring of R.

Suppose that R satisfies K_1^*. By [2] Theorem 13.14, S has a decomposition $S=S_1\times S_2$ such that S_1 is strongly regular ring and S_2 has no non-zero central abelian idempotent. Since $S_1\cap R\subset_e S_1$ as right R-module, it is sufficient to show that $S_2\cap R\subset_e S_2$ as right R-modules. By [2] Theorem 13.16, S_2 is generated as a ring by all its idempotents. For a given idempotent e in S_2, put $I=Se\cap R$ and $J=S(1-e)\cap R$. Then $I\oplus J$ is an essential left ideal of R. Since $r(I)\oplus r(J)=(1-e)\,S\cap R\oplus eS\cap R$ is essential in R_R, it follows that $(e^{\cdot}.R)\supset r(I)\oplus r(J)$ are essential right ideals of R. Therefore S_R is an essential extention of R_R.

Conversely, suppose that S is a right quotient ring of R, i.e., S is a subring of Q. Let $I\oplus J$ be an essential left ideal of R. There exists an idempotent f in S such that $I\subset_e Sf$ and $J\subset_e S(1-f)$. Then $fQ\cap R\oplus(1-f)\,Q\cap R$ is essential in R_R. Now $fQ\cap R\supset fS\cap R\supset fR\cap R$. While we have $fQ\cap R\subset fR\cap R$ from $f(fQ\cap R)=fQ\cap R$. So $r(J)=fS\cap R=fQ\cap R$. Similarily $r(I)=(1-f)\,Q\cap R$. Therefore $r(I)\oplus r(J)$ is essential in R_R. Thus R satisfies K_1^*.

REMARK. For a regular ring, the condition K_1^* implies the condition K_1. For, let R be a regular ring satisfying K_1^*. Suppose that I is a non essential left ideal of R with $r(I)=0$. Let J be a nonzero left ideal of R such that $I\oplus J$ is essential in R_R. Then $r(I)\oplus r(J)=r(J)$ is an essential right ideal. Since R_R is nonsingular, it follows that $J=0$. This is a contradiction.

We don't know whether the converse hold or not.

Here we consider the same problem as Theorem 17 for a regular ring with a rank function (Corollary 1) and for a prime regular ring (Corollary 2). The equivalence 2)$\Leftrightarrow$4) in Corollary 1 was proved by A. Vogel [8].

Collorary 1. *For a regular ring R with a rank function N, the following conditions are equivalent.*

1). *The maximal left quotient ring S of R is a right quotient ring of R and* $\sup\{N(x)\,|\,x\in I\}=1$ *for all essential left ideal I of R.*

2). *The N-completion $\bar{R}$ of R is the maximal right quotient ring of R*

3). *The maximal right quotient ring Q of R is right and left self-injective and there exists a rank function $\bar{N}$ on Q such that $\bar{N}$ is an extention of N and Q is complete in the $\bar{N}$-metric.*

4). *For every left ideal I of R,* $\sup\{N(x)\,|\,x\in I\}+\sup\{N(x)\,|\,x\in r(I)\}=1$.

Proof. 1)$\Rightarrow$2): Since S is right quotient ring, we have $Q_R\supset_e S_R\supset_e R_R$, where Q is the maximal right quotient ring of R. Let E be an injective hull of S_S. For every $a\in E$, $(a^{\cdot}.S)_S=\{x\in S\,|\,ax\in S\}$ is an essential right ideal of S, so $(a^{\cdot}.S)\cap R$ is an essential right ideal of R. Then we have $E_{R\,e}\supset S_R$, so we consider $Q\supset E$. For every $q\in Q$, we have $(q^{\cdot}.S)_S\supset(q^{\cdot}.R)_R$ and $(q^{\cdot}.R)_R$ is essential right ideal of R, so $E\supset Q$. Then Q is maximal right quotient ring of S. From Corollary 1 of Theorem 16, $\bar{R}$ satisfies 2).

2)$\Rightarrow$3): It is clear from [2] Theorem 19.6.

3)$\Rightarrow$4): Let $\{e_i\}_{i=1}^{\infty}$ be pairwise orthogonal idempotents with $\sum_{i=1}^{\infty} Re_i\,{}_e\!\subset R$. Set $f=\lim_n \sum_{i=1}^{n} e_i$ in Q. Then $0=r(\sum Re_i)\supset(1-f)\,Q\cap R$ implies $1-f=0$. From [2] Theorem 21.7, $\bar{N}$ is countably additive on Q. Thus we obtain $1=\sum\bar{N}(e_i)=\sum N(e_i)$, i.e., $1=\sup\{N(x)\,|\,x\in I\}$ for all essential left ideals I of R. While, from [2] Theorem 21.7, we have $1=\sup\{N(x)\,|\,x\in I'\}$ for all essential right ideals I' of R.

For a given left ideal I of R, set $I\oplus J\,{}_e\!\subset R$ and ${}_RJ\subset R$. Then $1=\sup\{N(x)\,|\,x\in I\}+\sup\{N(x)\,|\,x\in J\}$ Then $r(I)\oplus r(J)\,{}_e\!\subset R_R$ from Theorem 17. So $1=\sup\{N(x)\,|\,x\in r(I)\}+\sup\{N(x)\,|\,x\in r(J)\}$. From $(1-f)\,R\supset r(I)$ for every $f^2=f\in I$, we have $1-N(f)\geqq\sup\{N(x)\,|\,x\in r(I)\}$ for every $f\in I$, i.e., $1-\sup\{N(x)\,|\,x\in I\}\geqq\sup\{N(x)\,|\,x\in r(I)\}$. Similarly, $1-\sup\{N(x)\,|\,x\in J\}\geqq\sup\{N(x)\,|\,x\in r(J)\}$. Then $1\geqq\sup\{N(x)\,|\,x\in I\}+\sup\{N(x)\,|\,x\in r(I)\}=\sup\{N(x)\,|\,x\in I\}+1-\sup\{N(x)\,|\,x\in r(J)\}\geqq\sup\{N(x)\,|\,x\in I\}+\sup\{N(x)\,|\,x\in J\}=1$. Thus R satisfies 4).

4)$\Rightarrow$1): By 4), $\sup\{N(x)\,|\,x\in I\}=1$ for every essential left ideal I of R. So we have $\sup\{N(x)\,|\,x\in J\}+\sup\{N(x)\,|\,x\in J'\}=1$ for every essential left ideal $J\oplus J'$ of R. From 4), $\sup\{N(x)\,|\,x\in J\}+\sup\{N(x)\,|\,x\in r(J)\}=1$ and $\sup\{N(x)\,|\,x\in J'\}+\sup\{N(x)\,|\,x\in r(J')\}=1$. Hence we have $\sup\{N(x)\,|\,x\in$

$r(J)\} + \sup \{N(x) \mid x \in r(J')\} = 1$, so $r(J) \oplus r(J')$ is essential in R_R, i,e. R satisfies 3) of Theorem 17. Thus the maximal left quotient ring of R is right quotient ring of R and $1 = \sup \{N(x) \mid x \in I\}$ for all essential left ideal I of R.

Collorary 2. *For a prime regular ring R, following conditions are equivalent.*

1). *The maximal left quotient ring of R is a right quotient ring of R and directly finite.*

2). *There exists a rank function N on R such that the N-completion of R is the maximal right quotient ring of R.*

3). *The maximal right quotient ring of R is left and right self-injective.*

4). *There exists a rank function N on R such that* $1 = \sup \{N(x) \mid x \in I\} + \sup \{N(x) \mid x \in r(I)\}$ *for every left ideal I of R.*

Proof. 1) $\Rightarrow$ 2): From [2] Corollary 21.19, there exists a rank function N on R such that $\sup \{N(x) \mid x \in I\} = 1$ for every essential left ideal I of R. By Corollary 1 of Theorem 17, R satisfies 2) by the rank function N.

2) $\Rightarrow$ 3): [2] Theorem 19.6.

3) $\Rightarrow$ 4): By Corollary 21.14, there exists a rank function $\bar{N}$ of the maximal right quotient ring Q of R such that Q is complete in the $\bar{N}$-metric. From Corollary 1 of Theorem 17, R satisfies 4).

4) $\Rightarrow$ 1): It is clear from Corollary 1 of Theorem 17 and [2] Corollary 21.19.

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Kii School of General Education
Wakayama Medical College
Hironishi 651
Wakayama 649–63 Japan