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ON THE MORSE INDEX OF COMPLETE MINIMAL SURFACES IN EUCLIDEAN SPACE

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Introduction

Let M be a minimal surface in $\boldsymbol{R}^n$. The minimality of M is equivalent to the property that the area of M is critical for all compactly supported variations of M. So problems on the index of the Jacobi operator associated to M arise naturally. We define here the index of M as the supremum of the indices of the Jacobi operator on relatively compact domains in M. The condition that M be stable is just the condition that the index of M be equal to zero. Fischer-Colbrie [3], Gulliver and Lawson [4], [5] have proved independently that a complete oriented minimal surface in $\boldsymbol{R}^3$ has finite index if and only if it has finite total curvature. More recently Lopez and Ros [10] have proved that the catenoid and the Enneper's surface are the only complete oriented minimal surfaces in $\boldsymbol{R}^3$ with index one. This result has also been obtained by Cheng and Tysk [1] in the case of embedded ends.

The results mentioned so far deal with surfaces in $\boldsymbol{R}^3$. In this paper we study the index of minimal surfaces in $\boldsymbol{R}^n$. We first prove in Theorem 1 that *if a complete oriented minimal surface in $\boldsymbol{R}^n$ has finite total curvature, then it has finite index.* Thus we generalize, to the higher codimensional case, the "if" part of the result mentioned above due to Fischer-Colbrie and aslo due to Gulliver and Lawson.

We now restrict our attention to $n=4$. In view of Theorem 1 and the well-known fact that holomorphic curves in $\boldsymbol{C}^2$ $(=\boldsymbol{R}^4)$ are stable, we propose here the following question. *Suppose M is a complete oriented minimal surface in $\boldsymbol{R}^4$ with finite index. Is M of finite total curvature or a holomorphic curve with respect to some orthogonal complex structure on $\boldsymbol{R}^4$?* Related to this question we point out the work of Micallef [11], who proved, among other things, that any complete orineted parabolic stable minimal surface in $\boldsymbol{R}^4$ is a holomorphic curve with respect to some orthogonal complex structure on $\boldsymbol{R}^4$.

Following Osserman we say that the Gauss map of an oriented surface in $\boldsymbol{R}^n$ is degenerate if the Gauss image lies in a hyperplane of $\boldsymbol{P}^{n-1}(\boldsymbol{C})$ (see §3). Theorem 2 provides an affirmative answer to the question mentioned above *when* *the Gauss map of M is degenerate and M is of finite type* (that is, conformally equivalent to a compact Riemann surface with finitely many punctures). As a corollary of Theorem 2 we obtain a result on the index of entire 2-dimensional minimal graphs in $\boldsymbol{R}^4$.

In view of the result of Lopez and Ros, complete minimal surfaces with index one are of particular interest. Our final result is to give examples of such surfaces which lie fully in $\boldsymbol{R}^4$. They are obtained by deforming the catenoid and the Enneper's surface in $\boldsymbol{R}^3$.

If M is an oriented surface in $\boldsymbol{R}^n$, then isothermal coordinates for the induced metric together with the orientation give rise to a complex structure on M. Throughout this paper we consider M as a Riemann surface with the complex structure just mentioned.

This paper is divided into four sections. The first section is devoted to the preliminary discussion about the index of the Jacobi operator. In the second and the third sections we prove Theorem 1 and Theorem 2 respectively. In the last section examples of index one minimal surfaces in $\boldsymbol{R}^4$ are given.

After the completion of this paper the author learned that Ejiri [14] has proved that the index of a complete oriented minimal surface in $\boldsymbol{R}^n$ is bounded by a constant (which depends only on n) times the total curvature.

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1. Preliminaries

Let M be a minimal surface in $\boldsymbol{R}^n$. We denote by L the Jacobi operator associated to M, which is a differential operator acting on sections of the normal bundle NM of M. Let $\mathcal{B}$ denote an endomorphism of NM defined by

$$\mathcal{B}s = \sum_{i,j=1}^{2}(B(e_i, e_j)\cdot s)\, B(e_i, e_j),\ s\in\Gamma(NM)\,,$$

where B is the second fundamental form of M, $\{e_1, e_2\}$ is a local orthonormal frame for the tangent bundle of M and $s\cdot t$ is the inner product of $s, t\in\Gamma(NM)$ taken with respect to the fiber metric on NM. Then L is given by $L=-\Delta-\mathcal{B}$, where Δ is the Laplacian in NM defined by means of the normal connection. We denote by Q the quadratic form associated to L. Thus, for $s\in\Gamma(NM)$ with compact support

$$Q(s, s)=\int_M (Ls\cdot s)\, dA\,,$$

where dA is the area element corresponding to the induced metric on M.

For any relatively compact domain Ω in M, we define $\mathrm{Ind}\,(\Omega)$, the *index* of Ω, as the number of negative eigenvalues (counted with multiplicities) of the Dirichlet eigenvalue problem

$$Ls = \lambda s \text{ in } \Omega,\ s = 0 \text{ on } \partial\Omega\,.$$

We note that $\mathrm{Ind}(\Omega)$ is finite. $\mathrm{Ind}(\Omega)$ can also be defined as the maximal dimension of a subspace of $\Gamma_0(\Omega, NM)$, the space of sections of NM with compact support in Ω, on which Q is negative definite. We now define $\mathrm{Ind}(M)$, the *index* of M, as the supremum of the numbers $\mathrm{Ind}(\Omega)$ over all relatively compact domains in M. We note that $\mathrm{Ind}(M)=0$ if and only if M is stable, and also that if $\mathrm{Ind}(M)$ is finite, then there exists a compact set C in M such that $M-C$ is stable (see Gulliver and Lawson [5, Remark 3.18, p. 228], the proof is similar).

2. Complete minimal surfaces in $\boldsymbol{R}^n$ with finite total curvature

In this section we prove the following theorem.

Theorem 1. *Let M be a complete oriented minimal surface in $\boldsymbol{R}^n$ with finite total curvature. Then M has finite index.*

We first make preliminary observations which will be needed for the proof of Theorem 1. Let $G_{2,n}$ denote the Grassmannian of oriented 2-dimensional subspaces of $\boldsymbol{R}^n$. Let M be an oriented surface in $\boldsymbol{R}^n$ and $F\colon M\to\boldsymbol{R}^n$ the immersion. The (generalized) Gauss map $G\colon M\to G_{2,n}$ is defined by $G(p)=F_*(T_pM)$, where T_pM is the oriented tangent space of M at p and $F_*(T_pM)$ is translated from $F(p)$ to the origin of $\boldsymbol{R}^n$. We now let M be a complete oriented minimal surface in $\boldsymbol{R}^n$ with finite total curvature. By a theorem of Chern and Osserman [2], M is conformally equivalent to a compact Riemann surface with finitely many punctures. Moreover, the Gauss map G extends smoothly (in fact, as a holomorphic map) to the compactified surface.

We now recall that over $G_{2,n}$ we have the tautological 2-plane bundle $\gamma_{2,n}$ whose fiber over a point Π consists of the vectors in $\boldsymbol{R}^n$ which lie in Π. We also have the $(n-2)$-plane bundle $\gamma_{2,n}^\perp$ which is the orthogonal complement of $\gamma_{2,n}$ in $G_{2,n}\times\boldsymbol{R}^n$. The tangent and normal bundles of an oriented surface in $\boldsymbol{R}^n$ are isomorphic (geometrically) to the induced bundles $G^*(\gamma_{2,n})$ and $G^*(\gamma_{2,n}^\perp)$ respectively, where G is the Gauss map of the surface. Thus, in our case, the tangent and normal bundles of M extend with their fiber metrics to the vector bundles $\tilde{G}^*(\gamma_{2,n})$ and $\tilde{G}^*(\gamma_{2,n}^\perp)$ on the compactified surface $\tilde{M}$ respectively, where $\tilde{G}$ is the extension of G to $\tilde{M}$. We let $\tau=\tilde{G}^*(\gamma_{2,n})$ and $\nu=\tilde{G}^*(\gamma_{2,n}^\perp)$. We note that the direct sum $\tau\oplus\nu$ is isomorphic to the trivial bundle $\tilde{M}\times\boldsymbol{R}^n$. Let N denote the orthogonal projection from $\tilde{M}\times\boldsymbol{R}^n$ onto ν.

We define a metric connection $\tilde{D}$ on ν by

$$\tilde{D}_Y s = (d_Y s)^N,\ X\in\Gamma(T\tilde{M}),\ s\in\Gamma(\nu)\,,$$

where d_X denotes the derivative in the direction X acting on vector-valued functions on $\tilde{M}$. We also define a bilinear form $\tilde{B}$ on $T\tilde{M}\times\tau$ with values in ν by

$$\tilde{B}(X, t) = (d_X t)^N,\ X \in \Gamma(T\tilde{M}),\ t \in \Gamma(\tau).$$

Now let $d\tilde{s}^2$ be an arbitrary conformal metric on $\tilde{M}$. We denote by $\tilde{\Delta}$ the rough Laplacian corresponding to the connection $\tilde{D}$. Thus, for $s \in \Gamma(\nu)$

$$\tilde{\Delta}s = \mathrm{trace}_{d\tilde{s}^2}\, \tilde{D}\tilde{D}s.$$

We define an endomorphism $\tilde{\mathcal{B}}$ of ν by

$$\tilde{\mathcal{B}}s = \sum_{i,j=1}^{2} (\tilde{B}(e_i, t_j)\cdot s)\, \tilde{B}(e_i, t_j),\ s \in \Gamma(\nu),$$

where $\{e_i, e_j\}$ is a local orthonormal frame for $T\tilde{M}$ with respect to the metric $d\tilde{s}^2$and $\{t_i, t_j\}$ is a local orthonormal frame for τ. We set

$$\tilde{L} = -\tilde{\Delta} - \tilde{\mathcal{B}}.$$

$\tilde{L}$ is a formally self-adjoint strongly elliptic differential operator acting on $\Gamma(\nu)$. We denote by $\tilde{Q}$ the quadratic form associated to $\tilde{L}$. Thus, for $s \in \Gamma(\nu)$

$$\tilde{Q}(s, s) = \int_{\tilde{M}} (\tilde{L}s\cdot s)\, d\tilde{A},$$

where $d\tilde{A}$ is the area element corresponding it the metric $d\tilde{s}^2$.

We can define the indices of M and $\tilde{M}$ with respect to $\tilde{L}$ as we did for L. We denote them by $\mathrm{Ind}(\tilde{L}, M)$ and $\mathrm{Ind}(\tilde{L}, \tilde{M})$ respectively. We note that $\mathrm{Ind}(\tilde{L}, \tilde{M})$ is finite and $\mathrm{Ind}(\tilde{L}, M) \leq \mathrm{Ind}(\tilde{L}, \tilde{M})$.

Proof of Theorem 1. Let ds^2 denote the original metric on M induced by the immersion. The metrics ds^2 and $d\tilde{s}^2$ are expressed lically as $\lambda|dz|^2$ and $\tilde{\lambda}|dz|^2$ respectively, where $z = x + iy$ is a local complex coordinate on M. From the definition of $\tilde{L}$ it is easy to verify that

$$\tilde{L}_{|M} = \frac{\lambda}{\tilde{\lambda}} L, \tag{2.1}$$

where $\tilde{L}_{|M}$ denote the restriction of $\tilde{L}$ to $\Gamma(NM)$. Therefore, if $s \in \Gamma(NM)$ has a compact support in M

$$\begin{aligned}\tilde{Q}(s, s) &= \int_M (\tilde{L}s\cdot s)\, d\tilde{A}\\ &= \int_M (Ls\cdot s)\, dA,\end{aligned}$$

because $d\tilde{A} = \tilde{\lambda}\, dxdy = \dfrac{\tilde{\lambda}}{\lambda}\, dA$. Thus we have proved that

$$\tilde{Q}(s, s) = Q(s, s)$$

for all such s, and this clearly implies that

$$\mathrm{Ind}(M) = \mathrm{Ind}(\tilde{L}, M).$$

Hence, by the remark preceding the proof, $\mathrm{Ind}(M)$ is finite. The proof of Theorem 1 is complete.

REMARK 1. Actually it turns out from the same argument as in Fischer-Colbrie [3, Corollary 2, p. 131] that $\mathrm{Ind}(M)=\mathrm{Ind}(\tilde{L}, \tilde{M})$.

REMARK 2. The definition of $\tilde{L}$ depends on the choice of a metric on $\tilde{M}$. Let $d\tilde{s}_1^2=\tilde{\lambda}_1|dz|^2$ be another conformal metric on $\tilde{M}$ and $\tilde{L}_1$ the corresponding operator. Then we have $\tilde{L}_1=\dfrac{\tilde{\lambda}}{\tilde{\lambda}_1}\tilde{L}$. Thus another choice of metric simply multiplies $\tilde{L}$ by a positive function on $\tilde{M}$.

3. Complete minimal surfaces in $\boldsymbol{R}^4$ with degenerate Gauss maps

Let M be an oriented surface in $\boldsymbol{R}^n$ and $F\colon M\to\boldsymbol{R}^n$ the immersion. We recall that $G_{2,n}$ may be identified with the quadric $Q_{n-2}\subset P^{n-1}(\boldsymbol{C})$ defined by $(w_1)^2+\cdots+(w_n)^2=0$, where $(w_1, \cdots, w_n)$ is a homogeneous coordinate for a point in $P^{n-1}(\boldsymbol{C})$. Thus, the Gauss map G of M can be considered as a map from M into $Q_{n-2}\subset P^{n-1}(\boldsymbol{C})$. If z is a local complex coordinate on M, then $F_z(p)$ is a homogeneous coordinate for $G(p)$ (see, for example, [6] or [9]). Following Osserman [12, p. 122] we say that the Gauss map of M is *degenerate* if the Gauss image lies in a hyperplane of $P^{n-1}(\boldsymbol{C})$. Thus, the Gauss map of M is degenerate if there exists a nonzero fixed vector $A\in\boldsymbol{C}^n$ such that $A\cdot F_z\equiv 0$, where $v\cdot w=\sum_j v_j w_j$ for $v=(v_1, \cdots, v_n)$, $w=(w_1, \cdots, w_n)\in\boldsymbol{C}^n$. We point out here that oriented surfaces in $\boldsymbol{R}^n$ which lie in an $(n-1)$-dimensional affine subspace and holomorphic curves in $\boldsymbol{C}^n(=\boldsymbol{R}^{2n})$ have degenerate Gauss maps.

A Riemann surface is said to be *of finite type* if it is conformally equivalent to a compact Riemann surface with finitely many punctures.

In this section we prove the following theorem.

Theorem 2. *Let M be a complete oriented minimal surface of finite type in $\boldsymbol{R}^4$. Suppose that M has finite index and that the Gauss map of M is degenerate. Then M is of finite total curvature or a holomorphic curve with respect to some orthogonal complex structure on $\boldsymbol{R}^4$.*

We recall that for an oriented surface M in $\boldsymbol{R}^4$, NM can be given a complex structure, namely, rotation by $90°$ in the anticlockwise direction with respect to a fixed orientation on NM. Let $N_{\boldsymbol{C}}M=N_{\boldsymbol{C}}M^{1,0}\oplus N_{\boldsymbol{C}}M^{0,1}$ be the splitting of the complexified normal bundle into $(1, 0)$ and $(0, 1)$ subbundles with respect to the complex structure just mentioned.

Throughout the proof of Theorem 2 below we shall use the following notations: for any vector $v\in\boldsymbol{C}^4$, let v^T, v^N, $v^{1,0}$ and $v^{0,1}$ denote the orthogonal projection of v onto $T_{\boldsymbol{C}}M$, $N_{\boldsymbol{C}}M$, $N_{\boldsymbol{C}}M^{1,0}$ and $N_{\boldsymbol{C}}M^{0,1}$ respectively. We let D denote the normal connection and z will be a local complex coordinate on M.

Proof of Theorem 2. Since the Gauss map of M is degenerate, there exists a nonzero fixed vector $A\in\boldsymbol{C}^4$ such that $A\cdot F_z\equiv 0$, where F denotes the immersion of M into $\boldsymbol{R}^4$. We note that under the usual action of $O(n)$, the real orthogonal group, any nonzero vector in $\boldsymbol{C}^n$ is equivalent, up to a constant multiple, to $(\alpha, i, 0, \cdots, 0)$, $0\leq\alpha\leq 1$ (see [6, Proposition 2.4, p. 28]). Therefore we may assume, without loss of generality, that $A=(\alpha, i, 0, 0)$, $0\leq\alpha\leq 1$. If $\alpha=1$ then the Gauss image of M lies in a projective line of Q_2. Hence, by a proposition of Lawson [9, Proposition 16, p. 165], M is a holomorphic curve with respect to some orthogonal complex structure on $\boldsymbol{R}^4$.

Nest we consider the case $0\leq\alpha<1$. Since M has finite index, there exists a compact set C in M such that $M-C$ is stable (see §1). Thus the stability inequality

$$\int_M |(\partial\sigma)^T|^2 \leq \int_M |(\bar{\partial}\sigma)^N|^2 \tag{3.1}$$

holds for any section σ of $N_{\boldsymbol{C}}M$ with compact support in $M-C$, where $\partial\sigma=(\partial_z\sigma)dz$ and $\bar{\partial}\sigma=(\partial_{\bar{z}}\sigma)d\bar{z}$ (see Micallef [11, pp. 60–61]). We now set $\sigma=fs$ in (3.1), where s is the section of $N_{\boldsymbol{C}}M^{1,0}$ defined by $s=A^{1,0}$ and f is an arbitrary smooth real valued function with compact support in $M-C$. Since $(\partial_{\bar{z}}s)^N=D_{\bar{z}}s=0$ (see [11, Claim(i) in the proof of Theorem III, p. 77]), we obtain

$$\int_M f^2|(\partial s)^T|^2 \leq \int_M |\bar{\partial}f|^2|s|^2 = \frac{1}{2}\int_M |df|^2|s|^2 . \tag{3.2}$$

Some parts of the following computation also can be found in [11]. We shall, however, repeat them for completeness. Using the Leibniz rule and the minimality of M, that is, $F_{z\bar{z}}=0$, we obtain

$$\begin{aligned}(\partial_z s)^T &= \frac{1}{|F_z|^2}(\partial_z s\cdot F_{\bar{z}})\,F_z + \frac{1}{|F_z|^2}(\partial_z s\cdot F_z)\,F_{\bar{z}} \\ &= -\frac{1}{|F_z|^2}(s\cdot F_{zz})\,F_{\bar{z}} .\end{aligned}$$

Hence

$$|(\partial s)^T|^2 = \frac{|s\cdot F_{zz}|^2}{|F_z|^4} = \frac{|A\cdot\bar{\varepsilon}|^2|\varepsilon\cdot F_{zz}|^2}{|F_z|^4} , \tag{3.3}$$

where ε is a local unit section of $N_{\boldsymbol{C}}M^{1,0}$. Since $A\cdot F_z\equiv 0$, we have $A\cdot F_{zz}\equiv 0$. But

$$F_{zz} = (F_{zz})^N + \frac{1}{|F_z|^2}(F_{zz}\cdot F_{\bar{z}})\,F_z$$

and so, $A\cdot(F_{zz})_N\equiv 0$, that is,

$$(A\cdot\bar{\varepsilon})\,(\varepsilon\cdot F_{zz}) + (A\cdot\varepsilon)\,(\bar{\varepsilon}\cdot F_{zz}) \equiv 0 . \tag{3.4}$$

We also have

$$A=\frac{1}{|F_z|^2}(A\cdot F_{\bar z})F_z+(A\cdot\bar\varepsilon)\varepsilon+(A\cdot\varepsilon)\bar\varepsilon\,.$$

Therefore

$$|A\cdot\bar\varepsilon|^2+|A\cdot\varepsilon|^2\le|A|^2 \tag{3.5}$$

and

$$2|A\cdot\bar\varepsilon|\,|A\cdot\varepsilon|=|A\cdot A|=1-\alpha^2>0\,. \tag{3.6}$$

In particular, $|A\cdot\bar\varepsilon|$ and $|A\cdot\varepsilon|$ never vanish. We now obtain, by (3.4),

$$\begin{aligned}|(F_{zz})^N|^2&=|F_{zz}\cdot\varepsilon|^2+|F_{zz}\cdot\bar\varepsilon|^2\\&=|F_{zz}\cdot\varepsilon|^2\left(1+\frac{|A\cdot\bar\varepsilon|^2}{|A\cdot\varepsilon|^2}\right).\end{aligned} \tag{3.7}$$

Finally, the Gauss equation can be written as

$$-K=\frac{|(F_{zz})^N|^2}{|F_z|^4}\,, \tag{3.8}$$

where K is the Gauss curvature of M. From (3.3), (3.7), (3.8) (3.5) and (3.6) it follows that

$$\begin{aligned}|(\partial s)^T|^2&=\frac{|A\cdot\bar\varepsilon|^2\,|A\cdot\varepsilon|^2}{|A\cdot\bar\varepsilon|^2+|A\cdot\varepsilon|^2}(-K)\\&\ge\frac{|A\cdot A|^2}{4|A|^2}(-K)\,.\end{aligned} \tag{3.9}$$

On the other hand, we have clearly

$$|s|^2\le|A|^?\,. \tag{3.10}$$

Using (3.9) and (3.10) in (3.2) yields

$$\frac{|A\cdot A|^2}{2|A|^4}\int_M f^2(-K)\le\int_M|df|^2\,. \tag{3.11}$$

M is of finite type, $-K\ge0$ and (3.11) holds for any smooth function f with compact support in $M-C$. Following the argument of Fischer-Colbrie [3, the proof of Corollary 1, p. 129], we can now conclude that M has finite total curvature. The proof of Theorem 2 is complete.

In the rest of this section we study the index of entire 2-dimensional minimal graphs in $\boldsymbol{R}^4$. Let M be a complete minimal surface in $\boldsymbol{R}^4$ which is a graph defined over the whole (x_1, x_2)-plane. We recall here a result of Osserman [12, pp. 37–42] which states that such M is conformally equivalent to the complex plane, hence of finite type. Moreover, M is of either of the following two types:

Type 1. M is a graph of a function $f(x_1, x_2)=(f_1(x_1, x_2), f_2(x_1, x_2))$, where f_1+if_2 is a holomorphic or antiholomorphic function of x_1+ix_2. In this case M is clearly a holomorphic curve with respect to some orthogonal complex structure on $\boldsymbol{R}^4$.

Type 2. M is given, up to translation, by a conformal minimal embedding

$$F(z) = \mathrm{Re}\int^z \left(1, c, \frac{1}{2}(e^{H(\zeta)}-de^{-H(\zeta)}), \frac{i}{2}(e^{H(\zeta)}+de^{-H(\zeta)})\right) d\zeta, \ z\in \boldsymbol{C}\,, \tag{3.12}$$

of the complex plane, where $d=1+c^2$, c is an arbitrary complex constant satisfying only $c\neq\pm i$, $\mathrm{Im}\, c\neq 0$ and H is an arbitrary nonconstant holomorphic function on $\boldsymbol{C}$. In this case M is not a holomorphic curve with respect to any orthogonal complex structure on $\boldsymbol{R}^4$.

Corollary. *Let M be a complete minimal surface in $\boldsymbol{R}^4$ which is a graph defined over the whole plane $\boldsymbol{R}^2$. If M is of Type 1, then M is stable, that is,* $\mathrm{Ind}(M)=0$. *If M is of Type 2, then* $\mathrm{Ind}(M)$ *is infinite.*

REMARK. Micallef [11, Corollary 5.1, p. 68] and Kawai [7] have proved independently that entire 2-dimensional minimal graphs in $\boldsymbol{R}^4$ of Type 2 are unstable.

Proof. The first assertion is well-known. To prove the second assertion we first note that the Gauss map of M is degenerate. In fact, $A\cdot F_z\equiv 0$ with $A=(c, -1, 0, 0)$, where F and c are as in (3.12). Moreover, it is easy to see from (3.12) that the Gauss map does not extend to $S^2=\boldsymbol{C}\cup\{\infty\}$ as a holomorphic map. Hence, by a thoerem of Chern and Osserman [2], the total curvature of M is infinite. The second assertion now follows as an immediate consequence of Theorem 2. This completes the proof.

4. Examples of complete minimal surfaces in $\boldsymbol{R}^4$ with index one

Let M be a complete oriented minimal surface in $\boldsymbol{R}^n$ with finite total curvature. We use the same notations as in §2. We recall that, given a conformal metric on the compactified surface $\tilde{M}$, we have the operator $\tilde{L}$ acting on sections of the extended normal bundle ν. First we study the dimension of $\mathrm{Ker}(\tilde{L})$, the kernel of $\tilde{L}$. It is worth noting that by Remark 2 in §2, $\mathrm{Ker}(\tilde{L})$ is independent of the particular choice of a metric on $\tilde{M}$. We begin with typical examples.

EXAMPLE 1. Let M be the plane $\boldsymbol{R}^2=\boldsymbol{R}^2\times\{0\}\subset\boldsymbol{R}^n$. Then we have $\dim \mathrm{Ker}(\tilde{L})=n-2$. In fact, $e_3^N, \cdots, e_n^N$ form a basis of $\mathrm{Ker}(\tilde{L})$, where $\{e_1, \cdots, e_n\}$ is the standard basis of $\boldsymbol{R}^n$ and for any vector $v\in\boldsymbol{R}^n$, v^N denotes the orthogonal projection of v onto ν.

Example 2. Let M be the catenoid or the Enneper's surface, which lies in a 3-dimensional affine subspace of $\boldsymbol{R}^n$. Then we have dim $\mathrm{Ker}(\tilde{L})=n$. In fact, $e_1^N, \cdots, e_n^N$ form a basis of $\mathrm{Ker}(\tilde{L})$.

In general we have the following

Proposition. *Let M be a complete oriented minimal surface in $\boldsymbol{R}^n$ with finite total curvature. Then* dim *Ker*$(\tilde{L})$ *is at least* $n-2$. *Moreover, if* dim Ker $(\tilde{L})$ *is less than n, then M is a plane.*

Proof. For any vector $v\in\boldsymbol{R}^n$, we have $Lv^N=0$ on M (see Simons [13, Corollary 3.3.1, p. 74]). By (2.1) in the proof of Theorem 1 we have $\tilde{L}v^N=0$ on $\tilde{M}$, since v^N is defined over the whole of $\tilde{M}$. We define a vector subspace V of $\mathrm{Ker}(\tilde{L})$ by $V=\{v^N\,|\,v\in\boldsymbol{R}^n\}$. To prove the first assertion it is sufficient to prove that dim $V\geq n-2$. Let p be a point in $\tilde{M}$ and $\{v_1, \cdots, v_{n-2}\}$ a basis of ν_p, the fiber of ν over p. Clearly $v_1^N, \cdots, v_{n-2}^N$ are linearly independent over $\boldsymbol{R}$, where we consider $v_1, \cdots, v_{n-2}$ as vectors in $\boldsymbol{R}^n$. Therefore we obtain dim $V\geq n-2$ as desired.

Next we prove the second assertion. By the assumption we have dim $V<n$. Hence there exists a unit vector $v\in\boldsymbol{R}^n$ such that $v^N\equiv 0$. This means that, when considered as a constant vector field along M, v is everywhere tangent to M. We fix an arbitrary point $p\in M$. Let $\{e_1, e_2\}$ be an orthonormal tangnet frame at p with respect to the original metric on M such that $e_1=v$. For any tangnet vector u at p we obtain

$$B(u, e_1) = (d_u\, v^T)^N = (d_u\, v)^N = 0\,,$$

where v^T denotes the orthogonal porjection of v onto TM. In particular, we have that $B(e_1, e_1)=0$ and $B(e_1, e_2)=B(e_2, e_1)=0$. By the minimality of M we also have that $B(e_1, e_1)+B(e_2, e_2)=0$, and therefore that $B(e_2, e_2)=0$. We have thus proved that B vanishes at p. Since p is arbitrary, B vanishes identically. M must therefore be a plane. This completes the proof.

We can now give examples of complete oriented minimal surfaces with index one which lie fully in $\boldsymbol{R}^4$. We first recall that the catenoid and the Enneper's surface in $\boldsymbol{R}^3$ have index one (see Fischer-Colbrie [3, p. 131]). It is not difficult to verify that when they are considered as minimal surfaces in $\boldsymbol{R}^4$ via the inclusion $\boldsymbol{R}^3=\boldsymbol{R}^3\times\{0\}\subset\boldsymbol{R}^4$, they still have index one.

EXAMPLE 3. For each α, $0\leq\alpha<1$, we define a conformal immersion F_α: $\boldsymbol{C}\to\boldsymbol{R}^4$ by

$$F_\alpha(z) = \mathrm{Re}\Big(z^2, i\alpha z^2, (1+\alpha)\,z-\frac{1-\alpha}{3}\,z^3, i\,\{(1+\alpha)\,z+\frac{1-\alpha}{3}\,z^3\}\Big)\,.$$

It is easy to verify that each F_α defines a complete oriented simply-connected minimal surface in $\boldsymbol{R}^4$ with total curvature 4π. Thus we obtain a smooth one-parameter family of such surfaces, which will be denoted by $\{M_\alpha\}_{0\leq\alpha<1}$. We note that M_0 is the Enneper's surface and that if $\alpha\neq 0$, M_α lies fully in $\boldsymbol{R}^4$. Moreover, if $\alpha\neq\alpha'$, then M_α and $M_{\alpha'}$ are not congruent to each other (see [6, p. 64]). In what follows we show that $\mathrm{Ind}(M_\alpha)=1$ *for sufficiently small* α.

As we observed in §2, the tangent and normal bundles of each M_α extend with their fiber metrics to the vector bundles τ_α and ν_α on $S^2=\boldsymbol{C}\cup\{\infty\}$ respectively. Thus we obtain two smooth families of vector bundles $\{\tau_\alpha\}_{0\leq\alpha<1}$ and $\{\nu_\alpha\}_{0\leq\alpha<1}$ on S^2. We now fix an arbitrary metric on S^2. Then for each α we can construct the differential operator $\tilde{L}_\alpha$ acting on sections of ν_α as in §2. We note that $\{\tilde{L}_\alpha\}_{0\leq\alpha<1}$ is a smooth family of differential operators (for the definition, see [8, Definition 7.5, p. 325]). Hence for each positive integer k, the k-th eigenvalue $\lambda_k(\alpha)$ of $\tilde{L}_\alpha$ varies continuously in α (see [8, Theorem 7.2, p. 326]). From Remark 1 in §2 and Example 2, we know that

$$\lambda_1(0)<0,\ \lambda_2(0)=\cdots=\lambda_5(0)=0,\ \lambda_6(0)>0\,.$$

On the other hand, the above proposition shows that the multiplicity of 0-eigenvalue of each $\tilde{L}_\alpha$ is not less than four. It follows therefore that

$$\lambda_1(\alpha)<0,\ \lambda_2(\alpha)=\cdots=\lambda_5(\alpha)=0,\ \lambda_6(\alpha)>0$$

for sufficiently small α. Thus, by Remark 1 in §2, we have that $\mathrm{Ind}(M_\alpha)=1$ for sufficiently small α.

EXAMPLE 4. For each α, $0\leq\alpha<1$, we define a conformal immersion F_α: $\boldsymbol{C}-\{0\}\rightarrow\boldsymbol{R}^4$ by

$$F_\alpha(z)=\mathrm{Re}\left(z+\frac{1}{z},\ i\alpha\left(z+\frac{1}{z}\right),\ 2\sqrt{1-\alpha^2}\log z,\ i\sqrt{1-\alpha^2}\left(z-\frac{1}{z}\right)\right).$$

Thus we obtain a smooth one-parameter family of complete oriented doubly-connected minimal surfaces in $\boldsymbol{R}^4$ with total curvature 4π, which will also be denoted by $\{M_\alpha\}_{0\leq\alpha<1}$. We note that M_0 is the catenoid and that if $\alpha\neq 0$, M_α lies fully in $\boldsymbol{R}^4$. Moreover, if $\alpha\neq\alpha'$, then M_α and $M_{\alpha'}$ are not congruent to each other. By the same argument as in Example 3, we have that $\mathrm{Ind}(M_\alpha)=1$ for sufficiently small α.

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