Miyadera, R.
Osaka J. Math.
27 (1990), 381–412

CHARACTERIZATIONS OF CONDITIONAL EXPECTATION OPERATORS FOR L_p-VALUED FUNCTIONS ON A GENERAL MEASURE SPACE

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(Received July 24, 1989)

Introduction. Let $(\Omega, \boldsymbol{A}, \mu)$ be a measure space, where $\boldsymbol{A}$ is a σ-ring and μ is a σ-finite measure on $\boldsymbol{A}$, (X, S, λ) a measure space and E a real Banach space. We consider semi-constant-preserving contractive projections of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself. If $(\Omega, \boldsymbol{A}, \mu)$ is a probability space and E is a strictly-convex Banach space, then Landers and Rogge [2] proved that such operators coincide precisely with the conditional expectation operators. If $(\Omega, \boldsymbol{A}, \mu)$ is a probability space and $E=L_p(X, S, \lambda)$, where $p=1$ or ∞, then Miyadera [3] and [4] proved that such operators coincide precisely with the conditional expectation operators under some additional conditions. In this paper we deal with the case when $(\Omega, \boldsymbol{A}, \mu)$ is a general measure space, where $\boldsymbol{A}$ is a σ-ring and λ is a σ-finite measure on $\boldsymbol{A}$. Substituting constant-preserving property by semi-constant-preserving property we can prove theorems which are generalizations of characterization theorems in Landers and Rogge [2], Miyadera [3] and [4].

1. Definitions and useful Lemmas. Let $(\Omega, \boldsymbol{A}, \mu)$ be a measure space, $\boldsymbol{A}(\mu)=\{A\in\boldsymbol{A};\ \mu(A)<\infty\}$ and E *a* real Banach space with the norm $\|\ \|$. Note that E can be the class $\boldsymbol{R}$ of real numbers. Let $\boldsymbol{N}$ be the class of natural numbers. For any $A, B\in\boldsymbol{A}$ we write $A\subset B$ if $\mu(A-B)=0$ aud $A=B$ if $\mu((A-B)\cup(B-A))=0$. $A, B\in\boldsymbol{A}$ are said to be disjoint if $\mu(A\cap B)=0$. We suppose that μ is σ-finite, i.e., for any $A\in\boldsymbol{A}$ there exists a sequence of sets $\{A_n;\ n\in\boldsymbol{N}\}$ such that $A_n\in\boldsymbol{A}(\mu)$ and $A=\cup\{A_n;\ n\in\boldsymbol{N}\}$. For any $A\in\boldsymbol{A}$ we denote by I_A the indicator function of A and by $A=\emptyset$ we mean $\mu(A)=0$. Let $L_1(\Omega, \boldsymbol{A}, \mu, E)$ be the calss of E-valued Bochner integrable functions, which is a Banach space with the norm $\|\ \|_L$ defined by

$$\|f\|_L=\int\|f(\omega)\|d\mu \qquad \text{for any} \quad f\in L_1(\Omega, \boldsymbol{A}, \mu, E)\,.$$

For any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ we denote $\{\omega;\ f(\omega)\neq 0\}$ by $s(f)$ and for any linear operator Q of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself we denote $S(Q)=\{A\in\boldsymbol{A}(\mu);$ there eixsts $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that $A \subset s(Q(f))\}$. For the definitions and properties of Bochner integral, see Hille and Phillips [1].

DEFINTION 1. Let $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$. For a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$, a function g is called the conditional expectation of f given $\boldsymbol{B}$ if $g \in L_1(\Omega, \boldsymbol{B}, \mu, E)$, and

$$\int_B g d\mu = \int_B f d\mu \qquad \text{for any} \quad B \in \boldsymbol{B},$$

where the integral is the Bochner integral. We denote by $f^{\boldsymbol{B}}$ the conditional expectation of f given $\boldsymbol{B}$. For any $\phi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ we define $\phi a \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ by $(\phi a)(\omega) = \phi(\omega)a$ for any $\omega \in \Omega$ and $a \in E$. Then it is clear that $(\phi a)^{\boldsymbol{B}} = \phi^{\boldsymbol{B}} a$.

DEFINITION 2. Let P be a linear operator of $L_1(\Omega\ \boldsymbol{A}, \mu, E)$ into itself. P is said to be *contractive* if

$$||P|| = \sup\{||P(f)||_L ; f \in L_1(\Omega, \boldsymbol{A}, \mu, E) \quad \text{and} \quad ||f||_L = 1\} \leqq 1,$$

semi-constant-preserving if for any $a \in E$, $\varepsilon > 0$, $A \in s(P)$ there exists $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$||I_A P(f) - I_A a||_L < \varepsilon,$$

and *a projection* if $P \circ P = P$, where $(P \circ P)(f) = P(P(f))$ for any $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$.

In this paper an operator P is said to satisfy Assumption 1 if
(1) P is a semi-constant-preserving contractive projection of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself.

Lemma 1.1. *Let $\boldsymbol{B}$ be a σ-subring of $\boldsymbol{A}$. Then for any $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ the conditional expectation $f^{\boldsymbol{B}}$ of f given $\boldsymbol{B}$ exists uniquely up to almost everywhere and the conditional expectation operator $(\ )^{\boldsymbol{B}}$ satisfies Assumption* 1.

Proof. Let $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$. If there exists $B \in \boldsymbol{B}$ such that $s(f) \subset B$, then by a theorem in Schwartz [5] $f^{\boldsymbol{B}}$ exists uniquely up to almost everywhere and $||f^{\boldsymbol{B}}||_L \leqq ||f||_L$ and $(f^{\boldsymbol{B}})^{\boldsymbol{B}} = f^{\boldsymbol{B}}$. For an arbitray $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ there exists $C \in \boldsymbol{B}$ such that

$$\int_C ||f|| d\mu = \sup\{\int_B ||f|| d\mu ; B \in \boldsymbol{B}\}.$$

Clearly $(I_{B-C} f)(\omega) = 0$ (*a.e.* ω) for any $B \in \boldsymbol{B}$. Since $s(I_C f) \subset C$, there exists $(I_C f)^{\boldsymbol{B}}$. For any $B \in \boldsymbol{B}$

$$\int_B f d\mu = \int_B I_C f d\mu + \int_{B-C} f d\mu = \int_B I_C f d\mu = \int_B (I_C f)^{\boldsymbol{B}} d\mu.$$

Therefore $(I_C f)^{\boldsymbol{B}} = f^{\boldsymbol{B}}$. The uniqueness of $f^{\boldsymbol{B}}$ is obvious from the properties of $(I_C f)^{\boldsymbol{B}}$.

$$\int \|f\| d\mu \geqq \int \|I_C f\| d\mu \geqq \int \|(I_C f)^B d\mu\| = \int \|f^B\| d\mu ,$$

and hence $(\ \ )^B$ is contractive. Since $s(f) \subset C$, $(\ \ )^B$ is a projection. Next we are going to prove that $(\ \ )^B$ is semi-constant-preserving. Suppose that there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $A \in \boldsymbol{A}(\mu)$ such that $A \subset s((f)^B)$. Let $a \in E$. Write

$$B_n = \{\omega ; \|f^B(\omega)\| > 1/n\} ,$$

then

$$s(f^B) = \cup \{B_n ; n \in \boldsymbol{N}\} .$$

For any positive number ε there exists $n \in \boldsymbol{N}$ such that

$$\|a\| \mu(A - B_n) < \varepsilon .$$

Then

$$\|I_A(I_{B_n} a)^B - I_A a\|_L = I\|_{B_n \cap A} a - I_A a\|_L = \|a\| \mu(A - B_n) < \varepsilon .$$

We have proved that $(\ \ )^B$ is semi-constant-preserving. Q.E.D.

Lemma 1.2. *Suppose that P is a contractive projection of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into istelf and $0 \leqq P(I_A)(\omega) \leqq 1$ (a.e.ω) for any $A \in \boldsymbol{A}(\mu)$. Then there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $P = (\ \ )^B$.*

For the proof see Wulbert [6].

Lemma 1.3. *Suppose that P is a contractive projection of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself. Then P is semi-constant-preserving and $\Omega \in s(P)$ iff P is constant-preserving in the sense used in* [2], [3] *and* [4], *i.e., $P(I_\Omega a) = I_\Omega a$ for any $a \in E$.*

Proof. First we suppose that $P(I_\Omega a) = I_\Omega a$ for any $a \in E$. It is clear that $\Omega \in s(P)$. For any $A \in s(P)$

$$\|I_A P(I_\Omega a) - I_A a\|_L = \|I_A a - I_A a\|_L = 0 .$$

Therefore P is semi-constant-preservig.

Conversely we suppose that P is semi-constant-preserving and $\Omega \in s(P)$. For any $n \in \boldsymbol{N}$ there exists $f_n \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$(2) \qquad \|P(f_n) - I_\Omega a\|_L < 1/n .$$

Since P is contractive,

$$\|P(f_n) - P(I_\Omega a)\|_L < 1/n ,$$

and hence by (2) and arbitrariness of n

$$P(I_\Omega a) = I_\Omega a .$$

Q.E.D.

In the remainder of this section we assume that Q satisfies Assumption 1.

Lemma 1.4. *Let K, $A\in \boldsymbol{A}(\mu)$, $K\cup A\in s(Q)$ and $a\in E$. Then*

$$||a-Q(I_A a)(\omega)||=||a||-||Q(I_A a)(\omega)|| \qquad (a.e.\omega) \text{ on } K.$$

Proof. Since $K\cup A\in s(Q)$ and Q is semi-constant-preserving, for any $\varepsilon>0$ there exists $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$(4) \qquad ||I_{A\cup K}Q(f)-I_{A\cup K}a||_L<\varepsilon .$$

Since Q is a contractive projection, by using (4) twice we have

$$\begin{aligned}
&||Q(f)-Q(I_A a)||_L\leqq||Q(f)-I_A a||_L\\
&\leqq||I_A Q(f)-I_A a||_L+||I_{\Omega-A}Q(f)||_L\\
&\leqq\varepsilon+||I_{\Omega-A}Q(f)||_L\\
&\leqq\varepsilon+||I_A Q(f)-I_A a||_L+||I_A Q(f)||_L-||I_A a||_L+||I_{\Omega-A}Q(f)||_L\\
&\leqq 2\varepsilon+||I_A Q(f)||_L-||I_A a||_L+||I_{\Omega-A}Q(f)||_L\\
&=2\varepsilon+||Q(f)||_L-||I_A a||_L\\
&\leqq 2\varepsilon+||Q(f)||_L-||Q(I_A a)||_L .
\end{aligned}$$

Therefore

$$(5) \qquad ||Q(f)-Q(I_A a)||_L\leqq 2\varepsilon+||Q(f)||_L-||Q(I_A a)||_L .$$

Since

$$||I_{\Omega-K}Q(f)-I_{\Omega-K}Q(I_A a)||_L\geqq||I_{\Omega-K}Q(f)||_L-||I_{\Omega-K}Q(I_A a)||_L ,$$

by (5) we get

$$(6) \qquad ||I_K Q(f)-I_K Q(I_A a)||_L\leqq 2\varepsilon+||I_K Q(f)||_L-||I_K Q(I_A a)||_L .$$

From (4) and (6) we get

$$||I_K a-I_K Q(I_A a)||_L\leqq 4\varepsilon+||I_K a||_L-||I_K Q(I_A a)||_L .$$

Since ε is an arbitrary positive number,

$$||I_K a-I_K Q(I_A a)||_L = ||I_K a||_L-||I_K Q(I_A a)||_L .$$

Therefore

$$||a-Q(I_A a)(\omega)|| = ||a||-||Q(I_A a)(\omega)|| \qquad (a.e.\omega) \text{ on } K.$$

Q.E.D.

Lemma 1.5. *Let $A\in s(Q)$ and $a\in E$. Then for any positive number ε there exist $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B\in s(Q)$ such that*

$$B\subset s(Q(f)),$$
$$||I_A a-I_B a||_L<\varepsilon ,$$
$$||I_{s(Q(f))}Q(I_B a)-Q(I_A a)||_L<\varepsilon ,$$
$$||I_{\Omega-s(Q(f))}Q(I_B a)||_L<\varepsilon ,$$

and

$$||a-Q(I_B a)(\omega)|| = ||a||-||Q(I_B a)(\omega)|| \qquad (a.e.\omega) \text{ on } s(Q(f)).$$

Proof. For any $\varepsilon>0$ we can choose a positive number δ such that $4\delta<\varepsilon$. Since Q is semi-constant-preserving, there exists $f\in L_1(\Omega, A, \mu, E)$ such that

$$(7) \qquad ||I_A Q(f)-I_A a||_L<\delta .$$

Write $B=A\cap s(Q(f))$. Therefore

$$(8) \qquad ||I_A a-I_B a||_L = ||I_A a-I_{A\cap s(Q(f))} a||_L$$
$$= ||I_{A-s(Q(g))} a||_L = ||I_{\Omega-s(Q(f))}(I_A Q(f)-I_A a)||_L<\delta<\varepsilon .$$

Since Q is contractive, by (8) and the triangle inequality

$$\begin{aligned}
&||I_{s(Q(f))}Q(I_B a)-Q(I_A a)||_L\\
&\leqq||I_{s(Q(f))}Q(I_B a)-I_{s(Q(f))}Q(I_A a)||_L+||I_{\Omega-s(Q(f))}Q(I_A a)||_L\\
&\leqq||I_B a-I_A a||_L+||I_{\Omega-s(Q(f))}Q(I_A a)||_L\\
&<\delta+||I_{\Omega-s(Q(f))}Q(I_A a)||_L\\
&=\delta+||I_{\Omega-s(Q(f))}Q(I_A a)-Q(f)||_L-||Q(f)||_L ,
\end{aligned}$$

where the last equality comes from the fact that

$$||I_{\Omega-s(Q(f))}Q(I_A a)-Q(f)||_L = ||I_{\Omega-s(Q(f))}Q(I_A a)||_L+||Q(f)||_L .$$

By the triangle inequality and the fact that Q is contractive,

$$\begin{aligned}
&\delta+||I_{\Omega-s(Q(f))}Q(I_A a)-Q(f)||_L-||Q(f)||_L\\
&\leqq\delta+||I_{\Omega-s(Q(f))}Q(I_A a)-Q(f)+I_{s(Q(f))}Q(I_A a)||_L+||I_{s(Q(f))}Q(I_A a)||_L-||Q(f)||_L\\
&\leqq\delta+||Q(I_A a)-Q(f)||_L+||I_{s(Q(f))}Q(I_A a)||_L-||Q(f)||_L\\
&\leqq\delta+||I_A a-Q(f)||_L+||I_A a||_L-||Q(f)||_L .
\end{aligned}$$

By (7)

$$\begin{aligned}
&\delta+||I_A a-Q(f)||_L+||I_A a||_L-||Q(f)||_L\\
&\leqq3\delta+||I_A Q(f)-Q(f)||_L+||I_A Q(f)||_L-||Q(f)||_L=3\delta<\varepsilon .
\end{aligned}$$

We have proved that

$$||I_{s(Q(f)}Q(I_B a)-Q(I_A a)||_L<3\delta<\varepsilon ,$$

and hence by (8)

$$\|I_{\Omega-s(Q(f))}Q(I_B a)\|_L = \|Q(I_B a)-I_{s(Q(f))}Q(I_B a)\|_L$$
$$\leqq \|Q(I_B a)-Q(I_A a)\|_L+\|Q(I_A a)-I_{s(Q(f))}Q(I_B a)\|_L$$
$$\leqq \|I_B a-I_A a\|_L+3\delta<\delta+3\delta<\varepsilon\,.$$

There exists a sequence $\{K_n;\, n\in \boldsymbol{N}\}$ such that $K_n\in \boldsymbol{A}(\mu)$ and $s(Q(f))=\cup\{K_n;\, n\in \boldsymbol{N}\}$. Since $B\cup K_n\in s(Q)$ for any $n\in \boldsymbol{N}$, by Lemma 1.4

$$\|a-Q(I_B a)(\omega) = \|a\|-\|Q(I_B a)(\omega)\| \qquad (a.e.\omega) \text{ on } K_n\,.$$

Therefore

$$\|a-Q(I_B a)(\omega)\| = \|a\|-\|Q(I_B a)(\omega)\| \qquad (a.e.\omega) \text{ on } s(Q(f))\,.$$

Q.E.D.

For any $A\in \boldsymbol{A}(\mu)$ let

$$k=\sup\{\mu(C);\, C\in \boldsymbol{A},\, C\subset A \text{ and } \mu(C\cap D)=0 \quad \text{for any} \quad D\in s(Q)\}\,.$$

Then there exists $E\in \boldsymbol{A}$ such that $E\subset A$, $\mu(E\cap D)=0$ for any $D\in s(Q)$ and $\mu(E)=k$. We write $N_Q(A)=E$. Clearly for any $A\in \boldsymbol{A}$ $N_Q(A)$ is unique up to sets of measure zero. When just one operator Q is under discussion, we omit the letter Q from symbols and write N instead of N_Q.

Lemma 1.6. *Let $A_n, B_m\in A(\mu)$ for any $n, m\in \boldsymbol{N}$ and $\cup\{A_n;\, n\in \boldsymbol{N}\}\subset \cup\{B_m;\, m\in \boldsymbol{N}\}$. Then $\cup\{N(A_n);\, n\in \boldsymbol{N}\}\subset \cup\{N(B_m);\, m\in \boldsymbol{N}\}$.*

Proof. For any $n, m\in \boldsymbol{N}$ $N(A_n)\cap B_m\in \boldsymbol{A}(\mu)$, $N(A_n)\cap B_m\subset B_m$ and $(N(A_n)\cap B_m)\cap D=\emptyset$ for any $D\in s(Q)$, and hence $N(A_n)\cap B_m\subset N(B_m)$. Therefore

$$\cup\{N(A_n);\, n\in \boldsymbol{N}\} = \cup\{N(A_n)\cap B_m;\, n, m\in \boldsymbol{N}\}\subset \cup\{N(B_m);\, m\in \boldsymbol{N}\}\,.$$

Q.E.D.

We can define $N(A)$ for any $A\in \boldsymbol{A}$, even if $\mu(A)=\infty$. Let $A_n\in \boldsymbol{A}(\mu)$ such that $A=\cup\{A_n;\, n\in \boldsymbol{N}\}$ and let $N(A)=\cup\{N(A_n);\, n\in \boldsymbol{N}\}$. By Lemma 1.6 $N(A)$ is independent of the choice of the sequence $\{A_n;\, n\in \boldsymbol{N}\}$. For any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ let $N(f)=I_{N(s(f))}f$, then N is a mapping of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself.

Lemma 1.7. *Let $A, B\in \boldsymbol{A}$ with $A\subset B$ and $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$. Then $N(A)=N(B)\cap A$, $N(A)\subset N(B)$, $N(N(A))=N(A)$ and $N(s(f))=s(N(f))$.*

Proof. We can choose sequences $\{A_n;\, n\in \boldsymbol{N}\}$ and $\{C_m;\, m\in \boldsymbol{N}\}$ such that $A_n, C_m\in \boldsymbol{A}(\mu)$ for any $n, m\in \boldsymbol{N}$ and $A=\cup\{A_n;\, n\in \boldsymbol{N}\}$ and $B-A=\cup\{C_m;\, m\in \boldsymbol{N}\}$. By the definition of N we have $N(B)\cap A=(\cup\{N(A_n)\cup N(C_m);\, n, m\in \boldsymbol{N}\})\cap A=\cup\{N(A_n);\, n\in \boldsymbol{N}\}=N(A)$, and hence $N(A)\subset N(B)$. Since $N(A)\subset A$, $N(N(A))=N(A)\cap N(A)=N(A)$. $N(f)=I_{N(s(f))}f$, and hence $s(N(f))=N(s(f))$. Q.E.D.

Lemma 1.8. *The family* $\{N(A);\ A\in \boldsymbol{A}\}$ *is a* σ*-subring of* A.

Proof. Let $A, B, A_n \in \boldsymbol{A}$ for any $n\in \boldsymbol{N}$ and let $C=\cup\{A_n;\ n\in\boldsymbol{N}\}\cup A\cup B$. Since $A, B, A-B\subset C$, by Lemma 1.7 $N(A)-N(B)=(A\cap N(C))-(B\cap N(C))=(A-B)\cap N(C)=N(A-B)$. $\cup\{A_n;\ n\in\boldsymbol{N}\}\subset C$, and hence $N(\cup\{A_n;\ n\in\boldsymbol{N}\})=\cup\{A_n;\ n\in\boldsymbol{N}\}\cap N(C)=\cup\{A_n\cap N(C);\ n\in\boldsymbol{N}\}=\cup\{N(A_n);\ n\in\boldsymbol{N}\}$. Q.E.D.

Lemma 1.9. *The operator* N *of* $L_1(\Omega, \boldsymbol{A}, \mu, E)$ *into itself is a contractive projection and* $\|f-N(f)\|_L\leqq\|f\|_L$ *for any* $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$.

Proof. First we will show that N is a linear operator. Since $s(af)=s(f)$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $a\in R$ with $a\neq 0$,

$$N(af)=I_{N(s(af))}af=aI_{N(s(f))}f=aN(f)\,.$$

For any $f, g\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ let $C=s(f)\cup s(g)$. Since $s(f), s(g), s(f+g)\subset C$, by Lemma 1.7 and the definition of N

$$\begin{aligned}N(f+g)&=I_{N(s(f+g))}(f+g)=I_{N(C)\cap s(f+g)}(f+g)=I_{N(C)}(f+g)\\&=I_{N(C)}f+I_{N(C)}g=I_{N(C)\cap s(f)}f+I_{N(C)\cap s(g)}g=N(f)+N(g)\,.\end{aligned}$$

Next we are going to show that N is a contractive projection. By Lemma 1.7

$$s(N(f))=N(s(f))\,. \tag{9}$$

By (9) and Lemma 1.7

$$\begin{aligned}N\circ N(f)&=I_{N(s(N(f)))}N(f)=I_{N(N(s(f)))}N(f)\\&=I_{N(s(f))}N(f)=I_{s(N(f))}N(f)=N(f)\,,\end{aligned}$$

and hence N is a projection.

$$\|N(f)\|_L=\|I_{N(s(f))}f\|_L\leqq\|f\|_L\,,$$

and hence N is contractive.

$$\|f-N(f)\|_L=\|f-I_{N(s(f))}f\|_L\leqq\|f\|_L\,. \qquad \text{Q.E.D.}$$

We define an operator Q^* of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself by $Q^*(f)=(Q-Q\circ N)(f)=Q(f-N(f))$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$. Since N is linear, Q^* is a linear operator.

Let $\boldsymbol{C}$ be a σ-subring of $\boldsymbol{A}$ and $\boldsymbol{P}$ the conditional expectation operator given $\boldsymbol{C}$. For any $A\in\boldsymbol{A}$ and $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ we denote $s(P)$, $N_P(A)$ and $N_P(f)$ by $s((\ )^{\boldsymbol{C}})$, $N_{\boldsymbol{C}}(A)$ and $N_{\boldsymbol{C}}(f)$ respectively. Let $\boldsymbol{A}_{\boldsymbol{C}}=\{N_{\boldsymbol{C}}(A);\ A\in\boldsymbol{A}\}$, then by Lemma 1.8 $\boldsymbol{A}_{\boldsymbol{C}}$ is σ-subring of $\boldsymbol{A}$. Note that for any $D\in\boldsymbol{A}$ we have $D\in s(\boldsymbol{P})$ iff there exists $C\in\boldsymbol{C}$ such that $D\subset C$.

Lemma 1.10. *Let $\boldsymbol{C}$ be a σ-subring of $\boldsymbol{A}$. Then*

$$(\ )^{\boldsymbol{C}}\circ N_{\boldsymbol{C}} = N_{\boldsymbol{C}}\circ(\ )^{\boldsymbol{C}}$$

Proof. Let $P=(\ )^{\boldsymbol{C}}$ and $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$. By the definition of $N_{\boldsymbol{C}}$ for any $A\in\boldsymbol{A}$ and $D\in s((\ ))^{\boldsymbol{C}}=s(P)$ we have $N_{\boldsymbol{C}}(A)\cap D=\emptyset$. $D\in s(P)$ iff there exists $C\in\boldsymbol{C}$ such that $D\subset C$, and hence for any $A\in\boldsymbol{A}$ and $C\in\boldsymbol{C}$

$$N_{\boldsymbol{C}}(A)\cap C=\emptyset\,. \tag{10}$$

$(N_{\boldsymbol{C}}(f))^{\boldsymbol{C}}=(I_{N_{\boldsymbol{C}}(s(f))}f)^{\boldsymbol{C}}=0$, since by (10) $N_{\boldsymbol{C}}(s(f))\cap C=\emptyset$ for any $C\in\boldsymbol{C}$. $s(f^{\boldsymbol{C}})\in\boldsymbol{C}$, and hence by (10) we have

$$N_{\boldsymbol{C}}(s(f^{\boldsymbol{C}}))=N_{\boldsymbol{C}}(s(f^{\boldsymbol{C}}))\cap s(f^{\boldsymbol{C}})=\emptyset\,.$$

Therefore

$$N_{\boldsymbol{C}}(f^{\boldsymbol{C}})=I_{N_{\boldsymbol{C}}(s(f^{\boldsymbol{C}}))}f^{\boldsymbol{C}}=0\,.$$ Q.E.D.

Lemma 1.11. *Operators Q, Q^* and N satisfy the conditions $N\circ Q=Q^*\circ N=0$, $Q^*\circ Q=Q$, $Q^*\circ Q^*=Q^*$ and $s(Q)=s(Q^*)$.*

Proof. By the definition of N we have $\mu(N(s(Q(f)))=0$, and hence

$$N\circ Q(f)=I_{N(s(Q(f)))}Q(f)=0\,. \tag{11}$$

By Lemma 1.9 N is a projection, i.e., $N\circ N=N$, and hence by the definition of Q^*

$$Q^*\circ N=(Q-Q\circ N)\circ N=Q\circ N-Q\circ N\circ N=0\,.$$

By (11)

$$Q^*\circ Q=(Q-Q\circ N)\circ Q=Q\circ Q-Q\circ(N\circ Q)=Q\circ Q=Q\,,$$

and hence

$$Q^*\circ Q^*=Q^*\circ(Q-Q\circ N)=(Q^*\circ Q^*)-(Q^*\circ Q)\circ N=Q-Q\circ N=Q^*\,.$$

By the definition of Q^* for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$

$$Q^*(f)=Q(f-N(f))\,, \tag{12}$$

and by the preceding part of this lemma $Q=Q^*\circ Q$, and hence

$$Q(f)=Q^*\circ Q(f)\,. \tag{13}$$

By (12) and (13) we have $s(Q)=s(Q^*)$. Q.E.D.

Lemma 1.12. *Q^* is semi-constant-preserving contractive projection and $Q(I_A a)=Q^*(I_A a)$ for any $A\in s(Q^*)$ and $a\in E$.*

Proof. Let $a\in E$, $\varepsilon>0$ and $A\in s(Q^*)$. By Lemma 1.11 $A\in s(Q)$, and hence by the fact that Q is semi-constant-preserving we can choose $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$||I_A Q(f)-I_A a||_L<\varepsilon .$$

By Lemma 1.11

$$Q(f) = Q^*\circ Q(f) ,$$

and hence

$$||I_A Q^*\circ Q(f)-I_A a||_L<\varepsilon .$$

Therefore Q^* is semi-constant-preserving. Since $A\in s(Q)$, $N(A)=\emptyset$. Therefore by Lemma 1.9

$$Q^*(I_A a) = Q(I_A a-N(I_A a)) = Q(I_A a) .$$

$||Q^*(f)||_L=||Q(f-N(f))||_L\leqq||f-N(f)||_L\leqq||f||_L$, and hence Q^* is contractive. By Lemma 1.11 $Q^*\circ Q^*=Q^*$, and hence Q^* is a projection. Q.E.D.

Lemma 1.13. *For any $A\in \boldsymbol{A}(\mu)$ there exists a pairwise disjoint sequence $\{A_n\in s(Q); n\in \boldsymbol{N}\}$ such that*

$$A-N(A) = \cup \{A_n; n\in \boldsymbol{N}\} .$$

Proof. Let $k=\sup\{\mu(C); C\in \boldsymbol{A}, C\subset A$ and there exists $C_n\in s(Q)$ for each $n\in \boldsymbol{N}$ such that $C\subset \cup \{C_n; n\in \boldsymbol{N}\}\}$. Then there exist $D\in \boldsymbol{A}$ and $D_n\in s(Q)$ for any $n\in \boldsymbol{N}$ such that $D\subset A$, $D\subset \cup \{D_n; n\in \boldsymbol{N}\}$ and $\mu(D)=k$. By the definition of k we have $\mu((A-D)\cap E)=0$ for any $E\in s(Q)$, and hence by Lemma 1.6 we have $A-D\subset N(A)$. Therefore

$$A-N(A)\subset D\subset \cup \{D_n; n\in \boldsymbol{N}\} .$$

Write $A_n=A\cap(D_n-\cup\{D_i; i\leqq n-1\})$. Since $A_n\in s(Q)$, $\mu(A_n\cap N(A))=0$. Hence the sequence $\{A_n; n\in \boldsymbol{N}\}$ consists of pairwise disjoint elements of $s(Q)$ and

$$A-N(A) = \cup \{A_n; n\in \boldsymbol{N}\} . \qquad \text{Q.E.D.}$$

In the remainder of this paper we assume that (S, X, λ) is a measure space, where S is a σ-ring and λ is a measure on S, and for any $K\in S$ we denote by J_K the indicator function of K. For any $K, H\in S$ we write $K\subset H$ if $\lambda(K-H)=0$, $K=\emptyset$ if $\lambda(K)=0$. K and H are said to be disjoint if $K\cap H=\emptyset$. For any real-valued measurable function $a(x)$, $b(x)$ on X we write $a\leqq b$ if $a(x)\leqq b(x)$ (a.e.x), i.e., $\lambda(\{x; a(x)>b(x)\})=0$ and $a=b$ if $a(x)=b(x)$ (a.e.x).

2. Lemmas for L_p-valued functions, where $1<p<\infty$. Let λ be a σ-finite measure on S. Throughout this section we assume that $E=L_p(X, S, \lambda, R)$ with $1<p<\infty$,

$$||a|| = \left(\int |a(x)|^p d\lambda\right)^{1/p} \qquad \text{for any} \quad a \in E$$

and that Q satisfies Assumption 1. (See (1).)

Lemma 2.1. *If $a, b \in E$ and $||a+b|| = ||a|| + ||b||$, then there exists a real number k such that $a = kb$ or $b = ka$.*

For the proof see Yosida [7] pp. 33 and 34.

Lemma 2.2. *Let $A \in s(Q)$, then there exists $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $Q(I_A a) = \psi a$ for any $a \in E$ and $0 \leqq \psi(\omega) \leqq 1$ (a.e.ω).*

Proof. By Lemma 1.5 for any $n \in \boldsymbol{N}$ there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B \in s(Q)$ such that

$$(14) \qquad ||I_{s(Q(f))} Q(I_B a) - Q(I_A a)||_L < 1/n\,,$$

and

$$||a - Q(I_B a)(\omega)|| = ||a|| - ||Q(I_B a)(\omega)|| \qquad (a.e.\omega) \text{ on } s(Q(f))\,.$$

Therefore by Lemma 2.1 there exists $\psi_n \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$I_{s(Q(f))} Q(I_B a) = \psi_n a$$

and

$$(15) \qquad 0 \leqq \psi_n(\omega) \leqq 1 \qquad (\text{a.e.}\omega)\,,$$

and hence by (14) we have

$$(16) \qquad ||Q(I_A a) - \psi_n a||_L < 1/n\,.$$

Since by (16) ψ_n is a Cauchy sequence, there exists $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$(17) \qquad ||\psi - \psi_n||_L \to 0 \qquad \text{as} \quad n \to \infty\,.$$

By (16) and (17) we have

$$Q(I_A a) = \psi a\,.$$

By (15) $0 \leqq \psi(\omega) \leqq 1$ (a.e.ω). Cleary ψ is independent of the choice of $a \in E$, since Q is a linear operator. Q.E.D.

3. Lemmas for L_1-valued functions. Let S be a σ-algebra and $S(\lambda) = \{K; K \in S \text{ and } \lambda(K) < \infty\}$.

DEFINITION 3. A measure space (X, S, λ) is said to be licalizable if any nonempty collection $\mathcal{V} \subset S(\lambda)$ has $\sup \mathcal{V} \in S$, in the sense that for any $K \in \mathcal{V}$, $\lambda(K - \sup \mathcal{V}) = 0$ and that if $H_1 \in S$ and $\lambda(K - H_1) = 0$ for any $K \in \mathcal{V}$, then

$\lambda(\sup \mathcal{V} - H_1) = 0$.

DEFINITION 4. We say that a measure space (X, S, λ) has the finite subset property if for any $K \in S$ with $\lambda(K) > 0$, there is $H \in S$ such that $H \subset K$ and $0 < \lambda(H) < \infty$.

DEFINITION 5. A class $\{f(x, K); K \in S(\lambda)\}$ of real-valued S-measureable functions on (X, S, λ) is called a cross-section if $f(x, K) = 0$ on K^c and for any $K, H \in S(\lambda)$ $J_{K \cap H}(x) f(x, K) = J_{K \cap H}(x) f(x, H)$ (a.e.x).

Lemma 3.1. *Suppose that a measure space (X, S, λ) is localizable. Then for any corss-section $\{f(x, K); K \in S(\lambda)\}$ there exists a real-valued S-measurable function f such that $J_K(x) f(x) = f(x, K)$ (a.e.x) for any $K \in S(\lambda)$.*

For the proof see Zaanen [8].

DEFINITION 6. Let T be a one-to-one transformation of (X, S, λ) into itself. Then T is called a bounded measurable transformation if T is a measurable transformation and there exists a positive number k such that $\lambda(T^{-1}(A)) \leqq k\lambda(A)$ for any $A \in S$.

DEFINITION 7. Let $\mathfrak{T}$ be a class of bounded measurable transformations T of X onto X such that $T^{-1}(S(\lambda) = S(\lambda)$ for any $T \in \mathfrak{T}$. Then $(X, S, \lambda, \mathfrak{T})$ is said to be ergodic if $A \in S$ and $\lambda(A \Delta T^{-1}(A)) = 0$ for any $T \in \mathfrak{T}$ imply $\lambda(A) = 0$ or $\lambda(A^c) = 0$.

Lemma 3.2. *If $(X, S, \lambda, \mathfrak{T})$ is an ergodic space, then for any bounded measurable function f on X, $f(x) = f(T(x))$ for any $T \in \mathfrak{T}$ imply that $f(x) = const.$*

For the proof see Miyadera [3].

Throughout this section we assume that $(X, S, \lambda, \mathfrak{T})$ is an ergodic localizable measure space with the finite subset property, $E = L_1(X, S, \lambda, R)$ with the norm

$$||a|| = \int |a(x)| d\lambda \qquad \text{for any} \quad a \in E$$

and Q satisfies Assumption 1. Let

$$E^+ = \{a; a \in E \text{ and } a(x) \geqq 0 \text{ (a.e.x.)}\} .$$

For any $a \in E$ we write $0 \leqq a$ if $a \in E^+$. For a real-valued measurable function $a(x)$, it is clear that $a(T(x))$ is also measurable, because of the measurability of T. If, in addition, $a \in E$, then $a(T(x)) \in E$. We shall write $T(a)(x) = a(T(x))$, and remark that T can be regarded as a bounded operator of E into istelf in the sense that there exists a real number k such that $||T(a)|| \leqq k||a||$ for any $a \in E$.

DEFINITION 8. Let Q be a transformation of $L_1(\Omega, A, \mu, E)$ into itself. Then Q is said to be covariant under $\mathfrak{T}$ if $Q(\psi T(a))(\omega) = T(Q(\psi(a)(\omega))$ (a.e.ω) for any $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$, $a \in E$ and $T \in \mathfrak{T}$.

Lenma 3.3. *Let $A \in s(Q)$ and $K \in S(\lambda)$. Then*

$$0 \leqq Q(I_A J_K)(\omega) \leqq J_K \qquad \text{(a.e.}\omega\text{)}.$$

Proof. By Lemma 1.5 for an arbitrary positive real number ε there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B \in s(Q)$ such that

$$\text{(18)} \qquad ||I_{s(Q(f))} Q(I_B J_K) - Q(I_A J_K)||_L < \varepsilon$$

and

$$||J_K - Q(I_B J_K)(\omega)|| = ||J_K|| - ||Q(I_B J_K)(\omega)|| \qquad \text{(a.e.w) on } s(Q(f)).$$

By the definition of the norm $||\quad||$

$$\text{(19)} \qquad \int |J_K - Q(I_B J_K)(\omega)|\, d\lambda = \int |J_K|\, d\lambda - \int |Q(I_B J_K)(\omega)|\, d\lambda$$
$$(a.e.\omega) \text{ on } a\ s(Q(f)),$$

which shows that

$$\text{(20)} \qquad 0 \leqq I_{s(Q(f))} Q(I_B J_K)(\omega) \leqq J_K \qquad (a.e.\omega).$$

Since ε is an arbitrary number, by (18) and (20) we have

$$0 \leqq Q(I_A J_K)(\omega) \leqq J_K \qquad (a.e.\omega). \qquad \text{Q.E.D.}$$

Lemma 3.4. *Let $A \in s(Q)$. Suppose that Q is covariant under $\mathfrak{T}$. Then there exists $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that $Q(I_A a) = \psi a$ for $a \in E$ and $0 \leqq \psi(\omega) \leqq 1$ (a.e.ω).*

Proof. Let $C \in \boldsymbol{A}(\mu)$. For any $K \in S(\lambda)$ write

$$e(K) = \int_C Q(I_A J_K)\, d\mu \in E.$$

By Lemma 3.3 for any $K \in S(\lambda)$

$$\text{(21)} \qquad 0 \leqq e(K) \leqq J_K \mu(C).$$

By (21) for any $K, H \in S(\lambda)$

$$J_{K \cap H} e(K) = J_{K \cap H}(e(K \cap H) + e(K - H)) = J_{K \cap H} e(K \cap H)$$
$$= J_{K \cap H}(e(K \cap H) + e(H - K)) = J_{K \cap H} e(H),$$

and hence $\{e(K); K \in S(\lambda)\}$ is a cross section. By Lemma 3.1 there exists a real-valued S-measurable function b on X such that

$$(22) \qquad J_K b = e(K) \qquad \text{for any} \quad K \in S(\lambda).$$

Since Q is covariant under $\mathfrak{T}$, for any $T \in \mathfrak{T}$

$$(23) \qquad J_{T^{-1}(K)} T(b) = T(J_K b) = T\left(\int_C Q(I_A J_K) d\mu\right)$$
$$= \int_C T(Q(I_A J_K)) d\mu = \int_C Q(I_A T(J_K)) d\mu = \int_C Q(I_A J_{T^{-1}(K)}) d\mu$$
$$= J_{T^{-1}(K)} b .$$

Since $(X, S, \lambda, \mathfrak{T})$ is ergodic, by the definition 7 $S(\lambda) = T^{-1}(S(\lambda))$. K is an arbitrary element of $S(\lambda)$, and hence (23) implies that $J_K T(b) = J_K b$ for any $K \in S(\lambda)$. By the finite subset property of (X, S, λ)

$$(24) \qquad T(b) = b .$$

By (21) and (22) b is a positive bounded function on X, and hence by Lemma 3.2 and (24) there exists a positive number $k(C)$ depending on C and A but not depending on K such that

$$b = J_X k(C) .$$

Therefore for any $C \in \boldsymbol{A}(\mu)$

$$\int_C Q(I_A J_K) d\mu = J_K k(C) .$$

Since μ is σ-finite, we can define a real-valued measure k on $\boldsymbol{A}$ by

$$J_K k(C) = \int_C Q(I_A J_K) d\mu \qquad \text{for any} \quad C \in \boldsymbol{A} .$$

Note that this integral is the Bochner integral, and hence $J_K k(C) \in E$. Therefore $0 \leqq k(C) < \infty$. Since k is absolutely continuous in the usual sense with respect to μ, there exists $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$, which may vary with A, such that

$$k(C) = \int_C \psi d\mu \qquad \text{for any} \quad C \in \boldsymbol{A} .$$

Thereofre for any $C \in \boldsymbol{A}$

$$\int_C Q(I_A J_K) d\mu = \int_C \psi J_K d\mu ,$$

and hence

$$Q(I_A J_K) = \psi J_K .$$

By Lemma 3.3 $0 \leqq \psi(\omega) \leqq 1$ $(a.e.\omega)$. Since $k(\ )$ is independent of the choice of

K, so is ψ. Any $a \in E$ can be approximated by a sequence of simple functions, and hence we have for any $a \in E$

$$Q(I_A a) = \psi a . \qquad t.\text{E.D.}$$

4. Lemmas for L_∞-valued functions. Throughout this section we assume that $E = L_\infty(X, S, \lambda, R)$, for $a \in E$

$$\|a\| = \text{ess. sup}\{|a(X)|\,;\, x \in X\}$$

and Q satisfies Assujption 1. Let

$$E^+ = \{a;\, a \in E \quad \text{and} \quad a(x) \geqq 0 \qquad (\text{a.e.x})\} .$$

Lemma 4.1. *For any $A \in s(Q)$ and $K \in S$,*

$$\|Q(I_A J_K)(\omega)\| \leqq 1 \qquad (a.e.\omega)$$

and

$$J_K Q(I_A J_K)(\omega) \in E^+ \qquad (a.e.\omega) .$$

Proof. For any arbitrary positive number ε by Lemma 1.5 there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B \in s(Q)$ such that

$$\|I_{s(Q(f))} Q(I_B J_K) - (I_A J_K)\|_L < \varepsilon \tag{25}$$

and

$$\|J_K - Q(I_B J_K)(\omega)\| = \|J_K\| - \|Q(I_B J_K)(\omega)\| \qquad (\text{a.e.}\omega) \text{ on } s(Q(f)) .$$

Therefore

$$\|I_{s(Q(f))} Q(I_B J_K)(\omega)\| \leqq 1 \qquad (\text{a.e.}\omega) \tag{26}$$

and

$$I_{s(Q(f))} J_K Q(I_B J_K)(\omega) \in E^+ \qquad (\text{a.e.}\omega) . \tag{27}$$

By (25), (26) and (27) we have

$$\|Q(I_A J_K)(\omega)\| \leqq 1 \qquad (\text{a.e.}\omega)$$

and

$$J_K Q(I_A J_K)(\omega) \in E^+ \qquad (\text{a.e.}\omega).$$

Lemma 4.2. *Let A, $B \in s(Q)$ and $A \subset B$. Suppose that there exists a pairwise disjoint class $\{K, L, M\}$ such that $\lambda(K) > 0$ and $\lambda(L \cup M) > 0$, where L can be a set of measure zero. Then for any natural number k*

$$\mu(B) \geqq \int_B \|Q(I_A J_K) + J_L + (-1)^k J_M\| d\mu - \int_{\Omega - B} \|Q(I_A J_K)\| d\mu . \tag{28}$$

Proof. Since Q is semi-constant-preserving, for an arbitrary positive number δ there exist $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$\|I_B Q(f) - I_B J_M\|_L < \delta \tag{29}$$

and

$$\|I_B Q(g) - I_B J_L\| < \delta . \tag{30}$$

Write

$$\varepsilon = \int_{\Omega - B} \|Q(I_A J_K)\| d\mu . \tag{31}$$

Therefore by (29), (30), (31) and the relation $A \subset B$

$$\begin{aligned}
\mu(B) &= \int_B \|I_A J_K + J_L + (-1)^k J_M\| d\mu \\
&\geqq \int_B \|I_A J_K + Q(g) + (-1)^k Q(f)\| d\mu - 2\delta \\
&= \int \|I_A J_K + Q(g) + (-1)^k Q(f)\| d\mu \\
&\quad - \int_{\Omega - B} \|I_A J_K + Q(g) + (-1)^k Q(f)\| d\mu - 2\delta \\
&\geqq \int_B \|Q(I_A J_K) + Q(g) + (-1)^k Q(f)\| d\mu \\
&\quad + \int_{\Omega - B} \|Q(I_A J_K) + Q(g) + (-1)^k Q(f)\| d\mu \\
&\quad - \int_{\Omega - B} \|I_A J_K + Q(g) + (-1)^k Q(f)\| d\mu - 2\delta \\
&\geqq \int_B \|Q(I_A J_K) + Q(g) + (-1)^k Q(f)\| d\mu \\
&\qquad + \int_{\Omega - B} \|Q(g) + (-1)^k Q(f)\| d\mu - \int_{\Omega - B} \|Q(g) + (-1)^k Q(f)\| d\mu - 2\delta - \varepsilon \\
&= \int_B \|Q(I_A J_K) + Q(g) + (-1)^k Q(f)\| d\mu - 2\delta - \varepsilon \\
&\geqq \int_B \|Q(I_A J_K) + J_L + (-1)^k J_M\| d\mu - 4\delta - \varepsilon .
\end{aligned}$$

We have proved (28), since δ is an arbitrary number. Q.E.D.

Lemma 4.3 *Let K and L be disjoint elements of S which are of positive meaaure. Then for any $A \in s(Q)$*

$$\int J_L Q(I_A J_K) d\mu = 0 .$$

Proof. Suppose that there exists a positive real number ε such that

$$\| \int J_L Q(I_A J_K) d\mu \| > 7\varepsilon . \tag{32}$$

By Lemma 1.5 there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B \in A(\mu)$ such that $B \subset s(Q(f))$,

$$\| I_{s(Q(f))} Q(I_B J_K) - Q(I_A J_K) \|_L < \varepsilon \tag{33}$$

and

$$\| I_{\Omega - s(Q(f))} Q(I_B J_K) \|_L < \varepsilon . \tag{34}$$

By (32) and (33)

$$\| \int I_{s(Q(f))} J_L Q(I_B J_K) d\mu \| > 6\varepsilon . \tag{35}$$

By (34) and (35) we can choose $C \in \boldsymbol{A}(\mu)$ such that $C \subset s(Q(f))$,

$$\| I_{\Omega - C} Q(I_B J_K) \|_L < 2\varepsilon \tag{36}$$

and

$$\| \int I_C J_L Q(I_B J_K) d\mu \| > 5\varepsilon . \tag{37}$$

By (37) and the definition of the norm $\| \ \|$ there exist $M \in S$ and a natural number k such that $M \subset L$,

$$(-1)^k \int I_C J_M Q(I_B J_K) d\mu \in E^+ \tag{38}$$

and

$$\| \int I_C J_M Q(I_B J_K) d\mu \| > 5\varepsilon . \tag{39}$$

$B \cup C \subset s(Q(f))$, and hence $B \cup C \in s(Q)$. By (36) we have

$$\int_{\Omega - (B \cup C)} \| Q(I_B J_K) \| d\mu < 2\varepsilon \tag{40}$$

and

$$\int_{B - C} \| Q(I_B J_K) \| d\mu < 2\varepsilon . \tag{41}$$

K and M are disjoint, and hence by Lemma 4.2, (38), (39), (40) and (41)

$$\begin{aligned} \mu(B \cup C) &= \int_{B \cup C} \| I_B J_K + (1 -{}^k) J_M \| d\mu \\ &\geqq \int_{B \cup C} \| Q(I_B J_K) + (-1)^k J_M \| d\mu - 2\varepsilon \end{aligned}$$

$$\geqq \int_{B\cup C} \|J_M Q(I_B J_K) + (-1)^k J_M\| d\mu - 2\varepsilon$$
$$\geqq \int_{B\cup C} \|I_C J_M Q(I_B J_K) + (-1)^k J_M\| d\mu - 4\varepsilon$$
$$\geqq \left\| \int_C J_M Q(I_B J_K) d\mu + (-1)^k \mu(B\cup C) J_M \right\| - 4\varepsilon$$
$$= \left\| (-1)^k \int_C J_M Q(I_B J_K) d\mu \right\| + \mu(B\cup C) - 4\varepsilon$$
$$> 5\varepsilon + \mu(B\cup C) - 4\varepsilon = \mu(B\cup C) + \varepsilon ,$$

which is a contradiction. Therefore

$$\int J_L Q(I_A J_K) d\mu = 0 . \qquad \text{Q.E.D.}$$

Lemma 4.4. *Suppose that* $f, g, h \in L_1(\Omega, \boldsymbol{A}, \mu, R)$, $f(\omega) \geqq 0$, $g(\omega) \geqq 0$ *and* $h(\omega) \geqq 0$ *(a.e.ω). Then we have*

$$\int (g \vee h) d\mu \leqq \int ((f \vee h) + (f \vee g - g) + (f \vee g - f)) d\mu .$$

Proof.

$$\int (g \vee h) d\mu \leqq \int (f + |f-g|) \vee h d\mu \leqq \int ((f \vee h) + |f-g|) d\mu$$
$$= \int ((f \vee h) + (f \vee g - g) + (f \vee g - f)) d\mu . \qquad \text{Q.E.D.}$$

DEFINITION 9. A class of subsets $\{K, L, M\}$ is said to be a *partition* of X if K, L and M are pairwise disjoint and $\lambda(K) > 0$, $\lambda(L) > 0$, $\lambda(M) > 0$ and $K \cup L \cup M = X$ (a.e.x).

Lemma 4.5. *Suppose that* $A \in s(Q)$ *and* $K \in S$. *If we can choose* $L, M \in S$ *such that* $X = K \cup L \cup M$ *(a.e.x),* $\lambda(L) > 0$, $\lambda(M) > 0$ *and* $\lambda(L \cap M) = 0$, *then* $J_{L\cup M} Q(I_A J_K) = 0$. *(Note that K may be a set of measure zero.)*

Proof. Suppose that

$$\mu(\{\omega ; \|J_L Q(I_A J_K)\| > 0\}) > 0 .$$

Then there exist positive real numbers δ and ε such that

$$\mu(\{\omega ; \|J_L Q(I_A J_K)\| > 4\delta\}) > 3\varepsilon .$$

Let

$$F = \{\omega ; \|J_L Q(I_A J_K)\| > 4\delta\} ,$$

then $\mu(F) > 3\varepsilon$. By Lemma 1.5 there exist $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B \in s(Q)$

such that $B\subset s(Q(f))$,

(42) $$||I_{\Omega-s(Q(f))}Q(I_BJ_K)||_L<\varepsilon\delta$$

and

(43) $$||Q(I_BJ_K)-Q(I_AJ_K)||_L<\varepsilon\delta\ .$$

By (42) we can choose $C\in\boldsymbol{A}(\mu)$ such that $C\subset s(Q(f))$ and

$$||I_{\Omega-C}Q(I_BJ_K)||_L<\varepsilon\delta\ .$$

Let

$$D=\{\omega;\ ||J_LQ(I_BJ_K)||>3\delta\}\ .$$

Then by (43)

$$\delta\mu(F-D)\leqq\int_{F-D}||Q(I_BJ_K)-Q(I_AJ_K)||d\mu<\varepsilon\delta\ ,$$

and hence $\mu(F-D)<\varepsilon$. Since $\mu(F)>3\varepsilon$, $\mu(D)>2\varepsilon$. Therefore

(44) $$\int_D||J_LQ(I_BJ_K)||d\mu>6\varepsilon\delta\ .$$

Then by (42) and (44)

$$\int_{D\cap s(Q(f))}||J_LQ(I_BJ_K)||d\mu>6\varepsilon\delta-\varepsilon\delta=5\varepsilon\delta\ .$$

Let $E=(D\cap s(Q(f)))\cup C\cup B$, then $E\subset s(Q(f))$,

(45) $$||I_EJ_LQ(I_BJ_K)||_L>5\varepsilon\delta\ .$$

and

(46) $$||I_{\Omega-E}Q(I_BJ_K)||_L<\varepsilon\delta\ .$$

By Lemma 4.2, Lemma 4.3 and (46) for any $k\in\boldsymbol{N}$

(47) $$\begin{aligned}\mu(E)&=\int_E||I_BJ_K+J_M+(-1)^kJ_L||d\mu\\&\geqq\int_E||Q(I_BJ_K)+J_M+(-1)^kJ_L||d\mu-\varepsilon\delta\\&\geqq\int_E||J_MQ(I_BJ_K)+J_M||\vee||J_LQ(I_BJ_K)+(-1)^kJ_L||d\mu-\varepsilon\delta\\&\geqq\int||J_MQ(I_BJ_K)+I_EJ_M||\vee||J_LQ(I_BJ_K)+(-1)^kI_EJ_L||d\mu-2\varepsilon\delta\\&\geqq\int||J_MQ(I_BJ_K)+I_EJ_M||d\mu\wedge\int||J_LQ(I_BJ_K)+(-1)^kI_EJ_L||d\mu-2\varepsilon\delta\\&\geqq||\int J_MQ(I_BJ_K)d\mu+\mu(E)J_M||\vee||\int J_LQ(I_BJ_K)d\mu+(-1)^k\mu(E)J_L||-2\varepsilon\delta\end{aligned}$$

$$=\|\mu(E)J_M\|\wedge\|(-1)^k\mu(E)J_L\|-2\varepsilon\delta=\mu(E)-2\varepsilon\delta\,,$$

where the last equation comes from the fact that $M\neq\emptyset$ and $L\neq\emptyset$. Therefore by Lemma 4.4, (47) and (45)

$$\mu(E)+4\varepsilon\delta\geqq\int\|J_LQ(I_BJ_K)+I_EJ_L\|\vee\|J_LQ(I_BJ_K)-I_EJ_L\|d\mu$$
$$=\int(\|J_LQ(I_BJ_K)\|+I_E)d\mu\geqq\mu(E)+5\varepsilon\delta\,,$$

which is a contradiction. Therefore

$$J_LQ(I_AJ_K)=0\,. \tag{48}$$

Similarly we can prove

$$J_MQ(I_AJ_K)=0\,. \tag{49}$$

By (48) and (49) we have

$$J_{L\cup M}Q(I_AJ_K)=0\,.$$ Q.E.D.

Lemma 4.6. *Suppose that $A\in s(Q)$ and there exists a partition $\{K, L, M\}$ of X. Then there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $0\leqq\psi(\omega)\leqq1$ (a.e.ω) and $Q(I_Aa)=\psi a$ for any $a\in E$.*

Proof. By Lemma 1.5 for any arbitrary number $\varepsilon>0$ there exist $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B\in s(Q)$ such that

$$\|I_{s(Q(f))}Q(I_BJ_X)-Q(I_AJ_X)\|_K<\varepsilon \tag{50}$$

and

$$\|J_X-Q(I_BJ_X)(\omega)\|=\|J_X\|-\|Q(I_BJ_X)(\omega)\| \quad \text{(a.e.}\omega\text{) on } s(Q(f))\,, \tag{51}$$

and hence

$$Q(I_BJ_X)(\omega)=\|Q(I_BJ_X)(\omega)\|J_X \text{ (a.e.x) on } s(Q(f))\,,$$

which implies

$$I_{s(Q(f))}Q(I_BJ_X)=\|Q(I_BJ_X)\|I_{s(Q(f))}J_X\,. \tag{52}$$

$\|Q(I_BJ_X)\|I_{s(Q(f))}\in L_1(\Omega, \boldsymbol{A}, \mu, R)$, and hence by (50) and (52) there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$Q(I_AJ_X)=\psi J_X\,. \tag{53}$$

By (51) $0\leqq\psi(\omega)\leqq1$ (a.e.ω). Let $N\in S$ and $\lambda(N)>0$. If $\lambda(K\cap N)>0$, then by the assumption that $\{K, L, M\}$ is a partition of X and Lemma 4.5 we have

$$J_{N\cap K}Q(I_AJ_L)=0\,,\quad J_{N\cap K}Q(I_AJ_M)=0\,,\quad J_{N\cap K}Q(I_AJ_{K-N})=0$$

and

$$J_{X-(N\cap K)}Q(I_AJ_{N\cap K})=0\,.$$

Therefore by (53)

$$Q(I_AJ_{N\cap K})=J_{N\cap K}Q(I_AJ_{N\cap K})=J_{N\cap K}Q(I_AJ_X)=\psi J_{N\cap K}\,. \tag{54}$$

If $\lambda(K\cap N)=0$, then (54) is trivial. Similarly we can prove that

$$Q(I_AJ_{N\cap L})=\psi J_{N\cap L} \tag{55}$$

and

$$Q(I_AJ_{N\cap M})=\psi J_{N\cap M}\,. \tag{56}$$

Therefore by (54), (55) and (56) we have $Q(I_AJ_N)=\psi J_N$ and ψ is independent of the choice of N. Since N is an arbitrary element of S and any $a\in E$ can be approximated by a sequence of simple functions, we have for any $a\in E$

$$Q(I_Aa)=\psi a\,. \qquad \text{Q.E.D.}$$

5. Semi-constant-preserving contracttve projections and conditional expectations. In this section an operator Q is said to satisfy Assumption 2 if

(57) for any $A\in s(Q)$ there exists $\psi\in L_1(\Omega,\, \boldsymbol{A},\, \mu,\, R)$ such that $0\leqq\psi(\omega)\leqq 1$ (a.e.ω) and $Q(I_Aa)=\psi a$ for any a$\in E$, where ψ is independent of the choice of a.

In Section 2, Section 3 and Section 4 we used the following conditions (58), (59) and (60) respectively.

(58) $E=L_p(X,\, S,\, \lambda,\, R)$, where $1<p<\infty$.

(59) $E=L_1(X,\, S,\, \lambda,\, R)$, where $(X,\, S,\, \lambda,\, \mathcal{G})$ is an ergodic licalizable measure space and Q is covariant under $\mathcal{G}$.

(60) $E=L_\infty(X,\, S,\, \lambda,\, R)$ and there exists a partition $\{K,\, L,\, M\}$ of X.

If Q satisfies Assumption 1 (See (1).) and one of the conditions (58), (59) and (60) is satisfied, then by Lemma 2.2, Lemma 3.4 and Lemma 4.6 Q satisfies Assumption 2.

Lemma 5.1. *Suppose that Q satisfies Assumption* 1 *and Assumption* 2, *then for any $\psi\in L_1(\Omega,\, \boldsymbol{A},\, \mu,\, R)$ there exists $\phi\in L_1(\Omega,\, \boldsymbol{A},\, \mu,\, R)$ such that for any $a\in E$*

$$Q^*(\psi a)=\phi a$$

and

$$\phi(\omega)\geqq 0\ \ (a.e.\omega)\ \ \textit{if}\ \ \phi(\omega)\geqq 0\ \ (a.e.\omega)\,.$$

Proof. It is sufficient to prove this Lemma for $\psi = I_A$ with $A \in \boldsymbol{A}(\mu)$. By Lemma 1.13 there exists a sequence $\{A_n; n \in \boldsymbol{N}\}$ of pairwise disjoint elements of $s(Q)$ such that

$$A - N(A) = \cup \{A_n; n \in \boldsymbol{N}\} .$$

By (57) for any n there exists $\phi_n \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that for any $a \in E$

$$Q(I_{A_n} a) = \phi a_n .$$

Since Q is contractive,

$$\|\phi_n\|_L \|a\| = \|\phi_n a\|_L \leqq \|I_{A_n} a\|_L = \mu(A_n)\|a\| ,$$

and hence

$$\sum \{\|\phi_n\|_L; n \in \boldsymbol{N}\} \leqq \mu(A) .$$

Therefore by writting $\phi = \sum \{\phi_n; n \in \boldsymbol{N}\}$ we have $\phi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$. $Q^*(I_A a) = \sum \{Q(I_{A_n} a); n \in \boldsymbol{N}\} = \phi a$ for any $a \in E$. Q.E.D.

Lemma 5.2. *If Q satisfies Assumption 1 and Assumption 2, then for any $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ there exists $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $\psi(\omega) \geqq 0$ (a.e.ω) and $s(Q^*(\psi a)) \supset s(Q^*(f))$ (a.e.ω) for any non-zero element a of E.*

Proof. First we suppose that f is a simple function and $f = I_{A_1} a_1 + \cdots + I_{A_n} a_n$, where $A_i \in \boldsymbol{A}(\mu)$, $A_i \cap A_j = \emptyset$ $(i \neq j)$ and $a_i \in E$ for $i = 1, 2, \cdots, n$. By Lemma 5.1 there exists $\phi_i \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ for any i such that $\phi_i(\omega) \geqq 0$ (a.e.ω) and $Q^*(I_{A_i} a_i) = \phi_i a_i$. Let $\psi = I_{A_1 \cup \cdots \cup A_n}$ and a an arbitrary non-zero element of E, then

$$s(Q^*(f)) = s(\phi_1 a_1 + \cdots + \phi_n a_n) \subset s(\phi_1 a + \cdots + \phi_n a) = s(Q^*(\psi a)) .$$

For an arbitrary $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $n \in \boldsymbol{N}$ there exists a simple function $f_n \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$\|f - f_n\|_L < 1/n . \tag{58}$$

In the preceding part of this proof we have proved that for any f_n there exists $\psi_n \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$s(Q^*(f_n)) \subset s(Q^*(\psi_n a)) \tag{59}$$

and

$$\psi_n(\omega) \geqq 0 \qquad \text{(a.e.}\omega\text{)} .$$

Let

$$\psi = \sum \{(\psi_n / (2^n \|\psi_n\|_L)); n \in \boldsymbol{N}\} .$$

Then

$$s(Q^*(\psi a)) = \cup \{s(Q^*(\psi_n a)); n \in \boldsymbol{N}\} . \tag{60}$$

By (58), (59) and (60) and the fact that Q^* is contractive

$$(61)\qquad \int_{s(Q^*(f))-s(Q^*(\psi a))}\|Q^*(f)\|d\mu \leqq \int_{s(Q^*(f))-\cup\{s(Q^*(f_n));\, n\in N\}}\|Q^*(f)\|d\mu$$
$$=\int_{s(Q^*(f))-\cup\{s(Q^*(f_n));\, n\in N\}}\|Q^*(f)-Q^*(f_n)\|d\mu \leqq \|f-f_n\|_L<1/n\,.$$

Since $\|Q^*(f)(\omega)\|>0$ for any $\omega\in s(Q^*(f))-s(Q^*(\psi a))$ and n is an arbitrary number, (61) implies that

$$\mu(s(Q^*(f))-s(Q^*(\psi a)))=0\,. \qquad \text{Q.E.D.}$$

Lemma 5.3. *Suppose that Q satisfiw satisfies Assumption 1 and Assumption 2 and $A_n\in s(Q)=s(Q^*)$ for any $n\in \boldsymbol{N}$. If $\cup\{A_n;\, n\in \boldsymbol{N}\}\in \boldsymbol{A}(\mu)$, then $\cup\{A_n;\, n\in \boldsymbol{N}\}\in s(Q)=s(Q^*)$.*

Proof. Since $A_n\in s(Q^*)$, by the definition of $s(Q^*)$ there exists $f_n\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that $A_n\subset s(Q^*(f_n))$. Therefore by Lemma 5.1 and 5.2 there exist ψ_n, $\phi_n\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ and $a\in E$ such that $\psi_n(\omega)\geqq 0$ (a.e.ω), $\phi_n(\omega)\geqq 0$ (a.e.ω), $Q^*(\psi_n a)=\phi_n a$ and

$$s(Q^*(f_n))\subset s(Q^*(\psi_n a))=s(\phi_n)\,,$$

where we can assume that $\|\psi_n\|_L=1/2^n$. Q^* is contractive, and hence $\|\phi_n\|_L\leqq 1/2^n$.

Write $\psi=\sum\{\psi_n;\, n\in \boldsymbol{N}\}$ and $\phi=\sum\{\phi_n;\, n\in \boldsymbol{N}\}$. Then ψ, $\phi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ and

$$s(Q^*(\psi a))=s(\phi)=\cup\{s(\phi_n);\, n\in \boldsymbol{N}\}\,.$$

Therefore $\cup\{A_n;\, n\in \boldsymbol{N}\}\subset s(Q^*(\psi a))$. Since $\cup\{A_n;\, n\in \boldsymbol{N}\}\in \boldsymbol{A}(\mu)$, by the definition of $s(Q^*)$ $\cup\{A_n;\, n\in \boldsymbol{N}\}\in s(Q^*)$. Q.E.D.

The following lemma is more delicate than Lemma 5.1.

Lemma 5.4. *Suppose that Q satisfies Assumption 1 and Assumption 2. Then for any $A\in \boldsymbol{A}(\mu)$ there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $0\leqq\psi(\omega)\leqq 1$ (a.e.ω) and $Q^*(I_A a)=\psi a$ for any $a\in E$.*

Proof. Let $A\in \boldsymbol{A}(\mu)$. Then by Lemma 1.13 there exists a sequence $\{A_n;\, n\in \boldsymbol{N}\}$ such that $A_n\in s(Q)$ and

$$A-N(A)=\cup\{A_n;\, n\in \boldsymbol{N}\}\,.$$

By Lemma 5.3 $\cup\{A_n;\, n\in \boldsymbol{N}\}\in s(Q)$, and hence

$$A-N(A)\in s(Q)\,.$$

By Assumption 2 there exists $\psi\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $0\leqq\psi(\omega)\leqq 1$ (a.e.ω)

and

$$Q(I_{A-N(A)}a) = \psi a .$$

Therefore

$$Q^*(I_A a) = Q(I_{A-N(A)}a) = \psi a . \qquad \text{Q.E.D.}$$

Lemma 5.5. *If Q satisfies Assumption* 1 *and Assumption* 2, *then there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that*

(i) $$Q^*(f) = f^{\boldsymbol{B}} ,$$

(ii) $$N_Q(f) = N_{\boldsymbol{B}}(f)$$

and

(iii) $$Q(f) \in L_1(\Omega, \boldsymbol{B}, \mu, E) \qquad \textit{for any} \quad f \in L_1(\Omega, \boldsymbol{A}, \mu, E) .$$

Proof. (i) By Lemma 5.4 for any $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ there exists $\phi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$Q^*(\psi a) = \phi a \qquad \text{for any} \quad a \in E ,$$

and that $0 \leqq \phi(\omega) \leqq 1$ (a.e.ω) if $\psi = I_A$ for some $A \in \boldsymbol{A}(\mu)$. If we fix a, Q^* can be regarded as an operator of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into itself, which satisfies the assumption of Lemma 1.2. Therefore there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $Q^*(\psi a) = \psi^{\boldsymbol{B}} a$ for any $\psi \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ and any $a \in E$. Since any $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ can be approximated by simple functions, $Q^*(f) = f^{\boldsymbol{B}}$ for any $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$.

(ii) It is sufficient to show that $s(Q) = s((\ )^{\boldsymbol{B}})$. If $A \in s(Q)$ then there exists $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

(62) $$A \subset s(Q(f)) .$$

By Lemma 1.11 and the preceding part of this proof

(63) $$Q(f) = Q^*(Q(f)) = Q(f)^{\boldsymbol{B}} .$$

By (62) and (63) we have $A \in s((\ )^{\boldsymbol{B}})$. On the other hand if $A \in s((\ )^{\boldsymbol{B}})$, then there exists $f \in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

(64) $$A \subset s(f^{\boldsymbol{B}}) .$$

By the definition of $Q^*$and the preceding part of this Lemma

(65) $$f^{\boldsymbol{B}} = Q^*(f) = Q(f - N_Q(f)) .$$

By (64) and (65) we have $A \in s(Q)$.

(iii) Since $Q(f) = Q^*(Q(f)) = Q(f^{\boldsymbol{B}})$, $Q(f) \in L_1(\Omega, \boldsymbol{B}, \mu, E)$ Q.E.D.

Theorem 1. (i) *If Q satisfies Assumption* 1 *and Assumption* 2, *then there*

exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that $Q(f)=f^{\boldsymbol{B}}+Q(N_Q(f))=f^{\boldsymbol{B}}+Q(N_{\boldsymbol{B}}(f))$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$.
(ii) *If there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ and a contractive linear operator $\boldsymbol{P}$ of $L_1(\Omega, \boldsymbol{A_B}, \mu, E)$ into $L_1(\Omega, \boldsymbol{B}, \mu, E)$, then the operator defined by $Q(f)=f^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f))$ for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ satisfies Assumption* 1 *and Assumption* 2.

Proof. (i) By Lemma 5.5 and the definitions of Q^*, N_Q and $N_{\boldsymbol{B}}$ there exists a σ-subring $\boldsymbol{B}$ of $\boldsymbol{A}$ such that

$$Q(f) = Q^*(f)+Q(N_Q(f)) = f^{\boldsymbol{B}}+Q(N_{\boldsymbol{B}}(f))\,.$$

(ii) By the fact that $P(f)\in L_1(\Omega, \boldsymbol{B}, \mu, E)$ for any $f\in L_1(\Omega, \boldsymbol{A_B}, \mu, E)$ and properties of operators $(\ )^{\boldsymbol{B}}$ and $N_{\boldsymbol{B}}$ and Lemma 1.10 we have

$$(\ )^{\boldsymbol{B}}\circ P = P\,, \tag{66}$$

$$N_{\boldsymbol{B}}\circ P = 0\,, \tag{67}$$

$$(\ )^{\boldsymbol{B}}\circ N_{\boldsymbol{B}} = 0\,, \tag{68}$$

and

$$N_{\boldsymbol{B}}\circ(\ )^{\boldsymbol{B}} = 0\,, \tag{69}$$

which imply that

$$Q\circ(\ )^{\boldsymbol{B}} = (\ )^{\boldsymbol{B}}\circ(\ )^{\boldsymbol{B}}+P\circ N_{\boldsymbol{B}}\circ(\ )^{\boldsymbol{B}} = (\ )^{\boldsymbol{B}}\,. \tag{70}$$

By (66), (67) and (69)

$$\begin{aligned} Q\circ Q(f) &= (f^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f)))^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f)))) \\ &= f^{\boldsymbol{B}}+P(N_{\boldsymbol{B}}(f)) = Q(f)\,. \end{aligned}$$

Therefore Q is a projection.
By (68) and the fact that $(\ )^{\boldsymbol{B}}$ and P are contractive

$$\begin{aligned} ||Q(f)||_L &\leqq ||f^{\boldsymbol{B}}||_L+||P(N_{\boldsymbol{B}}(f))||_L = ||f^{\boldsymbol{B}}-(N_{\boldsymbol{B}}(f))^{\boldsymbol{B}}||_L+||P(N_{\boldsymbol{B}}(f))||_L \\ &\leqq ||f-N_{\boldsymbol{B}}(f)||_L+||N_{\boldsymbol{B}}(f)||_L \\ &= ||I_{s(f)-N_{\boldsymbol{B}}(s(f))}f||_L+||I_{N_{\boldsymbol{B}}(s(f))}f||_L = ||f||_L\,, \end{aligned}$$

and hence Q is contractive.

Next we are going to show that Q is semi-constant-preserving and satisfies Assumption 2.
Let $A\in s(Q)$, $a\in E$ and $\varepsilon>0$. By the definition of $s(Q)$ there exists $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that $A\subset s(Q(f))$. By Lemma 5.5 $Q(f)\in L_1(\Omega, \boldsymbol{B}, \mu, E)$, and hence

$$A\subset s(Q(f)) = s((Q(f))^{\boldsymbol{B}})\,. \tag{71}$$

Conditional expectation operators are semi-constant-preserving, and hence by (71) there exists $g\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ such that

$$\|I_A g^B - I_A a\|_L < \varepsilon . \tag{72}$$

By (70) and (72)

$$\|I_A Q(g^B) - I_A a\|_L < \varepsilon ,$$

which implies that Q is semi-constant-preserving. Since by (71) and the definition of N_B $N_B(I_A a)=0$,

$$Q(I_A a) = (I_A a)^B + P(N_B(I_A a)) = (I_A a)^B = (I_A)^B a ,$$

and hence Q satisfies Assumption 2. Q.E.D.

6. $\boldsymbol{R}^2$-valued case. Let $E=L_\infty(X, S, \lambda, R)$. If we cannot choose K, L and M such that $\{K, L, M\}$ is a partition of X, then $E\cong R$ with the norm $\|x\|=|x|$ for $x\in R$ or $E\cong R^2$ with the norm $\|(x, y)\|=|x|\vee|y|$ for $(x, y)\in R^2$. If $E\cong R$, then we can use Lemma 2.2. Therefore our next aim is to consider the case when $E\cong R^2$. Throughout this section we assume that $E=R^2$ with the norm $\|(x, y)\|=|x|\vee|y|$ for $(x, y)\in R^2$. Note that for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ there exist $f_1, f_2\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $f(\omega)=(f_1(\omega), f_2(\omega))$. Throughout this section we assume that Q is a linear operator of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into itself.

Lemma 6.1. *Let Q satisfy Assumption 1 and $A\in s(Q)$. If $Q((I_A, I_A))=(f_1, f_2)$ and $Q((I_A, -I_A))=(g_1, g_2)$, then $f_1=f_2$, $g_1=-g_2$, $0\leqq f_1(\omega)\leqq 1$ (a.e.ω) and $0\leqq g_1(\omega)\leqq 1$ (a.e.ω).*

Proof. By Lemma 1.5 for any $\varepsilon>0$ there exist $f\in L_1(\Omega, \boldsymbol{A}, \mu, E)$ and $B\in \boldsymbol{A}(\mu)$ such that $B\subset s(Q(f))$,

$$\|I_{s(Q(f))}Q(I_B(1, 1)) - Q(I_A(1, 1))\|_L < \varepsilon \tag{73}$$

and

$$\begin{aligned} &\|(1, 1) - Q(I_B(1, 1))(\omega)\| \\ &= \|(1, 1)\| - \|Q(I_B(1, 1))(\omega)\| \qquad \text{(a.e.}\omega\text{) on } s(Q(f)) . \end{aligned} \tag{74}$$

Let $(h_1, h_2)=I_{s(Q(f))}Q(I_B(1, 1))$. Then by (74)

$$\|(1, 1)-(h_1, h_2)\| = \|(1, 1)\| - \|(h_1, h_2)\| ,$$

and hence we have

$$|1-h_1(\omega)|\vee|1-h_2(\omega)| = 1-|h_1(\omega)|\vee|h_2(\omega)| ,$$

which shows that $h_1=h_2$, $0\leqq h_1(\omega)\leqq 1$ (a.e.ω). Therefore by (73)

$$||(f_1, f_2)-(h_1, h_1)||_L<\varepsilon,$$

which shows that

$$f_1=f_2, \quad 0\leqq f_1(\omega)\leqq 1 \qquad (\text{a.e.}\omega),$$

since ε is an arbitrary number.
Similarly we can prove that $g_1=-g_2$ and $0\leqq g_1(\omega)\leqq 1$. Q.E.D.

If an operator Q satisfies Assumption 1, then by Lemma 6.1 we can define linear operator Q_1 and Q_2 of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into itself by

(75) $$Q^*(f, f)=(Q_1(f), Q_1(f))$$

and

(76) $$Q^*(f, -f)=(Q_2(f), -Q_2(f)).$$

Then by the definitions of Q_1 and Q_2

(77) $$\begin{aligned} Q^*(f, g) &= (1/2)Q^*(f+g+f-g, f+g-(f-g)) \\ &= (1/2)(Q_1(f+g)+Q_2(f-g), Q_1(f+g)-Q_2(f-g)). \end{aligned}$$

Lemma 6.2. *Let Q satisfy Assumption 1. Then Q_1 and Q_2 are contractive projections and for any $A\in s(Q)$ and $\varepsilon>0$ there exist $f, g\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that*

(78) $$||I_A Q_1(f)-I_A||_L<\varepsilon$$

and

(79) $$||I_A Q_2(g)-I_A||_L<\varepsilon.$$

In particular Q_1 and Q_2 are semi-constant-preserving.

Proof. Let $A\in s(Q)$ and $\varepsilon>0$. By Lemma 1.1 Q^* is a semi-constant-preserving contractive projection, and hence Q_1 and Q_2 are contractive projections and there exist $f', g'\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

(80) $$||I_A Q^*(f', g')-(I_A, I_A)||_L<\varepsilon.$$

By (77)

$$\int_A |Q_1((f'+g')/2)+Q_2((f'-g')/2)-1| \vee |Q_1((f'+g')/2) -Q_2((f'-g')/2)-1| d\mu<\varepsilon,$$

which implies that

$$\int_A |Q_1((f'+g')/2)-1| d\mu<\varepsilon,$$

and by writing $f=(f'+g')/2$ we have

(78) $$\|I_A Q_1(f)-I_A\|_L<\varepsilon .$$

Similarly we can prove that

(79) $$\|I_A Q_2(g)-I_A\|_L<\varepsilon .$$

Clearly $s(Q)=s(Q^*)\supset s(Q_1)$, $s(Q_2)$, and hence by (78) and (79) Q_1 and Q_2 are semi-constant-preserving. Q.E.D.

Since Q_1 and Q_2 are operators of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into itself we can use the result of Section 1 and Section 2 for Q_1 and Q_2.

Lemma 6.3. *Let Q satisfy Assumption 1. Then there exist σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ of $\boldsymbol{A}$ such that for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, R)$*

$$Q_1(f)=f^{\boldsymbol{B}},$$
$$Q_2(f)=f^{\boldsymbol{C}}$$

and

$$N_{\boldsymbol{B}}(A)=N_{\boldsymbol{C}}(A)=N_Q(A) \qquad \textit{for any} \quad A\in\boldsymbol{A}(\mu) .$$

Proof. By Lemma 6.2 Q_1 and Q_2 are semi-constant-preserving contractive projections of $L_1(\Omega, \boldsymbol{A}, \mu, R)$ into itself, and hence by Lemma 2.2 and Theorem 1 there exist σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ such that for any $f\in L_1(\Omega, \boldsymbol{A}, \mu, R)$

(81) $$Q_1(f)=f^{\boldsymbol{B}}+Q_1(N_{Q_1}(f)) ,$$

(82) $$Q_2(f)=f^{\boldsymbol{C}}+Q_2(N_{Q_2}(f)) ,$$

(83) $$N_{Q_1}(f)=N_{\boldsymbol{B}}(f)$$

and

(84) $$N_{Q_2}(f)=N_{\boldsymbol{C}}(f) .$$

Let $A\in s(Q)$. By (78) and (79) for any $n\in N$ there exist f_n, $g_n\in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$\|I_A Q_1(f_n)-I_A\|_L<1/n$$

and

$$\|I_A Q_2(g_n)-I_A\|_L<1/n .$$

Therefore

$$\mu(A-s(Q_1(f_n)))<1/n$$

and

$$\mu(A-s(Q_2(g_n)))<1/n .$$

Write $A_n=A\cap s(Q_1(f_n))$. Then $A_n\in s(Q_1)$ and

(85) $$A = \cup \{A_n;\, n \in \boldsymbol{N}\} \qquad (\text{a.e.}\omega)\,.$$

By Lemma 2.2 and Lemma 6.2 Q_1 satisfies Assumption 1 and Assumption 2, and hence by (85) and Lemma 5.3 $A \in s(Q_1)$. Since A is an arbitrary element of $s(Q)$, we have proved that $s(Q) \subset s(Q_1)$. By the definition of Q_1 and Lemma 1.11 $s(Q_1) \subset s(Q^*) = s(Q)$. Therefore we have

(86) $$s(Q) = s(Q_1)\,.$$

Similarly we can prove that

(87) $$s(Q) = s(Q_2)\,.$$

By (86) and (87) togehter with (83) and (84) we have

(88) $$N_Q(A) = N_{Q_1}(A) = N_{Q_3}(A) = N_{\boldsymbol{B}}(A) = N_{\boldsymbol{C}}(A)\,.$$

By Lemma 1.11 $Q^* \circ N_Q = 0$, and hence by (75) and (76)

(89) $$Q_1 \circ N_Q = 0$$

and

(90) $$Q_2 \circ N_Q = 0\,.$$

By (81), (82), (88), (89) and (90)

$$Q_1(f) = f^{\boldsymbol{B}}$$

and

$$Q_2(f) = f^{\boldsymbol{C}} \qquad \text{for any} \quad f \in L_1(\Omega, \boldsymbol{A}, \mu, R)\,.$$ Q.E.D.

By (77) and Lemma 6.3 we have

(91) $$Q^*(f, g) = (1/2)(f^{\boldsymbol{B}} + g^{\boldsymbol{B}} + f^{\boldsymbol{C}} - g^{\boldsymbol{C}},\, f^{\boldsymbol{B}} + g^{\boldsymbol{B}} - f^{\boldsymbol{C}} + g^{\boldsymbol{C}})\,.$$

Let us denote the operator, expressed in the right hand side of the above formula, by $F(\boldsymbol{B}, \boldsymbol{C})$.

Lemma 6.4. *For any σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ with $N_{\boldsymbol{B}} = N_{\boldsymbol{C}}$ the operator $F(\boldsymbol{B}, \boldsymbol{C})$ satisfies Assumption* 1.

Proof. It is clear that $F(\boldsymbol{B}, \boldsymbol{C}) \circ F(\boldsymbol{B}, \boldsymbol{C}) = F(\boldsymbol{B}, \boldsymbol{C})$, and hence $F(\boldsymbol{B}, \boldsymbol{C})$ is a projection. Next we are going to show that $F(\boldsymbol{B}, \boldsymbol{C})$ is semi-constant-preserving. Let $A \subset s(F(\boldsymbol{B}, \boldsymbol{C})(f, g))$ for some $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ and $a = (a_1, a_2) \in E$. Then by the definition of $F(\boldsymbol{B}, \boldsymbol{C})$ we can choose sequences $\{B_n \in \boldsymbol{B}(\mu);\, n \in \boldsymbol{N}\}$ and $\{C_n \in \boldsymbol{C}(\mu);\, n \in \boldsymbol{N}\}$ such that

$$s(F(\boldsymbol{B}, \boldsymbol{C})(f, g)) \subset \cup \{B_n;\, n \in \boldsymbol{N}\} \cup \{C_n;\, n \in \boldsymbol{N}\}\,.$$

Then $A \subset \cup \{B_n;\, n \in \boldsymbol{N}\} \cup \{C_n;\, n \in \boldsymbol{N}\}$. By the definition of $N_{\boldsymbol{C}}$ we have

$N_{\boldsymbol{C}}(A) \cap C_n = \emptyset$ for any $n \in \boldsymbol{N}$, and hence

$$N_{\boldsymbol{C}}(A) \subset \cup \{B_n;\ n \in \boldsymbol{N}\} .$$

Since $N_{\boldsymbol{B}}(A) = N_{\boldsymbol{C}}(A)$, $N_{\boldsymbol{B}}(A) = N_{\boldsymbol{C}}(A) \subset \cup \{B_n;\ n \in \boldsymbol{N}\}$. By the definition of $N_{\boldsymbol{B}}$ we have $N_{\boldsymbol{B}}(A) \cap B_n = \emptyset$ for any $n \in \boldsymbol{N}$, and hence

$$N_{\boldsymbol{C}}(A) = N_{\boldsymbol{B}}(A) = \emptyset \qquad (\text{a.e.}\omega) . \tag{92}$$

Therefore by (92) and the definitions of $N_{\boldsymbol{B}}(A)$ and $N_{\boldsymbol{C}}(A)$ for any $\varepsilon > 0$ there exist $B \in \boldsymbol{B}(\mu)$ and $C \in \boldsymbol{C}(\mu)$ such that

$$\mu(A-B) < \varepsilon/\|a\| \tag{93}$$

and

$$\mu(A-C) < \varepsilon/\|a\| . \tag{94}$$

By (93), (94) and the fact that $I_B(I_{B\cup C})^{\boldsymbol{B}} = I_B$ and $I_C(I_{B\cup C})^{\boldsymbol{C}} = I_C$ we have

$$\begin{aligned}
&\|I_A F(\boldsymbol{B}, \boldsymbol{C})(I_{B\cup C}a_1,\ I_{B\cup C}a_2) - I_A(a_1,\ a_2)\|_L \\
&= \|(1/2)(I_A(I_{B\cup C})^{\boldsymbol{B}}(a_1+a_2,\ a_1+a_2) + I_A(I_{B\cup C})^{\boldsymbol{C}}(a_1-a_2,\ -a_1+a_2)) \\
&\quad - I_A(a_1,\ a_2)\|_L \\
&\leqq \|(1/2)(I_A I_B(I_{A\cup C})^{\boldsymbol{B}}(a_1+a_2,\ a_1+a_2) + I_A I_C(I_{B\cup C})^{\boldsymbol{C}}(a_1-a_2,\ -a_1+a_2)) \\
&\quad - I_A(a_1,\ a)_2\|_L + 2\varepsilon \\
&= \|(1/2)(I_A I_B(a_1+a_2,\ a_1+a_2) + I_A I_C(a_1-a_2,\ -a_1+a_2)) - I_A(a_1,\ a_2)\|_L + 2\varepsilon \\
&\leqq \|(1/2)(I_A(a_1+a_2,\ a_1+a_2) + I_A(a_1-a_2,\ -a_1+a_2)) - I_A(a_1,\ a_2)\|_L + 4\varepsilon \\
&= 4\varepsilon ,
\end{aligned}$$

and hence $F(\boldsymbol{B}, \boldsymbol{C})$ is semi-constant-preserving, since ε is an arbitrary number.

Next we are going to show that $F(\boldsymbol{B}, \boldsymbol{C})$ is contractive. Since

$$|x| \vee |y| = (1/2)(|x+y| + |x-y|) \qquad \text{for any} \quad x, y \in \boldsymbol{R} ,$$

$$\begin{aligned}
\|F(\boldsymbol{B}, \boldsymbol{C})(f, g)\|_L &= (1/2) \int |f^{\boldsymbol{B}} + g^{\boldsymbol{B}} + f^{\boldsymbol{C}} - g^{\boldsymbol{C}}| \vee |f^{\boldsymbol{B}} + g^{\boldsymbol{B}} - f^{\boldsymbol{C}} + g^{\boldsymbol{C}}|\, d\mu \\
&= (1/2) \int (|f^{\boldsymbol{B}} + g^{\boldsymbol{B}}| + |f^{\boldsymbol{C}} - g^{\boldsymbol{C}}|) d\mu \\
&\leqq (1/2) \int (|f+g| + |f-g|) d\mu \\
&= \int |f| \vee |g|\, d\mu = \|(f, g)\|_L ,
\end{aligned}$$

which shows that $F(\boldsymbol{B}, \boldsymbol{C})$ is contractive. Q.E.D.

Obviously $L(\boldsymbol{B}, \boldsymbol{C}) = \{F(\boldsymbol{B}, \boldsymbol{C})(f, g);\ (f, g) \in L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)\}$ is a normed linear subspace of $L_1(\Omega,\ \boldsymbol{A},\ \mu,\ E)\}$.

Theorem 2. *Let Q be a linear operator of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into istelf. Then Q satisfies Assumption 1 if and only if there exist σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ of $\boldsymbol{A}$ with $N_{\boldsymbol{B}}=N_{\boldsymbol{C}}$ (As a consequence $\boldsymbol{A}_{\boldsymbol{B}}=\boldsymbol{A}_{\boldsymbol{C}}$.) and a contractive operator P of $L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E)$ into $L(\boldsymbol{B}, \boldsymbol{C})$ such that for any $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, R)$*

$$Q(f, g) = (1/2)(f^{\boldsymbol{B}}+g^{\boldsymbol{B}}+f^{\boldsymbol{C}}-g^{\boldsymbol{C}}, f^{\boldsymbol{B}}+g^{\boldsymbol{B}}-f^{\boldsymbol{C}}+g^{\boldsymbol{C}})+P(N_{\boldsymbol{B}}(f, g)) .$$

Proof. Suppose that Q satisfies Assumption 1. Then by Lemma 6.3 and the definitions of Q^* and N_Q we have

$$N_{\boldsymbol{B}} = N_{\boldsymbol{C}} = N_Q \tag{95}$$

and

$$\begin{aligned} Q(f, g) &= Q^*(f, g)+Q(N_Q(f, g)) \\ &= (1/2)(f^{\boldsymbol{B}}+g^{\boldsymbol{B}}+f^{\boldsymbol{C}}-g^{\boldsymbol{C}}, f^{\boldsymbol{B}}+g^{\boldsymbol{B}}-f^{\boldsymbol{C}}+g^{\boldsymbol{C}})+Q(N_{\boldsymbol{B}}(f, g)) . \end{aligned} \tag{96}$$

By (95) $\boldsymbol{A}_{\boldsymbol{B}}=\boldsymbol{A}_{\boldsymbol{C}}$, and hence

$$N_{\boldsymbol{B}}(f, g) \in L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E) . \tag{97}$$

By Lemma 1.11 and Lemma 6.3 for any $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, R)$

$$Q(f, g) = Q^* \circ Q(f, g) = F(\boldsymbol{B}, \boldsymbol{C}) \circ Q(f, g) \in L(\boldsymbol{B}, \boldsymbol{C}) . \tag{98}$$

Denote by P the restriction of Q to $L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E)$, then by (96), (97) and (98) P is a contractive operator of $L_1(\Omega, \boldsymbol{A}, \mu, E)$ into $L(\boldsymbol{B}, \boldsymbol{C})$ and

$$Q(f, g) = (1/2)(f^{\boldsymbol{B}}+g^{\boldsymbol{B}}+f^{\boldsymbol{C}}-g^{\boldsymbol{C}}, f^{\boldsymbol{B}}+g^{\boldsymbol{B}}-f^{\boldsymbol{C}}+g^{\boldsymbol{C}})+P(N_{\boldsymbol{B}}(f, g)) .$$

Conversely suppose that there exist σ-subrings $\boldsymbol{B}$ and $\boldsymbol{C}$ of $\boldsymbol{A}$ with $N_{\boldsymbol{B}}=N_{\boldsymbol{C}}$ and a contractive operator P of $L_1(\Omega, \boldsymbol{A}_{\boldsymbol{B}}, \mu, E)$ into $L(\boldsymbol{B}, \boldsymbol{C})$ such that

$$Q(f, g) = F(\boldsymbol{B}, \boldsymbol{C})(f, g)+P(N_{\boldsymbol{B}}(f, g)) .$$

Let $A \in s(Q)$, $a \in E$ and $\varepsilon > 0$. Since $F(\boldsymbol{B}, \boldsymbol{C}) \circ F(\boldsymbol{B}, \boldsymbol{C}) = F(\boldsymbol{B}, \boldsymbol{C})$,

$$F(\boldsymbol{B}, \boldsymbol{C})(f, g) = (f, g) \qquad \text{for any} \quad (f, g) \in L(\boldsymbol{B}, \boldsymbol{C}) . \tag{99}$$

Since $P(f, g) \in L(\boldsymbol{B}, \boldsymbol{C})$, by (99) we have

$$F(\boldsymbol{B}, \boldsymbol{C}) \circ P = P . \tag{100}$$

By the definition of $N_{\boldsymbol{B}}$ and $N_{\boldsymbol{C}}$ and the condition that $N_{\boldsymbol{B}}=N_{\boldsymbol{C}}$ we have

$$\begin{aligned} N_{\boldsymbol{B}} \circ (\ )^{\boldsymbol{C}} &= N_{\boldsymbol{C}} \circ (\ )^{\boldsymbol{C}} = 0 , \\ N_{\boldsymbol{C}} \circ (\ )^{\boldsymbol{B}} &= N_{\boldsymbol{B}} \circ (\ )^{\boldsymbol{B}} = 0 , \\ (\ )^{\boldsymbol{B}} \circ N_{\boldsymbol{C}} &= (\ )^{\boldsymbol{C}} \circ N_{\boldsymbol{C}} = 0 \end{aligned}$$

and

$$(\ )^{\boldsymbol{C}} \circ N_{\boldsymbol{B}} = (\ )^{\boldsymbol{C}} \circ N_{\boldsymbol{C}} = 0 ,$$

and hence by the definition and properties of $F(\boldsymbol{B}, \boldsymbol{C})$ and $\boldsymbol{P}$ we have

$$N_{\boldsymbol{B}} \circ F(\boldsymbol{B}, \boldsymbol{C}) = N_{\boldsymbol{C}} \circ F(\boldsymbol{B}, \boldsymbol{C}) = 0\,, \tag{101}$$

$$N_{\boldsymbol{B}} \circ P = N_{\boldsymbol{C}} \circ P = 0 \tag{102}$$

and

$$F(\boldsymbol{B}, \boldsymbol{C}) \circ N_{\boldsymbol{B}} = F(\boldsymbol{B}, \boldsymbol{C}) \circ N_{\boldsymbol{C}} = 0\,. \tag{103}$$

For convenience's sake we denote $F(\boldsymbol{B}, \boldsymbol{C})$ by F. By Lemma 6.4 and (100)

$$F \circ Q = F \circ (F + P \circ N_{\boldsymbol{B}}) = F \circ F + F \circ P \circ N_{\boldsymbol{B}} = F + \boldsymbol{P} \circ N_{\boldsymbol{B}} = Q\,. \tag{104}$$

By (101), (102) and (104)

$$Q \circ Q = F \circ Q + \boldsymbol{P} \circ N_{\boldsymbol{B}} \circ (F + \boldsymbol{P} \circ N_{\boldsymbol{B}}) = Q + P \circ N_{\boldsymbol{B}} \circ F + \boldsymbol{P} \circ N_{\boldsymbol{B}} \circ \boldsymbol{P} \circ N_{\boldsymbol{B}} = Q\,,$$

which shows that Q is a projection. By (103) and the fact that F and P are contractive we have

$$\begin{aligned} ||Q(f, g)||_L &= ||F(f, g) + P \circ N_{\boldsymbol{B}}(f, g)||_L \\ &= ||F((f, g) - N_{\boldsymbol{B}}(f, g)) + F \circ N_{\boldsymbol{B}}(f, g) + P \circ N_{\boldsymbol{B}}(f, g)||_L \\ &\leqq ||F((f, g) - N_{\boldsymbol{B}}(f, g))||_L + ||P \circ N_{\boldsymbol{B}}(f, g)||_L \\ &\leqq ||(f, g) - N_{\boldsymbol{B}}(f, g)||_L + ||N_{\boldsymbol{B}}(f, g)||_L = ||(f, g)||_L\,, \end{aligned}$$

which implies that Q is contractive. Next we are going to show that Q is semi-constant-preserving. Let $A \in a(Q)$, $a \in E$ and $\varepsilon > 0$. Then there exist $f, g \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that $A \subset s(Q(f, g))$. By (104)

$$A \subset s(Q(f, g)) = s(F \circ Q(f, g))\,,$$

and hence $A \in s(F)$. By Lemma 6.4 there exist $f', g' \in L_1(\Omega, \boldsymbol{A}, \mu, R)$ such that

$$||I_A F(\boldsymbol{B}, \boldsymbol{C})(f', g') - I_A a||_L < \varepsilon\,. \tag{105}$$

By Lemma 6.4 and (101)

$$Q \circ F = (F + \boldsymbol{P} \circ N_{\boldsymbol{B}}) \circ F = F \circ F + \boldsymbol{P} \circ N_{\boldsymbol{B}} \circ F = F + 0 = F\,,$$

and hence by (105)

$$||I_A Q(F(\boldsymbol{B}, C)(f', g')) - I_A a||_L < \varepsilon\,,$$

which shows that Q is semi-constant-preserving. Q.E.D.

Acknowledgement. The author would like to thank Professors Tsuyoshi Ando, Hirokichi Kudo and Teturo Kamae for their helpful suggestions. The author also would like to thank the referee for his helpful suggestions.

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